Reduction of linear systems based on Serre's theorem

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Smith forms and reduction problem

- When is a polynomial matrix equivalent to its Smith form

$$\mathrm{diag}(\gamma_1, \ldots, \gamma_q),$$

where $\gamma_i = \alpha_i / \alpha_{i-1}$, $\alpha_i =$ gcd of the $i \times i$-minors of R $(\alpha_0 = 1)$?

- Theorem: (Boudellioua, 05) Let $D = \mathbb{R}[x_1, \ldots, x_n]$, $R \in D^{p \times p}$ be a full row rank matrix. Then, the assertions are equivalent

1. There exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_p(D)$ satisfying:

$$U\,R\,V = \begin{pmatrix} I_{p-1} & 0 \\ 0 & \det R \end{pmatrix}.$$

2. There exists $\Lambda \in D^p$ admitting a left-inverse over D such that $P = (R \quad -\Lambda) \in D^{p \times (p+1)}$ admits a right-inverse over D.

Outline of the lecture

- The purpose of this lecture is to:

1. Explain the relations between the previous result and a Serre's theorem (Séminaire Dubreuil-Pisot 60-61) based on the concept of Baer extensions.
2. Simplify and generalize this result to a general full row rank matrix $R \in D^{q \times p}$ over an Ore algebra D.
3. Constructively solve the problem for important cases.

- Implementation in the forthcoming package SERRE.

- For the reduction problem, the next results are more efficient than the general ones obtained in Cluzeau-Q., LAA 08, based on idempotents of the endomorphism ring $\mathrm{end}_D(M)$, where:

$$M = D^{1\times p}/(D^{1\times q}\, R) \quad (\text{OREMORPHISMS}).$$

Generalization of a Serre's result

• Theorem: Let $R \in D^{q \times p}$ be a full row rank matrix, $\Lambda \in D^q$, $P = (R \quad -\Lambda)$ and the two left D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $E = D^{1\times (p+1)}/(D^{1\times q}\, P)$ defining an extension of D by M:

$$0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0.$$

We have the equivalent assertions:

1. E is stably free of rank $p+1-q$: $E \oplus D^{1\times q} \cong D^{1\times (p+1)}$.
2. $P = (R \quad -\Lambda)$ admits a right-inverse over D.
3. $\mathrm{ext}^1_D(E, D) = D^q/(P\, D^{p+1}) = 0$.
4. $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$ is the cyclic right D-module generated by $\rho(\Lambda)$, where ρ denotes the projection:

$$\rho : D^q \longrightarrow \mathrm{ext}^1_D(M, D) = D^q/(R\, D^p).$$

The previous equivalences only depend on the residue class $\rho(\Lambda)$.

Main result

- Theorem: Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ satisfying that there exists $U \in \mathrm{GL}_{p+1}(D)$ such that:

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

Let us denote by

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \in \mathrm{GL}_{p+1}(D),$$

where:

$$S_1 \in D^{p \times q}, \; S_2 \in D^{1 \times q}, \; Q_1 \in D^{p \times (p+1-q)}, \; Q_2 \in D^{1 \times (p+1-q)}.$$

Then, we have:

$$\boxed{M = D^{1 \times p}/(D^{1 \times q}\, R) \cong L = D^{1 \times (p+1-q)}/(D\, Q_2)}$$

The converse result also holds. These results only depend on:

$$\rho(\Lambda) \in \mathrm{ext}^1_D(M, D) = D^q/(R\, D^p).$$

Corollaries

- Corollary: We have the following isomorphism:

$$\begin{array}{rcl} \psi : M = D^{1\times p}/(D^{1\times q}\, R) & \longrightarrow & L = D^{1\times (p+1-q)}/(D\, Q_2) \\ \pi(\lambda) & \longmapsto & \kappa(\lambda\, Q_1). \end{array}$$

