Introduction to constructive algebraic analysis

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Mathematical systems theory

- Systems theory aims at studying general systems of equations
  1. ordinary/partial differential equations, difference equations, time-delay equations... = functional equations,
  2. determined, overdetermined, underdetermined (open, close),
  3. linear, time-varying, nonlinear systems...

coming from engineering sciences (e.g., electromagnetism, elasticity, hydrodynamics), mathematical physics, control theory...

- The purpose of these talks is to give a short introduction to algebraic systems theory which studies linear functional systems by means of constructive algebra, module theory, homological algebra.

- Algebraic systems theory can be traced back to the development of algebraic analysis by Malgrange, Sato, Kashiwara...

Outline

The purpose of this talk is to develop the following 3 ideas:

1. A large class of linear functional systems can be studied by means of a non-commutative polynomial approach over skew polynomial rings and Ore algebras of functional operators.

   Non-commutative Gröbner bases $\Rightarrow$ constructive approach.

2. Algebraic analysis is a natural mathematical framework for the intrinsic study of linear systems theory (module theory).

3. Constructive homological algebra allows us to develop algorithms and symbolic packages dedicated to the study of the structural properties of linear functional systems.

Matrices of differential operators

- Newton: Fluxion calculus (1666) ("dot-age")

$$\left\{\begin{array}{l} \ddot{x}_1(t) + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \ddot{x}_2(t) + \alpha\, x_2(t) - \alpha\, u(t) = 0, \end{array}\right. \qquad \alpha = g/l.$$

- Leibniz: Infinitesimal calculus (1676) ("d-ism")

$$\left\{\begin{array}{l} \frac{d^2 x_1(t)}{dt^2} + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \frac{d^2 x_2(t)}{dt^2} + \alpha\, x_2(t) - \alpha\, u(t) = 0. \end{array}\right.$$

- Boole: Operational calculus (1859-60)

$$\begin{pmatrix} \frac{d^2}{dt^2} + \alpha & 0 & -\alpha \\ 0 & \frac{d^2}{dt^2} + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} = 0.$$

$\Rightarrow$ Ring of differential operators $D = \mathbb{Q}(\alpha)\left[\frac{d}{dt}\right]$:

$$\sum_{i=0}^{n} a_i \left(\frac{d}{dt}\right)^i \in D, \quad a_i \in \mathbb{Q}(\alpha), \quad \left(\frac{d}{dt}\right)^i = \frac{d}{dt} \circ \ldots \circ \frac{d}{dt} = \frac{d^i}{dt^i}.$$

Functional operators

- Differential operators: $\left(\sum_{j=0}^{m} b_j(t)\, \frac{d^j}{dt^j}\right)\, \left(\sum_{i=0}^{n} a_i(t)\, \frac{d^i}{dt^i}\right)$

$$\frac{d}{dt}\,(a\,y) = a\,\frac{d}{dt}\,y + \left(\frac{da}{dt}\right)\,y \quad \Rightarrow \quad \frac{d}{dt}\,a\,\cdot = a\,\frac{d}{dt}\,\cdot + \frac{da}{dt}\,\cdot$$

- Shift operators: $\delta\, a(t) = a(t-h), \quad \sigma\, a_n = a_{n+1}$.

$$\delta\,(a(t)\,y(t)) = a(t-h)\,y(t-h) = \delta\,a\,\delta\,y \quad \Rightarrow \quad \delta\,a\,\cdot = (\delta\,a)\,\delta\,\cdot$$

$$\sigma\,(a_n\,y_n) = a_{n+1}\,y_{n+1} = \sigma\,a\,\sigma\,y \quad \Rightarrow \quad \sigma\,a\,\cdot = (\sigma\,a)\,\sigma\,\cdot$$

- Difference operators: $\Delta\, a(x) = a(x+1) - a(x)$.

$$\Delta\,a(x)\,\cdot = a(x+1)\,\Delta\,\cdot + (\Delta\,a)\,\cdot$$

- Divided difference operators: $d_{x_0} a(x) = \frac{a(x)-a(x_0)}{x-x_0}$.

$$d_{x_0}\,a(x)\,\cdot = a(x_0)\,d_{x_0}\,\cdot + (d_{x_0}\,a)\,\cdot$$

- q-difference, q-shift, q-dilation, Frobenius, Euler operators. . .

Skew polynomial rings (Ore, 1933)

- Definition: A skew polynomial ring $A[\partial; \alpha, \beta]$ is a non-commutative polynomial ring in ∂ with coefficients in A satisfying

$$\forall\, a \in A, \quad \boxed{\partial\, a = \alpha(a)\, \partial + \beta(a)}$$

where $\alpha : A \longrightarrow A$ and $\beta : A \longrightarrow A$ are such that:

$$\begin{cases} \alpha(1) = 1, \\ \alpha(a+b) = \alpha(a) + \alpha(b), \\ \alpha(a\, b) = \alpha(a)\, \alpha(b), \end{cases} \qquad \begin{cases} \beta(a+b) = \beta(a) + \beta(b), \\ \beta(a\, b) = \alpha(a)\, \beta(b) + \beta(a)\, b. \end{cases}$$

- $P \in A[\partial; \alpha, \beta]$ has a unique expression $P = \sum_{i=0}^{n} a_i\, \partial^i$, $a_i \in A$.
  - Ring of differential operators: $A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$.
  - Ring of shift operators: $A[\partial; \delta, 0]$, $A[\partial; \sigma, 0]$.
  - Ring of difference operators: $A[\partial; \tau, \tau - \mathrm{id}]$, $\tau a(x) = a(x+1)$.

