Exercises: Module theory

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Notation. Let D be a ring and:

$$\left\{ \begin{array}{l} D^q = \{(\mu_1 \ \dots \ \mu_q)^T \mid \mu_i \in D, \ i = 1, \dots, q\}, \\ D^{1\times p} = \{(\lambda_1 \ \dots \ \lambda_p) \mid \lambda_i \in D, \ i = 1, \dots, p\}. \end{array} \right.$$

Exercise 1 Let $R \in D^{q\times p}$ and write:

$$R = \left(\begin{array}{c} R_{1\bullet} \\ \vdots \\ R_{q\bullet} \end{array} \right), \quad R_{i\bullet} \in D^{1\times p}.$$

Consider the following left D-homomorphism (i.e., left D-linear map)

$$\begin{array}{ccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \\ \lambda = (\lambda_1 \ \dots \ \lambda_q) & \longmapsto & \lambda\, R = \lambda_1\, R_{1\bullet} + \dots + \lambda_q\, R_{q\bullet}, \end{array}$$

and its cokernel $M = D^{1\times p}/(D^{1\times q}\, R)$.

1. Show that we have the following exact sequence of left D-modules

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \tag{1}$$

where π denotes the left D-homomorphism which sends any element $\lambda \in D^{1\times p}$ onto its residue class $\pi(\lambda) \in M$, i.e., $\pi(\lambda) = \pi(\lambda')$ iff there exists $\mu \in D^{1\times q}$ such that $\lambda = \lambda' + \mu\, R$.

2. Let $\{f_j\}_{j=1,\dots,p}$ (resp., $\{e_i\}_{i=1,\dots,q}$) be the *standard basis* of the $D^{1\times p}$ (resp., $D^{1\times q}$), i.e., f_j (resp., e_i) is the row vector of length p (resp., q) with 1 at the j^{th} (resp., i^{th}) position and 0 elsewhere. Prove that $\{y_j = \pi(f_j)\}_{j=1,\dots,p}$ is a *set of generators* of M, namely, for any $m \in M$, there exist $d_1, \dots, d_p \in D$ such that $m = \sum_{i=1}^p d_j\, y_j$.

3. What is the value of $\pi(e_i\, R)$ in M? Express $e_i\, R$ in the standard basis of $D^{1\times p}$ and prove that the set of generators $\{y_j = \pi(f_j)\}_{j=1,\dots,p}$ satisfies the following *left D-linear relations*:

$$\sum_{j=1}^{p} R_{ij}\, y_j = 0, \quad i = 1, \dots, q. \tag{2}$$

We then say that M is defined by generators and relations and M is a *finitely presented* left D-module. Moreover, (1) is called a *finite presentation of M*.

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4. If $y = (y_1 \ \ldots \ y_p)^T$, then note that (2) can formally be rewritten as $R\, y = 0$.

Exercise 2 Let M and $\mathcal{F}$ be two left D-modules. We denote by $\hom_D(M, \mathcal{F})$ the set of left D-homomorphisms from M to $\mathcal{F}$.

1. If $f,\, g \in \hom_D(M, \mathcal{F})$, then check that the map $f+g$ defined by $(f+g)(m) = f(m)+g(m)$ is a left D-homomorphism, i.e., $f+g \in \hom_D(M, \mathcal{F})$.

2. Prove that $\hom_D(M, \mathcal{F})$ has generally no left D-module structure, i.e., $\hom_D(M, \mathcal{F})$ is generally only an abelian group.

   Hint. Consider $d,\, d' \in D$ and $f \in \hom_D(M, \mathcal{F})$ and suppose that $d\, f \in \hom_D(M, \mathcal{F})$ is defined by $(d\, f)(m) = f(d\, m)$, for all $m \in M$. Then, compute $((d'\, d)\, f)(m)$ and $(d'\, (d\, f))(m)$, for all $m \in M$. Check that the associativity condition is generally not satisfied.

3. If $\mathcal{F}$ is a left D-module and a right E module, then $\mathcal{F}$ is called a $D-E$-*bimodule* if $(d\, \eta)\, e = d\, (\eta\, e)$, for all $d \in D$, $e \in E$ and $\eta \in \mathcal{F}$. Prove that if $\mathcal{F}$ is a $D-E$-bimodule, then $\hom_D(M, \mathcal{F})$ has a right E-module structure defined by:

$$(f\, e)(m) = f(m)\, e, \quad \forall\, e \in E.$$

   Hint. The only non-trivial condition to check is the associativity one. Hence, compute $(f\, (e_1\, e_2))(m)$ and $((f\, e_1)\, e_2)(m)$, for all $e_1,\, e_2 \in E$ and $m \in M$, and compare the results.

