EXTENDED BÉZOUT IDENTITIES

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Abstract

We study primeness of multidimensional control systems defined in terms of algebraic properties of $D = k[\chi_1, \ldots, \chi_n]$-modules and show how to pass from one to another by inversion of a certain $\pi \in D$. We use these results to determine effectively extended Bézout identities of multidimensional control systems and the minimal number of χ_i contained in π.

1 Introduction

In [13], it has been shown that primeness of multidimensional systems, defined by a full row rank matrix R with entries in $D = k[\chi_1, \ldots, \chi_n]$, are linked with *extended Bézout identities*, i.e. the existence of a matrix S and $\pi \in D$ such that $R\,S = \pi\,I$, where I is the identity matrix. However, only two different types of primeness, *ZLP* and *MLP*, have been defined in [13], which correspond to the case $\pi = 1$ and π a polynomial containing $n-1$ variables χ_i.

To my knowledge, nothing has been done for the other cases until the work of Oberst [6], surely because the complexity of the matrices increases with the number n. The main contribution of [6] has been the introduction of *algebraic analysis* concepts [2] in the theory of multidimensional systems. Following an idea of Malgrange, it is shown in [6] how to associate with any multidimensional system a finitely presented D-module M. Then, the author shows that ZLP (resp. MLP) corresponds to a projective (resp. torsion-free) D-module M and he defines a new type of primeness, *WZLP*, which corresponds to π containing one variable.

In [12], it is shown that for a multidimensional control system, defined by a full row rank matrices R, there exist a one-to-one correspondence between the number of χ_i in π and a chain of n different type of primeness, defined by the dimension of the algebraic variety formed by the zeros of all the maximal order minors of R (this chain includes ZLP, WZLP and MLP).

Finally, using the classification of Palamodov-Kashiwara [2], it is shown in [8] that, for the ring $K[d_1, \ldots, d_n]$ of differential operators with coefficients in a differential field K ($K[d_1, \ldots, d_n] \cong k[\chi_1, \ldots, \chi_n]$ if $K = k$ is a constant field), there are one-to-one equivalences between the *chain of algebraic properties* of M (torsion-free $\subseteq$ reflexive $\subseteq \ldots \subseteq$ projective), the number of *successive parametrizations* of the multidimensional control system, the index i of the first non-zero $\mathrm{ext}^i_D(T(M), D)$ for $i \geq 1$, where $T(M)$ is the *transposed* module of M and *ext* is the extension functor, and, in the case of a full row rank matrix R, the dimension of the characteristic variety $\mathrm{char}(M)$ and the type of primeness obtained in [12] if in addition $K = k$.

In this paper, we show how to pass from an element of the previous equivalent chains to another one, by means of the inversion of a certain polynomial π in $D = k[\chi_1, \ldots, \chi_n]$ containing more or less χ_i. In particular, it is shown how to pass from a type of primeness to another one by localization of a polynomial π and how to obtain effectively extended Bézout identities for each type of primeness.

2 Modules and Extension functor

In the course of this paper, we shall note $D = k[\chi_1, \ldots, \chi_n]$, where $k = \mathbb{R}$ or $\mathbb{C}$.

Definition 1. [11] Let M be a finitely generated D-module. Then:

- M is a *free* D-module if $M \cong D^r$ for a certain $r \in \mathbb{Z}_+$,
- M is a *projective* D-module if $M \oplus P \cong D^r$ for a certain D-module P and $r \in \mathbb{Z}_+$,
- M is a *reflexive* D-module if the D-morphism defined by
$$\epsilon : M \to \hom_D(\hom_D(M, D), D),$$
$$\epsilon(m)(f) = f(m),\ \forall\, f \in \hom_D(M, D),\ \forall\, m \in M,$$
is an isomorphism,
- M is a *torsion-free* D-module if
$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D,\ P\,m = 0\} = 0.$$
- M is a *torsion* D-module if $t(M) = M$.

Theorem 1. *[11] We have the following assertions:*

- $\text{free} \subseteq \text{projective} \subseteq \ldots \subseteq \text{reflexive} \subseteq \text{torsion}-\text{free}.$
- *If $D = k[\chi_1]$, then any torsion-free D-module is free.*
- *Any projective $D = k[\chi_1, \ldots, \chi_n]$-module is free (Quillen-Suslin) [14].*

Definition 2. We have the following definitions (see e.g. [11] for more details):

- A *projective resolution* (resp. *free resolution*) of a D-module M is an exact sequence of the form

$$\ldots \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0, \tag{1}$$

  where P_i is a projective (resp. free) D-module and d_i is a D-morphism.