Its inverse $\psi^{-1} : L \longrightarrow M$ is defined by $\psi^{-1}(\kappa(\mu)) = \pi(\mu\, T_1)$:

$$U^{-1} = \left(\begin{array}{cc} R & -\Lambda \\ T_1 & T_2 \end{array}\right), \quad T_1 \in D^{(p+1-q)\times p}, \quad T_2 \in D^{(p+1-q)}.$$

- Corollary: Let $\mathcal{F}$ be a left D-module and the linear systems:

$$\left\{ \begin{array}{l} \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}, \\ \ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{p+1-q} \mid Q_2\, \zeta = 0\}. \end{array}\right.$$

Then, we have the isomorphism $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$ and:

$$\ker_{\mathcal{F}}(R.) = Q_1\, \ker_{\mathcal{F}}(Q_2.), \quad \ker_{\mathcal{F}}(Q_2.) = T_1\, \ker_{\mathcal{F}}(R.).$$

Ring conditions

- Proposition: Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that $P = (R \quad -\Lambda) \in D^{q \times (p+1)}$ admits a right-inverse over D. Moreover, if D is either a

  1. principal left ideal domain,
  2. commutative polynomial ring with coefficients in a field,
  3. Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - q \geq 1$,

  then there exists $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $P\,U = (I_q \quad 0)$.

- The matrix U can be obtained by means of:
  1. a Jacobson form (JACOBSON),
  2. the Quillen-Suslin theorem (QUILLENSUSLIN),
  3. Stafford's theorem (STAFFORD).

Example: Wind tunnel model

- The wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(a, k, \omega, \zeta)[\partial, \delta]$, the system matrix

$$R = \begin{pmatrix} \partial + a & -k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3\times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 3}\,R)$.

- The D-module $\mathrm{ext}^1_D(M, D) = D^3/(R\,D^4)$ is a $\mathbb{Q}(a, k, \omega, \zeta)$-vector space of dimension 1 and $\rho((1 \quad 0 \quad 0)^T)$ is a basis.

Example: Wind tunnel model

- Let us consider $\Lambda = (1 \quad 0 \quad 0)^T$ and $P = (R \quad -\Lambda)$.
- The matrix P admits the following right-inverse S:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{\partial + 2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} \\ -1 & 0 & 0 \end{pmatrix} \in D^{5 \times 3}.$$

- According to Quillen-Suslin theorem, $E = D^{1\times 5}/(D^{1\times 3}\,P)$ is free D-module of rank 2.

Example: Wind tunnel model

- Computing a basis of E, we obtain that $U \in \mathrm{GL}_5(D)$,

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & 0 & \omega^2\,\partial \\ 0 & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \partial^2+2\,\zeta\,\omega\,\partial+\omega^2 \\ -1 & 0 & 0 & -(\partial+a) & -\omega^2\,k\,a\,\delta \end{pmatrix},$$

satisfies that $P\,U = (I_3 \quad 0)$ (OreModules, QuillenSuslin).

- The wind tunnel model is equivalent to the sole equation:

$$(\partial+a)\,\zeta_1+\omega^2\,k\,a\,\delta\,\zeta_2=0$$

$$\Leftrightarrow \quad \dot{\zeta}_1(t)+a\,\zeta_1(t)+\omega^2\,k\,a\,\zeta_2(t-h)=0.$$

Algorithmic issue

1. Consider an ansatz $\Lambda \in D^q$ of a given order.
2. Compute a Gröbner basis of $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$.
3. Compute the normal form $\overline{\Lambda} \in D^q$ of $\rho(\Lambda)$.
4. Compute the obstructions to freeness of the left D-module $\overline{E} = D^{1\times(p+1)}/(D^{1\times q}\,(R \quad -\overline{\Lambda}))$ (π-polynomials).
5. Solve the systems in the arbitrary coefficients obtained by making the obstructions vanish.
6. If a solution $\Lambda_\star$ exists, then compute $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $(R \quad -\Lambda)\, U = (I_q \quad 0)$ and return $Q_2 \in D^{1\times(p+1-q)}$.

- Remark: If $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$ is 0-dimensional, then we take $\overline{\Lambda}$ to be a generic combination of a basis of $\mathrm{ext}^1_D(M, D)$.