Ore algebras (Chyzak-Salvy, 1996)

- We can iterate skew polynomial rings to obtain Ore extensions:

$$\boxed{A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_n; \alpha_n, \beta_n]}$$

- Definition: An Ore extension $A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_n; \alpha_n, \beta_n]$ is called an Ore algebras if the ∂_i's commute $\partial_i \, \partial_j = \partial_j \, \partial_i$, i.e., if we have

$$1 \leq j < i \leq m, \quad \alpha_i(\partial_j) = \partial_j, \quad \beta_i(\partial_j) = 0,$$

and the $\alpha_{i\,|_A}$'s and $\beta_{j\,|_A}$'s commute for $i \neq j$.

- Ring of differential operators: $A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$.
- Ring of differential delay operators: $A\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right] [\partial_2; \delta, 0]$.
- Ring of shift operators: $A[\partial_1; \sigma_1, 0] \ldots [\partial_n; \sigma_n, 0]$.

Matrix of functional operators

- The wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{cases} \qquad (\star)$$

- We first introduce the commutative Ore algebra of ordinary differential time-delay operators:

$$D = \mathbb{Q}(a, k, \omega, \zeta)\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \delta, 0].$$

- Then, the system $(\star)$ can be rewritten as:

$$\begin{pmatrix} \partial_1 + a & -k\,a\,\partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

Matrix of functional operators

- Linearization of the Navier-Stokes $\sim$ a parabolic Poiseuille profile

$$\left\{\begin{array}{ll} \partial_t\, u_1 + 4\, y\, (1-y)\, \partial_x\, u_1 - 4\, (2\, y - 1)\, u_2 - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2)\, u_1 + \partial_x\, p = 0, & \\ \partial_t\, u_2 + 4\, y\, (1-y)\, \partial_x\, u_2 - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2)\, u_2 + \partial_y\, p = 0, & (*) \\ \partial_x\, u_1 + \partial_y\, u_2 = 0. \qquad \text{(e.g., Vazquez-Krstic, IEEE TAC 07)} & \end{array}\right.$$

- Let us introduce the so-called Weyl algebra ($\partial_x\, x = x\, \partial_x + 1$):

$$D = \mathbb{Q}(Re)[t, x, y] \left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right] \left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right] \left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right].$$

- The system $(*)$ is defined by the matrix of PD operators:

$$\begin{pmatrix} \partial_t + 4\, y\, (1-y)\, \partial_x - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2) & -4\, (2\, y - 1) & \partial_x \\ 0 & \partial_t + 4\, y\, (1-y)\, \partial_x - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}.$$

Non-commutative Gröbner bases

- Let $D = A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_m; \alpha_m, \beta_m]$ be an Ore algebra.

- Theorem: (Kredel, 93) Let $A = k[x_1, \ldots, x_n]$ a commutative polynomial ring ($k = \mathbb{Q}$, $\mathbb{F}_p$) and D an Ore algebra satisfying

$$\alpha_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \beta_i(x_j) = c_{ij},$$

for certain $0 \neq a_{ij} \in k$, $b_{ij} \in k$, $c_{ij} \in A$ and $\deg(c_{ij}) \leq 1$. Then, a non-commutative version of Buchberger's algorithm terminates for any term order and its result is a Gröbner basis.

- Implementation in the Maple package Ore_algebra (Chyzak).

- Gröbner bases can be used to effectively compute over D.

- D is a left noetherian domain $\Rightarrow$ left Ore domain:

$$\forall\, a, b \in D \setminus \{0\},\ \exists\, c, d \in D \setminus \{0\} :\ c\, a = d\, b.$$

Algebraic analysis

- Let D be an Ore algebra and $R \in D^{q\times p}$.
- Let us consider the left D-homomorphism:

$$\begin{array}{ccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \\ \lambda = (\lambda_1 \ldots \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

- We introduce the finitely presented left D-module:

$$\boxed{M = D^{1\times p}/\mathrm{im}_D(.R) = D^{1\times p}/(D^{1\times q}\, R).}$$

- M is formed by the equivalence classes $\pi(\mu)$ of $\mu \in D^{1\times p}$ for the equivalence relation $\sim$ on $D^{1\times p}$ defined by:

$$\mu_1 \sim \mu_2 \;\Leftrightarrow\; \mu_1 - \mu_2 \in D^{1\times q}\, R.$$

1. Number theory: $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$, $\mathbb{Z}[i\sqrt{5}] = \mathbb{Z}[x]/(x^2+5)$.
2. Algebraic geometry: $\mathbb{C}[x, y]/(x^2 + y^2 - 1, x - y)$.