- If M is defined by a projective resolution (1), then the defects of exactness of

$$\ldots \xleftarrow{d_{i+1}^\star} P_i^\star \xleftarrow{d_i^\star} P_{i-1}^\star \xleftarrow{d_{i-1}^\star} \ldots \xleftarrow{d_2^\star} P_1^\star \xleftarrow{d_1^\star} P_0^\star \longleftarrow 0, \tag{2}$$

  where $P_i^\star = \hom_D(P_i, D)$ and $d_i^\star : P_{i-1}^\star \to P_i^\star$ is defined by $d_i^\star(f) = f \circ d_i, \ \forall f \in P_i^\star$, only depend on M and not on (1). They are called $\mathrm{ext}_D^i(M, D)$. Therefore:

$$\begin{cases} \mathrm{ext}_D^0(M, D) = \ker d_1^\star = \hom_D(M, D), \\ \mathrm{ext}_D^i(M, D) = \ker d_{i+1}^\star / \mathrm{im}\, d_i^\star. \end{cases}$$

Remark 1. If M is a finitely generated D-module, then M has a finite free resolution

$$\ldots \xrightarrow{.R_{i+1}} D^{l_i} \xrightarrow{.R_i} D^{l_{i-1}} \xrightarrow{.R_{i-1}} \ldots \xrightarrow{.R_1} D^{l_0} \longrightarrow M \longrightarrow 0, \tag{3}$$

where R_i is a $l_i \times l_{i-1}$ matrix with entries in D and $.R_i : D^{l_i} \to D^{l_{i-1}}$ is defined by letting operate a row vector of length l_i on the left of R_i to obtain a row vector of length l_{i-1}. Then, (2) is defined by

$$\ldots \xleftarrow{R_{i+1}.} D^{l_i} \xleftarrow{R_i.} D^{l_{i-1}} \xleftarrow{R_{i-1}.} \ldots \xleftarrow{R_2.} D^{l_1} \xleftarrow{R_1.} D^{l_0} \longleftarrow 0,$$

where $R_i. : D^{l_{i-1}} \to D^{l_i}$ is defined by letting operate a column vector of length l_{i-1} on the right of R_i to obtain a column vector of length l_i. Then, we have:

$$\mathrm{ext}_D^i(M, D) = \ker(R_{i+1}.)/\,\mathrm{im}(R_i.), \ \forall\, i \geq 1.$$

Definition 3. If M is a D-module defined by the following finite presentation

$$F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0 \tag{4}$$

then, its *transposed module* $N = T(M)$ is the D-module defined by $N = \mathrm{coker}\, d_1^\star$, i.e. N is the D-module defined by the following finite presentation:

$$0 \longleftarrow N \longleftarrow F_1^\star \xleftarrow{d_1^\star} F_0^\star. \tag{5}$$

We easily verify that for any finitely generated D-module we have:

$$T(T(M)) \cong M.$$

Proposition 1. *Let M be a finitely presented D-module and S a multiplicative set of D, then we have:*

$$T(S^{-1}D \otimes_D M) \cong T(M) \otimes_D S^{-1}D.$$

Proof. Taking the tensor product $S^{-1}D \otimes_D \cdot$ of (4), we obtain the exact sequence of $S^{-1}D$-modules [11]:

$$S^{-1}D \otimes_D F_1 \xrightarrow{\mathrm{id}_S \otimes d_1} S^{-1}D \otimes_D F_0 \longrightarrow S^{-1}D \otimes_D M \longrightarrow 0.$$

Then, $T(S^{-1}D \otimes_D M) = \mathrm{coker}(\mathrm{id}_S \otimes d_1)^\star$ is defined by the finite presentation defined in Figure 1. We have $\hom_{S^{-1}D}(S^{-1}D \otimes_D F_i, S^{-1}D) \cong F_i^\star \otimes_D S^{-1}D, \ i = 0, 1$, and $(\mathrm{id}_S \otimes d_1)^\star = d_1^\star \otimes \mathrm{id}_S$ because F_0 and F_1 are two finitely generated D-modules [11]. Moreover, if we take the tensor product of (5) by $S^{-1}D$, we obtain the following exact sequence:

$$0 \longleftarrow T(M) \otimes_D S^{-1}D \longleftarrow F_1^\star \otimes_D S^{-1}D \xleftarrow{d_1^\star \otimes \mathrm{id}_S} F_0^\star \otimes_D S^{-1}D.$$

