Constructive Algebraic Analysis & Algebraic Systems Theory

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Introduction

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  - ODEs, PDEs, difference equations, time-delay equations. . .
  - Determined, overdetermined and underdetermined systems.

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  - ODEs, PDEs, difference equations, time-delay equations...
  - Determined, overdetermined and underdetermined systems.
  - Determined: integration (closed-forms & numerical analysis).
  - Overdetermined: integrability & compatibility conditions (Cartan, Riquier, Janet, Spencer...)
  - Underdetermined: control theory, mathematical physics...

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  - We develop a dictionary between the properties of linear functional systems and the properties of modules.
  - We develop a constructive approach to homological algebra to check the module properties and thus the system properties.
  - We implement the algorithms in dedicated packages in computer algebra systems (Gröbner/Janet bases).

Stirred Tank model

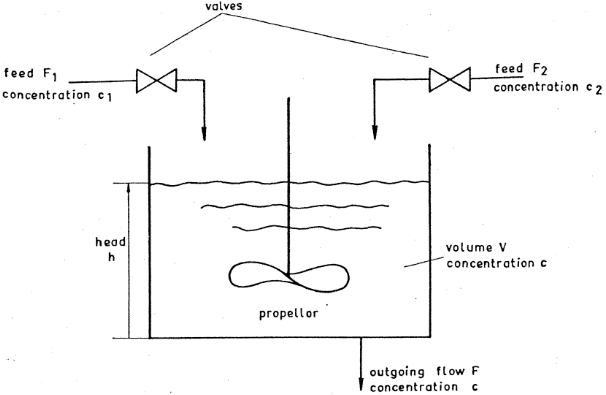


Fig. 1.3. A stirred tank.

H. Kwakernaak, R. Sivan,
Linear Optimal Control Systems, Wiley, 1972.

Stirred Tank model

- Non-linear model of a stirred tank (p. 7) (mass balance eqs):

$$\begin{cases} \dfrac{d\,V(t)}{dt} = -k\,\sqrt{\dfrac{V(t)}{S}} + F_1(t) + F_2(t), \\ \dfrac{d\,(c(t)\,V(t))}{dt} = -c(t)\,k\,\sqrt{\dfrac{V(t)}{S}} + c_1\,F_1(t) + c_2\,F_2(t). \end{cases}$$

- F_1, F_2: flow rates of two incoming flows feeding the tank,
- c_1, c_2: constant concentrations of dissolved materials,
- c: concentration in the tank,
- V: volume,
- k: experimental constant,
- S: constant cross-sectional area.

Stirred Tank model

- V_0: constant volume, c_0: constant concentration,

$$F_{10} = \frac{(c_2 - c_0)}{(c_2 - c_1)}\, k \sqrt{\frac{V_0}{S}}, \quad F_{20} = \frac{(c_0 - c_1)}{(c_2 - c_1)}\, k \sqrt{\frac{V_0}{S}}.$$

- Linearized model around the steady-state situation (p. 8-9):

$$V(t) = V_0 + x_1(t), \quad c(t) = c_0 + x_2(t),$$
$$F_1(t) = F_{10} + u_1(t), \quad F_2(t) = F_{20} + u_2(t).$$

$$\begin{cases} \dot{x}_1(t) = -\dfrac{1}{2\,\theta}\, x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\dfrac{1}{\theta}\, x_2(t) + \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t) + \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t), \end{cases}$$

$$y_1(t) = \frac{1}{2\,\theta}\, x_1(t), \quad y_2(t) = x_2(t).$$

where $F_0 = k\,\sqrt{(V_0/S)}$ and $\theta = V_0/F_0$ (holdup time of the tank).

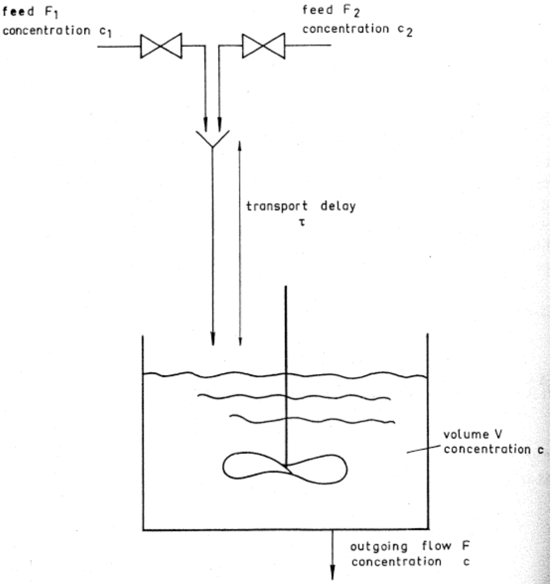
Stirred Tank model

• Discrete-time model: if the stirred tank is commanded by a process control computer, then the valve settings only change at discrete instants, remaining constant in between (p. 449) and

$$\begin{cases} x_1(n+1) = e^{-\frac{\Delta}{2\,\theta}}\, x_1(n) + 2\,\theta\,(1 - e^{-\frac{\Delta}{2\,\theta}})\,(u_1(n) + u_2(n)) \\ x_2(n+1) = e^{-\frac{\Delta}{\theta}}\, x_2(n) \\ \qquad\qquad + \dfrac{\theta\,(1 - e^{-\frac{\Delta}{\theta}})}{V_0}\,((c_1 - c_0)\,u_1(n) + (c_2 - c_0)\,u_2(n)) \end{cases}$$

where Δ is the constant length of time intervals.

Stirred Tank model with a transport delay



Stirred Tank model

• Differential time-delay model: if there is a transport delay of amplitude τ occuring in the pipe (p. 449-451), then

$$\begin{cases} \dot{x}_1(t) = -\dfrac{1}{2\,\theta}\, x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\dfrac{1}{\theta}\, x_2(t) + \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t-\tau) + \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t-\tau). \end{cases}$$

where $\tau > 0$ is the amplitude of the delay.

• Other models: PDEs, integro-differential systems...

Matrices of differential operators

- Newton: Fluxion calculus (1666) ("dot-age")

$$\left\{\begin{array}{l} \ddot{x}_1(t) + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \ddot{x}_2(t) + \alpha\, x_2(t) - \alpha\, u(t) = 0, \end{array}\right. \qquad \alpha = g/l.$$

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- Boole: Operational calculus (1859-60)

$$\left(\begin{array}{ccc} \frac{d^2}{dt^2} + \alpha & 0 & -\alpha \\ 0 & \frac{d^2}{dt^2} + \alpha & -\alpha \end{array}\right) \left(\begin{array}{c} x_1(t) \\ x_2(t) \\ u(t) \end{array}\right) = 0.$$

$\Rightarrow$ ring of differential operators $D = \mathbb{Q}(\alpha)\left[\frac{d}{dt}\right]$:

$$\sum_{i=0}^{n} a_i \left(\frac{d}{dt}\right)^i \in D, \quad a_i \in \mathbb{Q}(\alpha), \quad \left(\frac{d}{dt}\right)^i = \frac{d}{dt} \circ \ldots \circ \frac{d}{dt} = \frac{d^i}{dt^i}.$$

Functional operators

- Differential operator: $\left(\sum_{s=0}^{m} b_s(t)\,\partial^s\right)\left(\sum_{r=0}^{n} a_r(t)\,\partial^r\right)$

$$\partial\colon y \longmapsto \frac{dy}{dt}, \quad a = a(\cdot), \quad a\colon y \longmapsto a\,y,$$

$$(\partial\, a)(y) = \partial\,(a(y)) = \partial(a\,y) = \frac{d}{dt}(a\,y) = a\,\frac{dy}{dt} + \frac{da}{dt}\,y$$
$$= \left(a\,\partial + \frac{da}{dt}\right)(y).$$

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- Shift operator: $\partial \colon y_n \longmapsto \sigma(y_n) = y_{n+1}, \quad a \colon y_n \longmapsto a_n\,y_n,$

$$(\partial\, a)(y_n) = \partial(a(y_n)) = \partial(a_n\,y_n) = \sigma\,(a_n\,y_n) = a_{n+1}\,y_{n+1}$$
$$= (\sigma(a)\,\partial)(y_n).$$

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- Time-delay operator: $\partial\colon y \longmapsto \delta(y) = y(\cdot - \tau),$

$$(\partial\, a)(y) = \partial(a(y)) = \partial(a\,y) = \delta\,(a\,y) = a(\cdot - \tau)\,y(\cdot - \tau)$$
$$= (\delta(a)\,\partial)(y).$$

Functional operators

- Other functional operators: difference, divided difference, Eulerian, Frobenius, q-dilation, q-shift, q-difference... operators.
- Unique expansion: $P = \sum_{i=0}^{n} a_i \, \partial^i$, $a_i \in A$: ring of coeffs.
- Degree condition: $\partial \, a = \alpha \, \partial + \beta = \alpha(a) \, \partial + \beta(a)$, $a, \, b, \, c \in A$.