Generators and relations of $M = D^{1\times p}/(D^{1\times q}\, R)$

- Let $\{f_k\}_{k=1,\dots,p}$ the standard basis of $D^{1\times p}$ defined by:

$$f_1 = (1\ 0\ \dots\ 0), \quad f_k = (0\ \dots\ 1\ \dots\ 0), \quad f_p = (0\ \dots\ 0\ 1).$$

- Let $\pi : D^{1\times p} \longrightarrow M$ be the D-homomorphism sending μ to $\pi(\mu)$,

$$y_k = \pi(f_k), \quad k = 1, \dots, p.$$

- $\{y_k\}_{k=1,\dots,p}$ is a family of generators of M:

$$\forall\ m \in M,\ \exists\ \mu \in D^{1\times p} :\ m = \pi(\mu) = \textstyle\sum_{k=1}^{p} \mu_k\, \pi(f_k) = \sum_{k=1}^{p} \mu_k\, y_k.$$

- $\{y_k\}_{k=1,\dots,p}$ satisfies the relations:

$$\pi((R_{l1} \dots R_{lp})) = \pi\left(\sum_{k=1}^{p} R_{lk}\, f_k\right) = \sum_{k=1}^{p} R_{lk}\, y_k = 0,\ l = 1, \dots, q,$$

$\Rightarrow y = (y_1 \dots y_p)^T$ satisfies the relation $R\, y = 0$.

Duality modules — systems

- Let $\mathcal{F}$ be a left D-module and $\hom_D(M, \mathcal{F})$ the abelian group:

$$\hom_D(M, \mathcal{F}) = \{f : M \longrightarrow \mathcal{F} \mid f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2)\}.$$

- Applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

we obtain the following exact sequence of abelian groups:

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{\iota\circ\pi^\star} \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

- Theorem (Malgrange):

$$\boxed{\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}}$$

- Remark: $\hom_D(M, \mathcal{F})$ intrinsically characterizes $\ker_{\mathcal{F}}(R.)$ as it does not depend on the embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$.

Module theory

- Definition: 1. M is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^r$.

2. M is stably free if $\exists\, r, s \in \mathbb{Z}_+$ such that $M \oplus D^s \cong D^r$.

3. M is projective if $\exists\, r \in \mathbb{Z}_+$ and a D-module P such that:

$$M \oplus P \cong D^r.$$

4. M is reflexive if $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall\, m \in M, \quad \forall\, f \in \hom_D(M, D).$$

5. M is torsion-free if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

6. M is torsion if $t(M) = M$.

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0 \quad \text{exact sequence.}$$

Classification of modules

- Theorem: 1. We have the following implications:

$$\text{free} \Rightarrow \text{stably free} \Rightarrow \text{projective} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free}.$$

2. If D is a principal domain (e.g., $B_1(\mathbb{Q}) = \mathbb{Q}(t)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{free}.$$

3. If D is a hereditary ring (e.g., $A_1(\mathbb{Q}) = \mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{projective}.$$

4. If $D = k[x_1, \ldots, x_n]$ and k a field, then:

$$\text{projective} = \text{free} \quad \text{(Quillen-Suslin theorem)}.$$

5. If $D = A_n(k)$ or $B_n(k)$, k is a field of characteristic 0, then

$$\text{projective} = \text{free} \quad \text{(Stafford theorem)},$$

for modules of rank at least 2.

Free resolutions

• Definition: A sequence of D-morphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called a complex if $g \circ f = 0$, i.e., $\mathrm{im}\, f \subseteq \ker g$.

The defect of exactness at M is $H(M) = \ker g / \mathrm{im}\, f$.

The complex is exact at M if $\mathrm{im}\, f = \ker g$.

• Definition: A free resolution of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_i \in D^{l_i \times l_{i-1}}$ and:

$$\begin{array}{ccc} D^{1\times l_i} & \xrightarrow{.R_i} & D^{1\times l_{i-1}} \\ (P_1 \ \ldots \ P_{l_i}) & \longmapsto & (P_1 \ \ldots \ P_{l_i})\, R_i. \end{array}$$

• Algorithm: Find a basis of the compatibility conditions of the inhomogeneous system $R_i\, y = u$ by eliminating y (e.g., Gb):

$$\forall\ P \in \ker_D(.R_i), \quad P\,(R_i\, y) = P\, u \ \Rightarrow\ P\, u = 0.$$

Extension functor $\mathrm{ext}^i_D(\cdot, \mathcal{F})$

- We define the reduced free resolution of M by:

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \longrightarrow 0 \quad (\star).$$

- Let $\mathcal{F}$ be a left D-modules. Applying the functor $\hom_D(\cdot, \mathcal{F})$ to $(\star)$, we obtain the following complex:

$$\ldots \xleftarrow{R_3 .} \mathcal{F}^{l_2} \xleftarrow{R_2 .} \mathcal{F}^{l_1} \xleftarrow{R_1 .} \mathcal{F}^{l_0} \longleftarrow 0 \quad (\star\star)$$

$$\text{where} \quad \begin{array}{ccc} \mathcal{F}^{l_i} & \xleftarrow{R_i .} & \mathcal{F}^{l_{i-1}} \\ R_i\,\eta & \longleftarrow & \eta. \end{array}$$