Finally, we have the commutative exact diagram defined in Figure 2 which proves the proposition. □

Theorem 2. *If M is a finitely generated D-module and $N = T(M)$, then we have:*

1. $t(M) \cong \mathrm{ext}_D^1(N, D)$,
2. *M is a torsion-free D-module iff* $\mathrm{ext}_D^1(N, D) = 0$,
3. *M is a reflexive D-module iff* $\mathrm{ext}_D^i(N, D) = 0, \ i = 1, 2$,
4. *M is a projective D-module iff* $\mathrm{ext}_D^i(N, D) = 0, \ i = 1, \ldots, n$.

Proof. See [9] for the proves of 1 and 2. We have the following exact sequence

$$0 \longrightarrow \mathrm{ext}_D^1(N, D) \longrightarrow M \xrightarrow{\epsilon} M^{\star\star} \longrightarrow \mathrm{ext}_D^2(N, D) \longrightarrow 0 \tag{6}$$

(see [8] and its references for a proof) which proves 3. An algebraic proof of 4 can be easily adapted from the proof of Corollary 4 in [8]. □

Definition 4.

- The *grade* of a D-module M is defined by:

$$j(M) = \min_{i \geq 0} \{\, i \mid \mathrm{ext}_D^i(M, D) \neq 0 \,\} \in \{0, \ldots, n, +\infty\}.$$

- We call *dimension* of a D-module M the *Krull dimension* $d(M)$ of $D/\mathrm{ann}\,(M)$ (with the convention that $d(0) = -1$) [11].

Theorem 3. *[1, 2] Let M be a finitely generated D-module, then we have:*

$$j(M) = \mathrm{cd}(M) := n - d(M) \ \text{ if } \ M \neq 0 \ \ (+\infty \ \text{ if } \ M = 0).$$

$$0 \longleftarrow T(S^{-1}D \otimes_D M) \longleftarrow \hom_{S^{-1}D}(S^{-1}D \otimes_D F_1, S^{-1}D) \overset{(\mathrm{id}_S \otimes d_1)^\star}{\longleftarrow} \hom_{S^{-1}D}(S^{-1}D \otimes_D F_0, S^{-1}D).$$

Figure 1: Exact sequence

$$\begin{array}{ccccccc} 0 \longleftarrow & T(S^{-1}D \otimes_D M) & \longleftarrow & \hom_D(S^{-1}D \otimes_D F_1, S^{-1}D) & \overset{(\mathrm{id}_S \otimes d_1)^\star}{\longleftarrow} & \hom_D(S^{-1}D \otimes_D F_0, S^{-1}D) \\ & & & \parallel & & \parallel \\ 0 \longleftarrow & T(M) \otimes_D S^{-1}D & \longleftarrow & F_1^\star \otimes_D S^{-1}D & \overset{d_1^\star \otimes \mathrm{id}_S}{\longleftarrow} & F_0^\star \otimes_D S^{-1}D. \end{array}$$

Figure 2: Commutative exact diagram

3 Main results

3.1 General case

Definition 5. *Let M be a finitely generated D-module and $N = T(M)$, then we define:*

$$i(M) = \min_{i \geq 1} \{\, i-1 \mid \ \mathrm{ext}^i_D(N, D) \neq 0 \,\} \in \{0, ..., n-1, +\infty\}.$$

Remark 2. The notation $i(M)$ is justified by the fact that N only depends on M up to a *projective equivalence*, and thus, $\mathrm{ext}^k_D(N, D),\ k \geq 1$, only depends on M [10]. Moreover, by Theorem 2, $t(M) \neq 0 \Leftrightarrow i(M) = 0$, $t(M) = 0 \Leftrightarrow i(M) = 1$, M reflexive $\Leftrightarrow i(M) = 2$, ..., M projective $\Leftrightarrow i(M) = +\infty$.

We shall denote by S_n the group of permutations of n elements.

