Exercises: Monge problem and basis computation

Alban Quadrat*

Exercise 1 If D is a left principal ideal domain and $R \in D^{q \times p}$ a full row rank matrix, then, using the Jacobson normal form of R, find an easy way to compute a Monge parametrization of a linear system defined by R.

Exercise 2 We consider the following time-invariant linear OD system

$$\left\{ \begin{array}{l} \ddot{x}_1(t) + \frac{g}{l}\, x_1(t) - \frac{g}{l}\, u(t) = 0, \\ \ddot{x}_2(t) + \frac{g}{l}\, x_2(t) - \frac{g}{l}\, u(t) = 0, \end{array} \right. \tag{1}$$

describing the linearization of a bipendulum of lengths l moving horizontally. Compute a Monge parametrization of (2).

Exercise 3 We consider the following time-varying OD linear system

$$\left\{ \begin{array}{l} \rho\, \frac{\partial \theta}{\partial \rho} + \frac{1}{2}\,(\theta + K) + \frac{(\lambda + \mu)}{2} \left(\rho\, \frac{\partial \sigma}{\partial \rho} - \sigma \right) = 0, \\ 2\,\rho\, \frac{\partial \theta}{\partial \rho} + \rho\, \frac{\partial K}{\partial \rho} + 3\, K - (3\,\lambda + 2\,\mu)\, \sigma = 0, \\ \lambda\, \sigma + 2\,\mu \left(G + \rho\, \frac{\partial G}{\partial \rho} \right) - \rho\, \frac{\partial \theta}{\partial \rho} - K = 0, \end{array} \right. \tag{2}$$

where σ, G, θ and K are functions of $\rho = \sqrt{x^2 + y^2 + z^2}$ and λ and μ are real parameters, considered in J. Hadamard, “Sur l’équilibre des plaques élastiques circulaires libres ou appuyées et celui de la sphère isotrope”, *Annales Scientifiques de l’Ecole Normale Supérieure*, 18 (1901), 313-324. Compute a Monge parametrization of (2).

Exercise 4 Compute a Monge parametrization of the model of a flexible rod with a torque

$$\left\{ \begin{array}{l} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\, \dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{array} \right. \tag{3}$$

studied in H. Mounier, J. Rudolph, M. Petitot, M. Fliess, “A flexible rod as a linear delay system”, in *Proceedings of* 3^{rd} *European Control Conference*, Rome (Italy), 1995.

*INRIA Sophia Antipolis, 2004, Route des Lucioles, BP 93, 06902 Sophia Antipolis cedex, France. Alban.Quadrat@sophia.inria.fr, http://www-sop.inria.fr/members/Alban.Quadrat/index.html

Exercise 5 Let us consider the linear OD time-delay system

$$\begin{cases} y_1(t-2\,h)+y_2(t)-2\,\dot{u}(t-h)=0, \\ y_1(t)+y_2(t-2\,h)-2\,\dot{u}(t-h)=0, \end{cases} \tag{4}$$

describing a model of a tank containing a fluid and subjected to a one-dimensional horizontal (F. Dubois, N. Petit, P. Rouchon, "Motion planning and nonlinear simulations for a tank containing a fluid", in the proceedings of the 5^{th} *European Control Conference*, Karlsruhe (Germany), 1999). Compute a Monge parametrization of (4).

Exercise 6 We consider the linear PD system $\operatorname{grad}(\operatorname{div}\vec{v}) = \vec{0}$ appearing in mathematical physics, where $\operatorname{grad} = (\partial_1 \quad \partial_2 \quad \partial_3)^T$ (resp., $\operatorname{div} = \operatorname{grad}^T$) denotes the gradient (resp., divergence) operator in $\mathbb{R}^3$, or equivalently:

$$\begin{cases} \partial_1\,(\partial_1\,v_1+\partial_2\,v_2+\partial_3\,v_3)=0, \\ \partial_2\,(\partial_1\,v_1+\partial_2\,v_2+\partial_3\,v_3)=0, \\ \partial_3\,(\partial_1\,v_1+\partial_2\,v_2+\partial_3\,v_3)=0. \end{cases} \tag{5}$$

For instance, in acoustic, the speed $\vec{v}$ satisfies the PD linear system $\partial_t\,\vec{v}/c^2 - \operatorname{grad}(\operatorname{div}\vec{v}) = \vec{0}$, where c denotes the speed of sound. Compute a Monge parametrization of (5).

