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An Algebraic Interpretation to the Operator-Theoretic Approach to Stabilizability. Part I: SISO Systems

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Abstract. The purpose of this paper is to show that a duality exists between the fractional ideal approach [23, 26] and the operator-theoretic approach [4, 6, 8, 9, 33, 34] to stabilization problems. In particular, this duality helps us to understand how the algebraic properties of systems are reflected by the operator-theoretic approach and conversely. In terms of modules, we characterize the domain and the graph of an internally stabilizable plant or that of a plant which admits a (weakly) coprime factorization. Moreover, we prove that internal stabilizability implies that the graph of the plant and the graph of a stabilizing controller are direct summands of the global signal space. These results generalize those obtained in [6, 8, 9, 33, 34]. Finally, we exhibit a class of signal spaces over which internal stabilizability is equivalent to the existence of a bounded inverse for the linear operator mapping the errors e_1 and e_2 of the closed-loop system to the inputs u_1 and u_2.

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1. Introduction

The *behavioural approach* to linear control systems – defined by ordinary differential equations – has underlined the role of the system trajectories in the study of the system structural properties [18]. Based on the mathematical work of Malgrange [14], Oberst has shown in [16] that a certain duality existed between the behavioural approach and the *module-theoretic* approach to multidimensional linear systems [19, 36, 38]. The main idea of [16] is to use the standard module duality [1, 7, 29] in order to pass from the module-theoretic viewpoint to the behavioural one. Moreover, if the signal space to which the system trajectories lie satisfies certain properties, we can also recover the module-theoretic approach from the behavioural one. This duality gives a dictionary between these two frameworks and allows us to understand which properties of a system come from algebraic properties or are inherited from the signal space. See [36] for a nice introduction as well as [17, 19, 21, 30, 38]. In particular, this duality explains why, in both approaches, we recover the common philosophy of studying a system in its whole, that is to say, inputs and outputs together and not as a map from the inputs to the outputs.

A more classical algebraic approach to linear control systems is based on transfer matrices. In [16, 20], it is shown how we can recover this approach for multidimensional linear systems using the concept of localization of modules (i.e., using vector spaces obtained by inverting nonzero elements of the domain). However, recovering the transfer matrix approach to linear systems, we are condemned to work with fields and no more rings. Therefore, we may wonder if a mathematical framework exists which mimics module theory for transfer matrices.

For single input single output (SISO) systems, we have explained in [23] how the *theory of fractional ideals* [2, 7] was a natural algebraic machinery for the study of transfer functions using the techniques of module theory. In [24, 25], we show that the *lattice theory* [2] generalizes the fractional ideal theory to multi input multi output (MIMO) systems. The main motivation of [23] was to study *the fractional representation approach to analysis and synthesis problems* of linear systems [3, 4, 35]. However, the same techniques can be applied to the study of multidimensional linear systems [17, 19, 21, 30, 36, 38].

As for multidimensional linear systems, using the fact that the fractional ideal approach to transfer functions is a module theory, we can try to dualize this approach. The purpose of this paper is to give a system interpretation of the resulting theory. The main result of this paper is to show that the dual theory of the fractional ideal approach is an algebraic interpretation to the *operator-theoretic approach* [5, 8, 9, 34, 35]. More precisely, we first define the concepts of the *domain* and the *graph of a linear operator* defined by multiplying a transfer function $p = n/d$, $0 \neq d, n \in A$ (e.g., $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$ or $\hat{\mathcal{A}}$ [3]), to elements of a signal space $\mathcal{F}$, i.e., an A-module $\mathcal{F}$ (e.g., $\mathcal{F} = H_2(\mathbb{C}_+)$ [3]). We explain why these concepts generalize the ones classically used in [5, 8, 9, 34, 35] and when we recover them (conditions on the A-module $\mathcal{F}$). In particular, we show that these new definitions are justified by considering a simple example of a linear ordinary differential control system ($A = \mathbb{R}[\frac{d}{dt}]$, $\mathcal{F} = C^\infty(\mathbb{R})$). As in the module-theoretic and the behavioural approaches to multidimensional linear systems, we find that the fractional ideal and the operator-theoretic approaches also study a system in its whole. Indeed, we shall see that the structural properties of a system, defined by means of a transfer function $p = n/d$, $0 \neq d, n \in A$, can be characterized by means of the fractional ideal of the integral domain A

$$J = A + A p = \{a + b p \mid a, b \in A\},$$

which corresponds to the system $(1 \; -p)(y \; u)^{\mathrm{T}} = 0$, or by means of its graph in the A-module $\mathcal{F}$

$$\mathrm{graph}_{\mathcal{F}}(p) = \{(u \quad pu)^{\mathrm{T}} \in \mathcal{F} \times \overline{\mathcal{F}} \mid u \in \mathrm{dom}_{\mathcal{F}}(p)\},$$

where $\overline{\mathcal{F}} = \mathcal{F}/\{x \in \mathcal{F} \mid \mathrm{d}x = 0, \forall \mathrm{d} \in A : \mathrm{d}p \in A\}$ and $\mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid pu \in \overline{\mathcal{F}}\}$.

As in [23], we focus on the fractional representation approach to analysis and synthesis problems [3, 4, 35] and show how to recover and generalize some results of the operator-theoretic approach to synthesis problems developed in [5, 8, 9, 34, 35]. For a general A-module $\mathcal{F}$, we completely characterize the domains and the graphs of a stabilizable plant p and of a stabilizing controller c of p. Moreover, we prove that there exists a *split exact sequence* [1, 29] connecting the projections of these graphs onto a certain A-module $\underline{\mathcal{F}} \times \overline{\mathcal{F}}$ built using $\mathcal{F}$, p and c. In particular, this result shows that these projections of the graphs of p and c are two direct summands of $\underline{\mathcal{F}} \times \overline{\mathcal{F}}$.

In these results, we do not assume that the stabilizable p admits a coprime factorization. Indeed, it is well known that internal stabilizability is generally not equivalent to the existence of a coprime factorization for a plant [22–24, 31, 33]. For instance, the above equivalence is still open for some important classes of systems such as the ring $\hat{\mathcal{A}}$ of BIBO-stable infinite-dimensional linear systems [3]. When the plant p admits a (weakly) coprime factorization, we exhibit its domain and graph. These results generalize the ones obtained by Vidyasagar, Georgiou, Smith and others for $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$ (resp., $\mathcal{A}$) and $\mathcal{F} = H_2(\mathbb{C}_+)$ (resp., $L_q(\mathbb{R}_+), q \in [1, +\infty]$) [6, 8, 9, 34, 35]. We note that for $A = \mathcal{A}$ and $\mathcal{F} = L_q(\mathbb{R}_+), q \in [1, +\infty]$, we do not need to assume that the stabilizable plant p admits a coprime factorization as it is done in [34]. Then, depending on the properties of the A-module $\mathcal{F}$, we discuss how the previous results simplify. In particular, we prove that $H_2(\mathbb{C}_+)$ is a *flat* $H_\infty(\mathbb{C}_+)$*-module* [1, 2, 7, 29] and then show that the definition of the graph in $H_2(\mathbb{C}_+)^2$ of a *weakly coprime* transfer function $p = n/d, 0 \neq d, n \in H_\infty(\mathbb{C}_+)$, given in Section VII of [8] is justified.

Finally, it is known that internal stabilizability implies $\mathcal{F}$*-stabilizability* for every torsion-free A-module $\mathcal{F}$, namely, for every u_1 and $u_2 \in \mathcal{F}$, we have e_1, e_2, y_1 and $y_2 \in \mathcal{F}$ (see Figure 1), i.e., every signal in the closed-loop is $\mathcal{F}$-stable. The converse problem consisting in finding the signal spaces $\mathcal{F}$ for which internal stabilizability is equivalent to $\mathcal{F}$-stabilizability is an important issue in stabilization problems. We prove that if $\mathcal{F}$ is a *faithfully flat A-module* [1, 2, 7, 29], then internal stabilizability is equivalent to $\mathcal{F}$-stabilizability, i.e., there exists a bounded inverse to the linear operator mapping the errors $(e_1\ e_2)^{\mathrm{T}} \in \mathrm{dom}_{\mathcal{F}}(p) \times \mathrm{dom}_{\mathcal{F}}(c)$ of the closed-loop to the inputs $(u_1\ u_2)^{\mathrm{T}} \in \underline{\mathcal{F}} \times \overline{\mathcal{F}}$ (see Figure 1).

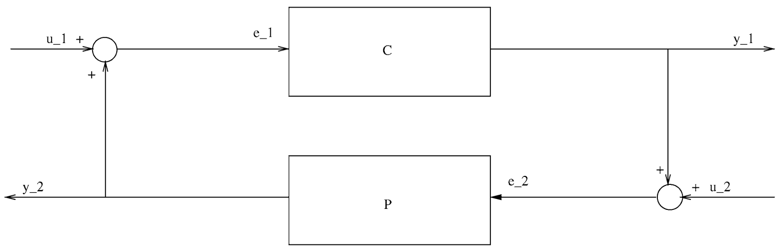

Figure 1. Closed-loop system.

Notations. We shall always denote by A a commutative integral domain, namely, a unital ring such that, for all a and $b \in A$, $ab = ba$ and $ab = 0, a \neq 0 \Rightarrow b = 0$ and by $Q(A) = \{n/d \mid 0 \neq d, n \in A\}$ the field of fractions of A. $A^{q \times p}$ will denote the set of $q \times p$ matrices with entries in A and I_p the identity matrix of $A^{p \times p}$. Moreover, $(a_1, \ldots, a_n)$ will be the A-module defined by $Aa_1 + \cdots + Aa_n$ and $(a_1 \ \cdots \ a_n)$ the row vector of $A^{1 \times n}$. If M and N are two A-modules, $M \cong N$ means that M and N are isomorphic as A-modules. If I is an ideal of A and $\mathcal{F}$ an A-module, then $I\mathcal{F}$ denotes the A-module defined by $\{\sum_{i=1}^{n} a_i x_i \mid a_i \in I, x_i \in \mathcal{F}, n \in \mathbb{Z}_+\}$. Finally, $\hom_A(M, N)$ denotes the A-module of A-morphisms (A-linear maps) from M to N and $\triangleq$ means 'by definition'.

2. The Fractional Representation Approach

2.1. INTRODUCTION TO ANALYSIS AND SYNTHESIS PROBLEMS

We briefly recall the fractional representation approach to analysis and synthesis problems [4, 35]. Within the fractional representation approach, time-invariant linear systems are defined by means of transfer functions which are elements of the *quotient field* $Q(A) = \{n/d \mid 0 \neq d, n \in A\}$ of an *integral domain* A *of single input single output (SISO) stable plants*. Let us give some examples of integral domains of SISO stable plants commonly used in the literature.

EXAMPLE 1.

1. $RH_\infty = \{n/d \in \mathbb{R}(s) \mid 0 \neq d, n \in \mathbb{R}[s], \deg n \leqslant \deg d, d(s^\star) = 0 \Rightarrow \operatorname{Re} s^\star < 0\}$ is the ring of proper and stable real rational functions [35]. Then, $p \in RH_\infty$ iff p is the transfer function of an exponentially-stable time-invariant finite-dimensional linear system.
2. $H_\infty(\mathbb{C}_+) = \{f \in \mathcal{H}(\mathbb{C}_+) \mid \|f\|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}$ is the ring of bounded holomorphic functions defined in the open right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ [3], where $\mathcal{H}(\mathbb{C}_+)$ denotes the ring of holomorphic functions in $\mathbb{C}_+$. Then, $p \in H_\infty(\mathbb{C}_+)$ iff p is the transfer function of a $L_2(\mathbb{R}_+) - L_2(\mathbb{R}_+)$-stable time-invariant infinite-dimensional linear system.
3. $\mathcal{A} = \{f(t) + \sum_{i=0}^{+\infty} a_i \delta(t - t_i) \mid f \in L_1(\mathbb{R}_+), (a_i)_{i \geqslant 0} \in l_1(\mathbb{Z}_+), 0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots\}$ is the ring of BIBO-stable time-invariant infinite-dimensional systems [3]. Then, $h \in \mathcal{A}$ iff h is the impulse response of a $L_\infty(\mathbb{R}_+) - L_\infty(\mathbb{R}_+)$-stable time-invariant infinite-dimensional linear system. If we denote by $\mathcal{L}(f)$ the Laplace transform of f, then $\hat{\mathcal{A}} = \{\mathcal{L}(f) \mid f \in \mathcal{A}\}$ is the integral domain formed by the Laplace transform of the elements of $\mathcal{A}$.
4. $W_+ = \{f(z) = \sum_{n=0}^{+\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) \mid \sum_{n=0}^{+\infty} |a_n| < +\infty\}$ is the ring of analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ whose Taylor series converge absolutely. Then, $p(z) \in A$ iff $p(z^{-1})$ is the z-transform of a $l_\infty(\mathbb{Z}_+) - l_\infty(\mathbb{Z}_+)$-stable linear filter [35]. We shall also denote by $l_1(\mathbb{Z}_+)$ the integral domain of the absolutely summable sequences.

5. $\mathbb{R}(x_1, \ldots, x_n)_S = \{r/s \mid 0 \neq s, r \in \mathbb{R}[x_1, \ldots, x_n], s(\underline{x}) = 0 \Rightarrow \underline{x} \notin \mathbb{D}^n\}$ is the ring of nD systems with structural stability [31] where $\mathbb{D}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leqslant 1, i = 1, \ldots, n\}$ denotes the closed unit polydisc of $\mathbb{C}^n$. See [3, 4, 11, 35] for more examples.

For instance, the transfer function $p = 1/(s-1)$ has an unstable pole at $1 \in \mathbb{C}_+$, and thus, p does not belong to RH_∞. But, we have $p = n/d$ where $n = 1/(s+1)$ and $d = (s-1)/(s+1) \in RH_\infty$, i.e., p belongs to the quotient field $Q(RH_\infty) = \mathbb{R}(s)$. Similarly, the transfer function $p = \mathrm{e}^{-s}/(s-1)$ does not belong to $H_\infty(\mathbb{C}_+)$ but to its quotient field $Q(H_\infty(\mathbb{C}_+))$ because we have $p = n/d$, where $n = \mathrm{e}^{-s}/(s+1)$, $d = (s-1)/(s+1) \in H_\infty(\mathbb{C}_+)$. In this approach, the problem of checking the stability of a transfer function p is equivalent to the *membership problem* "$p \in A$ or $p \notin A$".

We recall some definitions that will play important roles in what follows.

DEFINITION 1 [4, 22, 23, 35]. Let A be an integral domain of SISO stable plants and $K = Q(A)$ its quotient field. Then, we have the following definitions:

- A *fractional representation* of $p \in K$ is a representation of the form $p = n/d$ where $0 \neq d, n \in A$.
- A plant $p \in K$ is said to admit a *weakly coprime factorization* if there exist $0 \neq d, n \in A$ such that $p = n/d$ and, for all $k \in K$ satisfying $kn, kd \in A$, we then have $k \in A$.
- A plant $p \in K$ is said to admit a *coprime factorization* if there exist four elements $0 \neq d, n, x, y \in A$ such that $p = n/d$ and $dx - ny = 1$.
- A plant $p \in K$ is said to be *internally stabilizable* if there exists a controller $c \in K$ which satisfies

$$H(p, c) = \begin{pmatrix} 1 & c \\ p & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{1-pc} & -\dfrac{c}{1-pc} \\ -\dfrac{p}{1-pc} & \dfrac{1}{1-pc} \end{pmatrix} \in A^{2\times 2}, \tag{1}$$

  i.e., the four entries of the transfer matrix $H(p, c)$ from $(u_1\ u_2)^{\mathrm{T}}$ to $(e_1\ e_2)^{\mathrm{T}}$ (see Figure 1) belong to A. Such a controller c is then called a *stabilizing controller* of p.

We refer the reader to [3, 35] for the definitions of *strong* (resp., *bistably*, *simultaneous*, *robust*, *optimal*) *stabilization* as well as to [33] for a detailed treatment of properness in this algebraic context.

2.2. INTRODUCTION TO THE THEORY OF FRACTIONAL IDEALS

We give a basic introduction to the theory of *fractional ideals*. We refer to [2, 7, 23] for more details. In Section 2.3, we shall use this theory in order to characterize the concepts introduced in Definition 1.

DEFINITION 2 [2, 7, 29]. Let A be an integral domain and $K = Q(A)$ its quotient field. Then, we have the following definitions:

- A *fractional ideal* I of A is an A-submodule of K such that there exists $0 \neq a \in A$ satisfying $(a)I \triangleq \{ai \mid i \in I\} \subseteq A$. The set of nonzero fractional ideals of A is denoted by $\mathcal{F}(A)$.
- If $I, J \in \mathcal{F}(A)$, then their intersections, sums, products and residuals, namely,

$$I \cap J = \{a \in I, a \in J\}, \qquad I + J = \{a + b \mid a \in I, b \in J\},$$
$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J, n \in \mathbb{Z}_+ \right\},$$
$$I : J = \{k \in K \mid (k)J \subseteq I\},$$

  belong to $\mathcal{F}(A)$.
- A fractional ideal I of A is *principal* if there exists $k \in K$ such that $I = (k)$.
- A fractional ideal $I \subseteq A$ is called *integral ideal* of A.
- A fractional ideal I of A is *invertible* if there exists $J \in \mathcal{F}(A)$ such that $IJ = A$.

We note that if I is a fractional ideal, then there exists $0 \neq d \in A$ such that $(d)I \subseteq A$. Therefore, we have $I \subseteq (d^{-1})$, i.e., every element $i \in I$ has the form $i = a/d$ for some $a \in A$.

EXAMPLE 2. Let $p \in K = Q(A)$ and $J = (1, p) \triangleq A + Ap = \{a + bp \mid a, b \in A\}$. Using the fact that $p \in K$, then there exist $0 \neq d, n \in A$ such that $p = n/d$. Therefore, we have $(d)J = (d, n) \subseteq A$, i.e., J is a nonzero fractional ideal of A.

The next proposition plays an important role in what follows.

PROPOSITION 1 [2, 7, 29]. *If I is an invertible fractional ideal of A, then we have:*

1. *I admits a unique inverse denoted by I^{-1} and we have $I^{-1} = A : I = \{k \in K \mid (k)I \subseteq A\}$.*
2. *I^{-1} is also invertible and we have $(I^{-1})^{-1} = I$.*
3. *I is a finitely generated projective A-module, namely, there exist $r \in \mathbb{Z}_+$ and a finitely generated A-module P satisfying $I \oplus P \cong A^r$. Then, the A-module P is also projective.*
4. *If $I = \sum_{i=1}^{n} Af_i$, $f_i \in K$, then, for every maximal ideal $\mathfrak{m}$ of A, then the ideal $I_{\mathfrak{m}} \triangleq \sum_{i=1}^{n} A_{\mathfrak{m}} f_i$ of $A_{\mathfrak{m}} \triangleq \{n/d \mid n \in A, d \in A \backslash \mathfrak{m}\}$ is a principal ideal, i.e., there exists $g \in A_{\mathfrak{m}}$ such that $I_{\mathfrak{m}} = A_{\mathfrak{m}} g$ (we recall that a maximal ideal $\mathfrak{m}$ of A is an ideal which is not strictly contained in proper ideals of A).*

If I is an invertible fractional ideal of A, then we easily check that I^n is also invertible and its inverse, denoted by I^{-n}, satisfies $I^{-n} = A : I^n = (I^{-1})^n$ for all integer $n \geqslant 1$.

Finally, we shall need the next two arithmetic rules holding in $\mathcal{F}(A)$ (see [7] for more rules).