Theorem 4. *Let M be a finitely generated $D = k[\chi_1, \ldots, \chi_n]$-module and for all $\sigma \in S_n$:*

$$\begin{cases} D^\sigma_{n-i(M)} = k[\chi_{\sigma(1)}, \ldots, \chi_{\sigma(n-i(M))}], & 0 \leq i(M) \leq n-1, \\ D^\sigma_{-\infty} = k, & i(M) = +\infty. \end{cases} \tag{7}$$

Then, for all integer $l \geq 0$, there exists $\pi^\sigma_{n-i(M)} \in D^\sigma_{n-i(M)}$ such that

$$i(D_{\pi^\sigma_{n-i(M)}} \otimes_D M) \geq i(M) + l, \tag{8}$$

where $S_{\pi^\sigma_{n-i(M)}} = \{1, \pi^\sigma_{n-i(M)}, (\pi^\sigma_{n-i(M)})^2, \ldots\}$ is the multiplicative set formed by $\pi^\sigma_{n-i(M)}$ and $D_{\pi^\sigma_{n-i(M)}} = S^{-1}_{\pi^\sigma_{n-i(M)}} D$. In particular, for all $\sigma \in S_n$, there exists $\pi^\sigma_{n-i(M)} \in D^\sigma_{n-i(M)}$ such that $D_{\pi^\sigma_{n-i(M)}} \otimes_D M$ is a projective $D_{\pi^\sigma_{n-i(M)}}$-module.

Proof. First of all, let us notice that if $i(M) = +\infty$ or $l = 0$, then the result is trivial (take $\pi^\sigma_{n-i(M)} \in k$). In the following of the proof, we suppose $l \geq 1,\ 0 \leq i(M) \leq n-1$, and note:

$$K^\sigma_{i(M)} = (D^\sigma_{n-i(M)})^{-1} D, \quad 0 \leq i(M) \leq n-1,$$

that is to say, $K^\sigma_0 = k(\chi_1, \ldots, \chi_n)$ and, for $1 \leq i(M) \leq n-1$:

$$K^\sigma_{i(M)} = k(\chi_{\sigma(1)}, \ldots, \chi_{\sigma(n-i(M))})[\chi_{\sigma(n-i(M)+1)}, \ldots, \chi_{\sigma(n)}].$$

Therefore, we have [11]:

$$\begin{array}{c} \mathrm{gl}\dim(K^\sigma_{i(M)}) = i(M) \\ \Rightarrow\ \mathrm{ext}^j_{K^\sigma_{i(M)}}(K^\sigma_{i(M)} \otimes_D N, K^\sigma_{i(M)}) = 0, \ \ \forall\, j \geq i(M) + 1. \end{array}$$

Moreover, $K^\sigma_{i(M)}$ is a flat D-module and N is finitely presented, then we have $\forall\, j \geq 0$ [11]:

$$\mathrm{ext}^j_{K^\sigma_{i(M)}}(K^\sigma_{i(M)} \otimes_D N, K^\sigma_{i(M)}) \cong K^\sigma_{i(M)} \otimes_D \mathrm{ext}^j_D(N, D). \tag{9}$$

Hence, we obtain $\mathrm{ext}^j_{K^\sigma_{i(M)}}(K^\sigma_{i(M)} \otimes_D N, K^\sigma_{i(M)}) = 0,\ \forall\, j \geq 1$, i.e. $K^\sigma_{i(M)} \otimes_D N$ is a projective $K^\sigma_{i(M)}$-module. Finally, the right member of the isomorphism in (9) for $j \geq i(M) + 1$, combined with the fact that $\mathrm{ext}^j_D(N, D)$ is a torsion D-module [9] for $j \geq 1$, implies that we have $\forall\, j \geq i(M) + 1$:

$$I^{\sigma\, j}_{n-i(M)} := \mathrm{ann}(\mathrm{ext}^j_D(N, D)) \cap D^\sigma_{n-i(M)} \neq 0.$$

For $j \geq i(M) + 1$, let us take

$$\pi^{\sigma\, j}_{n-i(M)} \in \mathrm{ann}(\mathrm{ext}^j_D(N, D)) \cap D^\sigma_{n-i(M)}$$

and define:

$$\pi^\sigma_{n-i(M)} = \Pi_{\{i(M)+1 \leq j \leq i(M)+l,\ \pi^{\sigma\, j}_{n-i(M)} \neq 0\}}\ \pi^{\sigma\, j}_{n-i(M)}.$$

We have $\pi^\sigma_{n-i(M)} \in D^\sigma_{n-i(M)}$ and:

$$\pi^\sigma_{n-i(M)}\, \mathrm{ext}^j_D(N, D) = 0, \ \ i(M) + 1 \leq j \leq i(M) + l.$$