4. Deduce that $\hom_D(M, D)$ is a right D-module.

5. Let $R \in D^{q \times p}$, $D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$ a finite presentation of $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $\mathcal{F}$ a left D-module. Show that π induces a $\mathbb{Z}$-homomorphism $\pi^\star$ defined by:

$$\begin{array}{rcl} \pi^\star : \hom_D(M, \mathcal{F}) & \longrightarrow & \hom_D(D^{1 \times p}, \mathcal{F}) \\ \phi & \longmapsto & \phi \circ \pi. \end{array}$$

   Similarly, prove that $.R$ induces a $\mathbb{Z}$-homomorphism $(.R)^\star$ defined by:

$$\begin{array}{rcl} (.R)^\star : \hom_D(D^{1 \times p}, \mathcal{F}) & \longrightarrow & \hom_D(D^{1 \times q}, \mathcal{F}) \\ \psi & \longmapsto & \psi \circ (.R). \end{array}$$

6. Prove that $(.R)^\star \circ \pi^\star = 0$, i.e., we have the following complex:

$$\hom_D(D^{1 \times q}, \mathcal{F}) \xleftarrow{(.R)^\star} \hom_D(D^{1 \times p}, \mathcal{F}) \xleftarrow{\pi^\star} \hom_D(M, \mathcal{F}) \longleftarrow 0. \quad (3)$$

7. Check that $\pi^\star$ is injective and $\ker (.R)^\star = \operatorname{im} \pi^\star$ and conclude that (3) is an exact sequence of abelian groups.

8. Using the standard basis $\{f_j\}_{j=1,\ldots,p}$ (resp., $\{e_i\}_{i=1,\ldots,q}$) of $D^{1 \times p}$ (resp., $D^{1 \times q}$), show that an element $\psi \in \hom_D(D^{1 \times p}, \mathcal{F})$ (resp., $\varphi \in \hom_D(D^{1 \times q}, \mathcal{F})$) is totally defined by the knowledge of the vector $(\psi(f_1) \ \ldots \ \psi(f_p))^T \in \mathcal{F}^p$ (resp., $(\varphi(e_1) \ \ldots \ \varphi(e_q))^T \in \mathcal{F}^q$). In other words, we have the following abelian group isomorphisms:

$$\begin{array}{rcl} \iota_p : \hom_D(D^{1 \times p}, \mathcal{F}) & \longrightarrow & \mathcal{F}^p \\ \psi & \longmapsto & \left(\begin{array}{c} \psi(f_1) \\ \vdots \\ \psi(f_p) \end{array}\right), \end{array} \qquad \begin{array}{rcl} \iota_q : \hom_D(D^{1 \times q}, \mathcal{F}) & \longrightarrow & \mathcal{F}^q \\ \varphi & \longmapsto & \left(\begin{array}{c} \varphi(e_1) \\ \vdots \\ \varphi(e_q) \end{array}\right). \end{array}$$

9. For any $\eta = (\eta_1 \ \dots \ \eta_p)^T \in \mathcal{F}^p$, show that $\iota_p^{-1}(\eta) = \psi$, where $\psi \in \hom_D(D^{1\times p}, \mathcal{F})$ is defined by $\psi(f_j) = \eta_j$, for $j = 1, \dots, p$.

10. Check the following computations:

$$(\iota_q \circ (.R)^\star \circ \iota_p^{-1})(\eta) = \iota_q(\psi \circ (.R)) = \begin{pmatrix} \psi(e_1\, R) \\ \vdots \\ \psi(e_q\, R) \end{pmatrix} = \begin{pmatrix} \psi\left(\sum_{j=1}^p R_{1j}\, f_j\right) \\ \vdots \\ \psi\left(\sum_{j=1}^p R_{qj}\, f_j\right) \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{j=1}^p R_{1j}\, \eta_j \\ \vdots \\ \sum_{j=1}^p R_{qj}\, \eta_j \end{pmatrix} = R\, \eta.$$

11. Deduce that we get the following exact sequence of abelian groups

$$\begin{array}{ccccc} \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \xleftarrow{\iota_p \circ \pi^\star} & \hom_D(M, \mathcal{F}) \longleftarrow 0, \\ R\,\eta & \longleftarrow\!\shortmid & \eta & & \end{array} \tag{4}$$

which proves that $\ker_{\mathcal{F}}(R.) = (\iota_p \circ \pi^\star)(\hom_D(M, \mathcal{F}))$ and:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}).$$

Explain two interests of the previous isomorphism.

This isomorphism was first used by Malgrange in the study of linear systems of PDEs with constant coefficients and then by the Japanese school of Sato (Sato, Kashiwara, Kawai...) for linear systems of PDEs with analytic coefficients.