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$$\left\{ \begin{array}{l} \partial \, (a+b) = \alpha(a+b) \, \partial + \beta(a+b), \\ \partial \, a = \alpha(a) \, \partial + \beta(a), \\ \partial \, b = \alpha(b) \, \partial + \beta(b), \end{array} \right.$$

$$\partial(a+b) = \partial \, a + \partial \, b \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \alpha(a+b) = \alpha(a) + \alpha(b), \\ \beta(a+b) = \beta(a) + \beta(b). \end{array} \right.$$

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$$\begin{aligned}
\partial \, (a \, b) &= \alpha(a \, b) \, \partial + \beta(a \, b) \\
\partial \, (a \, b) &= (\partial \, a) \, b = (\alpha(a) \, \partial + \beta(a)) \, b \\
&= \alpha(a) \, (\alpha(b) \, \partial + \beta(b)) + \beta(a) \, b,
\end{aligned}$$

$$\Leftrightarrow \quad \begin{cases} \alpha(a \, b) = \alpha(a) \, \alpha(b), \\ \beta(a \, b) = \alpha(a) \, \beta(b) + \beta(a) \, b. \end{cases}$$

- α is an endomorphism of A and β is a α-derivation of A.

Skew polynomial rings (Ore, 1933)

- Definition: A skew polynomial ring $A[\partial;\alpha,\beta]$ is a non-commutative polynomial ring in ∂ with coefficients in A satisfying

$$\forall\, a \in A, \quad \boxed{\partial\, a = \alpha(a)\,\partial + \beta(a)}$$

where $\alpha : A \longrightarrow A$ and $\beta : A \longrightarrow A$ are such that:

$$\begin{cases} \alpha(1) = 1, \\ \alpha(a+b) = \alpha(a) + \alpha(b), \\ \alpha(a\,b) = \alpha(a)\,\alpha(b), \end{cases} \qquad \begin{cases} \beta(a+b) = \beta(a) + \beta(b), \\ \beta(a\,b) = \alpha(a)\,\beta(b) + \beta(a)\,b. \end{cases}$$

- $P \in A[\partial;\alpha,\beta]$ has a unique form $P = \sum_{i=0}^{n} a_i\,\partial^i,\ a_i \in A$.
  - Ring of differential operators: $A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$.
  - Ring of shift operators: $A[\partial;\sigma,0]$, $A[\partial;\delta,0]$.
  - Ring of difference operators: $A[\partial;\tau,\tau-\mathrm{id}]$, $\tau a(x) = a(x+1)$.

Ore algebras (Chyzak-Salvy, 1996)

- We can iterate skew polynomial rings to get Ore extensions:

$$\boxed{A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_n; \alpha_n, \beta_n]}$$

- Definition: An Ore extension $A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_n; \alpha_n, \beta_n]$ is called an Ore algebra if the ∂_i's commute, i.e., if we have

$$1 \leq j < i \leq m, \quad \alpha_i(\partial_j) = \partial_j, \quad \beta_i(\partial_j) = 0,$$

and the $\alpha_{i\,|_A}$'s and $\beta_{j\,|_A}$'s commute for $i \neq j$.

- Ring of differential operators: $A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$.
- Ring of differential delay operators: $A\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right] [\partial_2; \delta, 0]$.
- Ring of shift operators: $A[\partial_1; \sigma_1, 0] \ldots [\partial_n; \sigma_n, 0]$.

Matrix of functional operators

- The stirred tank model (Kwakernaak-Sivan, 72):

$$\begin{cases} \dot{x}_1(t) + \dfrac{1}{2\,\theta}\, x_1(t) - u_1(t) - u_2(t) = 0, \\ \dot{x}_2(t) + \dfrac{1}{\theta}\, x_2(t) - \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t-\tau) - \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t-\tau) = 0. \end{cases} \qquad (\star)$$

- We introduce the commutative Ore algebra:

$$D = \mathbb{Q}(\theta, c_0, c_1, c_2, V_0)\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \delta, 0].$$

- The linear functional system $(\star)$ can be rewritten as:

$$\begin{pmatrix} \partial_1 + \dfrac{1}{2\,\theta} & 0 & -1 & -1 \\ 0 & \partial_1 + \dfrac{1}{\theta} & -\left(\dfrac{c_1 - c_0}{V_0}\right)\partial_2 & -\left(\dfrac{c_2 - c_0}{V_0}\right)\partial_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u_1(t) \\ u_2(t) \end{pmatrix} = 0.$$

Matrix of functional operators

- Linearization of the Navier-Stokes $\sim$ a parabolic Poiseuille profile

$$\left\{\begin{array}{l} \partial_t\, u_1 + 4\, y\, (1-y)\, \partial_x\, u_1 - 4\, (2\, y - 1)\, u_2 - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2)\, u_1 + \partial_x\, p = 0, \\ \partial_t\, u_2 + 4\, y\, (1-y)\, \partial_x\, u_2 - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2)\, u_2 + \partial_y\, p = 0, \qquad (*) \\ \partial_x\, u_1 + \partial_y\, u_2 = 0. \end{array}\right.$$

(e.g., Vazquez-Krstic, IEEE 07)

- Let us introduce the so-called Weyl algebra $A_3(\mathbb{Q}(Re))$

$$D = \mathbb{Q}(Re)[t, x, y]\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right].$$

$$(\partial_x\, y = y\, \partial_x,\ \partial_x\, x = x\, \partial_x + 1,\ \partial_x\, \partial_y = \partial_y\, \partial_x \ldots):$$

- The system $(*)$ is defined by the matrix of PD operators:

$$\left(\begin{array}{ccc} \partial_t + 4\, y\, (1-y)\, \partial_x - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2) & -4\, (2\, y - 1) & \partial_x \\ 0 & \partial_t + 4\, y\, (1-y)\, \partial_x - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{array}\right).$$

Noncommutative Gröbner bases

- Let $D = A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_m; \alpha_m, \beta_m]$ be an Ore algebra.

- Theorem: (Kredel, 93) Let $A = k[x_1, \ldots, x_n]$ a commutative polynomial ring ($k = \mathbb{Q}$, $\mathbb{F}_p$) and D an Ore algebra satisfying

$$\alpha_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \beta_i(x_j) = c_{ij},$$

for certain $0 \neq a_{ij} \in k$, $b_{ij} \in k$, $c_{ij} \in A$ and $\deg(c_{ij}) \leq 1$. Then, a non-commutative version of Buchberger's algorithm terminates for any term order and its result is a Gröbner basis.

- Implementation in the Maple package Ore_algebra (Chyzak)

  (Singular:Plural, Macaulay 2, NCAlgebra, JanetOre...).

- Gröbner bases can be used to effectively compute over $D^{1\times p}/F$.

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D-module $\mathcal{F}$:

$$\forall \, d_1, \, d_2 \in D, \quad \forall \, f_1, \, f_2 \in \mathcal{F}, \quad d_1 \, f_1 + d_2 \, f_2 \in \mathcal{F}.$$

- Let us consider the system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R \, \eta = 0\}$.
- Let us consider the left D-homomorphism (left D-linear map):

$$\begin{array}{rcl} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} \\ \lambda = (\lambda_1 \, \ldots \, \lambda_q) & \longmapsto & \lambda \, R. \end{array}$$

- As in number theory or algebraic geometry, we associate with the linear system $\ker_{\mathcal{F}}(R.)$ the finitely presented left D-module:

$$M = \mathrm{coker}_D(.R) = D^{1 \times p}/(D^{1 \times q} \, R).$$

Examples: algebraic geometry & number theory

- Cauchy's definition of complex numbers: $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$

$$D = \mathbb{R}[x], \; R = x^2+1, \; M = D/(D\,R) = D/(x^2+1) = \mathbb{C}.$$

- Affine coordinate rings: $A = k[x,y]/(x^2+y^2-1, x-y)$

$$D = k[x,y], \; R = \begin{pmatrix} x^2+y^2-1 \\ x-y \end{pmatrix},$$

$$D^{1\times 2}\,R = (x^2+y^2-1, x-y), \; M = D/(D^{1\times 2}\,R) = A.$$

Example: linear system theory

- Let us consider $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]$ and the curl operator defined by:

$$R = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \in D^{3\times 3}.$$

- Let us consider the D-morphism (D-linear map)

$$\begin{array}{ccc} D^{1\times 3} & \xrightarrow{.R} & D^{1\times 3} \\ \lambda & \longmapsto & (\lambda_2\,\partial_3 - \lambda_3\,\partial_2 \quad -\lambda_1\,\partial_3 + \lambda_3\,\partial_1 \quad \lambda_2\,\partial_2 - \lambda_2\,\partial_1), \end{array}$$

and the D-module $M = \mathrm{coker}_D(.R) = D^{1\times 3}/(D^{1\times 3}\,R)$.