Therefore, for $\ i(M) + 1 \leq j \leq i(M) + l$, we have:

$$\begin{array}{l} \mathrm{ext}^j_{D_{\pi^\sigma_{n-i(M)}}}(D_{\pi^\sigma_{n-i(M)}} \otimes_D N, D_{\pi^\sigma_{n-i(M)}}) \\ \cong D_{\pi^\sigma_{n-i(M)}} \otimes_D \mathrm{ext}^j_D(N, D) = 0. \end{array}$$

By Theorem 2 and Proposition 1 (i.e. $T(D_{\pi^\sigma_{n-i(M)}} \otimes_D M) = N \otimes_D D_{\pi^\sigma_{n-i(M)}}$), we obtain:

$$i(D_{\pi^\sigma_{n-i(M)}} \otimes_D M) \geq i(M) + l.$$

If we take $l = n - i(M)$, then $D_{\pi^\sigma_{n-i(M)}} \otimes_D M$ is a projective $D_{\pi^\sigma_{n-i(M)}}$-module. □

Example 1. Let us consider the $D = k[\chi_1, \chi_2, \chi_3]$-module M defined by the matrix

$$R = \begin{pmatrix} 0 & -\chi_3 & \chi_2 \\ \chi_3 & 0 & -\chi_1 \\ -\chi_2 & \chi_1 & 0 \end{pmatrix},$$

corresponding to the curl operator in $\mathbb{R}^3$. Thus, we have the following free resolution of M

$$0 \longrightarrow D \xrightarrow{.R_1} D^3 \xrightarrow{.R} D^3 \longrightarrow M \longrightarrow 0,$$

where the matrix $R_1 = (\chi_1\ \chi_2\ \chi_3)$ corresponds to the divergence operator. Then, the D-module $N = T(M)$ is defined by $0 \longleftarrow N \longleftarrow D^3 \xleftarrow{R.} D^3$. We easily check that we have

$$\begin{cases} \mathrm{ext}^1_D(N, D) = 0, \\ \mathrm{ext}^2_D(N, D) = D/D^3 R_{-1} \neq 0, \\ \mathrm{ext}^j_D(N, D) = 0, \ \forall\, j \geq 3, \end{cases}$$

where $R_{-1} = R_1^t$. Thus, we obtain $i(M) = 2 - 1 = 1$ and $3 - i(M) = 2$. Moreover, $\mathrm{ext}^2_D(N, D) = D/D^3 R_{-1}$ is defined by the following equations

$$\begin{cases} \chi_1\, z = 0, \\ \chi_2\, z = 0, \\ \chi_3\, z = 0, \end{cases}$$

and we verify that $\forall\, \sigma \in S_3$:

$$I_2^{\sigma\, 2} = \mathrm{ann}(\mathrm{ext}^2_D(N, D)) \cap k[\chi_{\sigma(1)}, \chi_{\sigma(2)}] = (\chi_{\sigma(1)}, \chi_{\sigma(2)}).$$

But, $\chi^{-1}_{\sigma(1)}, \chi^{-1}_{\sigma(2)} \in K_1^\sigma = k(\chi_{\sigma(1)}, \chi_{\sigma(2)})[\chi_{\sigma(3)}]$, and thus, we have:

$$K_1^\sigma \otimes_D \mathrm{ext}^2_D(N, D) = \mathrm{ext}^2_{K_1^\sigma}(K_1^\sigma \otimes_D N, K_1^\sigma) = 0.$$

Moreover, we have $\mathrm{ext}^j_{K_1^\sigma}(K_1^\sigma \otimes_D N, K_1^\sigma) = 0, \ \forall\, j \geq 1$, which implies that $K_1^\sigma \otimes_D M$ is a projective K_1^σ-module. Finally, if we note $\pi_2^\sigma = \chi_{\sigma(1)}$, then $D_{\pi_2^\sigma} \otimes_D M$ is a projective $D_{\pi_2^\sigma}$-module, where $D_{\pi_2^\sigma} = S^{-1}_{\pi_2^\sigma} D$ with $S_{\pi_2^\sigma} = \{1, \pi_2^\sigma, (\pi_2^\sigma)^2, \dots\}$. By Theorem 1, $D_{\pi_2^\sigma} \otimes_D M$ is a free $D_{\pi_2^\sigma}$-module and we easily verify that a basis is given by $y_{\sigma(1)}$, where $y = (y_1\ y_2\ y_3)^t$ satisfies $R\, y = 0$ and $\sigma \in S_3$, because we have $y_{\sigma(i)} = (\chi_{\sigma(i)}/\chi_{\sigma(1)})\, y_{\sigma(1)}, \ i = 2, 3$.