PROPOSITION 2 [7]. *Let I, J and L be fractional ideals of A. Then, we have the following equalities:*

1. $I(J+L) = IJ + IL$.
2. $(I : J) : L = I : (JL) = (I : L) : J$.

2.3. A FRACTIONAL IDEAL APPROACH TO STABILIZATION PROBLEMS

We want to point out the striking similarities in the denominations *fractional representation approach to analysis and synthesis problems* (control theory) and *fractional ideal approach* (mathematical theory). They are even more amazing when we know that the theory of fractional ideals can be used in order to obtain simple and tractable solutions to stabilization problems as it was shown in [23, 27].

Within the fractional ideal approach to stabilization problems [23, 27], the fractional ideal of A defined by $J = (1, -p) = \{a - bp \mid a, b \in A\}$ is associated with the SISO system defined by:

$$y - pu = 0 \quad \Leftrightarrow \quad (1 \quad -p)\begin{pmatrix} y \\ u \end{pmatrix} = 0, \quad p \in K = Q(A). \tag{2}$$

In particular, we note that the whole system is considered, i.e., we do not separate the system variables (i.e., input and output). This point of view is similar to the *module theory* approach to stabilization problems [22, 33] and to the *behavioural approach* [16–18, 21]. Moreover, we also note that we have $J = (1, -p) = (1, p)$, and thus, for simplicity reasons, we shall only use $J = (1, p)$ in what follows.

THEOREM 1 [23]. *Let A be an integral domain of SISO stable plants, $K = Q(A)$ its quotient field, $p \in K$ and $J = (1, p) = A + Ap$. Then, we have the following results:*

1. *p is stable, i.e., $p \in A$, iff $J = A$ or, equivalently, iff $A : J = A$.*
2. *p admits a weakly coprime factorization iff the fractional ideal $A : J$ is a principal integral ideal of A, i.e., iff there exists $0 \neq d \in A$ such that $A : J = (d)$. Then, $p = n/d$, where $n = pd \in A$, is a weakly coprime factorization of p.*
3. *p is internally stabilizable iff the fractional ideal J is invertible, i.e., iff we have $(A : J)J = A$ or, equivalently, iff there exist $a, b \in A$ satisfying the following conditions:*

$$\begin{cases} a - pb = 1, \\ pa \in A. \end{cases} \tag{3}$$

(a) *If* $a \neq 0$, *then* $c = b/a$ *is a stabilizing controller of* p *and* $J^{-1} = A : J = (a, b)$.
(b) *If* $a = 0$, *then* $p = (-1)/b$ *is a coprime factorization of* p, $c = 1 - b$ *is a stabilizing controller of* p, $J = (1/b)$ *and* $J^{-1} = (b)$.

Then, we have $a = 1/(1 - pc)$ *and* $b = c/(1 - pc)$.

4. $c \in K$ *internally stabilizes* $p \in K$ *iff the following equality of fractional ideals holds:*

$$(1, p)(1, c) = (1 - pc). \tag{4}$$

5. p *admits a coprime factorization iff the fractional ideal* J *is principal fractional ideal of* A, *i.e., iff there exists* $0 \neq d \in A$ *such that* $J = (1/d)$. *Then,* $p = n/d$, *where* $n = dp \in A$, *is a coprime factorization of* p.

Equivalent conditions for internal stabilizability and the existence of coprime factorizations were firstly obtained by Shankar and Sule in [31, 33] using integral ideals over A. We refer the reader to [23, 27, 31] for characterizations of strong, bistably, simultaneous and robust stabilizations [35].

It is important to note that the existence of a coprime factorization implies internal stabilizability (i.e., if $J = (k)$, with $0 \neq k \in K$, then J is invertible and we have $J^{-1} = (k^{-1})$) but the converse is generally not true (invertible fractional ideals are not necessarily principal over a general ring A). In particular, it is still not known whether or not internal stabilizability is equivalent to the existence of coprime factorizations over the classes $\mathcal{A}$, $\hat{\mathcal{A}}$ and W_+ [3, 4, 22, 23, 35]. This equivalence holds for RH_∞ [35], $H_\infty(\mathbb{C}_+)$ [13, 32] and $\mathbb{R}(x_1, \ldots, x_n)_S$ [24, 26, 31].

2.4. PARAMETRIZATION OF ALL STABILIZING CONTROLLERS

An important issue in stabilization problems is to characterize the set of all stabilizing controllers of an internally stabilizable plant. If a plant admits a coprime factorization, then a parametrization of all stabilizing controllers exists and is called *Youla–Kučera parametrization* [4, 35]. Such parametrization has played an important role in the development of the H_∞ and H_2-optimal problems as it is a linear fractional transformation of the free parameter, and thus, it can be used in order to transform such nonlinear optimization problems into affine, and thus, convex ones [3, 35].

Within the fractional ideal approach to stabilization problems, we have recently shown in [23, 24] how to parametrize all stabilizing controllers of an internally stabilizable which does not necessarily admit coprime factorizations. This new parametrization generalizes the Youla–Kučera parametrization. We shall call it *Q-parametrization of all stabilizing controllers* as it is a generalization the Q-parametrization – developed by Zames and Francis in the eighties [37] – for internally stabilizable plants which do not necessarily admit coprime factorizations [28].

We give a new proof of the Q-parametrization of all stabilizing controllers using a mathematical approach which will be used in what follows.

LEMMA 1. *Let $p \in Q(A)$ and $J = (1, p)$ be the fractional ideal of A defined by 1 and p. Then, we have the following exact sequence*

$$0 \longleftarrow J \xleftarrow{f} A^{1\times 2} \xleftarrow{g} A : J \longleftarrow 0, \tag{5}$$

where the A-morphisms f and g are defined by:

$$\begin{array}{rcl} f : A^{1\times 2} & \longrightarrow & J, \\ (a_1 \quad a_2) & \longmapsto & f((a_1 \quad a_2)) = a_1 + a_2 p, \\ g : (A : J) & \longrightarrow & A^{1\times 2}, \\ l & \longmapsto & g(l) = (-lp \quad l). \end{array} \tag{6}$$

Proof. We have $A : J = \{k \in K \mid k, kp \in A\} = \{d \in A \mid dp \in A\}$, and thus, we obtain:

$$\begin{aligned} \ker f &= \{(a_1 \quad a_2) \in A^{1\times 2} \mid a_1 = -a_2 p\} \\ &= \{(-a_2 p \quad a_2) \in A^{1\times 2} \mid a_2 \in A, a_2 p \in A\} \\ &= \{a_2(-p \quad 1) \mid a_2 \in A : J\} = g(A : J). \end{aligned}$$

Moreover, if $l \in \ker g$, then we have $(-lp\ l) = (0\ 0)$, and thus, $l = 0$, i.e., $\ker g = 0$. Finally, we have $f(A^{1\times 2}) = (1, p) = J$, showing that f is surjective, and thus, (5) is a short exact sequence. □

The exact sequence (5) will play an important role as well as the following lemma.

LEMMA 2. *Let $p \in Q(A)$ and $J = (1, p)$. Then, the following assertions are equivalent:*

1. *p is internally stabilizable, i.e., J is invertible and $J^{-1} = (a, b)$ where $a, b \in A$ satisfy* (3).
2. *The short exact sequence* (5) *splits, namely, there exist two A-morphisms h: $J \longrightarrow A^{1\times 2}$ and k: $A^{1\times 2} \longrightarrow A : J$ which satisfy the conditions $f \circ h = id_J$, $k \circ g = id_{A:J}$ and $h \circ f + g \circ k = id_{A^{1\times 2}}$.*

Then, we have the following direct sums

$$h(J) \oplus g(J^{-1}) = A^{1\times 2} \quad \Leftrightarrow \quad J \oplus J^{-1} \cong A^{1\times 2}, \tag{7}$$

and we denote the split exact sequence (5) *by*

$$\begin{array}{c} 0 \longleftarrow J \overset{f}{\longleftarrow} A^{1\times 2} \overset{g}{\longleftarrow} J^{-1} \longleftarrow 0. \\ \quad\ \ \xrightarrow{h} \qquad\quad \xrightarrow{k} \end{array} \tag{8}$$

Proof. From 3 of Theorem 1, p is internally stabilizable iff there exist a and $b \in A$ satisfying (3), i.e., iff the fractional ideal $J = (1, p)$ of A is invertible. Then, we know that $J^{-1} = A : J = (a, b)$.

$1 \Rightarrow 2$. Let a and $b \in A$ satisfy (3) and let us define the following A-morphisms:

$$\begin{array}{rcl} h\colon J & \longrightarrow & A^{1\times 2}, \\ j & \longmapsto & h(j) = (ja \quad -bj), \end{array} \qquad \begin{array}{rcl} k\colon A^{1\times 2} & \longrightarrow & A : J, \\ v & \longmapsto & k(v) = v(b \quad a)^{\mathrm{T}}. \end{array} \tag{9}$$

The A-morphisms h and k are well-defined as, for every $j \in J$, we have $h(j) \in A^{1\times 2}$ because ja and $jb \in A$, and $k(A^{1\times 2}) = (a, b) = A : J$. Now, we have $(f \circ h)(j) = j(a - bp) = j$ for all $j \in J$, i.e., $f \circ h = id_J$. Similarly, we have $(k \circ g)(l) = l(-pb + a) = l$ for all $l \in A : J$, i.e., $k \circ g = id_{A:J}$. Finally, we have $h \circ f + g \circ k = id_{A^{1\times 2}}$ as, for all $(a_1\ a_2) \in A^{1\times 2}$, we have

$$\begin{aligned} (h \circ f + g \circ k)((a_1 \quad a_2)) &= (a_1 + a_2 p)(a \quad -b) + (a_1 b + a_2 a)(-p \quad 1) \\ &= (a_1(a - bp) \quad a_2(a - bp)) = (a_1 \quad a_2). \end{aligned}$$

Therefore, the exact sequence (5) splits, and thus, we easily obtain (7).

$2 \Rightarrow 1$. If the exact sequence (5) splits, then there exists an A-morphism $h\colon J \longrightarrow A^{1\times 2}$ such that $f \circ h = id_J$. Using the fact that J is generated as an A-module by 1 and p, then h is completely determined by $h(1) = (\alpha \ -\beta) \in A^{1\times 2}$ and $h(p) = (\gamma \ -\theta) \in A^{1\times 2}$. But, for all $d \in A : J$, we have $dh(p) = (d\gamma \ -d\theta)$ and $dh(p) = h(dp) = (dp)h(1)$ as $dp \in A$ and h is an A-morphism. Therefore, we obtain $(d\gamma \ -d\theta) = (dp\alpha \ -dp\beta)$ for all $d \in A : J$, and thus, we have $\gamma = p\alpha \in A$ and $\theta = p\beta \in A$. Hence, α and $\beta \in A : J$ and, for all $j = u + vp \in J$, u and $v \in A$, we have

$$\begin{aligned} h(j) &= uh(1) + vh(p) = u(\alpha \quad -\beta) + v(p\alpha \quad -p\beta) \\ &= (u + pv)(\alpha \quad -\beta) = j(\alpha \quad -\beta). \end{aligned}$$

Finally, using $f \circ h = id_J$, for all $j \in J$, we have $f(h(j)) = j$, i.e., $j(\alpha - \beta p) = j$ for all $j \in J$. Taking $j = 1 \in J$, we obtain $\alpha - \beta p = 1$, and thus, p is internally stabilizable from 3 of Theorem 1. □

We call *right-inverse* of the A-morphism $f\colon A^{1\times 2} \longrightarrow J$ any A-morphism $h\colon J \longrightarrow A^{1\times 2}$ such that $f \circ h = id_J$. We define the set $L = \{h\colon J \longrightarrow A^{1\times 2} \mid f \circ h = id_J\}$ of right-inverses of f and the set $S = \{(a - b) \in A^{1\times 2} \mid ap \in A,\ a - bp = 1\}$ of a and $b \in A$ satisfying (3). Then, the proof of Lemma 2 shows that there is a one-to-one correspondence between L and S defined by:

$$\begin{array}{rcl} L & \longleftrightarrow & S \\ h & \longmapsto & h(1) \\ h(j) = j(a \quad -b) & \longleftarrow & (a \quad -b). \end{array} \tag{10}$$

The next lemma is a standard result in the theory of the fractional ideals.

LEMMA 3. [7, 23]. *Let I and J be two fractional ideals of A and ϕ: $I \longrightarrow J$ an A-morphism from I to J. Then, there exists $q \in Q(A)$ such that $\phi(a) = qa$ for all $a \in I$. In particular, for every $0 \neq i \in I$, we have $q = \phi(i)/i \in J : I$, and thus, we obtain the following isomorphism:*

$$\hom_A(I, J) \cong J : I. \tag{11}$$

The next lemma gives a parametrization of all right-inverses of f.

LEMMA 4 *Let h: $J \longrightarrow A^{1\times 2}$ be a right-inverse of f: $A^{1\times 2} \longrightarrow J$. Then, every right-inverse of the A-morphism f has the form $h + g \circ \phi$, where ϕ is any A-morphism from J to J^{-1}, i.e., $\phi \in \hom_A(J, J^{-1}) \cong J^{-2}$.*

In other words, every right-inverse of f has the following form:

$$\begin{aligned} h_q\colon J &\longrightarrow A^{1\times 2} \\ \lambda &\longmapsto (\lambda(a - qp) \quad -\lambda(b - q)), \quad \forall q \in J^{-2}. \end{aligned} \tag{12}$$

Proof. Let us suppose that there exist h, h': $J \longrightarrow A^{1\times 2}$ such that $f \circ h = id_J$ and $f \circ h' = id_J$. Then, we have $f \circ (h' - h) = 0$, i.e., $f((h' - h)(a)) = 0$ for all $a \in J$. Using the fact that (5) is a short exact sequence, i.e., $\ker f = \operatorname{im} g$ and g is injective, then, for all $a \in J$, there exists a unique $b \in J^{-1}$ such that $h'(a) - h(a) = g(b)$. If we define the A-morphism ϕ: $J \longrightarrow J^{-1}$ by $\phi(a) = b$, then we obtain $h' = h + g \circ \phi$.

Conversely, if we have $f \circ h = id_J$ and we define $h' = h + g \circ \phi$ where ϕ: $J \longrightarrow J^{-1}$ is any A-morphism, then, using the fact that $f \circ g = 0$, we obtain

$$f \circ h' = f \circ h + f \circ g \circ \phi = f \circ h = id_J.$$

Therefore, every right inverse of f has the form $h' = h + g \circ \phi$, where h is a particular right-inverse of f and ϕ: $J \longrightarrow J^{-1}$ is any A-morphism. Now, from 2 of Proposition 2 and (11), we have

$$\hom_A(J, J^{-1}) \cong J^{-1} : J = (A : J) : J = A : J^2 = J^{-2},$$

where the isomorphism is defined by $\phi \in \hom_A(J, J^{-1}) \longmapsto \phi(j)/j \in J^{-2}$ and $0 \neq j \in J$ is a fixed element (see Lemma 3). Thus, every element $\phi \in \hom_A(J, J^{-1})$ corresponds to the multiplication map by $q = \phi(j)/j \in J^{-2}$. Finally, for all $\lambda \in J$, we obtain

$$\begin{aligned} h'(\lambda) &= h(\lambda) + g(q\lambda) = (\lambda a \quad -\lambda b) + (-q\lambda p \quad q\lambda) \\ &= (\lambda(a - qp) \quad -\lambda(b - q)). \end{aligned}$$ □

Let $J = (1, p)$ be an invertible fractional ideal of A and $J^{-1} = (a, b)$, where a and $b \in A$ satisfy (3). Then, we have $J^2 = (1, p, p^2) = (1, p^2)$ as, using (3), we obtain $p = (ap) - bp^2 \in (1, p^2)$. Therefore, on the one hand, we have

$$J^{-2} = A : (1, p^2) = \{l \in A \mid lp^2 \in A\} = (r_1, r_2),$$

where r_1 and $r_2 \in A$ are such that $r_1 - r_2 p^2 = 1$ and $r_1 p^2 \in A$ and, on the other hand, we have

$$J^{-2} = (J^{-1})^2 = (a^2, ab, b^2) = (a^2, b^2),$$

as, using (3), we obtain $ab = ba^2 - (ap)b^2 \in (a^2, b^2)$.

We are now in position to give the general parametrization of all stabilizing controllers of an internally stabilizable plant. We refer to [23] for a different proof.

THEOREM 2 *Let A be an integral domain of SISO stable plants, $K = Q(A)$ its quotient field, $p \in K$ a plant and the fractional ideal $J = (1, p)$ of A. If p is internally stabilizable and $c_\star = b/a$ is a stabilizing controller of p, i.e., $0 \neq a, b \in A$ satisfy* (3), *$a = 1/(1 - pc_\star)$ and $b = c_\star/(1 - pc_\star)$, then all stabilizing controllers of p are defined by*

$$c(q_1, q_2) = \frac{b + q_1 r_1 + q_2 r_2}{a + (q_1 r_1 + q_2 r_2)p} = \frac{c_\star + (q_1 r_1 + q_2 r_2)(1 - pc_\star)}{1 + (q_1 r_1 + q_2 r_2)p(1 - pc_\star)}, \tag{13}$$

where r_1 and $r_2 \in A$ are such that $J^{-2} = (r_1, r_2)$ and q_1 and q_2 are any elements of A satisfying:

$$a + pq_1 r_1 + q_2 r_2 \neq 0 \quad or \quad 1 + (q_1 r_1 + q_2 r_2)p(1 - pc_\star) \neq 0.$$

In particular, we can take $r_1 = a^2 = 1/(1 - pc_\star)^2$ and $r_2 = b^2 = c_\star^2/(1 - pc_\star)^2$ and the parametrization (13) *of all stabilizing controllers of p becomes*

$$c(q_1, q_2) = \frac{b + q_1 a^2 + q_2 b^2}{a + (q_1 a^2 + q_2 b^2)p} = \frac{(1 - pc_\star)c_\star + q_1 + q_2 c_\star^2}{(1 - pc_\star) + (q_1 + q_2 c_\star^2)p}, \tag{14}$$

where q_1 and q_2 are any elements of A such that $a + q_1 pa^2 + q_2 pb^2 \neq 0$ or $(1 - pc_\star) + q_1 p + q_2 pc_\star^2 \neq 0$.

Proof. Let $c_\star$ be a stabilizing controller of p and let us define $a = 1/(1 - pc_\star)$ and $b = c_\star/(1 - pc_\star) \in A$. Then, we have $c_\star = b/a$ where a and $b \in A$ satisfy (3). From Lemma 4, it follows that all right-inverses h_q of $f\colon A^{1\times 2} \longrightarrow J$ are defined by (12), where $q \in J^{-2}$, i.e., $h_q(\lambda) = \lambda(a - qp \quad -(b - q))$ for all $\lambda \in J$. By 3 of Theorem 1, we know that all stabilizing controllers c of p are of the form $c = \beta/\alpha$, where α and $\beta \in A$ satisfy $\alpha - \beta p = 1$ and $\alpha p \in A$. Therefore, using (10), we finally obtain that all stabilizing controllers of p have the form:

$$c(q) = \frac{b + q}{a + qp}, \quad \forall q \in J^{-2} = (r_1, r_2) = (a^2, b^2), \quad a + qp \neq 0. \tag{15}$$

Using the fact that $q \in J^{-2}$ has the form $q = q_1 r_1 + q_2 r_2$ or $q = q_1 a^2 + q_2 b^2$ for some q_1 and $q_2 \in A$, we finally obtain (13) and (14), which proves the result. □

If p admits a coprime factorization $p = n/d$, $dx - ny = 1$, $x, y \in A$, then, by 5 of Theorem 1, we obtain that $J = (1/d)$, and thus, $J^{-2} = (d^2)$. Therefore, every element $q \in J^{-2}$ has the form $q = q_1 d^2$ where $q_1 \in A$. Moreover, we easily check that $a = dx$ and $b = dy$, and thus, we have $b + q = d(y + q_1 d)$ and $a + qp = d(x + q_1 n)$. By substituting these two expressions into (15), we find the Q-parametrization firstly obtained by Zames and Francis in [37]:

$$c(q_1) = \frac{d(y + q_1 d)}{d(x + q_1 n)} = \frac{y + q_1 d}{x + q_1 n}, \quad \forall q_1 \in A, \quad x + q_1 n \neq 0.$$

We note that the last member of the previous equalities is nothing else than the Youla–Kučera parametrization of all stabilizing controllers [4, 35].