Example: Transmission line

- Let us consider a general transmission line:

$$\begin{cases} \dfrac{\partial V}{\partial x} + L\,\dfrac{\partial I}{\partial t} + R'\,I = 0, \\ C\,\dfrac{\partial V}{\partial t} + G\,V + \dfrac{\partial I}{\partial x} = 0. \end{cases}$$

- Let $D = \mathbb{Q}(L, R', C, G)[\partial_t, \partial_x]$ and $M = D^{1\times 2}/(D^{1\times 2}\,R)$, where:

$$R = \begin{pmatrix} \partial_x & L\,\partial_t + R' \\ C\,\partial_t + G & \partial_x \end{pmatrix} \in D^{2\times 2}.$$

- We consider $A = D[\alpha, \beta]$, $\Lambda = (\alpha \quad \beta)^T$, $P = (R \quad -\Lambda) \in A^{2\times 3}$.
- If we denote by $N = A^{1\times 2}/(A^{1\times 3}\,P^T)$, then we have:

$$\mathrm{ext}^1_A(N, A) = 0, \quad \mathrm{ext}^2_A(N, A) = A/(L_1, L_2),$$

$$\begin{cases} L_1 = (C\,\alpha^2 - L\,\beta^2)\,\partial_t + G\,\alpha^2 - R'\,\beta^2, \\ L_2 = (C\,\alpha^2 - L\,\beta^2)\,\partial_x + (L\,G - R'\,C)\,\alpha\,\beta. \end{cases}$$

Example: Transmission line

- We consider $\beta = C \neq 0$, $\alpha^2 = L\,C \neq 0$ and $R'\,C - L\,G \neq 0$.
- Over $B = D[\alpha]/(\alpha^2 - L\,C)$, we have $\mathrm{ext}^2_B(B \otimes_D N, B) = 0$, i.e., $E = B^{1\times 3}/(B^{1\times 2}\,P)$ is a projective B-module, and thus, is free.
- Then, we have:

$$S = \frac{1}{R'\,C - L\,G} \begin{pmatrix} -\alpha & L \\ -C & \alpha \\ -\frac{(\alpha\,\partial_x + C\,L\,\partial_t + L\,G)}{\alpha} & \frac{(\alpha\,\partial_x + L\,C\,\partial_t + R'\,C)}{C} \end{pmatrix},$$

$$Q_1 = \alpha\,\partial_x - L\,C\,\partial_t - R'\,C \quad C\,\partial_x - \alpha\,C\,\partial_t - \alpha\,G,$$

$$Q_2 = \partial_x^2 - L\,C\,\partial_t^2 - (L\,C + R'\,C)\,\partial_t - R'\,G.$$

- The transmission line is equivalent to the sole equation:

$$(\partial_x^2 - L\,C\,\partial_t^2 - (L\,C + R'\,C)\,\partial_t - R'\,G)\,Z(t,x) = 0.$$

Torsion-free degree

- Theorem: $\mathrm{ext}^1_D(M, D)$ is 0-dimensional iff the torsion-free degree of M is $n-1$ (the last but one step before projectiveness).

1. $n = 2$, M is torsion-free,
2. $n = 3$, M is reflexive, ...

Then, we can constructively check whether or not M ($\ker_{\mathcal{F}}(R.)$) can be generated by 1 relation (1 equation)!

- If $M = D^{1\times p}/(D^{1\times q}\, R)$ is free of rank $p-q$, i.e., there exists $V \in \mathrm{GL}_p(D)$ satisfying that $R\,V = (I_q \quad 0)$, then we have:

$$(R \quad 0)\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} = (I_q \quad 0),$$

$$\Rightarrow \; M \cong D^{1\times(p+1-q)}/(D\,(0 \; \ldots \; 1)) \cong \; D^{1\times(p-q)} \quad (0 \text{ equation!}).$$

Example: String with an interior mass

- Model of a string with an interior mass (Fliess et al, COCV 98):

$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\, \phi_1(t) - \eta_1\, \psi_1(t) - \eta_2\, \phi_2(t) + \eta_2\, \psi_2(t) = 0, \\ \phi_1(t - 2\, h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\, h_2) - v(t - h_2) = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(\eta_1, \eta_2)[\partial, \sigma_1, \sigma_2]$, the system matrix

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial + \eta_1 & \partial - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} \in D^{4 \times 6},$$

and the finitely presented D-module $M = D^{1 \times 6} / (D^{1 \times 4}\, R)$.