- If $\mathcal{F} = C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$... is a D-module, then:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^3 \mid \vec{\nabla} \wedge \eta = R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}).$$

Linear systems of equations

- $M = D^{1\times p}/(D^{1\times q}\, R)$ can be defined by generators and relations:
- Let $\{e_k\}_{k=1,\dots,p}$ be the standard basis of $D^{1\times p}$:

$$e_k = (0 \dots 1 \dots 0).$$

- Let $\pi : D^{1\times p} \longrightarrow M$ be the left D-morphism sending μ to $\pi(\mu)$.

$$\forall\, m \in M,\ \exists\, \mu = (\mu_1 \ \dots \ \mu_p) \in D^{1\times p} : \ m = \pi(\mu) = \sum_{k=1}^{p} \mu_k\, \pi(e_k),$$

$\Rightarrow \{y_k = \pi(e_k)\}_{k=1,\dots,p}$ is a family of generators of M.

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$\Rightarrow \{y_k = \pi(e_k)\}_{k=1,\dots,p}$ is a family of generators of M.

$$\pi((R_{l1}\ \dots R\ _{lp})) = \pi\left(\sum_{k=1}^{p} R_{lk}\, e_k\right) = \sum_{k=1}^{p} R_{lk}\, y_k = 0,\ l = 1, \dots, q,$$

$\Rightarrow y = (y_1\ \dots\ y_p)^T$ satisfies the relation $R\,y = 0$.

Duality modules — systems

- Let $\mathcal{F}$ be a left D-module and $\hom_D(M, \mathcal{F})$ the abelian group:

$$\hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F} \mid f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2)\}.$$

- Applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

we obtain the following exact sequence of abelian groups:

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{\iota\circ\pi^\star} \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

- Theorem (Malgrange):

$$\boxed{\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}}$$

- Remark: $\hom_D(M, \mathcal{F})$ intrinsically characterizes $\ker_{\mathcal{F}}(R.)$ as it does not depend on the embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$.

Linear functional systems

- Let $\mathcal{F}$ be a left D-module and $M = D^{1\times p}/(D^{1\times q}\,R)$.
- Let $f : M \longrightarrow \mathcal{F}$ be a left D-morphism. Then, we have:

$$\begin{array}{rcl} f : M & \longrightarrow & \mathcal{F} \\ y_k = \pi(e_k) & \longmapsto & \eta_k, \quad k = 1, \ldots, p, \end{array} \qquad f(0) = 0.$$

Linear functional systems

- Let $\mathcal{F}$ be a left D-module and $M = D^{1\times p}/(D^{1\times q}\, R)$.
- Let $f : M \longrightarrow \mathcal{F}$ be a left D-morphism. Then, we have:

$$\begin{array}{rcl} f : M & \longrightarrow & \mathcal{F} \\ y_k = \pi(e_k) & \longmapsto & \eta_k, \quad k = 1, \ldots, p, \end{array} \qquad f(0) = 0.$$

$$\pi((R_{l1} \ \ldots \ R_{lp})) = \pi\left(\sum_{k=1}^{p} R_{lk}\, e_k\right) = \sum_{k=1}^{p} R_{lk}\, y_k = 0.$$

$$f\left(\sum_{k=1}^{p} R_{lk}\, y_k\right) = \sum_{k=1}^{p} R_{lk}\, f(y_k) = \sum_{k=1}^{p} R_{lk}\, \eta_k = 0, \quad l = 1, \ldots, q.$$

$$\Rightarrow \quad \eta = (\eta_1 \ \ldots \ \eta_p)^T \in \mathcal{F}^p : \ R\, \eta = 0.$$

Monge problem (1784)

- Let D be a ring of functional operators.
- Let $\mathcal{F}$ be a left D-module:

$$\forall\, P_1,\, P_2 \in D,\ \forall\, y_1,\, y_2 \in \mathcal{F}:\ \ P_1\, y_1 + P_2\, y_2 \in \mathcal{F}.$$

Let us consider $R \in D^{q \times p}$ and the linear functional system:

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \,|\, R\,\eta = 0\}.$$

- Question: When does $Q \in D^{p \times m}$ exist such that:

$$\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.) \triangleq Q\,\mathcal{F}^m?$$

$\Rightarrow$ Q is called a parametrization of $\ker_{\mathcal{F}}(R.)$.

Module theory

- Definition: 1. M is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^r$.

2. M is stably free if $\exists\, r,\, s \in \mathbb{Z}_+$ such that $M \oplus D^s \cong D^r$.

3. M is projective if $\exists\, r \in \mathbb{Z}_+$ and a D-module P such that:

$$M \oplus P \cong D^r.$$

4. M is reflexive if $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall\, m \in M, \quad f \in \hom_D(M, D).$$

5. M is torsion-free if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

6. M is torsion if $t(M) = M$.

Classification of modules

- Theorem: 1. We have the following implications:

$$\text{free} \Rightarrow \text{stably free} \Rightarrow \text{projective} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free}.$$

2. If D is a principal domain (e.g., $K\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{free}.$$

3. If D is a hereditary ring (e.g., $\mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{projective}.$$

4. If $D = k[x_1, \ldots, x_n]$ and k a field, then:

$$\text{projective} = \text{free} \quad \text{(Quillen-Suslin theorem)}.$$

4. If $D = A_n(k)$ or $B_n(k)$, k is a field of characteristic 0, then

$$\text{projective} = \text{free} \quad \text{(Stafford theorem)},$$

for modules of rank at least 2.

Free resolutions

• Definition: A sequence of D-morphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called a complex if $g \circ f = 0$, i.e., $\mathrm{im}\, f \subseteq \ker g$.

The defect of exactness at M is $H(M) = \ker g / \mathrm{im}\, f$.

The complex is exact at M if $\mathrm{im}\, f = \ker g$.

• Definition: A free resolution of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1 \times l_2} \xrightarrow{.R_2} D^{1 \times l_1} \xrightarrow{.R_1} D^{1 \times l_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_i \in D^{l_i \times l_{i-1}}$ and:

$$\begin{array}{ccc} D^{1 \times l_i} & \xrightarrow{.R_i} & D^{1 \times l_{i-1}} \\ (P_1 \ \ldots \ P_{l_i}) & \longmapsto & (P_1 \ \ldots \ P_{l_i})\, R_i. \end{array}$$

• Algorithm: Find a basis of the compatibility conditions of the inhomogeneous system $R_i\, y = u$ by eliminating y (e.g., Gb):

$$\forall\ P \in \ker_D(.R_i), \quad P\,(R_i\, y) = P\, u \ \Rightarrow\ P\, u = 0.$$

Extension functor $\mathrm{ext}_D^i(\,\cdot\,,\mathcal{F})$

- We define the reduced free resolution of M by:

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \longrightarrow 0 \quad (\star).$$

- Let $\mathcal{F}$ be a left D-modules. Applying the functor $\hom_D(\cdot,\mathcal{F})$ to $(\star)$, we obtain the following complex:

$$\ldots \xleftarrow{R_3.} \mathcal{F}^{l_2} \xleftarrow{R_2.} \mathcal{F}^{l_1} \xleftarrow{R_1.} \mathcal{F}^{l_0} \longleftarrow 0 \quad (\star\star)$$

$$\text{where} \quad \begin{array}{ccc} \mathcal{F}^{l_i} & \xleftarrow{R_i.} & \mathcal{F}^{l_{i-1}} \\ R_i\,\eta & \longleftarrow\!\shortmid & \eta. \end{array}$$

- We denote the defects of exactness of $(\star\star)$ by:

$$\begin{cases} \mathrm{ext}_D^0(M,\mathcal{F}) = \hom_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \mathrm{ext}_D^i(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)\,/\,\mathrm{im}_{\mathcal{F}}(R_i.), \quad i \geq 1. \end{cases}$$

- Theorem: The abelian group $\mathrm{ext}_D^i(M,\mathcal{F})$ only depends on M and $\mathcal{F}$ but not on the resolution $(\star)$.

Auslander transpose

- Definition: Let $R \in D^{q \times p}$. If $M = D^{1 \times p}/(D^{1 \times q} R)$ denotes the left D-module finitely presented by R, then its Auslander transpose is the right D-module defined by $N = D^q/(R\, D^p)$.

- Proposition: The Auslander transpose $N = D^q/(R\, D^p)$ only depends on M up to a projective equivalence.

Hence, if we have $M = D^{1 \times p'}/(D^{1 \times q'} R')$ and $N' = D^{q'}/(R'\, D^{p'})$ denotes the corresponding Auslander transpose, then there exist two projective right D-modules P and P' such that:

$$N \oplus P \cong N' \oplus P'.$$

- Proposition: If P is a projective module, then we have:

$$\mathrm{ext}_D^i(P, \mathcal{F}) = 0, \quad \forall\, i \geq 1.$$

- Corollary: The $\mathrm{ext}_D^i(N, \mathcal{F})$'s, $i \geq 1$, only depend on M and $\mathcal{F}$.