We have already proved that $J^2 = (1, p^2)$, and thus, by 5 of Theorem 1, J^2 is a principal fractional ideal of A iff p^2 admits a coprime factorization $p^2 = s/r$, $0 \neq s, r \in A$. Then, $J^2 = (1/r)$, and thus, $J^{-2} = (r)$ and any $q \in J^{-2}$ has the form $q = q_1 r$ where $q_1 \in A$. We obtain the following corollary.

COROLLARY 1 [23]. *An internal stabilizable plant p admits a parametrization of all stabilizing controllers with a single free parameter iff p^2 admits a coprime factorization. Then, we have:*

1. *If p does not admit a coprime factorization but p^2 admits a coprime factorization $p^2 = s/r$, $0 \neq r, s \in A$, then all stabilizing controllers of p have the form*

$$c(q) = \frac{b + qr}{a + qrp} = \frac{c_\star + qr(1 - pc_\star)}{1 + qrp(1 - pc_\star)}, \tag{16}$$

where q is any element of A such that $a + qrp \neq 0$ or $1 + qrp(1 - pc_\star) \neq 0$, $c_\star = b/a$ is a stabilizing controller of p, $a = 1/(1 - pc_\star)$ and $b = c_\star/(1 - pc_\star) \in A$ satisfy (3).

2. *If p admits a coprime factorization $p = n/d$, $0 \neq d, n \in A$, $dx - ny = 1$ $(x, y \in A)$, then, all stabilizing controllers of p have the form*

$$c(q) = \frac{y + qd}{x + qn}, \quad \forall q \in A, \quad x + qn \neq 0. \tag{17}$$

We refer to [23] for more details and examples. Finally, we note that the parametrizations of all stabilizing controllers (13) and (14) (resp., (16) and (17)) are linear fractional transformations of the free parameters q_1 and $q_2 \in A$ (resp., $q \in A$). Therefore, we can use them in order to transform nonlinear optimal problems into affine, and thus, convex ones. See [26, 27] for more details.

3. An Algebraic Interpretation to the Operator-Theoretic Approach

If instead of (2), we consider the map $u \longmapsto y = pu$, then we are led to introduce the fractional ideal (p). But, as we have shown in Theorems 1, 2 and Corollary 1, the structural properties of the system only depend on the fractional ideal $J = (1, p)$ and not on (p). Therefore, the fractional ideal approach shows that the system must be thought as $\{(u\ y)^{\mathrm{T}} \mid y - pu = 0\}$, i.e., as the kernel of the map $(u\ y)^{\mathrm{T}} \longmapsto y - pu$. In [34], Vidyasagar has shown that the appropriated way to study a linear system defined by a (unbounded) linear operator $u \longmapsto pu = y$ was to consider its *graph*, i.e., u and y together. See [8–10, 35] and the references therein for more details.

The main goal of this paper is to show how the *operator-theoretic approach to stabilization problems* developed in [5, 8–10, 34, 35] can be obtained as a dual theory of the fractional ideal approach. To our knowledge, this interpretation is new and it allows us to find again, unify and generalize different results obtained in the literature.

3.1. DUALITY

Let $p \in K = Q(A)$ and $J = (1, p)$ be the fractional ideal of A defined by 1 and p. In Lemma 1, we proved that the following sequence was exact:

$$0 \longleftarrow J \xleftarrow{f} A^{1\times 2} \xleftarrow{g} A : J \longleftarrow 0,$$

where the A-morphisms f and g are defined by (6). Now, if $\mathcal{F}$ is an A-module (e.g., $\mathcal{F} = H_2(\mathbb{C}_+)$, $A = RH_\infty$ or $H_\infty(\mathbb{C}_+)$), then, applying the functor $\hom_A(\cdot, \mathcal{F})$ to the previous exact sequence, we obtain the following exact sequence [1, 29]

$$\begin{aligned} 0 \ &\longrightarrow\ \hom_A(J, \mathcal{F}) \xrightarrow{f^\star} \hom_A(A^{1\times 2}, \mathcal{F}) \xrightarrow{g^\star} \hom_A(A : J, \mathcal{F}) \\ &\longrightarrow\ \mathrm{ext}^1_A(J, \mathcal{F}) \longrightarrow 0, \end{aligned} \tag{18}$$

where $f^\star$ is defined by $f^\star(\phi) = \phi \circ f$ for all $\phi \in \hom_A(J, \mathcal{F})$ and similarly for $g^\star$.

The A-morphism $f^\star\colon \hom_A(J, \mathcal{F}) \longrightarrow \hom_A(A^{1\times 2}, \mathcal{F})$ is defined by $f^\star(\phi) = \phi \circ f$, i.e., for all $a = (a_1\ a_2) \in A^{1\times 2}$, we have

$$\begin{aligned} (f^\star(\phi))(a) &= \phi(f(a)) = \phi(a_1 + a_2 p) = a_1\phi(1) + a_2\phi(p) \\ &= a(\phi(1) \quad \phi(p))^{\mathrm{T}}. \end{aligned}$$

Using the isomorphism $\iota\colon \hom_A(A^{1\times 2}, \mathcal{F}) \longrightarrow \mathcal{F}^2$ defined by

$$\iota(\psi) = (\psi(e_1) \quad \psi(e_2))^{\mathrm{T}}, \quad \forall \psi \in \hom_A(A^{1\times 2}, \mathcal{F}),$$

where $\{e_1 = (1\ 0), e_2 = (0\ 1)\}$ is the standard basis of $A^{1\times 2}$, we finally find:

$$\iota(f^\star(\phi)) = (\phi(1) \quad \phi(p))^{\mathrm{T}}, \quad \forall \phi \in \hom_A(J, \mathcal{F}).$$

We recall that $A : J = \{d \in A \mid dp \in A\}$. An A-morphism $\phi \in \hom_A(J, \mathcal{F})$ is completely defined by:

$$\begin{cases} \phi(1) \in \mathcal{F}, \\ \phi(p) \in \mathcal{F}, \\ \phi(dp) = (dp)\phi(1), \quad \forall d \in A : J, \\ \phi(dp) = d\phi(p), \quad \forall d \in A : J, \end{cases} \iff \begin{cases} \phi(1) \in \mathcal{F}, \\ \phi(p) \in \mathcal{F}, \\ d\phi(p) = (dp)\phi(1), \\ \quad \forall d \in A : J. \end{cases} \tag{19}$$

The A-morphism $g^\star$: $\hom_A(A^{1\times 2}, \mathcal{F}) \longrightarrow \hom_A(A:J, \mathcal{F})$ is defined by

$$\begin{aligned} g^\star(\psi)(d) &= \psi(g(d)) = \psi((-dp \quad d)) = d(-p \quad 1)(\psi(e_1) \quad \psi(e_2))^{\mathrm{T}} \\ &= d(-p \quad 1)\iota(\psi), \quad \forall d \in A:J. \end{aligned}$$

Hence, the exact sequence (18) implies the following exact sequence:

$$\begin{array}{l} 0 \longrightarrow \hom_A(J, \mathcal{F}) \xrightarrow{\iota\circ f^\star} \mathcal{F}^2 \xrightarrow{g^\star\circ\iota^{-1}} \hom_A(A:J, \mathcal{F}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow \operatorname{ext}^1_A(J, \mathcal{F}) \longrightarrow 0. \\ \qquad\qquad \phi \longmapsto \begin{pmatrix} \phi(1) \\ \phi(p) \end{pmatrix} \\ \qquad\qquad\qquad \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto y - p\,u \end{array} \tag{20}$$

From general arguments of homological algebra, we already know that the sequences (18) and (20) are exact. Let us give a direct proof. Using (19), we obtain that $\iota \circ f^\star$ is injective because we have:

$$\begin{aligned} \phi(1) = \phi(p) = 0 \;\Rightarrow\; & \forall \lambda_1, \lambda_2 \in A, \\ & \phi(\lambda_1 + \lambda_2 p) = \lambda_1\phi(1) + \lambda_2\phi(p) = 0 \Rightarrow \phi = 0. \end{aligned}$$

Let us check the exactness of (20) at $\mathcal{F}^2$. Firstly, for all $\phi \in \hom_A(J, \mathcal{F})$, the A-morphism defined by

$$\begin{aligned} ((g^\star \circ \iota^{-1}) \circ (\iota \circ f^\star))(\phi) &= (g^\star \circ f^\star)(\phi) \\ &= \phi(p) - p\phi(1) \in \hom_A(A:J, \mathcal{F}) \end{aligned}$$

satisfies, for all $d \in A:J$, $(\phi(p) - p\phi(1))(d) = d\phi(p) - (dp)\phi(1) = 0$ as we have (19). Therefore, we have $g^\star \circ f^\star = 0$, and thus, we obtain $\operatorname{im}(\iota \circ f^\star) \subseteq \ker(g^\star \circ \iota^{-1})$.

Secondly, if $(u \;\; y)^{\mathrm{T}} \in \ker(g^\star \circ \iota^{-1})$, then $u, y \in \mathcal{F}$ are such that $y - py = 0$ as an A-morphism of $\hom_A(A:J, \mathcal{F})$, i.e., for all $d \in A:J$, we have $dy = (dp)u$. Then, using (19), we obtain that the A-morphism defined by $\phi(1) = u$ and $\phi(p) = y$ belongs to $\phi \in \hom_A(J, \mathcal{F})$ and we have $(\iota \circ f^\star)(\phi) = (u \;\; y)^{\mathrm{T}}$, which proves the exactness of the sequence (20) at $\mathcal{F}^2$.

3.2. DOMAINS AND GRAPHS OF LINEAR OPERATORS

In this section, from the exact sequence (20), we derive a new exact sequence which has an operator-theoretic interpretation. In order to do that, we first introduce a few definitions.

DEFINITION 3. Let $p \in K = Q(A)$, $J = (1, p)$, $\mathcal{F}$ be an A-module and $u \in \mathcal{F}$.

- We use the following notations for the A-modules

$$\mathrm{ann}_{\mathcal{F}}(A:J) = \{y \in \mathcal{F} \mid \forall d \in A:J, dy = 0\},$$
$$\overline{\mathcal{F}} = \mathcal{F}/\mathrm{ann}_{\mathcal{F}}(A:J),$$

and $\overline{y}$ denotes the residue class of $y \in \mathcal{F}$ modulo $\mathrm{ann}_{\mathcal{F}}(A:J)$.

- pu denotes the residue class $\overline{y} \in \overline{\mathcal{F}}$ of the elements $y \in \mathcal{F}$ which satisfy:

$$dy = (dp)u, \quad \forall d \in A:J = \{d \in A \mid dp \in A\}. \tag{21}$$

- We denote by $pu \in \overline{\mathcal{F}}$ the fact that there exists $\overline{y} \in \overline{\mathcal{F}}$ such that $\overline{y} = pu$.
- The domain of the linear operator $p\colon \mathcal{F} \longrightarrow \overline{\mathcal{F}}$ is defined by

$$\mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid pu \in \overline{\mathcal{F}}\}. \tag{22}$$

- The graph of the linear operator $p\colon \mathcal{F} \longrightarrow \overline{\mathcal{F}}$ is defined by

$$\mathrm{graph}_{\mathcal{F}}(p) = \{(u \quad pu)^{\mathrm{T}} \in \mathcal{F} \times \overline{\mathcal{F}} \mid u \in \mathrm{dom}_{\mathcal{F}}(p)\}. \tag{23}$$

We note that pu is well-defined because if there exist $y_1, y_2 \in \mathcal{F}$ such that $dy_i = dpu, i = 1, 2$, then $d(y_1 - y_2) = 0$ for all $d \in A:J$, and thus, $\overline{y_1} = \overline{y_2} = pu$. We now explain on an example why pu does not generally belong to $\mathcal{F}$.

EXAMPLE 3. Let us consider $A = \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$ be the ring of differential operators in $\frac{\mathrm{d}}{\mathrm{d}t}$ with coefficients in $\mathbb{R}$, $K = Q(A) = \mathbb{R}(\frac{\mathrm{d}}{\mathrm{d}t})$, $p = (\frac{\mathrm{d}}{\mathrm{d}t})^{-1} \in K$, $J = (1, p)$ and $\mathcal{F} = C^{\infty}(\mathbb{R})$. Then, we have $A:J = (\frac{\mathrm{d}}{\mathrm{d}t})$.

If, instead of (22), we had chosen the definition $\mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid pu \in \mathcal{F}\}$, where $pu \in \mathcal{F}$ means that there exists $y \in \mathcal{F}$ such that $dy = (dp)u$ for all $d \in A:J = \{d \in A \mid dp \in A\}$, then the map $p\colon \mathrm{dom}_{\mathcal{F}}(p) \longrightarrow \mathcal{F}$ would have become a multivalued function. Indeed, $p0 \in \mathcal{F}$ would have meant that there exists $y \in \mathcal{F}$ such that $dy = 0$ for all $d \in A:J$, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}y = 0$. Therefore, if $y = c$ is any constant function, i.e., $y(t) = c$ for all $t \in \mathbb{R}$, then $p0 = c$ for all $c \in \mathbb{R}$, which shows that p is not a well-defined function. However, if we use Definition 3, then $\mathrm{ann}_{\mathcal{F}}(A:J) = \{y \in \mathcal{F} \mid y = c, c \in \mathbb{R}\}$, $p0 = \overline{c} = \overline{0}$ and $p\colon \mathrm{dom}_{\mathcal{F}}(p) \longrightarrow \overline{\mathcal{F}}$ becomes a well-defined function, where $\overline{\mathcal{F}} = \mathcal{F}/\{y \in \mathcal{F} \mid y = c, c \in \mathbb{R}\}$. We note that we have $\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}$ because, for every $u \in \mathcal{F}$, there always exists $y \in \mathcal{F}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}y = u$ (see Section 4.2 and Example 4). Finally, $(u \ \overline{y})^{\mathrm{T}} \in \mathrm{graph}_{\mathcal{F}}(p)$ means that $u \in \mathcal{F}$ and $\overline{y} \in \overline{\mathcal{F}}$ satisfy $\frac{\mathrm{d}}{\mathrm{d}t}y = u$. Integrating the ordinary differential equation, we obtain $y(t) = \int_0^t u(\tau)\,\mathrm{d}\tau + y(0)$, and thus, we have

$$\mathrm{graph}_{\mathcal{F}}(p) = \left\{ \left(u \quad \overline{\int_0^t u(\tau)\,\mathrm{d}\tau} \right)^{\mathrm{T}} \in \mathcal{F} \times \overline{\mathcal{F}} \mid u \in \mathcal{F} \right\}.$$

Let us state the following useful lemma.

LEMMA 5.

1. *The domain* $\mathrm{dom}_{\mathcal{F}}(p)$ *of* p *is an* A*-module.*
2. *If* $u_1, u_2 \in \mathrm{dom}_{\mathcal{F}}(p)$ *and* $a_1, a_2 \in A$*, then we have*

$$p(a_1u_1 + a_2u_2) = a_1pu_1 + a_2pu_2,$$

and thus, $\mathrm{graph}_{\mathcal{F}}(p)$ *is an* A*-module.*

3. *If* $p \in Q(A)$*,* $J = (1, p)$*,* $x \in \mathcal{F}$ *and* $d \in A : J = \{d \in A \mid dp \in A\}$*, then we have*

$$p(dx) = \overline{(dp)x} = \overline{nx},$$

where $n = dp \in A$*, i.e.,* $p = n/d,\ 0 \neq d, n \in A$*, is the fractional representation of* p*. In particular, we have*

$$(A : J)\mathcal{F} \subseteq \mathrm{dom}_{\mathcal{F}}(p).$$

Proof. 1 & 2. Let $u_1, u_2 \in \mathrm{dom}_{\mathcal{F}}(p)$ and $a_1, a_2 \in A$. Then, there exist $y_1, y_2 \in \mathcal{F}$ such that $\overline{y_i} = pu_i$, i.e., we have $dy_i = (dp)u_i$ for all $d \in A : J$ and $i = 1, 2$. Therefore, we have

$$\begin{aligned} d(a_1y_1 + a_2y_2) &= a_1dy_1 + a_2dy_2 = a_1(d\,p)u_1 + a_2(dp)u_2 \\ &= (dp)(a_1u_1 + a_2u_2), \end{aligned}$$

for all $d \in A : J$ and $a_1y_1 + a_2y_2 \in \mathcal{F}$, which shows that $a_1u_1 + a_2u_2 \in \mathrm{dom}_{\mathcal{F}}(p)$. Moreover, from the previous equality, we obtain:

$$p(a_1u_1 + a_2u_2) = \overline{a_1y_1 + a_2y_2} = a_1\overline{y_1} + a_2\overline{y_2} = a_1(pu_1) + a_2(pu_2).$$

This result proves that $\mathrm{graph}_{\mathcal{F}}(p)$ is an A-module.

3. Let $d \in A : J$, $n = dp \in A$ and $x \in \mathcal{F}$. For every $d' \in A : J$, we denote by $n' = d'p \in A$ and we have $p = n/d = n'/d'$, i.e., $dn' = d'n$. Therefore, for all $d' \in A : J$, we have

$$\begin{aligned} (d'p)(dx) &= n'(dx) = (n'd)x = (d'n)x = d'(nx) \\ &\Rightarrow\ p(dx) = \overline{nx} = \overline{(dp)x}. \end{aligned}$$

□

Within the fractional ideal approach, we have just developed the concepts of the domain and graph of a linear operator. These concepts generalize the ones used in [6, 8, 9, 34] and we shall explain in Corollary 4 when we can find them again.