Example: String with an interior mass

- We can prove that M is a reflexive D-module (OreModules)

$\Rightarrow$ the D-module $\mathrm{ext}^1_D(M, D) = D^4/(R\, D^6)$ is a $\mathbb{Q}(\eta_1, \eta_2)$-vector space of dimension 1 and $\rho((0 \quad 1 \quad 0 \quad 0)^T)$ is a basis.

- Let us consider $\Lambda = (0 \quad 1 \quad 0 \quad 0)^T$ and $P = (R \quad -\Lambda)$.

- The matrix P admits the following right-inverse S:

$$S = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & -\sigma_1 & 0 \\ -\sigma_2 & 0 & 0 & -\sigma_2 \\ -\eta_2 & -1 & -2\,\eta_1 & -2\,\eta_2 \end{pmatrix} \in D^{7\times 4}.$$

$\Rightarrow$ the D-module $E = D^{1\times 7}/(D^{1\times 4}\, P)$ is free of rank 3.

Example: String with an interior mass

- Computing a basis of N, we obtain that $U \in \mathrm{GL}_7(D)$,

$$U = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & \sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\sigma_2 \\ -1 & 0 & 0 & -1 & -1 & 0 & \sigma_2 \\ 0 & 0 & -\sigma_1 & 0 & -\sigma_1 & \sigma_1^2 - 1 & 0 \\ -\sigma_2 & 0 & 0 & -\sigma_2 & -\sigma_2 & 0 & \sigma_2^2 - 1 \\ -\eta_2 & -1 & -2\,\eta_1 & -2\,\eta_2 & -(\partial + \eta_1 + \eta_2) & 2\,\eta_1\,\sigma_1 & 2\,\eta_2\,\sigma_2 \end{pmatrix},$$

satisfies that $P\,U = (I_4 \quad 0)$ (OreModules, QuillenSuslin).

- The string model is then equivalent to the sole equation:

$$(\partial + \eta_1 + \eta_2)\,\zeta_1 - 2\,\eta_1\,\sigma_1\,\zeta_2 - 2\,\eta_2\,\sigma_2\,\zeta_3 = 0$$

$$\Leftrightarrow\ \dot{\zeta}_1(t) + (\eta_1 + \eta_2)\,\zeta_1(t) - 2\,\eta_1\,\zeta_2(t - h_1) - 2\,\eta_2\,\zeta_3(t - h_2) = 0.$$

Example: Stress tensor (elasticity)

- Let $D = \mathbb{Q}[\partial_x, \partial_y]$ and $M = D^{1\times 3}/(D^{1\times 2}\, R)$, where:

$$R = \begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix} \in D^{2\times 3}.$$

- The D-module $\mathrm{ext}^1_D(M, D) = D^2/(R\, D^3)$ is a $\mathbb{Q}$-vector space of dimension 3 with basis $\{\rho((1 \quad 0)^T), \rho((0 \quad 1)^T), \rho((0 \quad \partial_x)^T)\}$.
- Let us consider $\Lambda = (a \quad b + c\,\partial_x)^T$, $P = (R \quad -\Lambda)$.
- If we denote by $A = D[a, b, c]$ and $N = A^2/(P\, A^4)$, then we get:

$$\mathrm{ext}^1_A(N, A) = 0, \quad \mathrm{ext}^2_A(N, A) = A/(\partial_x, \partial_y).$$

- Hence, $E = A^{1\times 4}/(A^{1\times 2}\, P)$ is never a projective A-module and

$$\begin{cases} \partial_x\, \sigma^{11} + \partial_y\, \sigma^{12} = 0, \\ \partial_x\, \sigma^{12} + \partial_y\, \sigma^{22} = 0, \end{cases}$$

cannot be defined by a sole equation! $(\mu(M) = 3)$.