Module M	**Homological algebra**	$\mathcal{F}$ **injective cogenerator**
with torsion	$t(M) \cong \mathrm{ext}^1_D(N, D)$	$\emptyset$
torsion-free	$\mathrm{ext}^1_D(N, D) = 0$	$\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^{l_1}$
reflexive	$\mathrm{ext}^i_D(N, D) = 0$ $i = 1, 2$	$\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{l_2}$
projective $=$ stably free	$\mathrm{ext}^i_D(N, D) = 0$ $1 \leq i \leq n = \mathrm{gld}(D)$	$\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{l_2}$ $\ldots$ $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n\, \mathcal{F}^{l_n}$
free	$\exists\ Q \in D^{p \times m},\ T \in D^{m \times p}$, $\ker_D(.Q) = D^{1 \times q}\, R,\ T\, Q = I_m$	$\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^m$, $\exists\ T \in D^{m \times p}:\ \ T\, Q = I_m$

Parametrizability problem

$$\begin{array}{ccccccccccc}
 & & & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
0 & \longleftarrow & N & \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{\pi^\star} & \hom_D(M, D) & \longleftarrow & 0 \\
0 & \longleftarrow & N & \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{Q.} & D^m & & \\
 & & & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & &
\end{array}$$

- Let $R' \in D^{q' \times p}$: $\ker_D(.Q) = D^{1\times q'}\, R'$:

$$t(M) \cong \mathrm{ext}^1_D(N, D) \cong \ker_D(.Q)/(D^{1\times q}\, R) = (D^{1\times q'}\, R')/(D^{1\times q}\, R),$$

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0, \quad M = D^{1\times p}/(D^{1\times q}\, R),$$

$$\Rightarrow \; M/t(M) \cong D^{1\times p}/(D^{1\times q'}\, R').$$

$t(M)$ and $M/t(M)$

$$\mathrm{ext}^1_D(N, D) \cong t(M) = (D^{1\times q'}\, R')/(D^{1\times q}\, R),$$
$$M/t(M) \cong D^{1\times p}/(D^{1\times q'}\, R').$$

- $D^{1\times q}\, R \subseteq D^{1\times q'}\, R' \;\Rightarrow\; \exists\, R'' \in D^{q\times q'}$:

$$R = R''\, R'.$$

- Since $(D^{1\times q'}\, R')/(D^{1\times q}\, R)$ is a torsion left D-module, then:

$$\exists\, P_i \in D : \; P_i\, \pi(R'_{i\bullet}) = 0 \;\Leftrightarrow\; \pi(P_i\, R'_{i\bullet}) = 0$$

$$\Rightarrow\; \exists\, \mu_i \in D^{1\times q'} : \; P_i\, R'_{i\bullet} = \mu_i\, R \;\Leftrightarrow\; (P_i \quad -\mu_i) \left(\begin{array}{c} R'_{i\bullet} \\ R \end{array}\right) = 0.$$

$\Rightarrow$ Find the compatibility conditions of

$$\left\{\begin{array}{l} R'_{i\bullet}\, \eta = \tau_i, \\ R\, \eta = 0. \end{array}\right. \;\Rightarrow\; P_{ik}\, \tau_i = 0, \; k = 1, \ldots, m_i.$$

Presentation of $t(M)$

- If $R_2' \in D^{r' \times q'}$ is such that $\ker_D(.R') = D^{1 \times r'} R_2'$, then:

$$t(M) \cong (D^{1\times q'} R')/(D^{1\times q} R) \qquad (R = R'' R')$$
$$\cong (D^{1\times q'} R')/\left(D^{1\times (q+r')} \begin{pmatrix} R'' R' \\ R_2' R' \end{pmatrix}\right)$$
$$\cong D^{1\times q'}/\left(D^{1\times (q+r')} \begin{pmatrix} R'' \\ R_2' \end{pmatrix}\right).$$

- $t(M)$ admits the following finite presentation

$$D^{1\times (q+r')} \xrightarrow{.\begin{pmatrix} R'' \\ R_2' \end{pmatrix}} D^{1\times q'} \xrightarrow{\sigma} t(M) \longrightarrow 0,$$

i.e., the torsion elements satisfy the following equations:

$$\begin{cases} R''\,\tau = 0, \\ R_2'\,\tau = 0. \end{cases} \qquad (\tau = R'\,\eta \;\&\; R\,\eta = 0).$$

Parametrization of torsion free modules

- We have the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & & \\ \| & & \| & & \uparrow \phi & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow & 0, \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

where $\phi(\pi'(\lambda)) = \lambda\, Q$, for all $\lambda \in D^{1\times p}$

$$\Rightarrow \; M/t(M) \cong \phi(M/t(M)) = D^{1\times p}\, Q \subseteq D^{1\times m}.$$

Hence, every element m' of $\phi(M/t(M))$ has the form:

$$m' = \sum_{i=1}^{p} \nu_i\, Q_i,$$

for some $\nu_i \in D^{1\times p}$ and $i = 1, \ldots, p$. Q is a parametrization of M.

Free modules

$$\begin{array}{ccccccc} D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & & \\ \| & & \| & & \uparrow \phi & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow & 0, \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

where $\phi(\pi'(\lambda)) = \lambda\, Q$, for all $\lambda \in D^{1\times p}$

$$\Rightarrow \; M/t(M) \cong \phi(M/t(M)) = D^{1\times p}\, Q \subseteq D^{1\times m}.$$

- $D^{1\times p} \xrightarrow{.Q} D^{1\times m} \longrightarrow 0 \;\Leftrightarrow\; \exists\; T \in D^{m\times p}: \; T\, Q = I_m$, then:

$$M/t(M) \cong \phi(M/t(M)) = D^{1\times p}\, Q = D^{1\times m}.$$

$\Rightarrow M/t(M)$ is a free left D-module of rank m.

Free Modules

$$\begin{array}{ccccccc}
 & \overset{.S}{\longleftarrow} & & \overset{.T}{\longleftarrow} & & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & \longrightarrow & 0 \\
\| & & \| & & \uparrow \phi \downarrow \phi^{-1} & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{\pi} & M/t(M) & \longrightarrow & 0,
\end{array}$$

The isomorphisms ϕ and ϕ^{-1} are defined by:

$$\begin{array}{rcl}
\phi : M/t(M) & \longrightarrow & D^{1\times (p-q)} \\
\pi'(\lambda) & \longmapsto & \lambda\, Q,
\end{array}
\qquad
\begin{array}{rcl}
\phi^{-1} : D^{1\times m} & \longrightarrow & M/t(M) \\
\mu & \longmapsto & \pi'(\mu\, T).
\end{array}$$

- If we denote by $\{h_k\}_{k=1,\dots,m}$ the standard basis of $D^{1\times m}$, then $\{\phi^{-1}(h_k) = \pi(h_k\, T) = \pi(T_{k\bullet})\}_{k=1,\dots,m}$ is a basis of $M/t(M)$

$\Rightarrow$ the residue classes of the rows of T in $M/t(M)$ is a basis..

Minimal parametrizations

- We generally have $m \geq \mathrm{rank}_D(M)$ (e.g., $\ker_D(Q.) \neq 0$).
- Theorem: Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a torsion-free left D-module. Then, there exists a parametrization $P \in D^{p\times l}$ of M such that $l = \mathrm{rank}_D(M)$, i.e., $L = D^{1\times l}/(D^{1\times p}\, P)$ is torsion.
- P is called a minimal parametrization of M.
- Algorithm: P can be obtained by selecting $\mathrm{rank}_D(M)$ right D-linearly independent columns of a parametrization Q of M.
- Heuristic method for computing basis of a free left D-modules:
  1. Compute $Q \in D^{p\times m}$ such that $\ker_D(R.) = Q\, D^m$.
  2. Define $P \in D^{p\times \mathrm{rank}_D(M)}$ by selecting $\mathrm{rank}_D(M)$ right D-linearly independent columns of Q.
  3. Check whether or not P admits a left-inverse over D.

Example

We consider the $D = \mathbb{Q}[x_1, x_2]$-module $M = D^{1\times 3}/(D\,R)$, where:

$$R = (x_1\,x_2^2 + 1 \quad 3\,x_2/2 + x_1 - 1 \quad 2\,x_1\,x_2), \qquad \mathrm{rank}_D(M) = 2.$$

- Checking $\mathrm{ext}^1_D(D/(D^{1\times 3}\,R^T), D) = 0$, we obtain that

$$Q = \left(\begin{array}{ccc} -4\,x_1 - 6\,x_2 + 4 & 6\,x_2^2 - 4\,x_2 & 0 \\ 4 & -4\,x_2 & 4\,x_1\,x_2 \\ 2\,x_1\,x_2 + 3\,x_2^2 - 2\,x_2 & -3\,x_2^3 + 2\,x_2^2 + 2 & -2\,x_1 - 3\,x_2 + 2 \end{array}\right),$$

is a parametrization of M, i.e., $M \cong D^{1\times 3}\,Q \subseteq D^{1\times 3}$.