Let us define the following A-morphism:

$$\begin{aligned} \rho\colon\ \hom_A(J, \mathcal{F}) &\longrightarrow \mathrm{dom}_{\mathcal{F}}(p), \\ \phi &\longmapsto \phi(1). \end{aligned} \tag{24}$$

This A-morphism is well-defined because, from (19), $\phi \in \hom_A(J, \mathcal{F})$ is defined by $\phi(1) \in \mathcal{F}$ and $\phi(p) \in \mathcal{F}$ which satisfy $d\phi(p) = (dp)\phi(1)$ for all

$d \in A : J$, a fact showing that $\phi(1) \in \mathrm{dom}_{\mathcal{F}}(p)$ and $\overline{\phi(p)} = p\phi(1)$. Moreover, the A-morphism ρ is surjective: for any $u \in \mathrm{dom}_{\mathcal{F}}(p)$, there exists $y \in \mathcal{F}$ satisfying $dy = (dp)u$ for all $d \in A : J$, and thus, $\phi_u(1) = u \in \mathcal{F}$ and $\phi_u(p) = y \in \mathcal{F}$ define $\phi_u \in \hom_A(J, \mathcal{F})$ which satisfies $\rho(\phi_u) = \phi_u(1) = u$. Therefore, we have the following exact sequence

$$0 \longrightarrow \ker\rho \xrightarrow{i} \hom_A(J, \mathcal{F}) \xrightarrow{\rho} \mathrm{dom}_{\mathcal{F}}(p) \longrightarrow 0, \tag{25}$$

where $\ker\rho = \{\phi \in \hom_A(J, \mathcal{F}) \mid \phi(1) = 0, d\phi(p) = 0, \forall d \in A : J\}$ and we have:

$$\mathrm{dom}_{\mathcal{F}}(p) = \rho(\hom_A(J, \mathcal{F})) \cong \hom_A(J, \mathcal{F})/\ker\rho. \tag{26}$$

Using the exact sequences (20) and (25), we obtain the commutative exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \ker\rho & \xrightarrow{\iota\circ f^\star} & \mathcal{F}^2 & \xrightarrow{\pi} & \mathcal{F}^2/(\iota\circ f^\star)(\ker\rho) \longrightarrow 0 \\
 & & \downarrow & & \| & & \\
0 \longrightarrow & & \hom_A(J, \mathcal{F}) & \xrightarrow{\iota\circ f^\star} & \mathcal{F}^2 & \xrightarrow{g^\star\circ\iota^{-1}} & \hom_A(A : J, \mathcal{F}) \\
 & & & & & & \longrightarrow \mathrm{ext}^1_A(J, \mathcal{F}) \longrightarrow 0. \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{dom}_{\mathcal{F}}(p) & & 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

Then, using the fact

$$\begin{aligned}
\mathcal{F}^2/(\iota\circ f^\star)(\ker\rho) &= \mathcal{F}^2/\{(0 \quad y)^{\mathrm{T}} \in \mathcal{F}^2 \mid dy = 0, \forall d \in A : J\} \\
&= \mathcal{F} \times (\mathcal{F}/\{y \in \mathcal{F} \mid dy = 0, \forall d \in A : J\}) = \mathcal{F} \times \overline{\mathcal{F}},
\end{aligned}$$

we obtain $\pi((u\ y)^{\mathrm{T}}) = (u\ \overline{y})^{\mathrm{T}}$ for all u and $y \in \mathcal{F}$. Then, a chase in the previous commutative exact diagram shows that we have the following exact sequence

$$\begin{array}{ccccc}
0 \longrightarrow \mathrm{dom}_{\mathcal{F}}(p) & \xrightarrow{\delta} & \mathcal{F} \times \overline{\mathcal{F}} & \xrightarrow{\epsilon} & \hom_A(A : J, \mathcal{F}) \\
 & & & & \longrightarrow \mathrm{ext}^1_A(J, \mathcal{F}) \longrightarrow 0, \\
u & \longmapsto & (u \quad p\,u)^{\mathrm{T}} & & \\
 & & (u \quad \overline{y})^{\mathrm{T}} & \longmapsto & y - p\,u
\end{array} \tag{27}$$

where δ: $\mathrm{dom}_{\mathcal{F}}(p) \longrightarrow \mathcal{F} \times \overline{\mathcal{F}}$ is defined by $\delta(u) = (\pi\circ\iota\circ f^\star)(\phi)$ and ϕ is any element of $\hom_A(J, \mathcal{F})$ such that $\rho(\phi) = \phi(1) = u$. The value $\delta(u)$ does not depend on a particular choice of ϕ satisfying $\rho(\phi) = u$: if $\phi_1, \phi_2 \in \hom_A(J, \mathcal{F})$ are such that $\rho(\phi_1) = \rho(\phi_2) = u$, then we have

$$\phi_1 - \phi_2 \in \ker\rho \Rightarrow (\pi\circ\iota\circ f^\star)(\phi_1 - \phi_2) = 0 \Rightarrow \delta(\phi_1) = \delta(\phi_2).$$

Then, we have $\delta(u) = (\pi \circ \iota \circ f^\star)(\phi_u)$, where the A-morphism $\phi_u \in \hom_A(J, \mathcal{F})$ is defined by $\phi_u(1) = u$ and $\phi_u(p) = y$ where y is any element of $\mathcal{F}$ satisfying $dy = (dp)u$ for all $d \in A : J$. Therefore, we have $\delta(u) = (u \; \overline{y})^{\mathrm{T}}$ where $\overline{y}$ is the residue class of any element $y \in \mathcal{F}$ modulo $\mathrm{ann}_{\mathcal{F}}(A : J)$ satisfying $dy = (dp)u$ for all $d \in A : J$, i.e., $\overline{y} \in \overline{\mathcal{F}}$ or, in other words:

$$\delta(u) = (u \quad pu)^{\mathrm{T}} \quad \forall u \in \mathrm{dom}_{\mathcal{F}}(u). \tag{28}$$

Finally, using (27), we obtain

$$\mathrm{graph}_{\mathcal{F}}(p) = \{(u \quad pu)^{\mathrm{T}} \in \mathcal{F} \times \overline{\mathcal{F}} \mid u \in \mathrm{dom}_{\mathcal{F}}(p)\} = \delta(\mathrm{dom}_{\mathcal{F}}(p)). \tag{29}$$

The A-morphism $\epsilon\colon \mathcal{F} \times \overline{\mathcal{F}} \longrightarrow \hom_A(A : J, \mathcal{F})$ defined by $(u \; \overline{y})^{\mathrm{T}} \longmapsto y - pu$, is well-defined because it does not depend on the choice of the element $y' \in \mathcal{F}$ such that $\overline{y'} = \overline{y}$. Indeed, if $y' \in \mathcal{F}$ is such that $\overline{y'} = \overline{y}$, then we have $y - y' \in \mathrm{ann}_{\mathcal{F}}(A : J)$, i.e., $d(y - y') = 0$ for all $d \in A : J$. Moreover, we have

$$(y - pu) - (y' - pu) = y - y' \in \hom_A(A : J, \mathcal{F}).$$

Therefore, we have $(y - y')(d) = d(y - y') = 0$ for all $d \in A : J$, which shows that $y - y' = 0$ as an element of $\hom_A(A : J, \mathcal{F})$, and thus, we have $y - pu = y' - pu \in \hom_A(A : J, \mathcal{F})$.

4. An Operator-Theoretic Approach to Internal Stabilizability

Using the operator-theoretic interpretation obtained in the previous section, we now precisely determine the graph of a stabilizable plant and explain the links between the graph of a plant p and the graph of any stabilizing controller c of p. These results generalize those obtained for $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$ and $\mathcal{F} = H_2(\mathbb{C}_+)$ [6, 8, 9, 34, 35]. In particular, we do not assume that the internally stabilizable plant p admits a coprime factorization.

4.1. MAIN RESULTS ON THE GRAPH OF A STABILIZABLE PLANT

We shall need the following proposition.

PROPOSITION 3 (Lemma 3.59 of [29]). *If M is a finitely generated projective A-module* (*see* 3 *of Proposition* 1) *and $\mathcal{F}$ is an A-module, then we have the following isomorphism*

$$\sigma\colon M \otimes_A \mathcal{F} \cong \hom_A(\hom_A(M, A), \mathcal{F}),$$

where $\sigma(m \otimes f)(g) = g(m) f$, for all $g \in \hom_A(M, A)$, $m \in M$ and $f \in \mathcal{F}$.

We have the following proposition.

PROPOSITION 4. *Let $p \in Q(A)$, $J = (1, p)$ and $\mathcal{F}$ be an A-module.*

1. *If p is stable, i.e., $p \in A$, then we have* $\mathrm{ann}_{\mathcal{F}}(A : J) = 0$, $\overline{\mathcal{F}} = \mathcal{F}$ *and*

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}, \\ \mathrm{graph}_{\mathcal{F}}(p) = \{(u \quad pu)^{\mathrm{T}} \in \mathcal{F}^2 \mid u \in \mathcal{F}\}, \\ \hom_A(A : J, \mathcal{F}) \cong \mathcal{F}, \\ \mathrm{ext}^1_A(J, \mathcal{F}) = 0. \end{cases}$$

2. *If p admits a weakly coprime factorization $p = n/d$, $0 \neq d, n \in A$, then we have*

$$\begin{cases} \overline{\mathcal{F}} = \mathcal{F}/\{y \in \mathcal{F} \mid dy = 0\}, \\ \mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid \exists y \in \mathcal{F}, dy = nu\}, \\ \hom_A(A : J, \mathcal{F}) \cong (d^{-1}) \otimes_A \mathcal{F}. \end{cases}$$

Proof. 1. If p is stable, i.e., $p \in A$, then, by 1 of Theorem 1, we have $J = (1, p) = A$, $A : J = A$, $\mathrm{ann}_{\mathcal{F}}(A : J) = 0$ and $\overline{\mathcal{F}} = \mathcal{F}/0 = \mathcal{F}$. Now, using the isomorphism $\hom_A(A, \mathcal{F}) \cong \mathcal{F}$ defined by $\phi \longmapsto \phi(1)$, i.e., $\rho(\phi) = \phi(1)$, where ρ is defined in (24), we obtain

$$\begin{cases} \hom_A(J, \mathcal{F}) = \hom_A(A, \mathcal{F}) \cong \mathcal{F} \Rightarrow \mathrm{dom}_{\mathcal{F}}(p) = \rho(\hom_A(A, \mathcal{F})) = \mathcal{F}, \\ \hom_A(A : J, \mathcal{F}) = \hom_A(A, \mathcal{F}) \cong \mathcal{F}, \\ \mathrm{ext}^1_A(J, \mathcal{F}) = \mathrm{ext}^1_A(A, \mathcal{F}) = 0 \quad [1,\ 29]. \end{cases}$$

2. If p admits a weakly coprime factorization, then, by 2 of Theorem 1, there exists $0 \neq d \in A$ such that $A : J = (d)$, and thus, we obtain $\overline{\mathcal{F}} = \mathcal{F}/\{y \in \mathcal{F} \mid dy = 0\}$. Moreover, we have

$$\mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid \exists y \in \mathcal{F}, dy = nu\}.$$

Finally, as $A : J$ is a principal ideal of A, $A : J$ is a free A-module (A is an integral domain), and thus, a finitely generated projective A-module [2, 29]. Therefore, by Proposition 3 and Lemma 3, we have

$$\begin{aligned} \hom_A(A : J, \mathcal{F}) &= \hom_A((d), \mathcal{F}) \\ &\cong \hom_A((d), A) \otimes_A \mathcal{F} \cong (d^{-1}) \otimes_A \mathcal{F}. \end{aligned}$$ □

We shall need the following lemma in what follows.

LEMMA 6. *Let $p \in Q(A)$ be an internally stabilizable plant, $J = (1, p)$, a and b two elements of A satisfying* (3) *and $\mathcal{F}$ an A-module. Then, the exact sequence* (20) *becomes the split exact sequence*

$$0 \longrightarrow J^{-1} \otimes_A \mathcal{F} \underset{\xleftarrow{\eta}}{\xrightarrow{\phi}} \mathcal{F}^2 \underset{\xleftarrow{\kappa}}{\xrightarrow{\gamma}} J \otimes_A \mathcal{F} \longrightarrow 0, \qquad (30)$$

where the A-morphisms are defined as follows:

$$
\begin{array}{rcl}
J^{-1} \otimes_A \mathcal{F} & \xrightarrow{\phi} & \mathcal{F}^2, \\
a \otimes x_1 - b \otimes x_2 & \longmapsto & \begin{pmatrix} a x_1 - b x_2 \\ (ap) x_1 - (bp) x_2 \end{pmatrix}, \\
\mathcal{F}^2 & \xrightarrow{\gamma} & J \otimes_A \mathcal{F}, \\
\begin{pmatrix} u \\ y \end{pmatrix} & \longmapsto & 1 \otimes y - p \otimes u, \\
\mathcal{F}^2 & \xrightarrow{\eta} & J^{-1} \otimes_A \mathcal{F}, \\
\begin{pmatrix} u \\ y \end{pmatrix} & \longmapsto & a \otimes u - b \otimes y, \\
J \otimes_A \mathcal{F} & \xrightarrow{\kappa} & \mathcal{F}^2, \\
1 \otimes x_1 + p \otimes x_2 & \longmapsto & \begin{pmatrix} b x_1 + (bp) x_2 \\ a x_1 + (ap) x_2 \end{pmatrix}.
\end{array} \tag{31}
$$

Proof. Let us suppose that p is an internally stabilizable plant. Then, by Lemma 2, the split exact sequence (8) holds. Then, applying the functor $\hom_A(\cdot, \mathcal{F})$ to the split exact sequence (8), we obtain the following split exact sequence [1, 29]:

$$
0 \longrightarrow \hom_A(J, \mathcal{F}) \underset{\xleftarrow{h^\star \circ \iota^{-1}}}{\xrightarrow{\iota \circ f^\star}} \mathcal{F}^2 \underset{\xleftarrow{\iota \circ k^\star}}{\xrightarrow{g^\star \circ \iota^{-1}}} \hom_A(J^{-1}, \mathcal{F}) \longrightarrow 0. \tag{32}
$$

Now, using the fact that J and J^{-1} are two invertible fractional ideals of A, and thus, by 3 of Proposition 1, two finitely generated projective A-modules, by Proposition 3, we obtain

$$
\begin{cases}
\hom_A(J, \mathcal{F}) \cong \hom_A(J, A) \otimes_A \mathcal{F} \cong J^{-1} \otimes_A \mathcal{F}, \\
\hom_A(J^{-1}, \mathcal{F}) \cong \hom_A(J^{-1}, A) \otimes_A \mathcal{F} \cong J \otimes_A \mathcal{F}.
\end{cases}
$$

Let us explicitly describe the previous isomorphisms: let $J^{-1} = (a, b)$, i.e., $a, b \in A$ are such that $a - bp = 1$, $ap \in A$ and let us define the following A-morphisms

$$
\begin{array}{rcl}
\varphi_J \colon \hom_A(J, \mathcal{F}) & \longrightarrow & J^{-1} \otimes_A \mathcal{F}, \\
\phi & \longmapsto & a \otimes \phi(1) - b \otimes \phi(p), \\
\varphi_{J^{-1}} \colon \hom_A(J^{-1}, \mathcal{F}) & \longrightarrow & J \otimes_A \mathcal{F}, \\
\phi & \longmapsto & 1 \otimes \phi(a) - p \otimes \phi(b),
\end{array} \tag{33}
$$

$$
\begin{array}{rcl}
\psi_J \colon J \otimes_A \mathcal{F} & \longrightarrow & \hom_A(J^{-1}, \mathcal{F}), \\
1 \otimes x_1 + p \otimes x_2 & \longmapsto & \psi_J(1 \otimes x_1 + p \otimes x_2), \\
\psi_{J^{-1}} \colon J^{-1} \otimes_A \mathcal{F} & \longrightarrow & \hom_A(J, \mathcal{F}), \\
a \otimes x_1 - b \otimes x_2 & \longmapsto & \psi_{J^{-1}}(a \otimes x_1 - b \otimes x_2),
\end{array} \tag{34}
$$

where ψ_J and $\psi_{J^{-1}}$ are defined by

$$\begin{cases} \psi_J(1 \otimes x_1 + p \otimes x_2)(d) = d x_1 + (dp) x_2, & \forall d \in J^{-1}, \\ \psi_{J^{-1}}(a \otimes x_1 - b \otimes x_2)(k) = (ak) x_1 - (bk) x_2, & \forall k \in J. \end{cases}$$

Then, we easily check that $\psi_J^{-1} = \varphi_{J^{-1}}$ and $\psi_{J^{-1}} = \varphi_J^{-1}$. Finally, using (9) and (20), we can verified that the A-morphisms defined by $\phi = \iota \circ f^\star \circ \psi_{J^{-1}}$, $\gamma = \varphi_{J^{-1}} \circ g^\star \circ \iota^{-1}$, $\eta = \varphi_J \circ h^\star \circ \iota^{-1}$ and $\kappa = \iota \circ k^\star \circ \psi_J$ are exactly given by (31). We note that we can also check that $\eta \circ \phi = id_{J^{-1} \otimes_A \mathcal{F}}$, $\gamma \circ \kappa = id_{J \otimes_A \mathcal{F}}$ and $\phi \circ \eta + \kappa \circ \gamma = id_{\mathcal{F}^2}$ which proves again that (30) is a split exact sequence. □

We now define the domain and the graph of a stabilizing controller c of an internally stabilizable plant p. If c is a stabilizing controller of p, then $a = 1/(1 - pc)$ and $b = c/(1 - pc) \in A$ satisfy (3) and, by 3 of Theorem 1, we have $J^{-1} = (a, b)$. From (4), we obtain $(1, c)^{-1} = (a)(1, p) = (a, ap)$. Therefore, we have

$$\mathrm{ann}_{\mathcal{F}}((1, c)^{-1}) = \mathrm{ann}_{\mathcal{F}}((a, ap)) = \{u \in \mathcal{F} \mid au = 0, (ap)u = 0\}.$$

Let us denote by $\underline{\mathcal{F}} = \mathcal{F}/\mathrm{ann}_{\mathcal{F}}((1, c)^{-1})$ and $\underline{u} \in \underline{\mathcal{F}}$ the residue class of $u \in \mathcal{F}$ modulo $\mathrm{ann}_{\mathcal{F}}((1, c)^{-1})$. Then, we denote by $cy \in \underline{\mathcal{F}}$ the fact that there exists $\underline{u} \in \underline{\mathcal{F}}$ satisfying:

$$\forall d' \in (1, c)^{-1} = (a, ap), \quad d'u = (d'c)y. \tag{35}$$

As for p, we can define the A-morphism $\rho'\colon \hom_A((1, c), \mathcal{F}) \longrightarrow \mathrm{dom}_{\mathcal{F}}(c) = \{y \in \mathcal{F} \mid cy \in \underline{\mathcal{F}}\}$ by $\rho'(\psi) = \psi(1) = y$ for all $\psi \in \hom_A((1, c), \mathcal{F})$ and we have

$$\mathrm{dom}_{\mathcal{F}}(c) = \rho'(\hom_A((1, c), \mathcal{F})).$$

Using the fact that $J^{-1} = (a)(1, c)$, the A-morphism $a\colon (1, c) \longrightarrow J^{-1}$, defined by $a(u) = au$ for all $u \in (1, c)$, is invertible and its inverse $a^{-1}\colon J^{-1} \longrightarrow (1, c)$ is defined by $a^{-1}(v) = a^{-1}v$ for all $v \in J^{-1}$. Therefore, we obtain the isomorphism

$$a^\star\colon \hom_A(J^{-1}, \mathcal{F}) \longrightarrow \hom_A((1, c), \mathcal{F})$$

defined by

$$\begin{cases} a^\star(\phi)(u) = \phi(au), & \forall \phi \in \hom_A(J^{-1}, \mathcal{F}),\ \forall u \in (1, c), \\ (a^\star)^{-1}(\psi)(v) = \psi(a^{-1}v), & \forall \psi \in \hom_A((1, c), \mathcal{F}),\ \forall v \in J^{-1}. \end{cases}$$

We note that $\psi \in \hom_A(J^{-1}, \mathcal{F})$ is defined by

$$\psi(a) \in \mathcal{F}, \quad \psi(b) \in \mathcal{F}, \quad d'\psi(b) = (d'c)\psi(a), \forall d' \in (1, c)^{-1} = (a, ap).$$

Then, we have

$$\mathrm{dom}_{\mathcal{F}}(c) = \rho'(\hom_A((1, c), \mathcal{F})) = (\rho' \circ a^\star)(\hom_A(J^{-1}, \mathcal{F})), \tag{36}$$

and thus, we have the following exact sequence

$$0 \longrightarrow \ker(\rho' \circ a^\star) \longrightarrow \hom_A(J^{-1}, \mathcal{F}) \xrightarrow{\rho' \circ a^\star} \mathrm{dom}_{\mathcal{F}}(c) \longrightarrow 0, \tag{37}$$

where $(\rho' \circ a^\star)(\psi) = (a^\star \circ \psi)(1) = \psi(a)$ for all $\psi \in \hom_A(J^{-1}, \mathcal{F})$, and thus, we obtain

$$\begin{aligned} \ker(\rho' \circ a^\star) \; = \; & \{\psi \in \hom_A(J^{-1}, \mathcal{F}) \mid \psi(a) = 0, d'\psi(b) = 0, \\ & \forall d' \in (a, ap)\}. \end{aligned}$$

Using the definition of $k\colon\ A^{1\times 2} \longrightarrow J^{-1}$ defined in (9), we obtain

$$\begin{array}{rcl} \hom_A(J^{-1}, \mathcal{F}) & \xrightarrow{\iota \circ k^\star} & \mathcal{F}^2 \\ \psi & \longmapsto & (\psi(b) \quad \psi(a))^{\mathrm{T}}, \end{array}$$

and thus, we have

$$(\iota \circ k^\star)(\ker(\rho' \circ a^\star)) = \{(u \quad 0)^{\mathrm{T}} \in \mathcal{F}^2 \mid d'u = 0, \forall d' \in (a, ap)\}$$

and $\mathcal{F}^2/(\iota \circ k^\star)(\ker(\rho' \circ a^\star)) = \mathcal{F}^2/\{(u\ 0)^{\mathrm{T}} \in \mathcal{F}^2 \mid d'u = 0, \forall d' \in (a, ap)\} = \underline{\mathcal{F}} \times \mathcal{F}$.