Remark 3. Let us notice that Theorem 4 does not predict the minimal number of independent variables χ_i in the polynomial $\pi^\sigma_{n-i(M)}$. Indeed, in the previous example, we only need to invert $\pi_2^\sigma = \chi_{\sigma(1)}$ which contains just one independent variable, whereas, from Theorem 4, we only know that we have to invert a polynomial $\pi_2^\sigma \in k[\chi_{\sigma(1)}, \chi_{\sigma(2)}]$ in two variables. The next theorem gives a more precise statement on the minimal number of χ_i in $\pi^\sigma_{n-i(M)}$.

Lemma 1. *Let M be a finitely generated D-module and $N = T(M)$. Then, M is a projective D-module iff N is a projective D-module, i.e. $i(M) = +\infty \Leftrightarrow i(N) = +\infty$.*

Proof. We have the following exact sequence $0 \longleftarrow N \longleftarrow F_1^\star \xleftarrow{d_1^\star} F_0^\star \longleftarrow M^\star \longleftarrow 0$. If N is projective, then this exact sequence splits [7, 11] and we obtain that $M^\star$ is projective. Thus, $M^{\star\star}$ is still projective [11]. Moreover, we have $\mathrm{ext}^1_D(N, D) = 0 = \mathrm{ext}^2_D(N, D)$, because N is projective, thus, using the exact sequence (6), we obtain that $M \cong M^{\star\star}$ is projective. Changing N into M, we obtain the converse result, which proves the lemma. □

Theorem 5. *Let M be a finitely generated $D = k[\chi_1, \dots, \chi_n]$-module, $N = T(M)$ and:*

$$h(M) = i(M) + i(N) \in \{0, \dots, n, +\infty\}.$$

Then, for all $\sigma \in S_n$ and $l \geq 0$, there exists $\pi^\sigma_{n-h(M)} \in D^\sigma_{n-h(M)}$ such that we have (8), where $D^\sigma_{n-h(M)}$ is defined in (7). In particular, there exists $\pi^\sigma_{n-h(M)} \in D^\sigma_{n-h(M)}$ such that $D_{\pi^\sigma_{n-h(M)}} \otimes_D M$ is a projective $D_{\pi^\sigma_{n-h(M)}}$-module.

Proof. If M is projective, then the result is trivial. Let us suppose that M is not a projective D-module. Then, by Lemma 1, we have $0 \leq i(N) \leq n - 1$. The D-module M has a projective resolution of the form:

$$\dots \longrightarrow P_{i(N)+1} \xrightarrow{d_{i(N)+1}} P_{i(N)} \longrightarrow \dots \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0.$$

Using the fact that $i(N) = \min_{i \geq 1}\{\, i - 1 \mid \mathrm{ext}^i_D(M, D) \neq 0 \,\}$, we obtain by duality the following exact sequence

$$0 \longleftarrow N_{i(N)} \longleftarrow P^\star_{i(N)+1} \xleftarrow{d^\star_{i(N)+1}} \dots \xleftarrow{d_2^\star} P_2^\star \longleftarrow N \longleftarrow 0, \tag{10}$$

where $N_{i(N)} = \mathrm{coker}\, d^\star_{i(N)+1}$. Let us note $M_{i(N)} = \mathrm{coker}\, d_{i(N)+1}$. From (10), we deduce that:

$$\mathrm{ext}^{i(N)+l}_D(N_{i(N)}, D) \cong \mathrm{ext}^l_D(N, D), \ \forall\, l \geq 1$$

$$\Rightarrow i(M_{i(N)}) = i(M) + i(N) = h(M).$$

Applying Theorem 4 to $M_{i(N)}$, then $\forall\, l \geq 0$, there exists $\pi^\sigma_{n-h(M)} \in D^\sigma_{n-h(M)}$ such that:

$$i(D_{\pi^\sigma_{n-h(M)}} \otimes_D M_{i(N)}) \geq i(M_{i(N)}) + l = h(M) + l.$$