- We denote the defects of exactness of $(\star\star)$ by:

$$\begin{cases} \mathrm{ext}^0_D(M, \mathcal{F}) = \hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1 .), \\ \mathrm{ext}^i_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1} .) / \mathrm{im}_{\mathcal{F}}(R_i .), \quad i \geq 1. \end{cases}$$

- Theorem: The abelian group $\mathrm{ext}^i_D(M, \mathcal{F})$ only depends on M and $\mathcal{F}$ and not on the resolution $(\star)$.

Auslander transpose

- Definition: Let $R \in D^{q \times p}$. If $M = D^{1 \times p}/(D^{1 \times q} R)$ denotes the left D-module finitely presented by R, then its Auslander transpose is the right D-module defined by $N = D^q/(R\, D^p)$.

- Proposition: The Auslander transpose $N = D^q/(R\, D^p)$ only depends on M up to a projective equivalence.

Hence, if we have $M = D^{1 \times p'}/(D^{1 \times q'} R')$ and $N' = D^{q'}/(R'\, D^{p'})$ denotes the corresponding Auslander transpose, then there exist two projective right D-modules P and P' such that:

$$N \oplus P \cong N' \oplus P'.$$

- Proposition: If P is a projective module, then we have:

$$\mathrm{ext}^i_D(P, \mathcal{F}) = 0, \quad \forall\, i \geq 1.$$

- Corollary: The $\mathrm{ext}^i_D(N, \mathcal{F})$'s, $i \geq 1$, only depend on M and $\mathcal{F}$.

Module M	**Homological algebra**	**$\mathcal{F}$ injective cogenerator**
with torsion	$t(M) \cong \text{ext}^1_D(N, D)$	$\emptyset$
torsion-free	$\text{ext}^1_D(N, D) = 0$	$\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^{l_1}$
reflexive	$\text{ext}^i_D(N, D) = 0$ $i = 1, 2$	$\ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\,\mathcal{F}^{l_2}$
projective $=$ stably free	$\text{ext}^i_D(N, D) = 0$ $i = 1, \ldots, n = \text{gld}(D)$	$\ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\,\mathcal{F}^{l_2}$ $\ldots$ $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n\,\mathcal{F}^{l_n}$
free	$\exists\ Q \in D^{p\times m},\ T \in D^{m\times p}$, $\ker_D(.Q) = D^{1\times q}\,R,\ T\,Q = I_m$	$\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$, $\exists\ T \in D^{m\times p}:\ T\,Q = I_m$

Parametrizability problem

$$\begin{array}{ccccccccccc} & & & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ 0 & \longleftarrow & N & \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{\pi^\star} & \hom_D(M,D) & \longleftarrow & 0 \\ 0 & \longleftarrow & N & \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{Q.} & D^{m} & & \\ & & & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & & (\star) \end{array}$$

$t(M) \cong \mathrm{ext}^1_D(N,D) \cong \ker_D(.Q)/(D^{1\times q}\,R) = (D^{1\times q'}\,R')/(D^{1\times q}\,R)$,
where R' is defined by $\ker_D(.Q) = D^{1\times q'}\,R'$, and:

$$M/t(M) \cong D^{1\times p}/(D^{1\times q'}\,R').$$

• $t(M) = 0$ iff $(\star)$ is an exact sequence. If so and if $\mathcal{F}$ is an injective cogenerator, then $(\star)$ is exact iff so is the complex:

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m, \text{ i.e., } \ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m.$$

Injective cogenerator modules

- Definition: A left D-module $\mathcal{F}$ is injective if

$$\forall\, q \geq 1, \quad \forall\, R \in D^q, \quad \forall\, \zeta \in \ker_{\mathcal{F}}(S.),$$

where $\ker_D(.R) = D^{1\times r}\, S$, there exists $\eta \in \mathcal{F}$ satisfying $R\,\eta = \zeta$.

- Definition: If $\mathcal{F}$ is a injective left D-module, then we have:

$$\mathrm{ext}^i_D(M, \mathcal{F}) = 0, \quad \forall\, i \geq 1.$$

- Definition: A left D-module $\mathcal{F}$ is cogenerator if:

$$\mathrm{hom}_D(M, \mathcal{F}) = 0 \;\Rightarrow\; M = 0.$$

- Proposition: Injective cogenerator left D-module always exists.

- Example: If Ω is an open convex subset of $\mathbb{R}^n$, then the $\mathbb{C}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$-modules $C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{O}(\Omega)$ and $\mathcal{B}(\Omega)$ are injective cogenerators.

Involutions and adjoints

- Definition: A k-linear map $\theta : D \longrightarrow D$ is an involution of D if:

$$\forall\, P,\, Q \in D: \ \theta(P\,Q) = \theta(Q)\,\theta(P), \quad \theta^2 = \mathrm{id}.$$

- Example: 1. If D is a commutative ring, then $\theta = \mathrm{id}$.

2. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$ is:

$$\forall\, a \in A, \quad \theta(a(x)) = a(x), \quad \theta(\partial_i) = -\partial_i, \quad i = 1, \ldots, n.$$

3. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \delta, 0]$ is defined by:

$$\forall\, a \in A, \quad \theta(a(t)) = a(-t), \quad \theta(\partial_i) = \partial_i, \quad i = 1,\, 2.$$

- The adjoint of $R \in D^{q \times p}$ is defined by $\theta(R) = (\theta(R_{ij}))^T \in D^{p \times q}$.
- $\widetilde{N} = D^{1 \times q}/(D^{1 \times p}\,\theta(R))$ is called the adjoint of M.

Extension functor $\mathrm{ext}^1_D(\,\cdot\,, D)$

- Parametrizability: $R\,y = 0 \overset{?}{\Longleftrightarrow} \exists\, Q \in D^{p\times m}: \; y = Q\,z$.

$$
\begin{array}{ccc}
4. \quad \theta(P)\,z = y & \Longrightarrow & R\,y = 0 \quad 1. \\
\Uparrow & & \Downarrow \\
\text{involution } \theta & & \text{involution } \theta \\
\Uparrow & & \Downarrow \\
3. \quad 0 = P\,\mu & \overset{\text{Gb}}{\Longleftrightarrow} & \theta(R)\,\lambda = \mu \quad 2.
\end{array}
$$

$$
P \circ \theta(R) = 0 \quad \Longrightarrow \quad \theta(P \circ \theta(R)) = \theta^2(R) \circ \theta(P) = R \circ \theta(P) = 0.
$$

5. $\theta(P)\,z = y \quad \overset{\text{Gb}}{\Longleftrightarrow} \quad R'\,y = 0, \quad R' \in D^{q'\times p}$.

$$
\boxed{\mathrm{ext}^1_D(N, D) \cong (D^{1\times q'}\,R')/(D^{1\times q}\,R)}
$$

6. Using Gb, we can test whether or not $\mathrm{ext}^1_D(N, D) = 0$

Wind tunnel model (Manitius, IEEE TAC 84)

1. The w.t.m. is defined by the underdetermined system:

$$\begin{pmatrix} \partial_1 + a & -k\, a\, \partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\begin{cases} (\partial_1 + a)\ \lambda_1 = \mu_1, \\ -k\, a\, \partial_2\, \lambda_1 + \partial_1\, \lambda_2 + \omega^2\, \lambda_3 = \mu_2, \\ -\lambda_2 + (\partial_1 + 2\,\zeta\,\omega)\ \lambda_3 = \mu_3, \\ -\omega^2\, \lambda_3 = \mu_4. \end{cases} \qquad (2)$$

(2) is overdetermined $\stackrel{\mathrm{Gb}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Wind tunnel model (Manitius, IEEE TAC 84)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$\omega^2\,k\,a\,\partial_2\,\mu_1 + \omega^2\,(\partial_1 - a)\,\mu_2 + \omega^2\,\left(\partial_1^2 + a\,\partial_1\right)\,\mu_3$$
$$+ \left(\partial_1^3 + 2\,\zeta\,\omega\,\partial_1^2 + a\,\partial_1^2 + \omega^2\,\partial_1 + 2\,a\,\zeta\,\omega\,\partial_1 + a\,\omega^2\right)\,\mu_4 = 0.$$

4. We consider the overdetermined system $P^T\,z = y$.

$$\begin{cases} \omega^2\,k\,a\,\partial_2\,z = x_1, \\ \omega^2\,(\partial_1 - a)\,z = x_2, \\ \omega^2\,(\partial_1^2 + a\,\partial_1)\,z = x_3, \\ (\partial^3 + (2\,\zeta\,\omega + a)\,\partial_1^2 + (\omega^2 + 2\,a\,\omega\,\zeta)\,\partial_1 + a\,\omega)\,z = u. \end{cases} \quad (4)$$

5. The compatibility conditions of $P^T\,z = y$ are exactly generated by $R\,y = 0$ and (4) is a parametrization of the w.t.m.

Moving tank (Petit, Rouchon, IEEE TAC 02)

1. The model of a moving tank is defined by:

$$\begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\left\{\begin{array}{l} \partial_1\,\lambda_1 + \partial_1\,\partial_2^2\,\lambda_2 = \mu_1, \\ -\partial_1\,\partial_2^2\,\lambda_1 - \partial_1\,\lambda_2 = \mu_2, \\ a\,\partial_1^2\,\partial_2\,\lambda_1 + a\,\partial_1^2\,\partial_2\,\lambda_2 = \mu_3. \end{array}\right. \qquad (2)$$

(2) is overdetermined $\stackrel{\mathrm{Gb}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Moving tank (Petit, Rouchon, IEEE TAC 02)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$a\,\partial_1\,\partial_2\,\mu_1 - a\,\partial_1\,\partial_2\,\mu_2 - (1+\partial_2^2)\,\mu_3 = 0.$$