12. A solution $\eta \in \ker_{\mathcal{F}}(R.)$ is said to be *generic* if $\iota_p^{-1}(\phi) \in \hom_D(M, \mathcal{F})$ is injective. Prove that η is a non-generic solution if there exists $S \in D^{1\times p}$ such that $S\,\eta = 0$ but $S \notin D^{1\times q}\, R$. Let $D = \mathbb{Q}\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, $R = (\partial^2 \quad -\partial) \in D^{1\times 2}$, $M = D^{1\times 2}/(D\, R)$ and $\mathcal{F} = \mathcal{D}$ the D-module of compactly supported smooth functions on $\mathbb{R}$. Check that the elements of $\ker_{\mathcal{F}}(R.)$ are not generic.

Exercise 3 Let $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial; \mathrm{id}, \frac{\partial}{\partial x_2}\right]$ be the commutative ring of polynomials and:

$$R = \begin{pmatrix} \partial_2^2 \\ \partial_1\, \partial_2 \end{pmatrix}.$$

1. Let $.R$ be the D-homomorphism (i.e., D-linear map) defined by:

$$\begin{array}{ccc} D^{1\times 2} & \xrightarrow{.R} & D \\ \lambda = (\lambda_1 \quad \lambda_2) & \longmapsto & \lambda\, R = \lambda_1\, \partial_2^2 + \lambda_2\, \partial_1\, \partial_2. \end{array}$$

Interpret $\mathrm{im}_D(.R) = D^{1\times 2}\, R$ as an ideal of D. Compute its cokernel $\mathrm{coker}_D(.R)$ and its kernel $\ker_D(.R)$.

Hint. Use the fact that D is a greatest common divisor domain. Deduce that the D-module M admits the following finite free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0, \tag{5}$$

where $R_2 = (\partial_1 \quad -\partial_2) \in D^{1\times 2}$.

2. Using the command INTEGRABILITY, compute a Gröbner basis of $R\,\eta = \zeta$ and deduce the compatibility conditions on ζ. Check again that last result by computing $\ker_D(.R)$ using the command SYZYGYMODULE.

3. Compute again a finite free resolution directly with FREERESOLUTION.

4. Describe the finitely presented D-module $M = D/(D^{1\times 2}\,R)$ in terms of generators and relations.

5. We recall that a module is called *cyclic* if it can be generated by one element. Is M cyclic?

6. We recall that the torsion D-submodule $t(M)$ of M is defined by:

$$t(M) = \{m \in M \mid \exists\, 0 \neq d \in D:\; d\,m = 0\}.$$

Compute $t(M)$. A D-module is said to be *torsion* if $t(M) = M$ and *torsion-free* if $t(M) = 0$. Is M a torsion (resp., torsion-free) D-module? Check that result using the command TORSIONELEMENTS of OREMODULES.

7. Let $\mathcal{F}$ be a D-module. For instance, prove that the ring $C^\infty(\mathbb{R}^2)$ of smooth functions on $\mathbb{R}^2$ is a D-module.

8. Since D is a commutative ring, prove that $\hom_D(M, \mathcal{F})$ is a D-module where the action of D on $\hom_D(M, \mathcal{F})$ is defined by:

$$\forall\, d \in D, \quad \forall\, f \in \hom_D(M, \mathcal{F}), \quad \forall\, m \in M: \quad (d\,f)(m) = f(d\,m).$$

9. Show that the possible defects of exactness of the following complexes at $\mathcal{F}^2$ and $\mathcal{F}$

$$0 \longleftarrow \mathcal{F} \xleftarrow{R_2.} \mathcal{F}^2 \xleftarrow{R.} \mathcal{F} \longleftarrow 0$$

are respectively defined by:

$$\begin{cases} \operatorname{ext}^0_D(M, \mathcal{F}) = \hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.), \\ \operatorname{ext}^1_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_2.)/\operatorname{im}_{\mathcal{F}}(R.), \\ \operatorname{ext}^2_D(M, \mathcal{F}) = \mathcal{F}/(R_2\,\mathcal{F}^2). \end{cases}$$

10. Give a necessary condition on $\zeta = (\zeta_1 \quad \zeta_2)^T \in \mathcal{F}^2$ for the existence of $\eta \in \mathcal{F}$ satisfying the inhomogeneous linear OD system $R\,\eta = \zeta$, namely:

$$\begin{cases} \partial_2^2\,\eta = \zeta_1, \\ \partial_1\,\partial_2\,\eta = \zeta_2. \end{cases}$$

In terms of $\operatorname{ext}^1_D(M, \mathcal{F})$, characterize when that necessary condition is also sufficient.

If $\mathcal{F} = C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{B}(\Omega)\ldots$, where Ω is an open convex subset of $\mathbb{R}^n$, then a deep result asserts that $\operatorname{ext}^i_D(M, \mathcal{F}) = 0$ for $i \geq 1$, for all $D = k\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$-module M ($k = \mathbb{R}$ or $\mathbb{C}$). In this case, we say that $\mathcal{F}$ is an *injective D-module*.