Thus, for $1 \leq m \leq i(M) + l$, we have:

$$\begin{aligned} &\mathrm{ext}^{i(N)+m}_{D_{\pi^\sigma_{n-h(M)}}}(D_{\pi^\sigma_{n-h(M)}} \otimes_D N_{i(N)}, D_{\pi^\sigma_{n-h(M)}}) = 0 \\ &\cong D_{\pi^\sigma_{n-h(M)}} \otimes_D \mathrm{ext}^{i(N)+m}_D(N_{i(M)}, D) \\ &\cong D_{\pi^\sigma_{n-h(M)}} \otimes_D \mathrm{ext}^m_D(N, D) \\ &\cong \mathrm{ext}^m_{D_{\pi^\sigma_{n-h(M)}}}(D_{\pi^\sigma_{n-h(M)}} \otimes_D N, D_{\pi^\sigma_{n-h(M)}}). \end{aligned}$$

Hence, we deduce that $i(D_{\pi^\sigma_{n-h(M)}} \otimes_D M) \geq i(M) + l$, which proves (8). □

Example 2. If we take again Example 1, we easily show that $i(M) = 1$ and $3 - h(M) = 3 - (1 + 1) = 1$. Thus, there exists $\pi_1^\sigma \in D_1^\sigma = k[\chi_{\sigma(1)}]$ such that $D_{\pi_1^\sigma} \otimes_D M$ is a projective $D_{\pi_1^\sigma}$-module. We have seen in Example 1 that $\pi_2^\sigma = \chi_{\sigma(1)}$. Theorem 5 predicts that there exists π_1^σ containing just one variable $\chi_{\sigma(1)}$, which gives an answer to Remark 3.

Example 3. If $M = D/(D\,\chi_1 + D\,\chi_2 + D\,\chi_3)$ is the $D = k[\chi_1, \chi_2, \chi_3]$-module defined by the gradient operator, then we easily check that $i(M) = 0$, $i(N) = 2$ and $3 - h(M) = 3 - 2 = 1$. Therefore, there exists $\pi_1^\sigma \in k[\chi_{\sigma(1)}]$ such that $D_{\pi_1^\sigma} \otimes_D M$ is a projective $D_{\pi_1^\sigma}$-module. We let the reader check by himself that we can choose $\pi_1^\sigma = \chi_{\sigma(1)}$ and $D_{\pi_1^\sigma} \otimes_D M = 0$.

Remark 4. If $n - h(M) = 1$, then, following the proof of Theorem 4, we obtain that the ideal $I_1^{\sigma\, i}$, defined by $I_1^{\sigma\, i} = \mathrm{ann}(\mathrm{ext}_D^i(N, D)) \cap k[\chi_{\sigma(1)}]$ is principal, for every $i \geq 1$ and $\sigma \in S_n$. Thus, up to a constant of k, there exists a unique lower degree polynomial $\pi_1^{\sigma\, i}$ such that $I_1^{\sigma\, i} = (\pi_1^{\sigma\, i})$ and $\pi_1^{\sigma} = \prod_{\{i \geq 1 \,\mid\, \pi_1^{\sigma\, i} \neq 0\}} \pi_1^{\sigma\, i}$. This is exactly the case for Examples 2 and 3.

3.2 Particular case

Lemma 2. *[5] If M is a D-module defined by the following finite presentation*

$$0 \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0, \qquad (11)$$

then M is projective iff $N = T(M) \cong \mathrm{ext}_D^1(M, D) = 0$, i.e. $i(M) = +\infty \Leftrightarrow N = 0$.

Theorem 6. *If M is a D-module defined by the exact sequence (11), then we have:*

$$h(M) = i(M) = j(N) - 1 = \mathrm{cd}(N) - 1, \qquad (12)$$

i.e.:

$$n - h(M) = \begin{cases} d(N) + 1, & N \neq 0, \\ -\infty, & N = 0. \end{cases}$$

Proof. If M is projective, then Lemma 1 shows that $i(M) = +\infty \Leftrightarrow i(N) = +\infty$, and thus, $h(M) = i(M)$. If M is not projective, Lemma 2 shows that $N \cong \mathrm{ext}_D^1(M, D) \neq 0$, i.e. $i(N) = 0$. This shows the first equality of (12). Moreover, M is defined by a full rank matrix, then $N = T(M)$ is a torsion D-module, and thus, $\mathrm{ext}_D^0(N, D) = 0$. Finally, we obtain:

$$i(M) = \min_{k \geq 0} \{\, k - 1 \mid \mathrm{ext}_D^k(N, D) \neq 0 \,\} = j(N) - 1.$$