4. We consider the overdetermined system $P^T\,z = y$.

$$\begin{cases} a\,\partial_1\,\partial_2\,z = y_1, \\ -a\,\partial_1\,\partial_2\,z = y_2, \\ -(1+\partial_2^2)\,z = y_3. \end{cases} \qquad (4)$$

5. The compatibility conditions of $P^T\,z = y$ are $R'\,y = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & (1+\partial_2^2) & -a\,\partial_1\,\partial_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0.$$

Moving tank (Petit, Rouchon, IEEE TAC 02)

$$t(M) \quad \cong \quad \mathrm{ext}^1_D(N, D) \quad \cong$$

$$\left(D^{1\times 2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+\partial_2^2 & -a\,\partial_1\,\partial_2 \end{pmatrix}\right) / \left(D^{1\times 2}\begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{pmatrix}\right)$$

$$\left\{\begin{array}{l} y_1 + y_2 = z_1, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{array}\right. \quad \stackrel{\mathrm{Gb}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_1 = 0.$$

$$\left\{\begin{array}{l} (1+\partial_2^2)\,y_2 - a\,\partial_1\,\partial_2\,y_3 = z_2, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{array}\right. \quad \stackrel{\mathrm{Gb}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_2 = 0.$$

$\Rightarrow$ z_1 and z_2 are non-trivial torsion elements.

Examples

- 2D Stokes equations:

$$\begin{pmatrix} -\nu\,(\partial_x^2+\partial_y^2) & 0 & \partial_x \\ 0 & -\nu\,(\partial_x^2+\partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0$$

$$\Rightarrow \left\{ \begin{array}{l} (\partial_x^2+\partial_y^2)^2\,u=0, \\ (\partial_x^2+\partial_y^2)^2\,v=0, \\ (\partial_x^2+\partial_y^2)\,p=0. \end{array} \right. \quad \text{torsion module}$$

- Moving tank (Petit, Rouchon, IEEE TAC 02):

$$\left\{ \begin{array}{l} \dot{y}_1(t)-\dot{y}_2(t-2\,h)+\alpha\,\ddot{y}_3(t-h)=0, \\ \dot{y}_1(t-2\,h)-\dot{y}_2(t)+\alpha\,\ddot{y}_3(t-h)=0, \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} z_1(t)=y_1(t)+y_2(t), \\ z_2(t)=y_2(t)+y_2(t-2\,h)-a\,\dot{y}_3(t-h), \\ \frac{d}{dt}\,(1-\delta^2)\,z_i(t)=0,\ i=1,\,2. \end{array} \right. \quad \text{module with torsion}$$

Examples: torsion-free modules

- Wind tunnel model (Manitius, IEEE TAC 84):

$$\left\{\begin{array}{l} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\, \zeta\, \omega\, x_3(t) - \omega^2\, u(t) = 0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} x_1(t) = \omega^2\, k\, a\, z(t-h), \\ x_2(t) = \omega^2\, \dot{z}(t) - a\, \omega^2\, z(t), \\ x_3(t) = \omega^2\, \ddot{z}(t) + \omega^2\, a\, \dot{z}(t), \\ u(t) = z(t)^{(3)} + (2\, \zeta\, \omega + a)\, \ddot{z}(t) + (\omega^2 + 2\, a\, \omega\, \zeta)\, \dot{z}(t) + a\, \omega\, z(t). \end{array}\right.$$

$\Rightarrow$ motion planning and tracking (Fliess et al).

- 2D stress tensor (elasticity theory):

$$\left\{\begin{array}{l} \partial_x\, \sigma^{11} + \partial_y\, \sigma^{12} = 0, \\ \partial_x\, \sigma^{12} + \partial_y\, \sigma^{22} = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \sigma^{11} = \partial_y^2\, \lambda, \\ \sigma^{12} = -\partial_x\, \partial_y\, \lambda, \quad \text{Airy function } \lambda. \\ \sigma^{22} = \partial_x^2\, \lambda, \end{array}\right.$$

Examples: reflexive modules

- div-curl-grad: $\vec{\nabla} . \vec{B} = 0 \Leftrightarrow \vec{B} = \vec{\nabla} \wedge \vec{A}$, $\vec{\nabla} \wedge \vec{A} = \vec{0} \Leftrightarrow \vec{A} = \vec{\nabla} f$.
- First group of Maxwell equations:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} . \vec{B} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla} V, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right.$$

$$\left\{ \begin{array}{l} -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla} V = \vec{0}, \\ \vec{\nabla} \wedge \vec{A} = \vec{0}, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{A} = \vec{\nabla} \xi, \\ V = -\dfrac{\partial \xi}{\partial t}. \end{array} \right.$$

- 3D stress tensor: Maxwell, Morera parametrizations...
- Linearized Einstein equations (system of PDEs 10×10)?