Equivalence

• Theorem: If $\Lambda \in D^q$ admits a left-inverse $\Gamma \in D^{1\times q}$, i.e., $\Gamma\,\Lambda = 1$, then Q_1 admits the left-inverse $T_1 + T_2\,\Gamma\,R \in D^{(p+1-q)\times p}$ and the left D-module $\ker_D(.Q_1)$ is stably free of rank $q-1$.

If the left D-module $\ker_D(.Q_1)$ is free, then $\exists\, Q_3 \in D^{p\times(q-1)}$ s.t.:

$$V = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D).$$

Then, we have $W = (R\,Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$,

$$W^{-1} = \begin{pmatrix} Y_3\,S_1 \\ -S_2 + Q_2\,Y_1\,S_1 \end{pmatrix},$$

with $V^{-1} = (Y_3^T \quad Y_1^T)^T$, $Y_3 \in D^{(q-1)\times p}$, $Y_1 \in D^{(p-q+1)\times p}$ and:

$$\boxed{W^{-1}\,R\,V = \begin{pmatrix} I_{q-1} & 0 \\ 0 & Q_2 \end{pmatrix}}$$

Example: Wind tunnel model

- The vector $\Lambda = (1 \quad 0 \quad 0)^T$ admits the left-inverse $\Gamma = \Lambda^T$.
- We compute $Q_3 \in D^{2\times 2}$ such that $V = (Q_3^T \quad Q_1^T) \in \mathrm{GL}_4(D)$:

$$V = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & \omega^2\,\partial \\ -\frac{1}{\omega^2} & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & 0 & \partial^2 + 2\,\zeta\,\omega\,\partial + \omega^2 \end{pmatrix}.$$

- We have $W = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(D)$ and:

$$W^{-1}\,R\,V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(\partial + a) & -\omega^2\,k\,a\,\delta \end{pmatrix}.$$

Example: String with an interior mass

- The vector $\Lambda = (0 \quad 1 \quad 0 \quad 0)^T$ admits the left-inverse $\Gamma = \Lambda^T$.
- We compute $Q_3 \in D^{6\times 3}$ such that $V = (Q_3^T \quad Q_1^T) \in \mathrm{GL}_6(D)$:

$$V = \begin{pmatrix} 1 & 0 & 0 & -1 & \sigma_1 & 0 \\ 0 & -1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma_2 \\ 0 & -1 & -1 & -1 & 0 & \sigma_2 \\ 0 & 0 & 0 & -\sigma_1 & \sigma_1^2 - 1 & 0 \\ 0 & -\sigma_2 & -\sigma_2 & -\sigma_2 & 0 & \sigma_2^2 - 1 \end{pmatrix}.$$

$$W = (R\, Q_3 \quad \Lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial + \eta_1 & -\partial + \eta_1 - \eta_2 & -2\,\eta_2 & 1 \\ \sigma_1^2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}_4(D).$$

$$\Rightarrow W^{-1}\, R\, V = \mathrm{diag}(1, 1, 1, (-(\partial + \eta_1 + \eta_2) \quad 2\,\eta_1\,\sigma_1 \quad 2\,\eta_2\,\sigma_2)).$$

Conclusion

- The previous results can be extended to the cases

$$M \cong L = D^{1\times(p-m)}/(D^{1\times(q-m)}\, Q_2),\ \ Q_2 \in D^{(q-m)\times(p-m)},$$
$$W^{-1}\, R\, V = \mathrm{diag}(I_m, \star),$$

using the homological algebraic classical result:

$$\mathrm{ext}^1_D(M, D^{1\times(q-m)}) \cong \mathrm{ext}^1_D(M, D) \otimes_D D^{1\times(q-m)}.$$

- We then consider $\Lambda \in D^{q-m}$, $P = (R \quad -\Lambda) \in D^{q\times(p+q-m)}$.