Conclude that the necessary condition is also sufficient and thus the search for necessary and sufficient conditions for the existence of $\mathcal{F}$-solutions of inhomogeneous linear PD systems is an algebraic problem (computation of free resolution of D-modules).

11. Taking $\mathcal{F} = D$, compute explicitly $\mathrm{ext}^0_D(M, D)$, $\mathrm{ext}^1_D(M, D)$ and $\mathrm{ext}^2_D(M, D)$. Which ones are zero? Prove that the non-zero ones are torsion D-modules. Compute them again using the command Exti of OreModules.

12. Let us now consider the finitely presented D-module $N = D^{1\times 2}/\left(D\, R^T\right)$. Describe N in terms of generators and relations by introducing the canonical projection $\kappa : D^{1\times 2} \longrightarrow N$ onto N and the standard basis $\{e_i\}_{i=1,2}$ of $D^{1\times 2}$.

13. Compute a finite free resolution of N and $\mathrm{ext}^0_D(N, D)$, $\mathrm{ext}^1_D(N, D)$ and $\mathrm{ext}^2_D(N, D)$. Check again your computations using the command Exti.

14. Check that $t(M) \cong \mathrm{ext}^1_D(N, D)$ and $t(N) \cong \mathrm{ext}^1_D(M, D)$ and deduce the module properties of M and N.

Exercise 4 1. Let $f \in \hom_D(M, N)$. We call the *coimage* of f, denoted by $\mathrm{coim}\, f$, the left/right D-module $\mathrm{coim}\, f = M/\ker f$. Give a short exact sequence involving $\ker f$ and $\mathrm{coim}\, f$. Give a short exact sequence involving $\ker f$ and $\mathrm{im}\, f$. Prove $\mathrm{coim}\, f \cong \mathrm{im}\, f$.

2. Let now consider an exact sequence of left/righ D-modules $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4$. Prove $\ker f_3 = \mathrm{im}\, f_2 \cong \mathrm{coim}\, f_2 = \mathrm{coker}\, f_1$.

3. Now, let consider a complex of left/right D-modules $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4$. Characterize the cohomology $H(M_2)$ in terms of a short exact sequence. Write the image-coker short exact sequence for f_1. Combining those two short exact sequences, obtain a commutative exact diagram. Using the snake lemma, prove that we have the following short exact sequence:
$$0 \longrightarrow H(M_2) \longrightarrow \mathrm{coker}\, f_1 \longrightarrow \mathrm{im}\, f_2 \longrightarrow 0.$$
Characterize the cohomology $H(M_3)$ as a short exact sequence. Combining the two last short exact sequences, prove that the following long exact sequence holds:
$$0 \longrightarrow H(M_2) \longrightarrow \mathrm{coker}\, f_1 \longrightarrow \ker\, f_3 \longrightarrow H(M_3) \longrightarrow 0.$$

4. Let D be a noetherian domain, $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module finitely presented by R and $N = D^q/(R\, D^p)$ the right D-module finitely presented by R. Writing the beginning of a finite free resolution of N
$$0 \longleftarrow N \xleftarrow{\kappa} D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m \xleftarrow{P.} D^l,$$
and dualizing it, we obtain the following complex:
$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m} \xrightarrow{.P} D^{1\times l}.$$
Characterize the left D-module $\mathrm{ext}^1_D(N, D)$ and $\mathrm{ext}^1_D(N, D)$. Using 3, prove that we have the following long exact sequence of left D-modules:
$$0 \longrightarrow \mathrm{ext}^1_D(N, D) \longrightarrow M \longrightarrow \ker_D(.P) \longrightarrow \mathrm{ext}^2_D(N, D) \longrightarrow 0.$$
Finally, using 2, show that $M^\star \cong \ker_D(R.) = \mathrm{im}_D(Q.) \cong \mathrm{coker}_D(P.)$, i.e., $M^\star = \hom_D(M, D)$ admits a finite free presentation of the form $0 \longleftarrow M^\star \xleftarrow{\sigma} D^m \xleftarrow{P.} D^l$. Dualizing it, show that $M^\star = \hom_D(\hom_D(M, D), D) \cong \ker_D(.P)$, which finally yields the following long exact sequence of left D-modules:
$$0 \longrightarrow \mathrm{ext}^1_D(N, D) \longrightarrow M \xrightarrow{\varepsilon} M^{\star\star} \longrightarrow \mathrm{ext}^2_D(N, D) \longrightarrow 0.$$

Exercise 5 Let us consider the 2-dimensional Stokes equations defined by

$$\begin{pmatrix} -\nu\,\Delta & 0 & \partial_x \\ 0 & -\nu\,\Delta & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0, \tag{6}$$

where $\Delta = \partial_x^2 + \partial_y^2$ denotes the Laplacian in $\mathbb{R}^2$, u and v are the two components of the speed and p is the pressure. Let $D = \mathbb{Q}(\nu)\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$ be the commutative polynomial ring of PD operators with coefficients in the field $\mathbb{Q}(\nu, \rho)$, R the matrix appearing in the left-hand side of (6) and the D-module $M = D^{1\times 3}/(D^{1\times 3}\,R)$ finitely presented by R.