$\Rightarrow$ OreModules (Chyzak, Q., Robertz)

http://wwwb.math.rwth-aachen.de/OreModules/

Dictionary systems — modules

Module M	Structural properties $\ker_{\mathcal{F}}(R.)$	Stabilization problems Optimal control
Torsion	Autonomous system Poles/zeros classifications	
With torsion	Existence of autonomous elements	
Torsion-free	No autonomous elements, Controllability, Parametrizability, π-freeness	Variational problem without constraints (Euler-Lagrange equations)
Projective	Bézout identities, Internal stabilizability	Computation of Lagrange parameters without integration Existence of a parametrization all stabilizing controllers
Free	Flatness, Poles placement, Doubly coprime factorization	Youla-Kučera parametrization Optimal controller

Oberst, Pommaret, Fliess, Q., ...

Projectiveness, observability and controllability

- Theorem: If $R \in D^{q \times p}$ has full row rank, then the left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$ is projective iff:

$$N = D^{1 \times q}/(D^{1 \times p} \, \theta(R)) = 0 \Leftrightarrow \exists \, S \in D^{p \times q} : \; R \, S = I_q.$$

- Let $D = \mathcal{A}(I) \left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $R = (\partial \, I_n - A \quad -B) \in D^{n \times (n+m)}$.

$M = D^{1 \times (n+m)}/(D^{1 \times n} R)$ is projective iff $\theta(R) \, \lambda = 0 \Leftrightarrow \lambda = 0$:

$$\left\{ \begin{array}{l} -\partial \, \lambda - A^T \, \lambda = 0, \\ -B^T \, \lambda = 0, \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} \partial \, \lambda = -A^T \, \lambda, \\ B^T \, \lambda = 0, \\ B^T \, \partial \, \lambda + \dot{B}^T \, \lambda = (-B^T \, A^T + \dot{B}^T) \, \lambda = 0. \end{array} \right.$$

Hence, M is projective iff, for all $t_0 \in I$, we have:

$$\mathrm{rank}_{\mathbb{R}}(B \mid A \, B - \dot{B} \mid A^2 \, B + \ldots \mid A^{n-1} \, B + \ldots \mid \ldots)(t_0) = n.$$

- $D^{1 \times p}/(D^{1 \times q} \, (P(\partial) \quad -Q(\partial))$ proj. iff $P(\partial) \, X(\partial) + Q(\partial) \, Y(\partial) = I_q$.

Controllability à la Willems and flatness

- We consider two pendula mounted on a car:

$$\left\{\begin{array}{l} m_1\, L_1\, \ddot{w}_1(t) + m_2\, L_2\, \ddot{w}_2(t) - w_3(t) + (M + m_1 + m_2)\, \ddot{w}_4(t) = 0, \\ m_1\, L_1^2\, \ddot{w}_1(t) - m_1\, L_1\, g\, w_1(t) + m_1\, L_1\, \ddot{w}_4(t) = 0, \\ m_2\, L_2^2\, \ddot{w}_2(t) - m_2\, L_2\, g\, w_2(t) + m_2\, L_2\, \ddot{w}_4(t) = 0. \end{array}\right. \qquad (\star)$$

- $(\star)$ is parametrizable iff $L_1 \neq L_2$.

- If $L_1 \neq L_2$ then a parametrization of $(\star)$ is defined by:

$$\left\{\begin{array}{l} w_1(t) = -L_2\, \xi^{(4)}(t) + g\, \ddot{\xi}(t), \\ w_2(t) = -L_1\, \xi^{(4)}(t) + g\, \ddot{\xi}(t), \\ w_3(t) = L_1\, L_2\, M\, \xi^{(6)}(t) - (L_1\, m_2 + L_2\, m_1 + g\, (L_1 + L_2)\, M)\, \xi^{(4)}(t) \\ \qquad\qquad + g^2\, (m_1 + m_2 + M)\, \xi^{(2)(t)} \\ w_4(t) = L_1\, L_2\, \xi^{(4)}(t) - g\, (L_1 + L_2)\, \ddot{\xi}(t) + g^2\, \xi(t). \end{array}\right.$$

- The parametrization of $(\star)$ is injective as we have:

$$\xi(t) = \frac{1}{g^2\, (L_1 - L_2)}\, (L_1^2\, w_1(t) - L_2^2\, w_2(t) + (L_1 - L_2)\, w_4(t)). \quad (\star\star)$$

Controllability à la Willems and flatness

- Patching problem $\Leftrightarrow$ controllability: $\quad T \geq 0$.

$w^p = (w_1^p, w_2^p, w_3^p, w_4^p)$ a past trajectory of $(\star)$ on $]-\infty, 0]$.