Then, as for p, we have the following commutative exact diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \underline{\mathcal{F}} \times \mathcal{F} & \xleftarrow{\pi'} & \mathcal{F}^2 & \xleftarrow{\iota \circ k^\star} & \ker(\rho' \circ a^\star) & \longleftarrow & 0 \\ & & & & \| & & \downarrow & & \\ 0 & \longleftarrow & \hom_A(J, \mathcal{F}) & \xleftarrow{h^\star \circ \iota^{-1}} & \mathcal{F}^2 & \xleftarrow{\iota \circ k^\star} & \hom_A(J^{-1}, \mathcal{F}) & \longleftarrow & 0, \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & \mathrm{dom}_{\mathcal{F}}(c) & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

where $\pi'\colon\ \mathcal{F}^2 \longrightarrow \underline{\mathcal{F}} \times \mathcal{F}$ is defined by $\pi'((u\ y)^{\mathrm{T}}) = (\underline{u}\ y)^{\mathrm{T}}$. Then, a chase in the previous commutative exact diagram shows that we have the following exact sequence

$$0 \longleftarrow \hom_A(J, \mathcal{F}) \longleftarrow \underline{\mathcal{F}} \times \mathcal{F} \xleftarrow{\delta'} \mathrm{dom}_{\mathcal{F}}(c) \longleftarrow 0,$$

where $\delta'\colon\ \mathrm{dom}_{\mathcal{F}}(c) \longrightarrow \underline{\mathcal{F}} \times \mathcal{F}$ is defined by $\delta'(y) = (\pi' \circ \iota \circ k^\star)(\psi)$ and $\psi \in \hom_A(J^{-1}, \mathcal{F})$ is such that $(\rho' \circ a^\star)(\psi) = \psi(a) = y$. Therefore, we finally obtain

$$\delta'(y) = (cy \quad y)^{\mathrm{T}}, \quad \forall y \in \mathrm{dom}_{\mathcal{F}}(c), \tag{38}$$

and we have

$$\mathrm{graph}_{\mathcal{F}}(c) = \{(cy \quad y)^{\mathrm{T}} \in \underline{\mathcal{F}} \times \mathcal{F} \mid y \in \mathrm{dom}_{\mathcal{F}}(c)\} = \delta'(\mathrm{dom}_{\mathcal{F}}(c)). \tag{39}$$

We are now in position to state the main theorem of this paper.

THEOREM 3. *Let $p \in Q(A)$ be an internally stabilizable plant, $J = (1, p)$ and $\mathcal{F}$ an A-module. Let c be a stabilizing controller of p, $a = 1/(1 - pc)$, $b = c/(1 - pc) \in A$ satisfying* (3) *and let us define:*

$$\begin{cases} \overline{\mathcal{F}} = \mathcal{F}/\{y \in \mathcal{F} \mid dy = 0, \forall d \in (1, p)^{-1}\} \\ \quad\;\, = \mathcal{F}/\{y \in \mathcal{F} \mid ay = 0, by = (ac)y = 0\}, \\ \underline{\mathcal{F}} = \mathcal{F}/\{u \in \mathcal{F} \mid d'u = 0, \forall d' \in (1, c)^{-1}\} \\ \quad\;\, = \mathcal{F}/\{u \in \mathcal{F} \mid au = 0, (ap)u = 0\}. \end{cases}$$

1. *Let us denote by $\overline{y}$ (resp., $\underline{y}$) the residue class of $y \in \mathcal{F}$ in $\overline{\mathcal{F}}$ (resp., $\underline{\mathcal{F}}$). Then, we have:*

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(p) = (1, p)^{-1}\mathcal{F} = \{ax_1 - bx_2 = ax_1 - (ac)x_2 \in \mathcal{F} \mid x_1, x_2 \in \mathcal{F}\}, \\ \mathrm{graph}_{\mathcal{F}}(p) = \{(ax_1 - bx_2 \quad \overline{(ap)x_1 - (bp)x_2})^{\mathrm{T}} \in \mathcal{F} \times \overline{\mathcal{F}} \mid x_1, x_2 \in \mathcal{F}\}, \\ \mathrm{dom}_{\mathcal{F}}(c) = (1, c)^{-1}\mathcal{F} = \{ax_1 + (ap)x_2 \in \mathcal{F} \mid x_1, x_2 \in \mathcal{F}\}, \\ \mathrm{graph}_{\mathcal{F}}(c) = \{(\underline{bx_1 + (bp)x_2} \quad ax_1 + (ap)x_2)^{\mathrm{T}} \in \underline{\mathcal{F}} \times \mathcal{F} \mid x_1, x_2 \in \mathcal{F}\}. \end{cases} \tag{40}$$

2. *We have the following split exact sequence*

$$0 \longrightarrow \mathrm{dom}_{\mathcal{F}}(p) \underset{\overset{\longleftarrow}{\tau'}}{\overset{\pi'\circ\delta}{\longrightarrow}} \underline{\mathcal{F}} \times \overline{\mathcal{F}} \underset{\overset{\longleftarrow}{\pi\circ\delta'}}{\overset{\tau}{\longrightarrow}} \mathrm{dom}_{\mathcal{F}}(c) \longrightarrow 0, \tag{41}$$

where δ (resp., δ') is defined by (28) *(resp.,* (38)*) and $\pi\colon \mathcal{F}^2 \longrightarrow \mathcal{F} \times \overline{\mathcal{F}}$ (resp., $\pi'\colon \mathcal{F}^2 \longrightarrow \underline{\mathcal{F}} \times \mathcal{F}$) is defined by $\pi((u \;\, y)^{\mathrm{T}}) = (u \;\, \overline{y})^{\mathrm{T}}$ (resp., $\pi'((u \;\, y)^{\mathrm{T}}) = (\underline{u} \;\, y)^{\mathrm{T}}$), showing that we have*

$$\pi'(\mathrm{graph}_{\mathcal{F}}(p)) \oplus \pi(\mathrm{graph}_{\mathcal{F}}(c)) = \underline{\mathcal{F}} \times \overline{\mathcal{F}}, \tag{42}$$

or equivalently:

$$\begin{aligned} &\left\{(\underline{ax_1 - bx_2} \quad \overline{(ap)x_1 - (bp)x_2})^{\mathrm{T}} \mid x_1, x_2 \in \mathcal{F}\right\} \\ &\quad \oplus \left\{(\underline{bx_1 + (bp)x_2} \quad \overline{ax_1 + (ap)x_2})^{\mathrm{T}} \mid x_1, x_2 \in \mathcal{F}\right\} = \underline{\mathcal{F}} \times \overline{\mathcal{F}}. \end{aligned}$$

Proof. 1. Let us compute the domains and the graphs of p and c. Using (26), (29) and (34), we obtain

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(p) = \rho(\hom_A(J, \mathcal{F})) = (\rho \circ \psi_{J^{-1}})(J^{-1} \otimes_A \mathcal{F}), \\ \mathrm{graph}_{\mathcal{F}}(p) = \delta(\mathrm{dom}_{\mathcal{F}}(p)) = (\delta \circ \rho \circ \psi_{J^{-1}})(J^{-1} \otimes_A \mathcal{F}). \end{cases}$$

The A-morphism $\rho \circ \psi_{J^{-1}}\colon J^{-1} \otimes_A \mathcal{F} \longrightarrow \mathrm{dom}_{\mathcal{F}}(p)$ is defined by

$$\begin{aligned} (\rho \circ \psi_{J^{-1}})(a \otimes x_1 - b \otimes x_2) &= (\psi_{J^{-1}}(a \otimes x_1 - b \otimes x_2))(1) \\ &= ax_1 - bx_2, \quad \forall x_1, x_2 \in \mathcal{F}, \end{aligned}$$

and thus, we obtain $\mathrm{dom}_{\mathcal{F}}(p) \subseteq J^{-1}\mathcal{F}$. From 3 of Lemma 5, we already know that $J^{-1}\mathcal{F} \subseteq \mathrm{dom}_{\mathcal{F}}(p)$, which proves that $\mathrm{dom}_{\mathcal{F}}(p) = J^{-1}\mathcal{F}$. Now, we have

$$\mathrm{ann}_{\mathcal{F}}(J^{-1}) = \mathrm{ann}_{\mathcal{F}}((a, b)) = \{y \in \mathcal{F} \mid ay = 0, by = 0\}.$$

Using 3 of Lemma 5, we obtain $p(ax_1 - bx_2) = \overline{(ap)x_1 - (bp)x_2}$, and thus, we have

$$\delta(ax_1 - bx_2) = (ax_1 - bx_2 \quad \overline{(ap)x_1 - (bp)x_2})^{\mathrm{T}},$$

which proves that we have $\mathrm{graph}_{\mathcal{F}}(p) = \{(ax_1 - bx_2 \quad \overline{(ap)x_1 - (bp)x_2})^{\mathrm{T}} \in \mathcal{F}^2 \mid x_1, x_2 \in \mathcal{F}\}$.

Similarly for c, using (36), (39) and (34), we have

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(c) = (\rho' \circ a^\star)(\hom_A(J^{-1}, \mathcal{F})) = (\rho' \circ a^\star \circ \psi_J)(J \otimes_A \mathcal{F}), \\ \mathrm{graph}_{\mathcal{F}}(c) = \delta'(\mathrm{dom}_{\mathcal{F}}(c)) = (\delta' \circ \rho' \circ a^\star \circ \psi_J)(J \otimes_A \mathcal{F}). \end{cases}$$

The A-morphism $\rho' \circ a^\star \circ \psi_J \colon J \otimes_A \mathcal{F} \longrightarrow \mathrm{dom}_{\mathcal{F}}(p)$ is defined by

$$\begin{aligned} (\rho' \circ a^\star \circ \psi_J)(1 \otimes x_1 + p \otimes x_2) &= \psi_J(1 \otimes x_1 + p \otimes x_2)(a) \\ &= ax_1 + (ap)x_2, \quad \forall x_1, x_2 \in \mathcal{F}, \end{aligned}$$

and thus, we have $\mathrm{dom}_{\mathcal{F}}(c) \subseteq (a, ap)\mathcal{F}$. If we use c instead of p, then, from 3 of Lemma 5, we already know that $(a, ap)\mathcal{F} \subseteq \mathrm{dom}_{\mathcal{F}}(c)$, which proves that $\mathrm{dom}_{\mathcal{F}}(c) = (a, ap)\mathcal{F}$. Similarly, using 3 of Lemma 5 with $c = b/a$, we obtain $c(ax_1 + (ap)x_2) = \underline{bx_1 + (bp)x_2}$ and we finally obtain

$$\mathrm{graph}_{\mathcal{F}}(c) = \{(\underline{bx_1 + (bp)x_2} \quad ax_1 + (ap)x_2)^{\mathrm{T}} \in \underline{\mathcal{F}} \times \mathcal{F} \mid x_1, x_2 \in \mathcal{F}\}.$$

2. Using the exact sequences (25), (32) and (37), then we obtain the following exact diagram:

$$\begin{tikzcd} & 0 \arrow[d] & & 0 \arrow[d] & \\ & \ker\rho \arrow[d] & & \ker(\rho' \circ a^\star) \arrow[d] & \\ 0 \arrow[r] & \hom_A(J, \mathcal{F}) \arrow[r, shift left, "\iota \circ f^\star"] \arrow[d] & \mathcal{F}^2 \arrow[l, shift left, "h^\star \circ \iota^{-1}"] \arrow[r, shift left, "g^\star \circ \iota^{-1}"] & \hom_A(J^{-1}, \mathcal{F}) \arrow[l, shift left, "\iota \circ k^\star"] \arrow[r] \arrow[d] & 0. \\ & \mathrm{dom}_{\mathcal{F}}(p) \arrow[d] & & \mathrm{dom}_{\mathcal{F}}(c) \arrow[d] & \\ & 0 & & 0 & \end{tikzcd}$$

Let $((\iota \circ f^\star) \ (\iota \circ k^\star)) \colon \ker\rho \oplus \ker(\rho' \circ a^\star) \longrightarrow \mathcal{F}^2$ be the A-morphism defined by

$$((\iota \circ f^\star) \quad (\iota \circ k^\star))(\psi \quad \phi)^{\mathrm{T}} = (\iota \circ f^\star)(\psi) + (\iota \circ k^\star)(\phi).$$

Then, we obtain

$$((\iota \circ f^\star) \quad (\iota \circ k^\star))(\ker \rho \oplus \ker(\rho' \circ a^\star))$$
$$= \{(u \quad y)^{\mathrm{T}} \in \mathcal{F}^2 \mid dy = 0, \forall d \in (a, ac), d'u = 0, \forall d' \in (a, ap)\},$$

and thus, we have

$$\operatorname{coker}(((\iota \circ f^\star) \quad (\iota \circ k^\star))) = \underline{\mathcal{F}} \times \overline{\mathcal{F}}.$$

Now, let us define $(\pi' \ \pi) : \mathcal{F}^2 \longrightarrow \underline{\mathcal{F}} \times \overline{\mathcal{F}}$ by $(\pi' \ \pi)((u \ y)^{\mathrm{T}}) = (\pi'(u) \ \ \pi(y))^{\mathrm{T}} = (\underline{u} \quad \overline{y})^{\mathrm{T}}$. Then, we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \rho & \longrightarrow & \ker \rho \oplus \ker(\rho' \circ a^\star) & \longrightarrow & \ker(\rho' \circ a^\star) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom_A(J, \mathcal{F}) & \underset{\overset{h^\star \circ \iota^{-1}}{\longleftarrow}}{\xrightarrow{\iota \circ f^\star}} & \mathcal{F}^2 & \underset{\overset{\iota \circ k^\star}{\longleftarrow}}{\xrightarrow{g^\star \circ \iota^{-1}}} & \hom_A(J^{-1}, \mathcal{F}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{dom}_{\mathcal{F}}(p) & \xrightarrow{\pi' \circ \delta} & \underline{\mathcal{F}} \times \overline{\mathcal{F}} & \xrightarrow{\tau} & \operatorname{dom}_{\mathcal{F}}(c) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\tau \colon \underline{\mathcal{F}} \times \overline{\mathcal{F}} \longrightarrow \operatorname{dom}_{\mathcal{F}}(c)$ is defined by

$$\tau((\underline{u} \quad \overline{y})^{\mathrm{T}}) = ((\rho' \circ a^\star) \circ (g^\star \circ \iota^{-1}))((u \quad y)^{\mathrm{T}}) = ay - (ap)u.$$

Now, the A-morphism $\tau \circ (\pi \circ \delta') \colon \operatorname{dom}_{\mathcal{F}}(c) \longrightarrow \operatorname{dom}_{\mathcal{F}}(c)$ is defined by

$$(\tau \circ \pi \circ \delta')(y) = (\tau \circ \pi)(cy \quad y)^{\mathrm{T}} = \tau((\underline{cy} \quad \overline{y})^{\mathrm{T}}) = ay - (ap)u,$$

where $\underline{u} = cy$. But, by (35), $\underline{u} = cy$ implies that $(ap)u = ((ap)c)y$ and, using the relation $a(1 - pc) = 1$, we finally obtain

$$(\tau \circ (\pi \circ \delta'))(y) = ay - ((ap)c)y = a(1 - pc)y = y.$$

Therefore, we have $\tau \circ (\pi \circ \delta') = id_{\operatorname{dom}_{\mathcal{F}}(c)}$, which shows that we have the following split exact sequence

$$0 \longrightarrow \operatorname{dom}_{\mathcal{F}}(p) \underset{\overset{\tau'}{\longleftarrow}}{\xrightarrow{\pi' \circ \delta}} \underline{\mathcal{F}} \times \overline{\mathcal{F}} \underset{\overset{\pi \circ \delta'}{\longleftarrow}}{\xrightarrow{\tau}} \operatorname{dom}_{\mathcal{F}}(c) \longrightarrow 0$$

where $\tau' \colon \underline{\mathcal{F}} \times \overline{\mathcal{F}} \longrightarrow \operatorname{dom}_{\mathcal{F}}(p)$ is defined by

$$\tau'((\underline{u} \quad \overline{y})^{\mathrm{T}}) = (\rho \circ h^\star \circ \iota^{-1})((u \quad y)^{\mathrm{T}}) = au - by,$$

and thus, using (29) and (39), we obtain

$$\pi'(\operatorname{graph}_{\mathcal{F}}(p)) \oplus \pi(\operatorname{graph}_{\mathcal{F}}(c)) = \underline{\mathcal{F}} \times \overline{\mathcal{F}}. \qquad \square$$

We have the following corollary.

COROLLARY 2. *Let $p \in Q(A)$ be a transfer function which admits a coprime factorization $p = n/d$, $0 \neq d, n \in A$, $dr - ns = 1$, $0 \neq r, s \in A$, $J = (1, p)$ and $\mathcal{F}$ an A-module. Let us denote by*

$$\begin{cases} \overline{\mathcal{F}} = \mathcal{F}/\{y \in \mathcal{F} \mid dy = 0\}, \\ \underline{\mathcal{F}} = \mathcal{F}/\{u \in \mathcal{F} \mid ru = 0\}, \end{cases} \tag{43}$$

by $\overline{y}$ (resp., $\underline{y}$) the residue class of $y \in \mathcal{F}$ modulo $\mathrm{ann}_{\mathcal{F}}((d))$ (resp., $\mathrm{ann}_{\mathcal{F}}((r))$) and $c = s/r$ a stabilizing controller of p. Then, we have

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(p) = (d)\mathcal{F} = \{dx \mid x \in \mathcal{F}\}, \\ \mathrm{graph}_{\mathcal{F}}(p) = \{(\underline{dx} \quad \overline{nx})^{\mathrm{T}} \in \underline{\mathcal{F}} \times \overline{\mathcal{F}} \mid x \in \mathcal{F}\}, \\ \mathrm{dom}_{\mathcal{F}}(c) = (r)\mathcal{F} = \{rx \mid x \in \mathcal{F}\}, \\ \mathrm{graph}_{\mathcal{F}}(c) = \{(\underline{sx} \quad \overline{rx})^{\mathrm{T}} \in \underline{\mathcal{F}} \times \overline{\mathcal{F}} \mid x \in \mathcal{F}\}. \end{cases} \tag{44}$$

In particular, we have the following split exact sequence

$$0 \longrightarrow (d)\mathcal{F} \underset{\xleftarrow{\tau'}}{\xrightarrow{\pi' \circ \delta}} \underline{\mathcal{F}} \times \overline{\mathcal{F}} \underset{\xleftarrow{\pi \circ \delta'}}{\xrightarrow{\tau}} (r)\mathcal{F} \longrightarrow 0,$$

and (42) becomes $\{(\underline{dx} \quad \overline{nx})^{\mathrm{T}} \mid x \in \mathcal{F}\} \oplus \{(\underline{sx} \quad \overline{rx})^{\mathrm{T}} \mid x \in \mathcal{F}\} = \underline{\mathcal{F}} \times \overline{\mathcal{F}}$.