1. Compute $\ker_D(.R)$ using SyzygyModule. Does $R\,\eta = \zeta$ admit compatibility conditions?
2. Deduce the rank of M and check it using the command OreRank. Conclude that M is a torsion D-module.
3. Using the command DimensionRat, compute the Krull dimension of M.
4. Compute the formal adjoint $\widetilde{R}$ of R by means of Involution.
5. Using the command Exti, check that M is a torsion D-module by computing the D-module $\mathrm{ext}_D^1(\widetilde{N}, D)$, where $\widetilde{N} = D^{1\times 3}/(D^{1\times 3}\,\widetilde{R})$.
6. Check that the system variables of (6) satisfy the following uncoupled PDEs:

$$\begin{cases} \Delta^2\,u = 0, \\ \Delta^2\,v = 0, \\ \Delta\,p = 0. \end{cases}$$

Exercise 6 Let us consider the commutative polynomial ring $D = \mathbb{Q}(l_1, l_2, g)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, the matrix

$$R = \begin{pmatrix} \partial^2 & l_1\,\partial^2 + g & 0 \\ \partial^2 & 0 & l_2\,\partial^2 + g \end{pmatrix} \in D^{2\times 3}$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$ finitely presented by R.

1. Using the command ApplyMatrix, compute the equations of the corresponding system.
2. Check that R defines a "good system" by means of the command Integrability. Deduce that $\ker_D(.R) = 0$, i.e., R has full row rank. Check that last result by using directly SyzygyModule.
3. Deduce the rank of M and compare it with the result obtained by OreRank.
4. Deduce that the Krull dimension of M is 1 and check that result using DimensionRat.
5. Using the command Involution, compute the formal adjoint $\widetilde{R} \in D^{3\times 2}$ of R.
6. Using the command Integrability, compute a Gröbner basis of the inhomogeneous system $\widetilde{R}\,\lambda = \mu$. Deduce the compatibility condition $P\,\mu = 0$ of $\widetilde{R}\,\lambda = \mu$.
7. Compute $\ker_D(.\widetilde{R})$ using SyzygyModule. Compare with the previous compatibility condition defined by P.

8. Compute a finite free resolution of $\widetilde{N}$ by means of FreeResolution.

9. Compute the formal adjoint $Q = \widetilde{P}$ of P.

10. Compute $\ker_D(.P)$ using SyzygyModule and call this new matrix R_2.

11. Compare R and R_2 and conclude about the generic vanishing of $t(M) = (D^{1\times 2}\, R_2)/(D^{1\times 2}\, R)$. We shall investigate the non-generic situations later on.

12. Check again that last result using Quotient.

13. Redo the last steps using directly the command Exti.

14. Deduce a parametrization of M.

15. Compute a right-inverse of R using RightInverse. Conclude that M is a generically a projective D-module. Determine values of the system parameters for which that last result is not valid. Deduce that M is a free D-module of rank 1.

16. Check that the parametrization of M admits a left-inverse T using the command LeftInverse. Deduce that the corresponding parametrization is injective Check again that M is generically free. Find a basis of the D-module M.

17. Let $U = (R^T \quad T^T)^T \in D^{3\times 3}$. Check that $U \in \mathrm{GL}_3(D)$, i.e., U admits an inverse U^{-1} over D. Interpret the matrix U^{-1}.

18. We now investigate the particular case where $l_1 = l_2$. Define the corresponding matrix using the Maple command `subs` and denote the result by R_2.

19. Compute the formal adjoint $\widetilde{R_2}$ of R_2.

20. Let $\widetilde{N_2} = D^{1\times 2}/(D^{1\times 2}\, \widetilde{R_2})$. Compute $\mathrm{ext}^1_D(\widetilde{N_2}, D)$. Is the D-module $M_2 = D^{1\times 3}/(D^{1\times 2}\, R_2)$ torsion-free?

21. If not, deduce a parametrization of $M_2/t(M_2)$.

22. Compute directly the torsion elements of M_2 using the command TorsionElements.

23. Integrate them using the command AutonomousElements.

24. Check that the linear OD system admits a first integral using the command FirstIntegral.

25. We shall see later on that even if M_2 does not admit a parametrization, we can parametrize the linear OD system $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F} = C^\infty(\mathbb{R}_+)$, by glueing the autonomous elements to the parametrization of $M_2/t(M_2)$. Such a parametrization will be called a *Monge parametrization*. Compute one using the command Parametrization and check that it gives solutions of $\ker_{\mathcal{F}}(R.)$. We shall prove that it defines all the $\mathcal{F}$-solutions.