- Selecting the first two columns of Q, we obtain that

$$P = \left(\begin{array}{cc} -4\,x_1 - 6\,x_2 + 4 & 6\,x_2^2 - 4\,x_2 \\ 4 & -4\,x_2 \\ 2\,x_1\,x_2 + 3\,x_2^2 - 2\,x_2 & -3\,x_2^3 + 2\,x_2^2 + 2 \end{array}\right),$$

is a minimal parametrization of M, i.e.:

$$M \cong D^{1\times 3}\,Q \subseteq D^{1\times 2} = D^{1\times \mathrm{rank}_D(M)}.$$

Example

- The parametrization P of M admits a left-inverse defined by:

$$T = \frac{1}{4} \left(\begin{array}{ccc} x_2^2 & 1 & 2\,x_2 \\ x_2 & 0 & 2 \end{array} \right).$$

- The set of generators $\{y_j = \pi(f_j)\}_{j=1,2,3}$ of M satisfies

$$(x_1\,x_2^2 + 1)\,y_1 + (3\,x_2/2 + x_1 - 1)\,y_2 + 2\,x_1\,x_2\,y_3 = 0,$$

and a basis $\{z_1, z_2\}$ of M is defined by:

$$\left\{ \begin{array}{l} z_1 = \frac{1}{4}\,(x_2^2\,y_1 + y_2 + 2\,x_2\,y_3), \\ z_2 = \frac{1}{4}\,(x_2\,y_1 + 2\,y_3). \end{array} \right.$$

- The generators $\{y_1, y_2, y_3\}$ can be written in the basis $\{z_1, z_2\}$:

$$\left\{ \begin{array}{l} y_1 = (-4\,x_1 - 6\,x_2 + 4)\,z_1 + (6\,x_2^2 - 4\,x_2)\,z_2, \\ y_2 = 4\,z_1 - 4\,x_2\,z_2, \\ y_3 = (2\,x_1\,x_2 + 3\,x_2^2 - 2\,x_2)\,z_1 + (-3\,x_2^3 + 2\,x_2^2 + 2)\,z_2. \end{array} \right.$$

Injective cogenerator modules

- Definition: A left D-module $\mathcal{F}$ is injective if

$$\forall\, q \geq 1, \quad \forall\, R \in D^q, \quad \forall\, \zeta \in \ker_{\mathcal{F}}(S.),$$

where $\ker_D(.R) = D^{1\times r}\, S$, there exists $\eta \in \mathcal{F}$ satisfying $R\,\eta = \zeta$.

- Definition: If $\mathcal{F}$ is a injective left D-module, then we have:

$$\mathrm{ext}^i_D(M, \mathcal{F}) = 0, \quad \forall\, i \geq 1.$$

- Definition: A left D-module $\mathcal{F}$ is cogenerator if:

$$\mathrm{hom}_D(M, \mathcal{F}) = 0 \;\Rightarrow\; M = 0.$$

- Proposition: Injective cogenerator left D-module always exists.

- Example: If Ω is an open convex subset of $\mathbb{R}^n$, then the $\mathbb{C}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$-modules $C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{O}(\Omega)$ and $\mathcal{B}(\Omega)$ are injective cogenerators.

Duality and parametrizations of systems

$$\begin{array}{ccccccccc}
 & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
0 \longleftarrow & N \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{\pi^\star} & \hom_D(M,D) & \longleftarrow & 0 \\
0 \longleftarrow & N \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{Q.} & D^{m} & & \\
 & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & & (\star)
\end{array}$$

- $t(M) = 0$ iff $(\star)$ is an exact sequence and $Q\,\mathcal{F}^m \subseteq \ker_{\mathcal{F}}(R.)$.
- If $t(M) = 0$ and $\mathcal{F}$ is injective, then the exact sequence holds:

$$(\star\star) \quad \mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m, \quad \text{i.e.,} \quad \ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m.$$

- If $t(M) = 0$ and $\mathcal{F}$ is injective cogenerator, then:

$$(\star) \text{ is exact iff so is } (\star\star).$$

Involutions and adjoints

- Definition: A linear map $\theta : D \longrightarrow D$ is an involution of D if:

$$\forall\, P,\, Q \in D: \;\; \theta(P\,Q) = \theta(Q)\,\theta(P), \quad \theta^2 = \mathrm{id}.$$

- Example: 1. If D is a commutative ring, then $\theta = \mathrm{id}$.

2. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$ is:

$$\forall\, a \in A, \quad \theta(a(x)) = a(x), \quad \theta(\partial_i) = -\partial_i, \quad i = 1, \ldots, n.$$

3. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right]\,[\partial_2; \delta, 0]$ is defined by:

$$\forall\, a \in A, \quad \theta(a(t)) = a(-t), \quad \theta(\partial_i) = \partial_i, \quad i = 1,\, 2.$$

- The adjoint of $R \in D^{q \times p}$ is defined by $\theta(R) = (\theta(R_{ij}))^T \in D^{p \times q}$.
- $\widetilde{N} = D^{1 \times q}/(D^{1 \times p}\,\theta(R))$ is called the adjoint of M.

R has full row rank & PD case

Module M	**Homological algebra**	$\dim_{\mathbf{D}}(\widetilde{\mathbf{N}})$
with torsion	$t(M) \cong \mathrm{ext}^1_D(\widetilde{N}, D)_\theta$	$\dim(D) - 1$
torsion-free	$\mathrm{ext}^1_D(\widetilde{N}, D)_\theta = 0$	$\dim(D) - 2$
reflexive	$\mathrm{ext}^i_D(\widetilde{N}, D)_\theta = 0, \; i = 1, 2$	$\dim(D) - 3$
projective	$\mathrm{ext}^i_D(\widetilde{N}, D)_\theta = 0, \; i = 1, \ldots, \mathrm{gld}(D)$	-1
free	Quillen-Suslin & Stafford	$\emptyset$

Extension functor $\mathrm{ext}^1_D(N, D)$

$$4.\quad \theta(P)\, z = y \quad \Longrightarrow \quad R\, y = 0 \quad 1.$$

$$\Uparrow \qquad\qquad \Downarrow$$

$$\text{involution } \theta \qquad\qquad \text{involution } \theta$$

$$\Uparrow \qquad\qquad \Downarrow$$

$$3.\quad 0 = P\, \mu \quad \stackrel{\text{Gb}}{\Longleftrightarrow} \quad \theta(R)\, \lambda = \mu \quad 2.$$

$$P \circ \theta(R) = 0 \quad \Longrightarrow \quad \theta(P \circ \theta(R)) = \theta^2(R) \circ \theta(P) = R \circ \theta(P) = 0.$$

5. $\theta(P)\, z = y \quad \stackrel{\text{Gb}}{\Longleftrightarrow} \quad R'\, y = 0, \quad R' \in D^{q' \times p}.$

$$\boxed{\mathrm{ext}^1_D(N, D) \cong (D^{1 \times q'}\, R')/(D^{1 \times q}\, R)}$$

6. Using Gb, we can test whether or not $\mathrm{ext}^1_D(N, D) = 0$.

$$\mathrm{ext}^1_D(N, D) = 0 \ \Rightarrow \ R\, y = 0 \ \Leftrightarrow \ y = Q\, z, \quad Q = \theta(P).$$

Wind tunnel model (Manitius, IEEE TAC 84)

1. The w.t.m. is defined by the underdetermined system:

$$\begin{pmatrix} \partial_1 + a & -k\,a\,\partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\left\{ \begin{array}{l} (\partial_1 + a)\ \lambda_1 = \mu_1, \\ -k\,a\,\partial_2\,\lambda_1 + \partial_1\,\lambda_2 + \omega^2\,\lambda_3 = \mu_2, \\ -\lambda_2 + (\partial_1 + 2\,\zeta\,\omega)\ \lambda_3 = \mu_3, \\ -\omega^2\,\lambda_3 = \mu_4. \end{array} \right. \qquad (2)$$

(2) is over-determined $\stackrel{\text{Gb}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Wind tunnel model (Manitius, IEEE TAC 84)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$\begin{pmatrix} \omega^2\, k\, a\, \partial_2 & \omega^2\, (\partial_1 - a) & \omega^2\, (\partial_1^2 + a\,\partial_1) \\ (\partial_1^3 + 2\,\zeta\,\omega\,\partial_1^2 + a\,\partial_1^2 + \omega^2\,\partial_1 + 2\,a\,\zeta\,\omega\,\partial_1 + a\,\omega^2) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_4 \end{pmatrix} = 0.$$

4. We consider the overdetermined system $P^T\,z = y$.

$$\begin{cases} \omega^2\,k\,a\,\partial_2\,z = x_1, \\ \omega^2\,(\partial_1 - a)\,z = x_2, \\ \omega^2\,(\partial_1^2 + a\,\partial_1)\,z = x_3, \\ (\partial^3 + (2\,\zeta\,\omega + a)\,\partial_1^2 + (\omega^2 + 2\,a\,\omega\,\zeta)\,\partial_1 + a\,\omega)\,z = u. \end{cases} \qquad (4)$$

5. The compatibility conditions of $P^T\,z = y$ are exactly generated by $R\,y = 0$ and (4) is a parametrization of the w.t.m.