Finally, (30) becomes the following split exact sequence

$$0 \longrightarrow (d) \otimes_A \mathcal{F} \underset{\xleftarrow{\eta}}{\xrightarrow{\phi}} \mathcal{F}^2 \underset{\xleftarrow{\kappa}}{\xrightarrow{\gamma}} (d^{-1}) \otimes_A \mathcal{F} \longrightarrow 0,$$

where the A-morphisms are defined by:

$$\begin{array}{llll} (d) \otimes_A \mathcal{F} \xrightarrow{\phi} \mathcal{F}^2, & & \mathcal{F}^2 \xrightarrow{\gamma} (d^{-1}) \otimes_A \mathcal{F}, & \\ d \otimes x \longmapsto \begin{pmatrix} dx \\ nx \end{pmatrix}, & & \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto d^{-1} \otimes (dy - nu), & \\ \mathcal{F}^2 \xrightarrow{\eta} (d) \otimes_A \mathcal{F}, & & (d^{-1}) \otimes_A \mathcal{F} \xrightarrow{\kappa} \mathcal{F}^2, & \\ \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto d \otimes (ru - sy), & & d^{-1} \otimes z \longmapsto \begin{pmatrix} sz \\ rz \end{pmatrix}. & \end{array} \tag{45}$$

Proof. If p admits a coprime factorization $p = n/d$, then, by 5 of Theorem 1, we have $J = (d^{-1})$, and thus, $J^{-1} = (d)$. If we denote by $a = dr$ and $b = ds$, then we have $ap = nr \in A$ and, from $dr - ns = 1$, we obtain $a - bp = 1$, i.e., by 3 of Theorem 1, $c = b/a = (ds)/(dr) = s/r$ is a stabilizing controller of p. Moreover, we have $(1, c)^{-1} = (a, ap) = (dr, nr) = (r)$ because $dr - ns = 1$. Then, we obtain

$\text{ann}_{\mathcal{F}}((1,p)^{-1}) = \{y \in \mathcal{F} \mid dy = 0\}$ and $\text{ann}_{\mathcal{F}}((1,c)^{-1}) = \{u \in \mathcal{F} \mid ru = 0\}$ and $\overline{\mathcal{F}}$ and $\underline{\mathcal{F}}$ are defined by (43). Moreover, using (40), we obtain

$$\begin{cases} \text{dom}_{\mathcal{F}}(c) = (1,p)^{-1}\mathcal{F} = (d)\mathcal{F}, \\ \text{dom}_{\mathcal{F}}(c) = (1,c)^{-1}\mathcal{F} = (r)\mathcal{F}, \end{cases}$$

and, using 3 of Lemma 5, we obtain

$$\begin{cases} \text{graph}_{\mathcal{F}}(p) = \{(dx \quad \overline{nx})^{\mathrm{T}} \in \mathcal{F} \times \overline{\mathcal{F}} \mid x \in \mathcal{F}\}, \\ \text{graph}_{\mathcal{F}}(c) = \{(\underline{sx} \quad rx)^{\mathrm{T}} \in \underline{\mathcal{F}} \times \mathcal{F} \mid x \in \mathcal{F}\}. \end{cases}$$

Finally, we have $1 \otimes y - p \otimes u = d^{-1} \otimes (dy - nu) \in J \otimes_A \mathcal{F}$. Using $a = dr$ and $b = ds$, we obtain $a \otimes x_1 - b \otimes x_2 = d \otimes (rx_1 - sx_2) = d \otimes x \in J^{-1} \otimes_A \mathcal{F}$, with the notation $x = (rx_1 - sx_2)$. Therefore, the A-morphisms ϕ and γ defined in (31) become the A-morphisms defined in (45). Finally, using the fact that $a \otimes u - b \otimes y = d \otimes (ru - sy) \in J^{-1} \otimes_A \mathcal{F}$ and $1 \otimes x_1 + p \otimes x_2 = d^{-1} \otimes z$, with the notation $z = (dx_1 + nx_2) \in J \otimes_A \mathcal{F}$, then, doing as previously, the A-morphisms η and κ defined in (31) become those given in (45). □

4.2. DOMAINS AND GRAPHS OVER CERTAIN A-MODULES $\mathcal{F}$

We have previously studied the influence of the structural properties of p (e.g., stability, existence of (weakly) coprime factorization, stabilizability), i.e., the algebraic properties of J, on the domain and the graph of the linear operator defined by p. In this section, we now study them in the case where no assumptions are made on p but on the A-module $\mathcal{F}$, i.e., on the signal space.

4.2.1. *Divisible and Injective A-Modules $\mathcal{F}$*

Let us introduce a few definitions.

DEFINITION 4.

- [1, 29] The A-module $\mathcal{F}$ is called *divisible* if, for every $0 \neq a \in A$ and $u \in \mathcal{F}$, there exists $y \in \mathcal{F}$ satisfying $ay = u$ or, equivalently, if, for every $0 \neq a \in A$, the A-morphism $a\colon \mathcal{F} \longrightarrow \mathcal{F}$ defined by $a(y) = ay$, $y \in \mathcal{F}$, is surjective.
- The A-module $\mathcal{F}$ is called *injective* if one of the following equivalent assertions is satisfied:

  1. [7] Let U and V be two index sets of arbitrary cardinalities, $u_i \in \mathcal{F}$, $a_{ij} \in A$, $i \in U$, $j \in V$. Then, every *consistent system* $\sum_{j \in V} a_{ij} y_j = u_i$ in the unknowns y_j (for every fixed $i \in U$, almost all a_{ij} must vanish) – namely, if $\sum_{i \in U} b_i (\sum_{j \in V} a_{ij} y_j) = 0$, then $\sum_{i \in U} b_i u_i = 0$ (in the sums, almost all b_i must vanish) – is *solvable*, i.e., admits a solution $y_j \in \mathcal{F}$, $j \in V$.

2. [1, 29] For every exact sequence of A-modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0,$$

then we have the exact sequence

$$0 \longleftarrow \hom_A(M', \mathcal{F}) \xleftarrow{f^\star} \hom_A(M, \mathcal{F}) \xleftarrow{g^\star} \hom_A(M'', \mathcal{F}) \longleftarrow 0.$$

It is well known that an injective A-module is divisible. Moreover, if A is a *principal ideal domain*, namely every ideal of A can be generated by means of a single element of A, then any divisible A-module is injective. See [1, 7, 29] for more details.

EXAMPLE 4.

- The field of fraction $K = Q(A)$ of A is an injective A-module and, if A is a principal ideal domain (e.g., $A = RH_\infty$ or $k[s]$ with k a field), then K/A is also an injective A-module.
- We consider the ring $A = k[d_1, \ldots, d_n]$ of differential operators in $d_i = \partial/\partial x_i$ with coefficients in $k = \mathbb{R}, \mathbb{C}$, namely the ring of elements of the form $\sum_{0 \leqslant |\mu| \leqslant q} a_\mu d^\mu$, where $\mu = (\mu_1 \ldots \mu_n) \in \mathbb{Z}_+^n$ is a multi-index, $d^\mu = d_1^{\mu_1} \ldots d_n^{\mu_n}$, $a_\mu \in k$ and $|\mu| = \mu_1 + \cdots + \mu_n$. If Ω is an open convex subset of $\mathbb{R}^n$, then the k-vector space $\mathcal{F} = C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$) of smooth functions (resp., distributions, temperate distributions) in Ω are injective A-modules [14, 17, 16, 30].
- We consider the ring $A = \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}, \delta]$ of differential time-delay operators with coefficients in $\mathbb{R}$, namely the ring of elements of the form $\sum_{0 \leqslant |(i,j)| \leqslant q} a_{ij} \frac{\mathrm{d}^i}{\mathrm{d}t^i} \delta^j$, where $a_{ij} \in \mathbb{R}$, $\frac{\mathrm{d}}{\mathrm{d}t} f(t)$ is the time-derivative of f and $(\delta f)(t) = f(t-1)$ is the time delay operator. Then, $\mathcal{F} = C^\infty(\mathbb{R})$ is a divisible but not an injective A-module [11].

We have the following corollary of Theorem 3 and Corollary 2.

COROLLARY 3. *Let $p \in Q(A)$, $J = (1, p)$ and $\mathcal{F}$ be an A-module.*

1. *If $\mathcal{F}$ is an injective A-module, then* $\mathrm{ext}_A^1(J, \mathcal{F}) = 0$.
2. *If p admits a weakly coprime factorization and $\mathcal{F}$ is a divisible A-module, then* $\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}$.
3. *If p is internally stabilizable and $\mathcal{F}$ is an injective A-module, then*

$$\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}, \qquad \mathrm{dom}_{\mathcal{F}}(c) = \mathcal{F},$$

where c is a stabilizing controller of p.

4. *If p admits a coprime factorization and $\mathcal{F}$ is a divisible A-module, then*

$$\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}, \quad \mathrm{dom}_{\mathcal{F}}(c) = \mathcal{F},$$

where c is a stabilizing controller of p.

Proof. 1. A standard argument of homological algebra shows that, if $\mathcal{F}$ is an injective A-module, then $\mathrm{ext}^1_A(M, \mathcal{F}) = 0$ for every A-module M (see also 2 of Definition 4). Therefore, we have $\mathrm{ext}^1_A(J, \mathcal{F}) = 0$.

2. If $p = n/d$, $0 \neq d, n \in A$, is a weakly coprime factorization, then, by 2 of Proposition 4, we have $\mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid \exists y \in \mathcal{F}, dy = nu\}$. Now, using the fact that $\mathcal{F}$ is a divisible A-module, then, for every $u \in \mathcal{F}$, there exists $y \in \mathcal{F}$ such that $dy = nu$, i.e., $\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}$.

3. From Theorem 3, we know that if p is internally stabilizable and $c = b/a$ is a stabilizing controller, where $0 \neq a, b \in A$ satisfy (3), then we have:

$$\mathrm{dom}_{\mathcal{F}}(c) = \{ax_1 - bx_2 \in \mathcal{F} \mid x_1, x_2 \in \mathcal{F}\},$$
$$\mathrm{dom}_{\mathcal{F}}(c) = \{ax_1 + (ap)x_2 \in \mathcal{F} \mid x_1, x_2 \in \mathcal{F}\}.$$

Let $x_3 \in \mathcal{F}$ and let us consider the consistent equation $aX_1 - bX_2 = x_3$. Then, using the fact that $\mathcal{F}$ is an injective A-module, then there exist x_1 and $x_2 \in \mathcal{F}$ such that $ax_1 - bx_2 = x_3$, and thus, we have $\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}$. The same argument holds for $\mathrm{dom}_{\mathcal{F}}(c)$.

4. From Corollary 2, if $p = n/d$, $0 \neq d, n \in A$, is a coprime factorization with $dr - ns = 1$, $r, s \in A$, then $\mathrm{dom}_{\mathcal{F}}(p) = \{dx \mid x \in \mathcal{F}\}$. Now, using the fact that $\mathcal{F}$ is a divisible A-module, then the A-morphism $d\colon \mathcal{F} \longrightarrow \mathcal{F}$ defined by $d(x) = dx$ for all $x \in \mathcal{F}$ is surjective, and thus, $\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}$ and similarly for $\mathrm{dom}_{\mathcal{F}}(c)$. $\square$

We do not know yet non-trivial divisible or injective A-modules for the rings $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$, $\hat{\mathcal{A}}$ and W_+. However, the previous results can also be used for the study of linear multidimensional systems (see [16, 17, 20, 21, 30] and the references therein). Let us give an example.

EXAMPLE 5. Let us consider the ring $A = \mathbb{R}[d_1, d_2]$ of differential operators defined in Example 4, $K = \mathbb{R}(d_1, d_2)$, $\mathcal{F} = C^\infty(\mathbb{R}^2)$, $p = d_2^{-1} d_1 \in K$ and $J = (1, p)$. Using the fact that $p = d_2^{-1} d_1$ is a weakly coprime factorization of p, then we obtain

$$\begin{aligned} A : J = (d_2) \;&\Rightarrow\; \mathrm{ann}_{\mathcal{F}}(A : J) = \{y \in \mathcal{F} \mid d_2 y = 0\} \\ &\Rightarrow\; \overline{\mathcal{F}} = \mathcal{F}/\{y \in \mathcal{F} \mid y(x) = y(x_1)\}. \end{aligned}$$

As $\mathcal{F}$ is an injective A-module (see Example 4), then, by 2 of Corollary 3, we obtain

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid pu \in \overline{\mathcal{F}}\} = \{u \in \mathcal{F} \mid \exists y \in \mathcal{F}, d_2 y = d_1 u\} = \mathcal{F}, \\ \mathrm{graph}_{\mathcal{F}}(p) = \left\{ \begin{pmatrix} u & \overline{\displaystyle\int_0^{x_2} d_1 u(x)\,\mathrm{d}x_2} \end{pmatrix}^{\mathrm{T}} \in \mathcal{F} \times \overline{\mathcal{F}} \mid u \in \mathcal{F} \right\}. \end{cases}$$

4.2.2. *Torsion-Free A-Modules $\mathcal{F}$*

Let us introduce a few definitions.

DEFINITION 5. Let $\mathcal{F}$ be an A-module.

- The *torsion submodule* of an A-module $\mathcal{F}$ is defined by
  $t(\mathcal{F}) = \{x \in \mathcal{F} \mid \exists\, 0 \neq a \in A, ax = 0\}$.
- An A-module $\mathcal{F}$ is *torsion-free* if $t(\mathcal{F}) = 0$.

EXAMPLE 6.

- We shall prove in Proposition 8 that the Hilbert space $H_2(\mathbb{C}_+)$ of holomorphic functions in the open right half plane $\mathbb{C}_+$ which are bounded with respect to the norm
  $$\|f\|_2 = \left(\sup_{\operatorname{Re} x>0} \int_{-\infty}^{+\infty} |f(x+\mathrm{i}y)|^2\,\mathrm{d}y\right)^{1/2}$$
  is a torsion free $H_\infty(\mathbb{C}_+)$-module. Using the fact that $RH_\infty \subset \hat{\mathcal{A}} \subset H_\infty(\mathbb{C}_+)$, then we obtain that $H_2(\mathbb{C}_+)$ is also a torsion-free RH_∞-module (resp., $\hat{\mathcal{A}}$-module).
- For $q \in [1, +\infty]$, we can prove that $L_q(\mathbb{R}_+)$ is a torsion-free $\mathcal{A}$-module.

COROLLARY 4. *Let $p \in Q(A)$ be an internally stabilizable plant, $J = (1, p)$ and $\mathcal{F}$ a torsion-free A-module. Then, Theorem* 3 *and Corollary* 2 *hold for $\overline{\mathcal{F}} = \underline{\mathcal{F}} = \mathcal{F}$.*

Proof. Using the fact that $\mathcal{F}$ is a torsion-free A-module, we obtain

$$\operatorname{ann}_{\mathcal{F}}(A : J) = \{y \in \mathcal{F} \mid dy = 0, \forall d \in A : J\} = 0 \Rightarrow \overline{\mathcal{F}} = \mathcal{F}.$$

A similar result also holds for $\underline{\mathcal{F}}$. □

EXAMPLE 7. It is well known that every internally stabilizable plant $p \in Q(A)$ over $A = RH_\infty$ and $H_\infty(\mathbb{C}_+)$ admits a coprime factorization $p = n/d$, $0 \neq d, n \in A$, $dr - ns = 1$, $r, s \in A$ [13, 32, 35]. If $\mathcal{F} = H_2(\mathbb{C}_+)$ and $r \neq 0$, then $c = s/r$ internally stabilizes p and, using Corollary 4 and Example 6, we obtain:

$$\begin{cases} \operatorname{dom}_{\mathcal{F}}(p) = (d)\mathcal{F} = \{dx \mid x \in \mathcal{F}\}, \\ \operatorname{graph}_{\mathcal{F}}(p) = \{(dx \quad nx)^{\mathrm{T}} \in \mathcal{F}^2 \mid x \in \mathcal{F}\}, \\ \operatorname{dom}_{\mathcal{F}}(c) = (r)\mathcal{F} = \{rx \mid x \in \mathcal{F}\}, \\ \operatorname{graph}_{\mathcal{F}}(c) = \{(sx \quad rx)^{\mathrm{T}} \in \mathcal{F}^2 \mid x \in \mathcal{F}\}. \end{cases} \tag{46}$$

This result was proved in [8, 9]. We also note that the previous result holds if we take the torsion-free A-module $\mathcal{F} = A$. For $A = RH_\infty$, such a result was firstly obtained by Vidyasagar in Lemma 2 of Section 7.2 of [35] and it is used in order to define the *graph topology* [35].

However, for $A = \hat{\mathcal{A}}$, it is not known whether or not every internally stabilizable plant $p \in Q(A)$ admits a coprime factorization. Therefore, if $\mathcal{F} = H_2(\mathbb{C}_+)$ and $c \in Q(A)$ is a stabilizing controller of p, then defining $a = 1/(1 - pc)$ and $b = c/(1 - pc) \in A$ and using Corollary 4 and Example 6, we obtain:

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(p) = \{ax_1 - bx_2 \mid x_1, x_2 \in \mathcal{F}\}, \\ \mathrm{graph}_{\mathcal{F}}(p) = \{(ax_1 - bx_2 \quad (ap)x_1 - (bp)x_2)^{\mathrm{T}} \in \mathcal{F}^2 \mid x_1, x_2 \in \mathcal{F}\}, \\ \mathrm{dom}_{\mathcal{F}}(c) = \{ax_1 + (ap)x_2 \mid x_1, x_2 \in \mathcal{F}\}, \\ \mathrm{graph}_{\mathcal{F}}(c) = \{(bx_1 + (bp)x_2 \quad ax_1 + (ap)x_2)^{\mathrm{T}} \in \mathcal{F}^2 \mid x_1, x_2 \in \mathcal{F}\}. \end{cases}$$

A similar result holds for $A = \mathcal{A}$ and $\mathcal{F} = L_q(\mathbb{R}_+)$ with $q \in [1, +\infty]$ and for $A = l_1$ and $\mathcal{F} = l_p(\mathbb{Z}_+)$ with $q \in [1, +\infty]$. If we suppose that $p \in Q(\mathcal{A})$ (resp., $p \in Q(l_1)$) admits a coprime factorization, then we obtain (46). This result was obtained in Theorem 2 of [34].

To finish, let us give the following proposition.

PROPOSITION 5 [7]. *We have* (A *is an integral domain*)*:*

- *A torsion-free A-module $\mathcal{F}$ is injective iff it is divisible.*
- *A torsion-free divisible A-module $\mathcal{F}$ is a direct sum of copies of $Q(A)$.*

Applying Proposition 5 to the A-modules defined in Example 6, we obtain that they are not divisible, and thus, not injective A-modules. In particular, we know that there exists $0 \neq d \in A$ such that $\mathrm{dom}_{\mathcal{F}}(1/d) = (d)\mathcal{F} \subsetneq \mathcal{F}$. For instance, if we consider $A = RH_\infty$, $\mathcal{F} = H_2(\mathbb{C}_+)$ and $d = (s-1)/(s+1) \in A$, then we have $\mathrm{dom}_{\mathcal{F}}(1/d) = (d)\mathcal{F} \subsetneq \mathcal{F}$ as every element of $(d)\mathcal{F}$ must vanish at $s = 1$ but we have $1/(s+1) \in \mathcal{F}$. As we have $d \in \hat{\mathcal{A}}$ and $d \in H_\infty(\mathbb{C}_+)$, then the same example can be considered for $\hat{\mathcal{A}}$ (resp., $H_\infty(\mathbb{C}_+)$) and $\mathcal{F} = H_2(\mathbb{C}_+)$.