Exercise 7 Let us consider the following linear PD system:

$$\left\{\begin{array}{l} x_3\,\partial_1\,\xi_1 - x_1\,\partial_3\,\xi_1 + x_3\,\partial_2\,\xi_2 - x_2\,\partial_3\,\xi_2 - \xi_3 = 0, \\ -\xi_1 + x_1\,\partial_2\,\xi_2 - x_2\,\partial_1\,\xi_2 + x_1\,\partial_3\,\xi_3 - x_3\,\partial_1\,\xi_3 = 0, \\ x_2\,\partial_1\,\xi_1 - x_1\,\partial_2\,\xi_1 - \xi_2 + x_2\,\partial_3\,\xi_3 - x_3\,\partial_2\,\xi_3 = 0. \end{array}\right. \tag{7}$$

(C. M. Bender, G. V. Dunne, L. R. Mead, “Underdetermined systems of partial differential equations”, *Journal of Mathematical Physics*, vol. 41 no. 9 (2000), 6388-6398). We consider the first Weyl algebra $D = A_3(\mathbb{Q})$ and the system matrix R of (7) defined by

$$R = \begin{pmatrix} x_3\,\partial_1 - x_1\,\partial_3 & x_3\,\partial_2 - x_2\,\partial_3 & -1 \\ -1 & x_1\,\partial_2 - x_2\,\partial_1 & x_1\,\partial_3 - x_3\,\partial_1 \\ x_2\,\partial_1 - x_1\,\partial_2 & -1 & x_2\,\partial_3 - x_3\,\partial_2 \end{pmatrix} \in D^{3\times 3},$$

and the left D-module $M = D^{1\times 3}/(D^{1\times 3}\,R)$ finitely presented by R.

1. Compute a finite free resolution of M. Deduce the rank of M and check it again with OreRank.
2. Compute the left D-submodule $t(M)$ of M. Give a finite free presentation of $t(M)$.
3. Give a finite presentation of $M/t(M)$ and a parametrization of $M/t(M)$. Is the parametrization injective?
4. What are the module properties of $M/t(M)$?

Exercise 8 Let $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]$, $R = (\partial_1 \quad \partial_2 \quad \partial_3)$ the matrix of PD operators corresponding to the divergent operator in $\mathbb{R}^3$ and $M = D^{1\times 3}/(D\,R)$.

1. Compute a finite free resolution of M with FreeResolution.
2. Compute the formal adjoint $\widetilde{R}$ of R using Involution. Let $\widetilde{N} = D/(D^{1\times 3}\,\widetilde{R}) = D/(\partial_1, \partial_2, \partial_3)$ be the D-module defined by the gradient operator in $\mathbb{R}^3$.
3. Compute a finite free resolution of $\widetilde{N}$. Interpret the different matrices appearing in the finite free resolution.
4. Using Exti, compute $\mathrm{ext}_D^i(\widetilde{N}, D)$ for $i = 1, 2, 3$. Deduce that the D-module M is reflexive but not projective, i.e., not free by the Quillen-Suslin theorem.
5. Give a parametrization of M and recognize that it corresponds to the curl operator.
6. Using OreRank, compute the rank of M over D. Conclude that the curl parametrization is not minimal.
7. Using MinimalParametrizations, compute a few minimal parametrizations of M. Check again that they are parametrizations. Give one interest and one inconvenient of those minimal parametrizations.

Exercise 9 The purpose of the exercise is to comment pages 15-17 of K. Washizu, *Variational Methods in Elasticity & Plasticity*, Pergammon Press, 3^{rd}, 1982: “The necessary and sufficient conditions, that the six strain components can be derived from three single-valued functions as given in

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \\ &\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \end{aligned} \tag{8}$$

are called *the conditions of compatibility.* It is shown in Refs. 1 through 5, for example, that the conditions of compatibility are given in a matrix form as,

$$[R] = \begin{bmatrix} R_x & U_z & U_y \\ U_z & R_y & U_x \\ U_y & U_x & R_z \end{bmatrix} = 0,$$

where

$$\begin{aligned}
R_x &= \frac{\partial^2 \varepsilon_z}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial z^2} - \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z}, \\
R_y &= \frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} - \frac{\partial^2 \gamma_{zx}}{\partial z\, \partial x}, \\
R_z &= \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}, \\
U_x &= -\frac{\partial^2 \varepsilon_x}{\partial y\, \partial z} + \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \\
U_y &= -\frac{\partial^2 \varepsilon_y}{\partial z\, \partial x} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \\
U_z &= -\frac{\partial^2 \varepsilon_z}{\partial x\, \partial y} + \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right).
\end{aligned} \tag{9}$$