$w^f = (w_1^f, w_2^f, w_3^f, w_4^f)$ a future trajectory of $(\star)$ on $[T, +\infty[$.

$$\Rightarrow \exists\, w = (w_1, w_2, w_3, w_4) \in C^\infty(\mathbb{R})^4 \text{ trajectory of } (\star):$$

$$\begin{cases} w_{]-\infty,0]} = w^p, \\ w_{[T,+\infty[} = w^f. \end{cases}$$

- Using the flat output

$$\xi(t) = \frac{1}{g^2\,(L_1 - L_2)}\,(L_1^2\,w_1(t) - L_2^2\,w_2(t) + (L_1 - L_2)\,w_4(t))$$

and the parametrization, it is enough to find $\xi \in C^\infty(\mathbb{R})$ s.t.:

$$\xi_{]-\infty,0]} = \xi^p \quad \& \quad \xi_{[T,+\infty[} = \xi^f.$$

Motion planning

- Flexible rod with a torque (Mounier 95):

$$\left\{\begin{array}{l} \sigma^2 \, \frac{\partial^2 q(\tau,x)}{\partial \tau^2} - \frac{\partial^2 q(\tau,x)}{\partial x^2} = 0, \\ \frac{\partial q}{\partial x}(\tau, 0) = -u(\tau), \\ \frac{\partial q}{\partial x}(\tau, L) = -J \, \frac{\partial^2 q}{\partial \tau^2}(\tau, L), \\ y(\tau) = q(\tau, L). \end{array}\right. \qquad (\star)$$

- $q(\tau, x) = \phi(\tau + \sigma\, x) + \psi(\tau - \sigma\, x)$, $t = (\sigma/J)\,\tau$, $v = (2\,J/\sigma^2)\,u$,

$$(\star) \;\Rightarrow\; \ddot{y}(t+1) + \ddot{y}(t-1) + \dot{y}(t+1) - \dot{y}(t-1) = v(t)$$

$$\Leftrightarrow \left\{\begin{array}{l} y(t) = \xi(t-1), \\ v(t) = \ddot{\xi}(t) + \ddot{\xi}(t-2) + \dot{\xi}(t) - \dot{\xi}(t-2). \end{array}\right.$$

- If y_r is a desired trajectory then $\xi_r(t) = y_r(t+1)$ and we obtain the open-loop control law:

$$v_r(t) = \ddot{y}_r(t+1) + \ddot{y}_r(t-1) + \dot{y}_r(t+1) - \dot{y}_r(t-1).$$

Optimal control

- Let us minimize $\frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\, dt$ (1) under:

$$\dot{x}(t) + x(t) - u(t) = 0, \quad x(0) = x_0. \quad (2)$$

- (2) is parametrized by $\begin{cases} x(t) = \xi(t), \\ u(t) = \dot{\xi}(t) + \xi(t). \end{cases}$ (3)

- (1) & (3) $\Rightarrow$ min $\frac{1}{2}\int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\, dt$,

$\Rightarrow$ Euler-Lagrange equations $\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{cases}$

$$\Rightarrow u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\, e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\, e^{-\sqrt{2}(t-T)}}\, x(t).$$

- Let us extremize the electromagnetic action

$$\int \left(\frac{1}{2\,\mu_0} \parallel \vec{B} \parallel^2 - \frac{\epsilon_0}{2} \parallel \vec{E} \parallel^2 \right) dx_1\, dx_2\, dx_3\, dt, \quad (1)$$

where $\vec{B}$ and $\vec{E}$ satisfy the first group of Maxwell equations:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \,.\, \vec{B} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = -\vec{\nabla}\, V - \dfrac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right. \quad (3)$$

- Substituting (3) in (1) and using Lorentz gauge

$$\vec{\nabla} \,.\, \vec{A} + \frac{1}{c^2}\, \frac{\partial V}{\partial t} = 0, \quad c^2 = 1/(\epsilon_0\, \mu_0),$$

$$\Rightarrow \left\{ \begin{array}{l} \dfrac{1}{c^2}\, \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\, \vec{A} = 0, \\ \dfrac{1}{c^2}\, \dfrac{\partial^2 V}{\partial t^2} - \Delta\, V = 0. \end{array} \right. \quad \text{(electromagnetic waves)}.$$

- OreModules is a tool-box developed in *Maple*.
- OreModules uses *Ore_algebra* developed by Chyzak.
- OreModules handles linear systems of ODEs, PDEs, difference equations, differential time-delay equations...
- OreModules computes:

1. free resolutions, $\mathrm{ext}^i_D(\,\cdot\,, D)$, projective dim., Hilbert series,
2. torsion elements, autonomous elements,
3. parametrizations of underdetermined linear functional systems,
4. left-/right-/generalized inverses,
5. π-polynomials, bases, flat outputs,
6. first integrals of motion, Euler-Lagrange equations...

http://wwwb.math.rwth-aachen.de/OreModules/

Conclusion

- Based on algebraic analysis, module theory, constructive homological algebra and Ore algebras, we have developed a general non-commutative polynomial approach to functional linear systems.

- The different results are implemented in OreModules.

This new approach allowed us to:

1. Develop an intrinsic approach (independent of the form).
2. Develop generic algorithms and generic implementations.
3. Constructively study the parametrizability problem.
4. Solve conjectures in mathematical systems theory.

F. Chyzak, A. Quadrat, D. Robertz, "Effective algorithms for parametrizing linear control systems over Ore algebras", *Applicable Algebra in Engineering, Communications and Computing*, 16 (2005), 319-376.