4.2.3. *Flat A-Modules $\mathcal{F}$*

The concept of a *flat* A-module sharpens the concept of a torsion-free A-module.

DEFINITION 6 [2, 29]. An A-module $\mathcal{F}$ is called *flat* if, for every relation of the form $\sum_{i=1}^n a_i x_i = 0$, where $a_i \in A$ and $x_i \in \mathcal{F}$, then there exist $y_1, \ldots, y_m$ in $\mathcal{F}$ and $b = (b_{ij}) \in A^{n \times m}$ such that we have:

$$\begin{cases} x_i = \displaystyle\sum_{j=1}^m b_{ij} y_j, \quad 1 \leqslant i \leqslant n, \\ \displaystyle\sum_{k=1}^n a_k b_{kj} = 0, \quad 1 \leqslant j \leqslant m. \end{cases} \tag{47}$$

Equivalently, an A-module $\mathcal{F}$ is flat iff, for every exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

then we have the following exact sequence

$$0 \longrightarrow \mathcal{F} \otimes_A M' \longrightarrow \mathcal{F} \otimes_A M \longrightarrow \mathcal{F} \otimes_A M'' \longrightarrow 0.$$

EXAMPLE 8.

- Every finitely generated projective A-module is flat [1, 2, 29]. In particular, if $p \in Q(A)$ is an internally stabilizable plant, then, by 3 of Proposition 1, $J = (1, p)$ and J^{-1} are projective, and thus, flat A-modules.
- A *multiplicative set* S of A is a subset of A which satisfies that $1 \in S$ and, for all $s_1, s_2 \in S$, we have $s_1 s_2 \in S$. If S is a multiplicative subset of A, then $S^{-1}A \triangleq \{a/s \mid a \in A, s \in S\}$ is a flat A-module [2, 29]. In particular, if we take $S = A \backslash 0$, then $S^{-1}A = Q(A)$ is a flat A-module.

If I is an integral ideal of A, then we stress that the A-morphism

$$\sigma\colon I \otimes_A \mathcal{F} \longrightarrow I\mathcal{F} = \left\{ \sum_{i=1}^{n} a_i x_i \mid a_i \in I, x_i \in \mathcal{F}, n \in \mathbb{Z}_+ \right\},$$

defined by $\sigma(\sum_{i=1}^n d_i \otimes x_i) = \sum_{i=1}^n d_i x_i$, where $d_i \in I$ and $x_i \in \mathcal{F}$, is generally not injective. For instance, if $\mathcal{F}$ is not a torsion-free A-module, then there may exist $x \in \mathcal{F}$ which satisfies $dx = 0$ for a certain $0 \neq d \in A$, and thus, if we denote by $I = (d)$, then $\sigma(d \otimes x) = dx = 0$ (e.g., we can consider $A = \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$, $I = (\frac{\mathrm{d}}{\mathrm{d}t})$ and $\mathcal{F} = C^\infty(\mathbb{R})$). However, if $\mathcal{F}$ is a flat A-module, then, using the injection $0 \longrightarrow I \longrightarrow A$, we obtain the exact sequence $0 \longrightarrow I \otimes_A \mathcal{F} \xrightarrow{\sigma} A \otimes_A \mathcal{F} = \mathcal{F}$ and $\sigma(I \otimes_A \mathcal{F}) = I\mathcal{F}$. Therefore, we obtain $I \otimes_A \mathcal{F} \cong I\mathcal{F}$.

PROPOSITION 6.

1. [2, 29] *An A-module $\mathcal{F}$ is flat iff for every integral ideal I of A, we have*

$$I \otimes_A \mathcal{F} \cong I\mathcal{F}.$$

2. [2, 29] *A flat A-module $\mathcal{F}$ is torsion-free.*

We note that 2 of Proposition 6 can be proved as follows: if $\mathcal{F}$ is a flat A-module, then, from the exact sequence $0 \longrightarrow A \xrightarrow{d.} A$ where $0 \neq d \in A$ and $(d.)(\lambda) = d\lambda$ for all $\lambda \in A$, we obtain that the exact sequence $0 \longrightarrow A \otimes_A \mathcal{F} = \mathcal{F} \xrightarrow{d.} A \otimes_A \mathcal{F} = \mathcal{F}$, showing that $\mathcal{F}$ is a torsion-free A-module.

We have the following proposition.

PROPOSITION 7. *Let $\mathcal{F}$ be a flat A-module, $p \in Q(A)$ and $J = (1, p)$.*

1. *We have* $\mathrm{ann}_{\mathcal{F}}(A : J) = 0$*, and thus,* $\underline{\mathcal{F}} = \overline{\mathcal{F}} = \mathcal{F}$.

2. *We have the following exact sequence*

$$\begin{aligned} 0 &\longrightarrow (A:J)\mathcal{F} \xrightarrow{\delta} \mathcal{F}^2 \xrightarrow{\epsilon} (A:(A:J)) \otimes_A \mathcal{F} \\ &\longrightarrow ((A:(A:J))/J) \otimes_A \mathcal{F} \longrightarrow 0, \end{aligned} \tag{48}$$

where the A-morphisms δ and ϵ are defined by:

$$\begin{array}{ccccccc} (A:J)\mathcal{F} & \xrightarrow{\delta} & \mathcal{F}^2, & \quad & \mathcal{F}^2 & \xrightarrow{\epsilon} & (A:(A:J)) \otimes_A \mathcal{F}, \\ u & \longmapsto & \begin{pmatrix} u \\ pu \end{pmatrix}, & & \begin{pmatrix} u \\ y \end{pmatrix} & \longmapsto & 1 \otimes y - p \otimes u. \end{array}$$

In particular, we have

$$\begin{aligned} &\mathrm{dom}_{\mathcal{F}}(p) = (A:J)\mathcal{F}, \\ &\mathrm{graph}_{\mathcal{F}}(p) = \{(u \quad pu)^{\mathrm{T}} \in \mathcal{F}^2 \mid u \in \mathrm{dom}_{\mathcal{F}}(p)\}. \end{aligned}$$

3. *If p is an internally stabilizable plant, then Theorem* 3 *and Corollary* 2 *hold for $\overline{\mathcal{F}} = \underline{\mathcal{F}} = \mathcal{F}$.*

Proof. 1. Using 2 of Proposition 6, we obtain that $\mathcal{F}$ is a torsion-free A-module. Then, the result follows from Corollary 4.

2. Using the fact that $J \subseteq (A:(A:J))$, we obtain the exact sequence

$$0 \longrightarrow J \longrightarrow A:(A:J) \longrightarrow (A:(A:J))/J \longrightarrow 0$$

and combining it with the following exact sequence

$$0 \longrightarrow A:J \xrightarrow{(p \quad 1)^{\mathrm{T}}.} A^2 \xrightarrow{(1 \quad -p).} J \longrightarrow 0,$$

we finally obtain the exact sequence

$$0 \longrightarrow A:J \xrightarrow{(p \quad 1)^{\mathrm{T}}.} A^2 \xrightarrow{(1 \quad -p).} A:(A:J) \xrightarrow{\pi} (A:(A:J))/J \longrightarrow 0.$$

Now, using the fact that $\mathcal{F}$ is a flat A-module, we obtain the following exact sequence

$$\begin{aligned} 0 &\longrightarrow (A:J) \otimes_A \mathcal{F} \xrightarrow{(p \quad 1)^{\mathrm{T}}.\otimes id} \mathcal{F}^2 \xrightarrow{(1 \quad -p).\otimes id} (A:(A:J)) \otimes_A \mathcal{F} \\ &\xrightarrow{\pi \otimes id} ((A:(A:J))/J) \otimes_A \mathcal{F} \longrightarrow 0. \end{aligned}$$

Then, using the fact that $(A:J) \otimes_A \mathcal{F} \cong (A:J)\mathcal{F}$ (see 1 of Proposition 6), we finally obtain (48). In particular, we have $\mathrm{dom}_{\mathcal{F}}(p) = (A:J)\mathcal{F}$ and $\mathrm{graph}_{\mathcal{F}}(p) = \{(u \; pu)^{\mathrm{T}} \in \mathcal{F}^2 \mid u \in \mathrm{dom}_{\mathcal{F}}(p)\}$.

3. See the proof of Corollary 4. □

If $p \in Q(A)$ is internally stabilizable and c is a stabilizing controller of p, then, by Example 8, we know that $\mathcal{F} = (1, p)$, (a, b), $(1, c)$ and $(1, c)^{-1} = (a, ap)$ are flat A-modules. Using 3 of Proposition 7, we let the reader compute $\mathrm{dom}_{\mathcal{F}}(p)$, $\mathrm{graph}_{\mathcal{F}}(p)$, $\mathrm{dom}_{\mathcal{F}}(c)$ and $\mathrm{graph}_{\mathcal{F}}(c)$ for these A-modules $\mathcal{F}$.

COROLLARY 5. *Let $p \in Q(A)$ be a transfer function which admits a weakly coprime factorization $p = n/d$, $0 \neq d, n \in A$, and $\mathcal{F}$ a flat A-module. Then, we have*

$$\operatorname{dom}_{\mathcal{F}}(p) = (d)\mathcal{F} = \{dx \mid x \in \mathcal{F}\},$$
$$\operatorname{graph}_{\mathcal{F}}(p) = \{(dx \quad nx)^{\mathrm{T}} \in \mathcal{F}^2 \mid x \in \mathcal{F}\},$$

and the exact sequence (18) *becomes:*

$$0 \longrightarrow (d)\mathcal{F} \xrightarrow{\delta} \mathcal{F}^2 \xrightarrow{\epsilon} (d^{-1}) \otimes_A \mathcal{F} \longrightarrow ((d^{-1})/J) \otimes_A \mathcal{F} \longrightarrow 0. \quad (49)$$

Proof. By 2 of Theorem 1, if p admits a weakly coprime factorization $p = n/d$, $0 \neq d, n \in A$, then we have $A : J = (d)$, and thus, $A : (A : J) = (d^{-1})$. Then, by Proposition 7, the exact sequence (48) becomes (49) which proves the result. □

We have the following interesting result.

PROPOSITION 8. $H_2(\mathbb{C}_+)$ (*see Example* 6) *is a flat* $H_\infty(\mathbb{C}_+)$-*module.*

Proof. We denote by $A = H_\infty(\mathbb{C}_+)$, $\mathcal{F} = H_2(\mathbb{C}_+)$ and $a = (a_1 \ \dots \ a_n) \in A^{1\times n}$. Let us consider the equation $\sum_{i=1}^n a_i x_i = 0$, where $a_i \in A$ and $x_i \in \mathcal{F}$ and let us define the linear operator:

$$\begin{aligned} a.\colon \mathcal{F}^n &\longrightarrow \mathcal{F} \\ (x_1 \dots x_n)^{\mathrm{T}} &\longmapsto \sum_{i=1}^n a_i x_i. \end{aligned}$$

Then $\ker(a.) = \{x \in \mathcal{F}^n \mid \sum_{i=1}^n a_i x_i = 0\}$ is a closed shift-invariant subspace of $\mathcal{F}^n$. Therefore, by the Beurling–Lax Theorem [15], there exist $p \in \mathbb{Z}_+$ and an inner matrix $b = (b_{ij}) \in A^{n\times p}$ such that $\ker(a.) = b\mathcal{F}^p$. Therefore, if $x \in \ker(a.)$, then there exists $y = (y_1 \ \dots \ y_p)^{\mathrm{T}} \in \mathcal{F}^p$ such that $x = by$. In particular, for all $y \in \mathcal{F}^p$, we have $(ab)y = 0$, and thus, we obtain

$$\|(ab).\|_{\mathcal{L}(\mathcal{F}^p,\mathcal{F})} \triangleq \sup_{0\neq y\in\mathcal{F}^p} \frac{\|(ab)y\|_2}{\|y\|_2} = 0.$$

However, we know that $\|(ab).\|_{\mathcal{L}(\mathcal{F}^p,\mathcal{F})} = \|ab\|_\infty$ [9, 15], and thus, we obtain $\|ab\|_\infty$
$= 0$, i.e., $ab = 0$. Therefore, conditions (47) are satisfied, which proves the result. □

It is proved in [22] that the following equivalences hold:

- Every SISO system defined by means of transfer function $p \in K = Q(A)$ is internally stabilizable.
- Every MIMO system defined by means of a transfer matrix $P \in K^{q\times r}$ is internally stabilizable.

- A is a *Prüfer domain*, namely, for every $p \in K$ the fractional ideal $J = (1, p)$ is invertible [2, 29].

We have the following proposition.

PROPOSITION 9 (Theorem 9.10 of [7]). *If A is an integral domain, then A is a Prüfer domain iff torsion-freeness and flatness are equivalent for A-modules.*

In particular, this result holds if A is a Bézout domain, i.e., a domain over which every finitely generated ideal is principal (*e.g.*, RH_∞, $\mathcal{H}_0$ [11]).

COROLLARY 6. *$H_2(\mathbb{C}_+)$ is a flat $RH_\infty(\mathbb{C}_+)$-module and a torsion-free $\hat{\mathcal{A}}$-module.*

Proof. It is well known that RH_∞ is a principal ideal domain [35], and thus, a Prüfer domain [29]. By Proposition 8, we know that $H_2(\mathbb{C}_+)$ is a flat $H_\infty(\mathbb{C}_+)$-module. Therefore, by 2 of Proposition 6, $H_2(\mathbb{C}_+)$ is a torsion-free $H_\infty(\mathbb{C}_+)$-module. Using the fact that $RH_\infty \subset H_\infty(\mathbb{C}_+)$ and $\hat{\mathcal{A}} \subset H_\infty(\mathbb{C}_+)$ [3], then $H_2(\mathbb{C}_+)$ is also a torsion-free RH_∞-module (resp., $\hat{\mathcal{A}}$-module), and thus, by Proposition 9, a flat RH_∞-module. □

EXAMPLE 9. It is shown in [22] that $A = RH_\infty$ and $H_\infty(\mathbb{C}_+)$ are *coherent Sylvester domains*, and thus, every transfer function $p \in Q(A)$ admits a weakly coprime factorization $p = n/d$, $0 \neq d, n \in A$ [22] (see also [13, 32]). Therefore, if $\mathcal{F} = H_2(\mathbb{C}_+)$ and $A = RH_\infty$ or $H_\infty(\mathbb{C}_+)$, then, by Corollary 5, we have

$$\mathrm{dom}_{\mathcal{F}}(p) = (d)\mathcal{F}, \qquad \mathrm{graph}_{\mathcal{F}}(p) = (d \quad n)^{\mathrm{T}}\mathcal{F}.$$

In Section VII of [8], for $A = H_\infty(\mathbb{C}_+)$ and $\mathcal{F} = H_2(\mathbb{C}_+)$, the graph of a plant $p \in Q(A)$ which admits a weakly coprime factorization of $p = n/d, 0 \neq d,\ n \in A$, was defined by $\mathrm{graph}_{\mathcal{F}}(p) = (d \quad n)^{\mathrm{T}}\mathcal{F}$. Hence, we have just proved that such a definition was justified.

To finish, we also show how the previous results can be used for the study of multidimensional systems.

EXAMPLE 10. We consider again the ring $A = k[d_1, \ldots, d_n]$ of differential operators in d_i with coefficients in $k = \mathbb{R}, \mathbb{C}$ (see Example 4). If Ω is an open convex set of $\mathbb{R}^n$, then the k-vector space $\mathcal{F} = \mathcal{D}(\Omega)$ (resp., $\mathcal{E}'(\Omega)$, $\mathcal{S}(\Omega)$) of compactly supported smooth functions (resp., of compactly supported distributions, of rapidly decreasing functions) in Ω are flat A-modules [14, 16, 30]. Thus, if we take again Example 5 and we consider one of the previous flat A-modules for $\mathcal{F}$, then, using the fact that $p = d_2^{-1} d_1 \in Q(A)$ is a weakly coprime factorization of p, by Corollary 5, we obtain

$$\mathrm{dom}_{\mathcal{F}}(p) = (d_2)\mathcal{F} = \{d_2 x \mid x \in \mathcal{F}\},$$
$$\mathrm{graph}_{\mathcal{F}}(p) = \{(d_2 x \quad d_1 x)^{\mathrm{T}} \in \mathcal{F}^2 \mid x \in \mathcal{F}\}.$$

5. $\mathcal{F}$-Stabilizability and Internal Stabilizability

It is well known that if we consider $A = \mathcal{A}$ and $\mathcal{F} = L_q(\mathbb{R}_+)$, $q \in [1, +\infty]$, then internal stabilizability implies $\mathcal{F}$-stability of the closed-loop system, namely, for all u_1 and $u_2 \in \mathcal{F}$, we have $e_1, e_2, y_1, y_2 \in \mathcal{F}$ (see Figure 1), showing that every signal in the closed-loop system is $\mathcal{F}$-stable [5]. A similar result holds for $A = H_\infty(\mathbb{C}_+)$ and $\mathcal{F} = H_2(\mathbb{C}_+)$ [9].

The converse problem consisting in finding the A-modules $\mathcal{F}$ for which internal stabilizability is equivalent to $\mathcal{F}$-stabilizability of the closed-loop system is an important issue in stabilization problems. In this last part of the paper, we shall study this problem using the mathematical approach developed previously. In particular, we shall identify a class of A-modules $\mathcal{F}$ for which internal stabilizability and $\mathcal{F}$-stabilizability are equivalent.

In order to do that, we need to generalize the definition of $\mathcal{F}$-stabilizability used in the literature so that we can also study general signal spaces $\mathcal{F}$. Indeed, the standard definition can only be used for a torsion-free A-module $\mathcal{F}$. See Section 3.1 for more details.

DEFINITION 7. We use the same notations as in Theorem 3. Let $\mathcal{F}$ be an A-module, $p, c \in Q(A)$ and let us define the following A-morphism:

$$\begin{aligned} \mathrm{dom}_{\mathcal{F}}(p) \times \mathrm{dom}_{\mathcal{F}}(c) &\xrightarrow{F} \underline{\mathcal{F}} \times \overline{\mathcal{F}} \\ \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} &\longmapsto \begin{pmatrix} \underline{u_1} \\ \overline{u_2} \end{pmatrix} = (\pi' \circ \delta \quad \pi \circ \delta') \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \end{aligned} \tag{50}$$

i.e.,

$$F\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \pi'\left(\begin{pmatrix} 1 \\ p \end{pmatrix} e_1\right) + \pi\left(\begin{pmatrix} c \\ 1 \end{pmatrix} e_2\right) = \begin{pmatrix} \underline{e_1 + ce_2} \\ \overline{pe_1 + e_2} \end{pmatrix}.$$

- A controller c is said to *$\mathcal{F}$-stabilize* the plant p if the A-morphism

  $$F\colon \mathrm{dom}_{\mathcal{F}}(p) \times \mathrm{dom}_{\mathcal{F}}(c) \longrightarrow \underline{\mathcal{F}} \times \overline{\mathcal{F}}$$

  defined by (50) is an *isomorphism*, i.e., if there exists an A-morphism

  $$G\colon \underline{\mathcal{F}} \times \overline{\mathcal{F}} \longrightarrow \mathrm{dom}_{\mathcal{F}}(p) \times \mathrm{dom}_{\mathcal{F}}(c)$$

  such that we have

  $$G \circ F = id_{\mathrm{dom}_{\mathcal{F}}(p) \times \mathrm{dom}_{\mathcal{F}}(c)}, \qquad F \circ G = id_{\underline{\mathcal{F}} \times \overline{\mathcal{F}}}. \tag{51}$$

- A plant p is said to be *$\mathcal{F}$-stabilizable* if there exists a controller $c \in Q(A)$ which $\mathcal{F}$-stabilizes p.