[$\cdots$] We know from Eqs. (1.4) that when the body forces are absent, the equations of equilibrium can be written as:

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0, \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0.
\end{aligned} \tag{10}$$

These equations are satisfied identically when stress components are expressed in terms of either Maxwell's stress functions χ_1, χ_2 and χ_3 defined by

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 \chi_3}{\partial y^2} + \frac{\partial^2 \chi_2}{\partial z^2}, \quad & \tau_{yz} &= -\frac{\partial^2 \chi_1}{\partial y\, \partial z}, \\
\sigma_y &= \frac{\partial^2 \chi_1}{\partial z^2} + \frac{\partial^2 \chi_3}{\partial x^2}, \quad & \tau_{zx} &= -\frac{\partial^2 \chi_2}{\partial z\, \partial x}, \\
\sigma_z &= \frac{\partial^2 \chi_2}{\partial x^2} + \frac{\partial^2 \chi_1}{\partial y^2}, \quad & \tau_{xy} &= -\frac{\partial^2 \chi_3}{\partial x\, \partial y},
\end{aligned} \tag{11}$$

or Morera's stress functions ψ_1, ψ_3 and ψ_3 defined by

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 \psi_1}{\partial y\, \partial z}, \quad & \tau_{yz} &= -\frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right), \\
\sigma_y &= \frac{\partial^2 \psi_2}{\partial z\, \partial x}, \quad & \tau_{zx} &= -\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right), \\
\sigma_z &= \frac{\partial^2 \psi_3}{\partial x\, \partial y}, \quad & \tau_{xy} &= -\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_3}{\partial z}\right).
\end{aligned} \tag{12}$$

It is interesting to note that, when these two kind of stress functions are combined such that

$$
\begin{aligned}
\sigma_x &= \frac{\partial^2 \chi_3}{\partial y^2} + \frac{\partial^2 \chi_2}{\partial z^2} - \frac{\partial^2 \psi_1}{\partial y\, \partial z}, \ldots, \\
\tau_{yz} &= -\frac{\partial^2 \chi_1}{\partial y\, \partial z} + \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right), \ldots,
\end{aligned}
\tag{13}
$$

the expressions (9) and (13) have similar forms."

1. Write the matrix R over $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]$ which corresponds to the right hand side of (8) and denote by $M = D^{1\times 3}/(D^{1\times 6}\, R)$ the corresponding D-module. Check your result by using the command ApplyMatrix.
2. Check that M is a torsion D-module by computing its rank. You can use OreRank.
3. Using the command DimensionRat, compute the Krull dimension of M. Deduce that M is a finite $\mathbb{Q}$-vector space.
4. Computing a basis of this $\mathbb{Q}$-vector space by means of KBasis. What is its dimension?
5. Using the command SyzygyModule, prove that the compatibility conditions of (8) are exactly those given in (9). Compute a finite free resolution of the D-module M.
6. What are the properties of the D-module M?
7. Deduce that M is parametrizable and give a parametrization Q.
8. Using OreRank, compute the rank of M. Is the parametrization Q minimal?
9. In the parametrization Q, select the three columns containing the factor 2. Prove that the corresponding matrix defines a minimal parametrization of M. Check that it corresponds to the Morera parametrization (12).
10. Select the three last columns of the parametrization Q of M and prove that the corresponding matrix defines a minimal parametrization of M. Check that it corresponds to the Maxwell parametrization (11).
11. Compute the formal adjoint $\widetilde{Q}$ of Q? Compare with (9).
12. What is the mathematical explanation of the last comment "It is interesting to note that, when these two kind of stress functions are combined such that (13), the expressions (9) and (13) have similar forms".

Exercise 10 Let us consider the first set of *Maxwell equations*, namely,

$$
\left\{
\begin{array}{l}
\dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\
\vec{\nabla} \,.\, \vec{B} = 0,
\end{array}
\right.
\tag{14}
$$

where $\vec{B}$ (resp., $\vec{E}$) denotes the *magnetic* (resp., *electric*) *field*. Let us consider the ring $D = \mathbb{Q}\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]$ of PD operators with rational constant coefficients, the presentation matrix R_1 of (14) defined by

$$R = \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_t & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_t & -\partial_2 & \partial_1 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 \end{pmatrix} \in D^{4\times 6},$$

and the finitely presented D-module $M = D^{1\times 6}/(D^{1\times 4}\, R)$.