Moving tank (Petit, Rouchon, IEEE TAC 02)

1. The model of a moving tank is defined by:

$$\left(\begin{array}{ccc} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{array}\right)\left(\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right) = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\left\{\begin{array}{l} \partial_1\,\lambda_1 + \partial_1\,\partial_2^2\,\lambda_2 = \mu_1, \\ -\partial_1\,\partial_2^2\,\lambda_1 - \partial_1\,\lambda_2 = \mu_2, \\ a\,\partial_1^2\,\partial_2\,\lambda_1 + a\,\partial_1^2\,\partial_2\,\lambda_2 = \mu_3. \end{array}\right. \qquad (2)$$

(2) is overdetermined $\stackrel{\text{Gb}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Moving tank (Petit, Rouchon, IEEE TAC 02)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$\begin{pmatrix} a\,\partial_1\,\partial_2 & -a\,\partial_1\,\partial_2 & -(1+\partial_2^2) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = 0.$$

4. We consider the overdetermined system $P^T\,z = y$.

$$\left\{ \begin{array}{l} a\,\partial_1\,\partial_2\,z = y_1, \\ -a\,\partial_1\,\partial_2\,z = y_2, \\ -(1+\partial_2^2)\,z = y_3. \end{array} \right. \qquad (4)$$

5. The compatibility conditions of $P^T\,z = y$ are $R'\,y = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & (1+\partial_2^2) & -a\,\partial_1\,\partial_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0.$$

Moving tank (Petit, Rouchon, IEEE TAC 02)

$$t(M) \quad \cong \quad \mathrm{ext}^1_D(N, D) \quad \cong$$

$$\left(D^{1\times 2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+\partial_2^2 & -a\,\partial_1\,\partial_2 \end{pmatrix}\right) \Big/ \left(D^{1\times 2} \begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{pmatrix}\right)$$

$$\begin{cases} y_1 + y_2 = z_1, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{cases} \quad \stackrel{\mathrm{Gb}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_1 = 0.$$

$$\begin{cases} (1+\partial_2^2)\,y_2 - a\,\partial_1\,\partial_2\,y_3 = z_2, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{cases} \quad \stackrel{\mathrm{Gb}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_2 = 0.$$

$\Rightarrow$ $z_1(t)$ and $z_2(t)$ are autonomous elements.

Examples: reflexive modules

- div-curl-grad: $\vec{\nabla}.\,\vec{B}=0 \Leftrightarrow \vec{B}=\vec{\nabla}\wedge\vec{A},\ \vec{\nabla}\wedge\vec{A}=\vec{0} \Leftrightarrow \vec{A}=\vec{\nabla} f.$

- First group of Maxwell equations:

$$\left\{\begin{array}{l} \dfrac{\partial \vec{B}}{\partial t}+\vec{\nabla}\wedge\vec{E}=\vec{0}, \\ \vec{\nabla}.\,\vec{B}=0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \vec{E}=-\dfrac{\partial \vec{A}}{\partial t}-\vec{\nabla} V, \\ \vec{B}=\vec{\nabla}\wedge\vec{A}. \end{array}\right.$$

$$\left\{\begin{array}{l} -\dfrac{\partial \vec{A}}{\partial t}-\vec{\nabla} V=\vec{0}, \\ \vec{\nabla}\wedge\vec{A}=\vec{0}, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \vec{A}=\vec{\nabla}\xi, \\ V=-\dfrac{\partial \xi}{\partial t}. \end{array}\right.$$

- 3D stress tensor: Maxwell, Morera parametrizations...

- Linearized Einstein equations (10×10 system of PDEs)?

$\Rightarrow$ OreModules (Chyzak, Robertz, Q.)

Constructive version of Quillen-Suslin theorem

- If M is a projective left D-module M which admits a finite free resolution, then M is stably free (Serre's theorem).

$\Rightarrow$ M can be rewritten as $M = D^{1\times p'}/(D^{1\times q'}\, R')$, $\quad R'\, S = I_{q'}$.

- M is free iff the following completion problem can be solved:

$$\begin{pmatrix} R' \\ T \end{pmatrix} \begin{pmatrix} S & Q \end{pmatrix} = \begin{pmatrix} I_{q'} & 0 \\ 0 & I_{p'-q'} \end{pmatrix} = I_{p'}.$$

- A constructive proof of the Quillen-Suslin theorem was implemented by Fabiańska in the package QuillenSuslin.

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\, x_2(t) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1(t) = \dot{z}(t) - z(t-h) + a\, z(t), \\ x_2(t) = z(t), \\ u(t) = \ddot{z}(t) + \dot{z}(t) - \dot{z}(t-h) - z(t-h) + a\, \dot{z}(t) + a\, z(t). \end{cases}$$

Constructive version of Stafford's theorem

- The time-varying linear control system (Sontag)

$$\begin{cases} \dot{x}_1(t) - t\,u_1(t) = 0, \\ \dot{x}_2(t) - u_2(t) = 0, \end{cases}$$

is injectively parametrized by (Stafford (Robertz, Q.))

$$\begin{cases} x_1(t) = t^2\,\xi_1(t) - t\,\dot{\xi}_2(t) + \xi_2(t), \\ x_2(t) = t\,(t+1)\,\xi_1(t) - (t+1)\,\dot{\xi}_2(t) + \xi_2(t), \\ u_1(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) - \ddot{\xi}_2(t), \\ u_2(t) = t\,(t+1)\,\dot{\xi}_1(t) + (2\,t+1)\,\xi_1(t) - (t+1)\,\ddot{\xi}_2(t), \end{cases}$$

and $\{\xi_1, \xi_2\}$ is a basis of the free left $A_1(\mathbb{Q})$-module M as:

$$\begin{cases} \xi_1(t) = (t+1)\,u_1(t) - u_2(t), \\ \xi_2(t) = (t+1)\,x_1(t) - t\,x_2(t). \end{cases}$$

- Idem for $\partial_1\,y_1 + \partial_2\,y_2 + \partial_3\,y_3 + x_3\,y_1 = 0$.

Dictionary systems — modules

Module $M = D^{1\times p}/(D^{1\times q}\, R)$	**Structural properties** $\ker_{\mathcal{F}}(R.)$ $\mathcal{F}$ injective cogenerator	**Stabilization problems** **Optimal control**
Torsion	Autonomous system Poles/zeros classifications	
With torsion	Existence of autonomous elements	
Torsion-free	No autonomous elements, Controllability, Parametrizability, π-freeness	Variational problem without constraints (Euler-Lagrange equations)
Projective	Bézout identities, Internal stabilizability	Computation of Lagrange multipliers without integration Existence of a parametrization all stabilizing controllers
Free	Flatness, Poles placement, Doubly coprime factorization	Youla-Kučera parametrization Optimal controller

Oberst, Pommaret, Fliess, Mounier, Shankar, Zerz, Wood . . . , Q.

Projectiveness, observability and controllability

- Theorem: If $R \in D^{q \times p}$ has full row rank, then the left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$ is projective iff:

$$N = D^{1 \times q}/(D^{1 \times p}\, \theta(R)) = 0 \;\Leftrightarrow\; \exists\, S \in D^{p \times q}: \; R\, S = I_q.$$

- Let $D = \mathcal{A}(I)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $R = (\partial\, I_n - A \quad -B) \in D^{n \times (n+m)}$.

$M = D^{1 \times (n+m)}/(D^{1 \times n} R)$ is projective iff $\theta(R)\, \lambda = 0 \Leftrightarrow \lambda = 0$:

$$\left\{ \begin{array}{l} -\partial\, \lambda - A^T\, \lambda = 0, \\ -B^T\, \lambda = 0, \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} \partial\, \lambda = -A^T\, \lambda, \\ B^T\, \lambda = 0, \\ B^T\, \partial\, \lambda + \dot{B}^T\, \lambda = (-B^T\, A^T + \dot{B}^T)\, \lambda = 0. \end{array} \right.$$

Hence, M is projective iff, for all $t_0 \in I$, we have:

$$\mathrm{rank}_{\mathbb{R}}(B \mid A\, B - \dot{B} \mid A^2\, B + \ldots \mid A^{n-1}\, B + \ldots \mid \ldots)(t_0) = n.$$

- $D^{1 \times p}/(D^{1 \times q}\, (P(\partial) \;\; -Q(\partial)))$ proj. iff $P(\partial)\, X(\partial) + Q(\partial)\, Y(\partial) = I_q$.