Let us study the links between internal stabilizability and $\mathcal{F}$-stabilizability.

PROPOSITION 10. *Let $\mathcal{F}$ be an A-module, p and $c \in Q(A)$. Then, we have:*

1. *A-stabilizability is equivalent to internal stabilizability.*
2. *The controller c $\mathcal{F}$-stabilizes p iff $\pi' \circ \delta$ and $\pi \circ \delta'$ are injective A-morphisms and*

$$\pi'(\mathrm{graph}_{\mathcal{F}}(p)) \oplus \pi(\mathrm{graph}_{\mathcal{F}}(c)) = \underline{\mathcal{F}} \times \overline{\mathcal{F}},$$

or, equivalently, iff the split exact sequence (41) *holds.*

Proof. 1. Let us suppose that p is internally stabilizable and c is a stabilizing controller of p. Then, we have (1). Moreover, using the fact that $\mathcal{F} = A$ is a torsion-free A-module, by Corollary 4, we obtain

$$\underline{A} = \overline{A} = A, \qquad \mathrm{dom}_A(p) = (a, b) = (1, p)^{-1},$$
$$\mathrm{dom}_A(c) = (a, ap) = (1, c)^{-1}.$$

Therefore, the A-morphism $F\colon\ \mathrm{dom}_A(p) \times \mathrm{dom}_A(c) \longrightarrow A^2$ defined by

$$F\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = (\delta \quad \delta')\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 & c \\ p & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

is invertible and its inverse $G\colon\ A^2 \longrightarrow \mathrm{dom}_A(p) \times \mathrm{dom}_A(c)$ is defined by

$$G\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1-pc} & -\dfrac{c}{1-pc} \\ -\dfrac{p}{1-pc} & \dfrac{1}{1-pc} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a & -b \\ -ap & a \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Therefore, c A-stabilizes p and p is A-stabilizable.

Conversely, if p is A-stabilizable, then there exists a controller $c \in Q(A)$ such that the A-morphism $F\colon\ \mathrm{dom}_A(p) \times \mathrm{dom}_A(c) \longrightarrow A^2$ ($\underline{\mathcal{F}} = \overline{\mathcal{F}} = A$), defined by

$$F\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 & c \\ p & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

is invertible, where, by (26) and (36), we have

$$\begin{cases} \mathrm{dom}_A(p) = \rho(\hom_A((1, p), A)) = A : (1, p), \\ \mathrm{dom}_A(c) = \rho'(\hom_A((1, c), A)) = A : (1, c), \end{cases} \tag{52}$$

because $\ker \rho = \ker \rho' = 0$. Therefore, using the fact that F is an A-isomorphism, then there exist unique $a_1, a_2 \in \mathrm{dom}_A(p)$ and $b_1, b_2 \in \mathrm{dom}_A(c)$ such that we have:

$$\begin{pmatrix} 1 & c \\ p & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & c \\ p & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} a_1 + b_1 c = 1, \\ b_1 = -a_1 p, \\ a_2 = -b_2 c, \\ b_2 + a_2 p = 1. \end{cases}$$

From the last system, we deduce that $(1-pc)a_1 = 1$ and $(1-pc)b_2 = 1$, a fact which shows that $a_1 \neq 0$, $b_2 \neq 0$, $1-pc \neq 0$ and $a_1 = b_2 = 1/(1-pc)$. Therefore, we obtain

$$\begin{cases} b_2 + a_2 p = 1, \\ b_2 p = a_1 p = -b_1 \in A, \\ a_2 = -b_2 c, \end{cases}$$

which, by 3 of Theorem 1, shows that $c' = -a_2/b_2 = c$ internally stabilizes p.

2. Let us suppose that p is $\mathcal{F}$-stabilizable, i.e., there exists $c \in Q(A)$ such that the A-morphism

$$F\colon \operatorname{dom}_{\mathcal{F}}(p) \times \operatorname{dom}_{\mathcal{F}}(c) \longrightarrow \underline{\mathcal{F}} \times \overline{\mathcal{F}},$$

defined by (7) is an A-isomorphism. Therefore, there exists

$$G\colon \underline{\mathcal{F}} \times \overline{\mathcal{F}} \longrightarrow \operatorname{dom}_{\mathcal{F}}(p) \times \operatorname{dom}_{\mathcal{F}}(c)$$

such that the identities (51) hold. Let us denote by $G = (\tau' \quad \tau)^{\mathrm{T}}$ where τ and τ' are defined by

$$\begin{cases} \tau' = \pi_1 \circ G, \\ \tau = \pi_2 \circ G, \end{cases} \quad \text{and} \quad \begin{cases} \pi_1\colon \operatorname{dom}_{\mathcal{F}}(p) \times \operatorname{dom}_{\mathcal{F}}(c) \longrightarrow \operatorname{dom}_{\mathcal{F}}(p), \\ \pi_2\colon \operatorname{dom}_{\mathcal{F}}(p) \times \operatorname{dom}_{\mathcal{F}}(c) \longrightarrow \operatorname{dom}_{\mathcal{F}}(c), \end{cases}$$

are respectively the projections onto $\operatorname{dom}_{\mathcal{F}}(p)$ and $\operatorname{dom}_{\mathcal{F}}(c)$. Then, from (51), we deduce that

$$\begin{cases} F \circ G = (\pi' \circ \delta \quad \pi \circ \delta') \circ \begin{pmatrix} \tau' \\ \tau \end{pmatrix} = (\pi' \circ \delta) \circ \tau' + (\pi \circ \delta') \circ \tau' \\ \qquad = id_{\underline{\mathcal{F}} \times \overline{\mathcal{F}}}, \\ G \circ F = \begin{pmatrix} \tau' \\ \tau \end{pmatrix} \circ (\pi' \circ \delta \quad \pi \circ \delta') = \begin{pmatrix} \tau' \circ (\pi' \circ \delta) & \tau' \circ (\pi \circ \delta') \\ \tau \circ (\pi' \circ \delta) & \tau \circ (\pi \circ \delta') \end{pmatrix} \\ \qquad = \begin{pmatrix} id_{\operatorname{dom}_{\mathcal{F}}(p)} & 0 \\ 0 & id_{\operatorname{dom}_{\mathcal{F}}(c)} \end{pmatrix}, \end{cases}$$

and thus, we obtain the split exact sequence (41), $\pi' \circ \delta$ and $\pi \circ \delta'$ are injective and (42) holds.

Conversely, let us suppose that the A-morphisms $\pi' \circ \delta$ and $\pi \circ \delta'$ are injective and (42) holds. Then, using (29) and (39), we obtain:

$$(\pi' \circ \delta)(\operatorname{dom}_{\mathcal{F}}(p)) \oplus (\pi \circ \delta')(\operatorname{dom}_{\mathcal{F}}(c)) = \underline{\mathcal{F}} \times \overline{\mathcal{F}}.$$

Using the fact that $\pi' \circ \delta$ and $\pi \circ \delta'$ are injective and the following standard split exact sequence

$$0 \longrightarrow \mathcal{F}_1 \underset{\xleftarrow{\pi_1}}{\xrightarrow{i_1}} \mathcal{F}_1 \oplus \mathcal{F}_2 \underset{\xleftarrow{i_2}}{\xrightarrow{\pi_2}} \mathcal{F}_2 \longrightarrow 0,$$

where $\mathcal{F}_1 = (\pi' \circ \delta)(\mathrm{dom}_{\mathcal{F}}(p))$ and $\mathcal{F}_2 = (\pi \circ \delta')(\mathrm{dom}_{\mathcal{F}}(c))$, then we obtain the following exact diagram

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & (\pi' \circ \delta)(\mathrm{dom}_{\mathcal{F}}(p)) & \underset{\xleftarrow{\pi_1}}{\xrightarrow{i_1}} & \underline{\mathcal{F}} \times \overline{\mathcal{F}} & \underset{\xleftarrow{i_2}}{\xrightarrow{\pi_2}} & (\pi \circ \delta')(\mathrm{dom}_{\mathcal{F}}(c)) & \longrightarrow & 0, \\ & & {\scriptstyle \pi' \circ \delta} \uparrow & & & & \uparrow {\scriptstyle \pi \circ \delta'} & & \\ & & \mathrm{dom}_{\mathcal{F}}(p) & & & & \mathrm{dom}_{\mathcal{F}}(c) & & \\ & & \uparrow & & & & \uparrow & & \\ & & 0 & & & & 0 & & \end{array}$$

and thus, we obtain the split exact sequence

$$0 \longrightarrow \mathrm{dom}_{\mathcal{F}}(p) \underset{\xleftarrow{\tau'}}{\xrightarrow{\pi' \circ \delta}} \underline{\mathcal{F}} \times \overline{\mathcal{F}} \underset{\xleftarrow{\pi \circ \delta'}}{\xrightarrow{\tau}} \mathrm{dom}_{\mathcal{F}}(c) \longrightarrow 0,$$

where the A-morphisms τ and τ' are explicitly defined in the proof of Theorem 3. Therefore, if we define G: $\underline{\mathcal{F}} \times \overline{\mathcal{F}} \longrightarrow \mathrm{dom}_{\mathcal{F}}(p) \times \mathrm{dom}_{\mathcal{F}}(c)$ by

$$G\begin{pmatrix} \underline{u} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \tau'((\underline{u} \quad \overline{y})^{\mathrm{T}}) \\ \tau((\underline{u} \quad \overline{y})^{\mathrm{T}}) \end{pmatrix},$$

then, using the fact that (41) is a split exact sequence, we obtain the relations

$$\begin{array}{ll} \tau' \circ (\pi' \circ \delta) = id_{\mathrm{dom}_{\mathcal{F}}(p)}, & \tau \circ (\pi \circ \delta') = id_{\mathrm{dom}_{\mathcal{F}}(c)}, \\ \tau \circ (\pi' \circ \delta) = 0, & \tau' \circ (\pi \circ \delta') = 0, \end{array}$$

and $(\pi' \circ \delta) \circ \tau' + (\pi \circ \delta') \circ \tau = id_{\underline{\mathcal{F}} \times \overline{\mathcal{F}}}$ which imply (51). Therefore, the A-morphism F: $\mathrm{dom}_{\mathcal{F}}(p) \times \mathrm{dom}_{\mathcal{F}}(c) \longrightarrow \underline{\mathcal{F}} \times \overline{\mathcal{F}}$ is an isomorphism, and thus, c $\mathcal{F}$-stabilizes p. □

We have the following corollary of Theorem 3 and 2 of Proposition 10.

COROLLARY 7. *If $p \in Q(A)$ is internally stabilizable, then p is $\mathcal{F}$-stabilizable for every A-module $\mathcal{F}$.*

To finish, we try to understand over which A-module $\mathcal{F}$ the converse result of Corollary 7 holds, i.e., for which A-module $\mathcal{F}$, $\mathcal{F}$-stabilizability of p implies that p is internally stabilizable.

In order to do that, we introduce the following definition.

DEFINITION 8. $\mathcal{F}$ is *a faithfully flat* A-module if one of following equivalent assertions is satisfied:

1. $\mathcal{F}$ is a flat A-module and, for A-modules M', M and M'', $M' \longrightarrow M \longrightarrow M''$ is an exact sequence whenever $\mathcal{F} \otimes_A M' \longrightarrow \mathcal{F} \otimes_A M \longrightarrow \mathcal{F} \otimes_A M''$ is an exact sequence.
2. $\mathcal{F}$ is a flat A-module and, for every maximal ideal $\mathfrak{m}$ of A, $\mathfrak{m}\mathcal{F}$ is strictly contained in $\mathcal{F}$, i.e., $\mathfrak{m}\mathcal{F} \subsetneq \mathcal{F}$.

EXAMPLE 11. We give a few examples of faithfully flat A-modules.

- Every finitely generated free A-module $\mathcal{F}$, namely, $\mathcal{F} \cong A^n$, $n \in \mathbb{Z}_+$, is faithfully flat.
- [2] If $(f_i)_{i\in I}$ is a finite family of elements of A such that $\sum_{i\in I} A f_i = A$, then $B = \prod_{i\in I} A_{f_i}$ is a faithfully flat A-module, where $A_{f_i} = \{a/f_i^n \mid a \in A, n \in \mathbb{Z}_+\}$.
- Let us consider the ring $A = k[d_1, \ldots, d_n]$ of differential operators in d_i with coefficients in $k = \mathbb{R}, \mathbb{C}$. If Ω is an open convex set of $\mathbb{R}^n$, then the A-module $\mathcal{D}(\Omega)$ (resp., $\mathcal{E}'(\Omega)$) of compactly supported smooth functions (resp., of compactly supported distributions) in Ω is a faithfully flat A-module [14, 30].

The next lemma gives conditions on the A-module $\mathcal{F}$ so that the *basic assumption* of [39], namely, $p \in Q(A)$ is a bounded linear operator from $\mathcal{F}$ to $\mathcal{F}$ iff $p \in A$, is satisfied. We first note the fact that p defines a linear operator from $\mathcal{F}$ to $\mathcal{F}$ implies that $\mathcal{F}$ is necessarily a torsion-free A-module.

LEMMA 7. *Let $p \in Q(A)$, $J = (1, p)$ and $\mathcal{F}$ be a faithfully flat A-module. Then, we have*

$$\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F} \Leftrightarrow p \in A.$$

Proof. $\Rightarrow$ By 2 of Proposition 7, we have $\mathrm{dom}_{\mathcal{F}}(p) = (A : J)\mathcal{F}$. Therefore, $\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F}$ implies that $(A : J)\mathcal{F} = \mathcal{F}$. But, $\mathcal{F}$ is a faithfully flat A-module, and thus, by 2 of Definition 8, $A : J$ is not contained in a maximal ideal of A, i.e., $A : J = A$ [2]. Then, by 1 of Theorem 1, we obtain $p \in A$.

$\Leftarrow$ This result has already been proved in 1 of Proposition 4. □

Remark 1. The set of the maximal ideals of $A = H_\infty(\mathbb{C}_+)$ is not very well understood [12]. Therefore, we do not know yet whether or not $\mathcal{F} = H_2(\mathbb{C}_+)$ is a faithfully flat $H_\infty(\mathbb{C}_+)$-module. However, using the fact that A is a coherent Sylvester domain [22], we know that every transfer function $p \in Q(A)$ admits a weakly coprime factorization $p = n/d$, $0 \neq d$, $n \in d$, and thus, by 2 of Theorem 1, we have $A : (1, p) = (d)$. Therefore, "$\mathrm{dom}_{\mathcal{F}}(p) = \mathcal{F} \Rightarrow p \in A$" is equivalent to "$(d)\mathcal{F} = \mathcal{F} \Rightarrow d \in \mathrm{U}(A)$", where $\mathrm{U}(A)$ denotes the set of invertible elements of A. Such a result holds for $H_\infty(\mathbb{C}_+)$ [40].

PROPOSITION 11. *$H_2(\mathbb{C}_+)$ is a faithfully flat RH_∞-module.*

Proof. By Corollary 6, we already know that $\mathcal{F} = H_2(\mathbb{C}_+)$ is a flat $A = RH_\infty$-module. Using the fact that A is a principal ideal domain [35], we check that a maximal ideal of A is of the form (d) where $d_\infty = 1/(s+1)$, $d_a = (s-a)/(s+1)$ with $a \in \mathbb{R}$ and $a \geqslant 0$ or $d_{a\pm ib} = ((s-a)^2 + b^2)/(s+1)^2$ with $a, b \in \mathbb{R}$ and $a \geqslant 0$ and $b > 0$. In all these cases, we easily check that $(d)\mathcal{F} \subsetneq \mathcal{F}$, which, by 2 of Definition 8, shows that $\mathcal{F}$ is a faithfully flat A-module. □

We are now in position to state the last theorem of this paper.

THEOREM 4. *Let $\mathcal{F}$ be a faithfully flat A-module, p and $c \in Q(A)$. Then, the controller c $\mathcal{F}$-stabilizes the plant p iff c internally stabilizes p.*

Proof. Using the fact that $\mathcal{F}$ is a faithfully flat A-module, by 2 of Proposition 7 and 1 of Proposition 6, we obtain

$$\mathrm{dom}_{\mathcal{F}}(p) = \mathrm{dom}_A(p)\mathcal{F} = \mathrm{dom}_A(p) \otimes_A \mathcal{F}.$$

Similarly, we have $\mathrm{dom}_{\mathcal{F}}(c) = \mathrm{dom}_A(c)\mathcal{F} = \mathrm{dom}_A(c) \otimes_A \mathcal{F}$. Moreover, by 1 of Proposition 7, we also have $\underline{\mathcal{F}} = \overline{\mathcal{F}} = \mathcal{F}$.

By 1 of Proposition 10, p is internally stabilized by c iff p is A-stabilized by c, i.e., by 2 of Proposition 10, iff we have the following split exact sequence:

$$0 \longrightarrow \mathrm{dom}_A(p) \underset{\overset{\tau'}{\longleftarrow}}{\overset{\delta}{\longrightarrow}} A^2 \underset{\overset{\delta'}{\longleftarrow}}{\overset{\tau}{\longrightarrow}} \mathrm{dom}_A(c) \longrightarrow 0.$$

Using that $\mathcal{F}$ is a faithfully flat A-module, the previous split exact sequence holds iff we have the following split exact sequence:

$$0 \longrightarrow \mathrm{dom}_A(p) \otimes_A \mathcal{F} \underset{\overset{\tau'\otimes id}{\longleftarrow}}{\overset{\delta\otimes id}{\longrightarrow}} \mathcal{F}^2 \underset{\overset{\delta'\otimes id}{\longleftarrow}}{\overset{\tau\otimes id}{\longrightarrow}} \mathrm{dom}_A(c) \otimes_A \mathcal{F} \longrightarrow 0,$$

i.e., iff we have the following split exact sequence:

$$0 \longrightarrow \mathrm{dom}_{\mathcal{F}}(p) \underset{\overset{\tau'}{\longleftarrow}}{\overset{\delta}{\longrightarrow}} \mathcal{F}^2 \underset{\overset{\delta'}{\longleftarrow}}{\overset{\tau}{\longrightarrow}} \mathrm{dom}_{\mathcal{F}}(c) \longrightarrow 0,$$

and thus, by 2 of Proposition 10, iff p is $\mathcal{F}$-stabilizes by c. □

6. Conclusion

We hope that we have convinced the reader that the fractional ideal approach to SISO systems is a natural mathematical framework for the study of analysis and synthesis problems. In particular, we have shown that a certain duality existed between this algebraic approach and the operator-theoretic one. This duality allows us to understand the forms of the domain and the graph of a transfer function depending on the structural properties of the system (e.g., stable, internally stabilizable, existence of a (weakly) coprime factorization) or of the signal space over which the transfer function acts (e.g., injective, torsion-free, flat or faithfully flat module). The fractional ideal approach can be extended to MIMO systems using the concept of lattice of vector spaces. We refer the reader to [24, 25] for more details. Hence, dualizing this approach, we can develop an operator-theoretic approach to analysis and synthesis problems for MIMO systems. Such an extension will be studied in the future. Finally, the problem of determining the graph of a stabilizable plant has crucial applications in the study of the robustness topology (e.g., gap, graph, ν-gap metrics ...) [8, 9, 31, 35, 39]. In a forthcoming publication, we shall use the results obtained in this paper in order to study the robustness topology for internally stabilizable plants which do not necessarily admit coprime factorizations (see [31]).

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