1. Using the command SyzygyModule, compute $\ker_D(.R)$.
2. Compute a finite free resolution of M using the command FreeResolution.
3. What is the rank of M? Compare it with the result obtained by means of OreRank.
4. Compute the formal adjoint $\widetilde{R}$ of R by means of the command Involution.
5. Compute the $\mathrm{ext}^i_D(\widetilde{N}, D)$ for $i = 1, \ldots, 4$ of the D-module $\widetilde{N} = D^{1\times 4}/(D^{1\times 6}\, \widetilde{R})$.
6. Deduce the properties of the D-module M and find successive parametrizations. In particular, check that
$$\begin{cases} -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V = \vec{E}, \\ \vec{\nabla} \wedge \vec{A} = \vec{B}, \end{cases}$$
is a parametrization of M.
7. Is the parametrization of M minimal? If not, compute some of minimal parametrizations using the command MinimalParametrizations. Check again that they are parametrizations of M.

Exercise 11 Study the algebraic properties of the following linear OD time-delay systems:

$$\begin{cases} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\,\zeta\,\omega\, x_3(t) - \omega^2\, u(t) = 0, \end{cases} \tag{15}$$

(A. Manitius, "Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulations", *IEEE Trans. Autom. Contr.*, 29 (1984), 1058-1068),

$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\, \phi_1(t) - \eta_1\, \psi_1(t) - \eta_2\, \phi_2(t) + \eta_2\, \psi_2(t) = 0, \\ \phi_1(t - 2\, h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\, h_2) - v(t - h_2) = 0, \end{cases} \tag{16}$$

(see H. Mounier, J. Rudolph, M. Fliess, P. Rouchon, "Tracking control of a vibrating string with an interior mass viewed as delay system", *ESAIM COCV*, 3 (1998), 315-321),

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{cases} \tag{17}$$

(H. Mounier, J. Rudolph, M. Petitot, M. Fliess, "A flexible rod as a linear delay system", in *Proceedings of* 3^{rd} *European Control Conference*, Rome (Italy), 1995).

Exercise 12 Let $D = A_1(\mathbb{Q})$ be the first Weyl algebra and $M = D^{1\times 2}/(D^{1\times 2}\, R)$ the left D-module finitely presented by the following matrix:

$$R = \begin{pmatrix} -t^2 & t\,\partial - 1 \\ -(t\,\partial + 2) & \partial^2 \end{pmatrix}.$$

1. Using the commands LeftInverse and RightInverse, check that R does not admit a left or a right-inverse over D. Deduce that M is not reduced to 0.
2. Using Free Resolution, check that M admits the following finite free resolution:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D^{1\times 2} \xrightarrow{\pi} M \longrightarrow 0, \quad R_2 = (\partial \quad -t).$$

3. Check that R_2 admits a right-inverse S_2 over D using RightInverse.
4. Check that there exists $S \in D^{2\times 2}$ such that $S_2\, R_2 + R\, S = I_2$. To do that, first compute $F = \widetilde{I_2 - S_2\, R_2}$ using Involution, then, using the command Factorize, compute a matrix $\widetilde{S} \in D^{2\times 2}$ such that $F = \widetilde{S}\, \widetilde{R}$ and finally compute S by means of Involution.
5. Post-multiplying the identity $S_2\, R_2 + R\, S = I_2$, deduce that R admits a generalized inverse S, namely, $R\, S\, R = R$. Check this last result by means of the command Generalized-Inverse. Deduce that M is a projective left D-module.
6. Check that last result by proving that $\mathrm{ext}^1_D(\widetilde{N}, D) = 0$, where $\widetilde{N} = D^{1\times 2}/(D^{1\times 2}\, \widetilde{R})$.
7. Compute a parametrization of M. Is it injective?
8. Deduce that M is a free left D-module of rank 1 and compute a basis.

Exercise 13 Let us consider the matrix of PD operators corresponding to the infinitesimal transformations of the Lie pseudogroup defining the contact transformations (see Pommaret's books)

$$R = \frac{1}{2}\begin{pmatrix} x_2\,\partial_1 & 2\,(x_2\,\partial_2 + 1) & 2\,x_2\,\partial_3 + \partial_1 \\ -x_2\,\partial_2 - 3 & 0 & \partial_2 \\ -2\,\partial_1 - x_2\,\partial_3 & -2\,\partial_2 & -\partial_3 \end{pmatrix} \in D^{3\times 3},$$

where $D = A_3(\mathbb{Q})$ is the first Weyl algebra. The purpose of this exercise is to study the module properties of M.

1. Compute the formal adjoint $\widetilde{R}$ of R by means of Involution.
2. Using the command Exti, compute the $\mathrm{ext}^i_D(\widetilde{N}, D)$, for $i = 1, 2, 3$, of the left D-module $\widetilde{N} = D^{1\times 3}/(D^{1\times 3}\, \widetilde{R})$. Conclude on the property of the left D-module M.
3. Give a parametrization of M.
4. Using the command Quotient or Factorize, check this last result.
5. Is the parametrization of M injective? If so, conclude on the properties of M and compute a basis of M.