Flatness: two pendula mounted on a car

- We consider two pendula mounted on a car:

$$\begin{cases} m_1\, L_1\, \ddot{w}_1(t) + m_2\, L_2\, \ddot{w}_2(t) - w_3(t) + (M + m_1 + m_2)\, \ddot{w}_4(t) = 0, \\ m_1\, L_1^2\, \ddot{w}_1(t) - m_1\, L_1\, g\, w_1(t) + m_1\, L_1\, \ddot{w}_4(t) = 0, \\ m_2\, L_2^2\, \ddot{w}_2(t) - m_2\, L_2\, g\, w_2(t) + m_2\, L_2\, \ddot{w}_4(t) = 0. \end{cases} \qquad (\star)$$

- $(\star)$ is parametrizable iff $L_1 \neq L_2$.

- If $L_1 \neq L_2$ then a parametrization of $(\star)$ is defined by:

$$\begin{cases} w_1(t) = -L_2\, \xi^{(4)}(t) + g\, \ddot{\xi}(t), \\ w_2(t) = -L_1\, \xi^{(4)}(t) + g\, \ddot{\xi}(t), \\ w_3(t) = L_1\, L_2\, M\, \xi^{(6)}(t) - (L_1\, m_2 + L_2\, m_1 + g\, (L_1 + L_2)\, M)\, \xi^{(4)}(t) \\ \qquad\quad + g^2\, (m_1 + m_2 + M)\, \xi^{(2)(t)} \\ w_4(t) = L_1\, L_2\, \xi^{(4)}(t) - g\, (L_1 + L_2)\, \ddot{\xi}(t) + g^2\, \xi(t). \end{cases}$$

- The parametrization of $(\star)$ is injective as we have:

$$\xi(t) = \frac{1}{g^2\, (L_1 - L_2)}\, (L_1^2\, w_1(t) - L_2^2\, w_2(t) + (L_1 - L_2)\, w_4(t)). \quad (\star\star)$$

Flatness: two pendula mounted on a car

- Patching problem $\Leftrightarrow$ controllability: $\quad T > 0$.

$w^p = (w_1^p, w_2^p, w_3^p, w_4^p)$ a past trajectory of $(\star)$ on $]-\infty, 0[$.

$w^f = (w_1^f, w_2^f, w_3^f, w_4^f)$ a future trajectory of $(\star)$ on $]T, +\infty[$.

$$\Rightarrow \exists \, w = (w_1, w_2, w_3, w_4) \in C^\infty(\mathbb{R})^4 \text{ trajectory of } (\star):$$

$$\begin{cases} w_{]-\infty,0[} = w^p, \\ w_{]T,+\infty[} = w^f. \end{cases}$$

- Using the flat output

$$\xi(t) = \frac{1}{g^2\,(L_1 - L_2)}\,(L_1^2\, w_1(t) - L_2^2\, w_2(t) + (L_1 - L_2)\, w_4(t))$$

and the parametrization, it is enough to find $\xi \in C^\infty(\mathbb{R})$ s.t.:

$$\xi_{]-\infty,0[} = \xi^p \quad \& \quad \xi_{]T,+\infty[} = \xi^f.$$

Controllability à la Willems

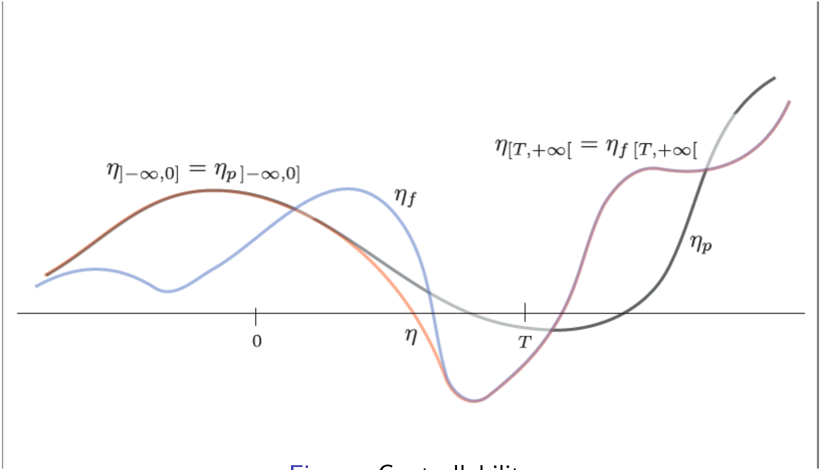

Figure: Controllability

Autonomous elements & controllability

- Stirred tank model with $c_2 = c_1 = c_0$

$$\begin{cases} \dot{x}_1(t) = -\dfrac{1}{2\,\theta}\, x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\dfrac{1}{\theta}\, x_2(t) + \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t-h) + \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t-h), \end{cases}$$

$$\dot{x}_2(t) + \frac{1}{\theta}\, x_2(t) = 0 \quad \Leftrightarrow \quad \left(\partial + \frac{1}{\theta}\right) x_2 = 0.$$

$\Rightarrow$ x_2 cannot be controlled using the inputs u_1 and u_2.

Autonomous elements & controllability

- Stirred tank model with $c_2 = c_1 = c_0$

$$\begin{cases} \dot{x}_1(t) = -\dfrac{1}{2\,\theta}\, x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\dfrac{1}{\theta}\, x_2(t) + \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t-h) + \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t-h), \end{cases}$$

$$\dot{x}_2(t) + \frac{1}{\theta}\, x_2(t) = 0 \quad \Leftrightarrow \quad \left(\partial + \frac{1}{\theta}\right) x_2 = 0.$$

$\Rightarrow$ x_2 cannot be controlled using the inputs u_1 and u_2.

- Moving tank model (Petit, Rouchon, IEEE TAC 02):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t - 2\,h) + \alpha\, \ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t - 2\,h) - \dot{y}_2(t) + \alpha\, \ddot{y}_3(t-h) = 0, \end{cases}$$

$$\Rightarrow \begin{cases} z_1(t) = y_1(t) + y_2(t), \\ z_2(t) = y_2(t) + y_2(t - 2\,h) - a\, \dot{y}_3(t-h), \end{cases} \qquad \begin{array}{c} \frac{d}{dt}\,(1 - \delta^2)\, z_i(t) = 0, \\ i = 1, 2. \end{array}$$

Flat linear functional systems

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\, x_2(t) = 0. \end{cases} \qquad (\star)$$

Flat linear functional systems

$$\left\{ \begin{array}{l} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\, x_2(t) = 0. \end{array} \right. \qquad (\star)$$

$$M = D^{1\times 3}/(D^{1\times 2}\, R): \quad \left\{ \begin{array}{l} \partial_1\, x_1 + x_1 - u = 0, \\ \partial_1\, x_2 - \partial_1\, \partial_2\, x_2 - x_1 + a\, x_2 = 0, \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} u = (\partial_1 + 1)\, x_1, \\ x_1 = (\partial_1 - \partial_1\, \partial_2 + a)\, x_2, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} u = (\partial_1 + 1)\, (\partial_1 - \partial_1\, \partial_2 + a)\, x_2, \\ x_1 = (\partial_1 - \partial_1\, \partial_2 + a)\, x_2. \end{array} \right.$$

$\Rightarrow$ M is a free $D = \mathbb{Q}(a)\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \delta, 0]$-module of basis $\{z_2\}$.

Flat linear functional systems

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\, x_2(t) = 0. \end{cases} \qquad (\star)$$

$$M = D^{1\times 3}/(D^{1\times 2}\, R) : \quad \begin{cases} \partial_1\, x_1 + x_1 - u = 0, \\ \partial_1\, x_2 - \partial_1\, \partial_2\, x_2 - x_1 + a\, x_2 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} u = (\partial_1 + 1)\, x_1, \\ x_1 = (\partial_1 - \partial_1\, \partial_2 + a)\, x_2, \end{cases} \Leftrightarrow \begin{cases} u = (\partial_1 + 1)\, (\partial_1 - \partial_1\, \partial_2 + a)\, x_2, \\ x_1 = (\partial_1 - \partial_1\, \partial_2 + a)\, x_2. \end{cases}$$

$\Rightarrow M$ is a free $D = \mathbb{Q}(a)\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \delta, 0]$-module of basis $\{z_2\}$.

$$(\star) \;\Leftrightarrow$$

$$\begin{cases} x_1(t) = \dot{x}_2(t) - x_2(t-h) + a\, x_2(t), \\ x_2(t) = x_2(t), \\ u(t) = \ddot{x}_2(t) + \dot{x}_2(t) - \dot{x}_2(t-h) - x_2(t-h) + a\, \dot{x}_2(t) + a\, x_2(t). \end{cases}$$

Flat linear functional systems

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\, x_2(t) = 0. \end{cases} \qquad (\star)$$

$$(\star) \Leftrightarrow$$

$$\begin{cases} x_1(t) = \dot{x}_2(t) - x_2(t-h) + a\, x_2(t), \\ x_2(t) = x_2(t), \\ u(t) = \ddot{x}_2(t) + \dot{x}_2(t) - \dot{x}_2(t-h) - x_2(t-h) + a\, \dot{x}_2(t) + a\, x_2(t). \end{cases}$$

- $y = x_2$: output $\Rightarrow$ $u(t) = \phi(y(t), \dot{y}(t), \ddot{y}(t), \dot{y}(t-h))$.