Exercise 7 Compute the projective dimension of the $D = \mathbb{Q}(x_1, x_2)$-module

$$M = D/(x_2^2, x_1\,x_2 - 1).$$

Conclude about the two finite presentations of the D-module M given in Jean-François' lectures.

Exercise 8 Using the command TwoGenerators of the package Stafford, compute two generators of the left ideal of $A_3(\mathbb{Q})$ defined by $I = D\,\partial_1 + D\,\partial_2 + D\,\partial_3$. Similar question with $I = D\,\partial_1^2 + D\,\partial_1\,\partial_2 + D\,\partial_2^2$.

Exercise 9 Using the commands InjectiveParametrization and BasisOfModule of the package Stafford, compute an injective parametrization and a basis of the left $A_1(\mathbb{Q})$-module associated with the following time-varying OD system:

$$\begin{cases} \dot{x}_2(t) - u_2(t) = 0, \\ \dot{x}_1(t) - t\,u_1(t) = 0. \end{cases}$$

Similar question with the PDE $\vec{\nabla}.\,\vec{\eta} + x_3\,\eta_1 = 0$, i.e., $\partial_1\,\eta_1 + \partial_2\,\eta_2 + \partial_3\,\eta_3 + x_3\,\eta_1 = 0$.

Exercise 10 Using the commands InjectiveParametrization and BasisOfCokernelModule of the package Stafford, compute an injective parametrization of the following linear OD time-delay system:

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\,x_2(t) = 0. \end{cases}$$

Similar question with the following linear OD time-delay system:

$$\begin{cases} \dot{y}_1(t) - y_1(t-h) + 2\,y_1(t) + 2\,y_2(t) - 2\,u(t-h) = 0, \\ \dot{y}_1(t) + \dot{y}_2(t) - \dot{u}(t-h) - u(t) = 0. \end{cases}$$

Similar question with the following matrices:

$$R = (x_1\,x_2^2 + 1 \quad 3\,x_2/2 + x_1 - 1 \quad 2\,x_1\,x_2),$$
$$R = (1 - x\,y \quad x^2 \quad y^2),$$
$$R = (x_1^2 - x_2^2 - 1 \quad x_1^2 + x_2^2 - 1 \quad x_1 - x_2).$$

Note. QuillenSuslin uses `LinearAlgebra`.

References:

- F. Chyzak, A. Quadrat, D. Robertz, "Effective algorithms for parametrizing linear control systems over Ore algebras", *Appl. Algebra Engrg. Comm. Comput.*, 16 (2005), 319-376.
- F. Chyzak, A. Quadrat, D. Robertz, "OreModules: A symbolic package for the study of multidimensional linear systems", in the book *Applications of Time-Delay Systems*, J. Chiasson and J. -J. Loiseau (Eds.), Lecture Notes in Control and Information Sciences (LNCIS) 352, Springer, 2007, 233-264, http://wwwb.math.rwth-aachen.de/OreModules.
- A. Fabiańska, A. Quadrat, "Applications of the Quillen-Suslin theorem in multidimensional systems theory", chapter of the book *Gröbner Bases in Control Theory and Signal Processing*, H. Park and G. Regensburger (Eds.), Radon Series on Computation and Applied Mathematics 3, de Gruyter publisher, 2007, 23-106, http://wwwb.math.rwth-aachen.de/QuillenSuslin.
- A. Quadrat, D. Robertz, "On the Monge problem and multidimensional optimal control", *Proceedings of MTNS06*, Kyoto (Japan), (20-24/07/06).
- A. Quadrat, D. Robertz, "Computation of bases of free modules over the Weyl algebras", *Journal of Symbolic Computation*, 42 (2007), 1113-1141, http://wwwb.math.rwth-aachen.de/OreModules.
- A. Quadrat, D. Robertz, "On the Baer extension problem for multidimensional linear systems", *INRIA report 6307*, 2007.
- A. Quadrat, D. Robertz, "Baer's extension problem for multidimensional linear systems", *Proceedings of MTNS08*, Virginia (USA), (28/07-01/08/08).