- The results only depend on the residue classes of the columns of Λ in the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$.

- If $\mathrm{ext}^1_D(M, D)$ is 0-dimensional, then a minimal presentation of M, i.e., a minimal representation of $\ker_{\mathcal{F}}(R.)$, can be computed

(constellations (Levandovskyy-Zerz 07)).

Appendix

A model of a two reflector antenna

Mounier, Rouchon, Rudolph, European Journal of Automation, 97.

$$R = \begin{pmatrix} \partial & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial + \frac{K_2}{T_e} & 0 & 0 & 0 & 0 & -\frac{K_p}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta \\ 0 & 0 & \partial & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial + \frac{K_2}{T_e} & 0 & 0 & -\frac{K_c}{T_e}\,\delta & -\frac{K_p}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta \\ 0 & 0 & 0 & 0 & \partial & -K_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial + \frac{K_2}{T_e} & -\frac{K_c}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta & -\frac{K_p}{T_e}\,\delta \end{pmatrix}$$

A model of a two reflector antenna

$$\overline{R} = V\,R\,W =$$

$$\left(\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta \end{array}\right)$$

A model of a two reflector antenna

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{T_e\,\partial + K_2}{K_1\,T_e} & 1 \end{pmatrix}$$

$$W = \begin{pmatrix} 0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 & 0 & 0 \\ -K_1^{-1} & 0 & 0 & T_e\,\partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 \\ 0 & -K_1^{-1} & 0 & 0 & 0 & T_e\,\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 \\ 0 & 0 & -K_1^{-1} & 0 & 0 & 0 & 0 & T_e\,\partial & 0 \\ 0 & 0 & 0 & 0 & T_e\,(K_p + K_c) & 0 & -K_c\,T_e & 0 & -K_c\,T_e \\ 0 & 0 & 0 & 0 & -K_c\,T_e & 0 & T_e\,(K_p + K_c) & 0 & -K_c\,T_e \\ 0 & 0 & 0 & 0 & -K_c\,T_e & 0 & -K_c\,T_e & 0 & T_e\,(K_p + K_c) \end{pmatrix}$$

Electric line (Mounier, PhD thesis 95)

$$R = \begin{pmatrix} \partial + a_0 & -(a_4\,\partial + a_0)\,\delta & -a_0 & 0 & -b_0\,\partial \\ -(a_5\,\partial + a_1)\,\delta & \partial + a_1 & 0 & a_1 & 0 \\ a_2 & -a_2\,a_4\,\delta & \partial & 0 & -a_2\,b_0 \\ a_3\,a_5\,\delta & -a_3 & 0 & \partial & 0 \end{pmatrix}$$

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \alpha & \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{cases} \alpha = a_0\,a_2\,(a_5\,\delta^2\,\partial^2 + a_1\,\delta^2\,\partial + a_1\,a_3\,a_5\,\delta^2 - \partial^2 - a_1\,\partial - a_1\,a_3), \\ \beta = (a_5\,\partial^2 + a_1\,\partial + a_1\,a_3\,a_5)\,(\partial^2 + a_0\,a_2). \end{cases}$$

Electric line (Mounier, PhD thesis 95)

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -a_0\,a_2 & a_0\,a_2\,\delta & \partial^2 + a_0\,a_2 & 0 & 0 \\ a_2 & -a_2\,a_4\,\delta & \partial & 0 & -a_2\,b_0 \\ (a_5\,\partial + a_1)\,\delta & -(\partial + a_1) & 0 & -a_1 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} -a_2\,\left(a_5\,\partial^2 + a_1\,\partial + a_1\,a_3\,a_5\right)\,\delta^2 & -a_0\,a_2\,\partial & (a_5\,\partial^2 + a_1\,\partial + a_1\,a_3\,a_5)\,\delta\,\partial & a_0\,a_1\,a_2 \\ -a_2 & 0 & \partial & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$