Motion planning

- Flexible rod with a torque (Mounier 95):

$$\left\{\begin{array}{l} \sigma^2\,\frac{\partial^2 q(\tau,x)}{\partial \tau^2} - \frac{\partial^2 q(\tau,x)}{\partial x^2} = 0, \\ \frac{\partial q}{\partial x}(\tau,0) = -u(\tau), \\ \frac{\partial q}{\partial x}(\tau,L) = -J\,\frac{\partial^2 q}{\partial \tau^2}(\tau,L), \\ y(\tau) = q(\tau,L). \end{array}\right. \qquad (\star)$$

- $q(\tau,x) = \phi(\tau + \sigma\,x) + \psi(\tau - \sigma\,x)$, $t = (\sigma/J)\,\tau$, $v = (2\,J/\sigma^2)\,u$,

$$(\star) \Rightarrow \ddot{y}(t+1) + \ddot{y}(t-1) + \dot{y}(t+1) - \dot{y}(t-1) = v(t)$$

$$\Leftrightarrow \left\{\begin{array}{l} y(t) = \xi(t-1), \\ v(t) = \ddot{\xi}(t) + \ddot{\xi}(t-2) + \dot{\xi}(t) - \dot{\xi}(t-2). \end{array}\right.$$

- If y_r is a desired trajectory then $\xi_r(t) = y_r(t+1)$ and we obtain the open-loop control law:

$$v_r(t) = \ddot{y}_r(t+1) + \ddot{y}_r(t-1) + \dot{y}_r(t+1) - \dot{y}_r(t-1).$$

Optimal control

- Let us minimize $\frac{1}{2} \int_0^T (x(t)^2 + u(t)^2)\, dt \quad (1) \quad$ under:

$$\dot{x}(t) + x(t) - u(t) = 0, \quad x(0) = x_0. \quad (2)$$

- (2) is parametrized by $\begin{cases} x(t) = \xi(t), \\ u(t) = \dot{\xi}(t) + \xi(t). \end{cases} \quad (3)$

- (1) & (3) $\Rightarrow$ min $\frac{1}{2} \int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\, dt$,

$$\Rightarrow \text{Euler-Lagrange equations} \quad \begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{cases}$$

$$\Rightarrow u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\, e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\, e^{-\sqrt{2}(t-T)}}\, x(t).$$

Variational problems

- Let us extremize the electromagnetic action

$$\int \left(\frac{1}{2\,\mu_0} \ \| \vec{B} \|^2 - \frac{\epsilon_0}{2} \ \| \vec{E} \|^2 \right) \, dx_1 \, dx_2 \, dx_3 \, dt, \qquad (1)$$

where $\vec{B}$ and $\vec{E}$ satisfy:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \,.\, \vec{B} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = -\vec{\nabla}\, V - \dfrac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right. \qquad (3)$$

- Substituting (3) in (1) and using Lorentz gauge

$$\vec{\nabla} \,.\, \vec{A} + \frac{1}{c^2} \, \frac{\partial V}{\partial t} = 0, \quad c^2 = 1/(\epsilon_0 \, \mu_0),$$

$$\Rightarrow \left\{ \begin{array}{l} \dfrac{1}{c^2} \, \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta \, \vec{A} = 0, \\ \dfrac{1}{c^2} \, \dfrac{\partial^2 V}{\partial t^2} - \Delta \, V = 0. \end{array} \right. \qquad \text{(electromagnetic waves).}$$

General Monge problem

- If $\mathcal{F}$ be a left D-module then we have the following equivalence:

$$R\,\eta = 0 \;\Leftrightarrow\; \begin{cases} R''\,\tau = 0, \\ R_2'\,\tau = 0, \\ R'\,\eta = \tau. \end{cases}$$

$\Rightarrow$ integration of $R\,\eta = 0$ in cascade:

1. Find the general solution $\overline{\tau} \in \mathcal{F}^p$ of the autonomous system:

$$\begin{cases} R''\,\tau = 0, \\ R_2'\,\tau = 0. \end{cases}$$

2. Find a particular solution $\eta^\star \in \mathcal{F}^p$ of $R'\,\eta = \overline{\tau}$.
3. If $\mathcal{F}$ is injective then $\ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m$ and:

$$\forall\,\eta \in \ker_{\mathcal{F}}(R.),\; \exists\,\xi \in \mathcal{F}^m : \; \eta = \eta^\star + Q\,\xi.$$

General Monge problem

- The following canonical short exact sequence splits,

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0, \text{ i.e., } \Leftrightarrow \; M \cong t(M) \oplus M/t(M),$$

iff there exist $X \in D^{p \times q'}$, $Y \in D^{q' \times q}$ and $Z \in D^{q' \times r'}$ such that:

$$R' X + (Y \quad Z) \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} = I_{q'}. \quad (\star)$$

- **Remark:** $(\star) \;\Leftrightarrow\; R' - R' X R' = Y R$, i.e., iff R' admits a generalized inverse modulo $D^{q' \times q} R$.
- **Remark:** Applying $(\star)$ to $\overline{\tau}$, we get:

$$\overline{\tau} = R'(X \overline{\tau}), \quad \text{i.e.,} \quad \eta^\star = X \overline{\tau}.$$

- $M/t(M)$ is projective iff R' admits a generalized inverse.
- If $(\star)$ or $(\star\star)$ is satisfied and $\mathcal{F}$ is injective, then:

$$\forall \, \eta \in \ker_{\mathcal{F}}(R.), \; \exists \, \xi \in \mathcal{F}^m : \; \eta = X \overline{\tau} + Q \, \xi.$$

Kronecker product

- Let D be a commutative polynomial ring.
- The Kronecker product of $E \in D^{q \times p}$ and $F \in D^{r \times s}$ is:

$$E \otimes F = \begin{pmatrix} E_{11}\, F & \dots & E_{1p}\, F \\ \vdots & \vdots & \vdots \\ E_{q1}\, F & \dots & E_{qp}\, F \end{pmatrix} \in D^{(q\, r) \times (p\, s)}.$$

Lemma: If $U \in D^{a \times b}$, $V \in D^{b \times c}$ and $W \in D^{c \times d}$, then we have:

$$\mathrm{row}(U\, V\, W) = \mathrm{row}(V)\, (U^T \otimes W),$$

- $\mathrm{row}(R'\, X) = \mathrm{row}(R'\, X\, I_{q'}) = \mathrm{row}(X)\, (R'^T \otimes I_{q'})$.
- $\mathrm{row}\left(I_{q'}\, (Y \quad Z) \begin{pmatrix} R'' \\ R_2' \end{pmatrix}\right) = \mathrm{row}(Y \quad Z) \left(I_{q'} \otimes \begin{pmatrix} R'' \\ R_2' \end{pmatrix}\right)$.

$$\Rightarrow \; \mathrm{row}(I_{q'}) = (\mathrm{row}(X) \quad \mathrm{row}(Y \quad Z)) \begin{pmatrix} R'^T \otimes I_{q'} \\ I_{q'} \otimes \begin{pmatrix} R'' \\ R_2' \end{pmatrix} \end{pmatrix} \; (\text{GB}).$$

OreModules (Chyzak, Q., Robertz)

- OreModules is a tool-box developed in *Maple*.
- OreModules uses *Ore_algebra* developed by Chyzak.
- OreModules handles linear systems of ODEs, PDEs, discrete equations, differential time-delay equations...
- OreModules computes:

1. free resolutions, $\text{ext}^i_D(\,\cdot\,, D)$, projective dim., Hilbert series,
2. torsion elements, autonomous elements,
3. parametrizations of under-determined systems,
4. left-/right-/generalized inverses,
5. bases, flat outputs, π-polynomials,
6. first integrals of motion, Euler-Lagrange equations...

http://wwwb.math.rwth-aachen.de/OreModules/

Conclusion

• Based on algebraic analysis, constructive homological algebra and Ore algebras, we have developed a general non-commutative polynomial approach to functional linear systems.

• The different results are implemented in the packages:

OreModules, Stafford, QuillenSuslin.

This approach allowed us to:

1. Develop an intrinsic approach (independent of the form).
2. Develop generic algorithms and generic implementations.
3. Constructively study the parametrizability problem.
4. Solve conjectures in mathematical systems theory.