By Theorem 3, we have $i(M) = \mathrm{cd}(N) - 1$, which proves the other equalities of (12). □

Example 4. Let M be the D-module defined by the matrix $R_1 = (\chi_1\ \chi_2\ \chi_3)$. We easily verify that

$$\begin{cases} \mathrm{ext}^i(N, D) = 0, \ \ 0 \leq i \leq 2, \\ \mathrm{ext}^3(N, D) = D/D^3 R_{-1} \neq 0, \\ \mathrm{ext}^i(N, D) = 0, \ \ i > 3, \end{cases}$$

where R_{-1} is defined in Example 1. Therefore, $j(N) = 3$, and, by Theorem 6, we obtain $3 - h(M) = 1$ and the existence of $\pi_1^\sigma \in k[\chi_{\sigma(1)}]$, with $\sigma \in S_3$, such that $M_{\pi_1^\sigma} = D_{\pi_1^\sigma} \otimes_D M$ is a projective $D_{\pi_1^\sigma} = S_{\pi_1^\sigma}^{-1} D$-module, with $S_{\pi_1^\sigma} = \{1, \pi_1^\sigma, (\pi_1^\sigma)^2, \ldots\}$ (we can take $\pi_1^\sigma = \chi_{\sigma(1)}$).

Corollary 1. *Let R be a full rank $q \times p$ matrix ($0 < q \leq p$) with entries in D, $M = D^p / D^q\, R$ and $N = T(M)$, then there exist $\pi_{n-h(M)}^\sigma \in D_{n-h(M)}^\sigma$, $R_{-1} \in D^{p \times (p-q)}$, $S \in D^{p \times q}$ and $S_{-1} \in D^{(p-q) \times p}$ and $\nu \in \mathbb{Z}_+$ such that we have the following extended Bézout identities for all $\sigma \in S_n$:*

1. $\begin{pmatrix} S & R_{-1} \end{pmatrix} \begin{pmatrix} R \\ S_{-1} \end{pmatrix} = (\pi_{n-h(M)}^\sigma)^\nu\, I_p,$

2. $\begin{pmatrix} R \\ S_{-1} \end{pmatrix} \begin{pmatrix} S & R_{-1} \end{pmatrix} = (\pi_{n-h(M)}^\sigma)^\nu \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix}.$

Proof. Applying Theorem 6 to M, then there exists $\pi_{n-h(M)}^\sigma$ such that $D_{\pi_{n-h(M)}^\sigma} \otimes_D M$ is a projective, and thus, free $D_{\pi_{n-h(M)}^\sigma}$-module by Theorem 1. Therefore, there exists an isomorphism $\phi : D_{\pi_{n-h(M)}^\sigma} \otimes_D M \longrightarrow D_{\pi_{n-h(M)}^\sigma}^{p-q}$. Using the fact that $D_{\pi_{n-h(M)}^\sigma}$ is a flat D-module [11], then we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow D_{\pi_{n-h(M)}^\sigma}^q & \xrightarrow{.R} & D_{\pi_{n-h(M)}^\sigma}^p & \xrightarrow{\mathrm{id} \otimes \pi} & D_{\pi_{n-h(M)}^\sigma} \otimes_D M \longrightarrow 0. \\ & & \| & & \downarrow \phi \\ & & D_{\pi_{n-h(M)}^\sigma}^p & \xrightarrow{\phi \circ (\mathrm{id} \otimes \pi)} & D_{\pi_{n-h(M)}^\sigma}^{p-q} \end{array}$$

Let us call $\overline{R}_{-1}$ the matrix corresponding to $\phi \circ (\mathrm{id} \otimes \pi)$ in the canonical basis of $D_{\pi_{n-h(M)}^\sigma}^p$ and $D_{\pi_{n-h(M)}^\sigma}^q$, then we obtain the following splitting exact sequence [7, 11]

$$0 \longrightarrow \quad D_{\pi_{n-h(M)}^\sigma}^q \quad \underset{\xleftarrow{.\overline{S}'}}{\xrightarrow{.R}} \quad D_{\pi_{n-h(M)}^\sigma}^p \quad \underset{\xleftarrow{.\overline{S}'_{-1}}}{\xrightarrow{.\overline{R}_{-1}}} \quad D_{\pi_{n-h(M)}^\sigma}^{p-q} \longrightarrow 0,$$

where $\overline{R}_{-1}$, $\overline{R}_{-1}$ and $\overline{S}'$ are matrices with entries in $D_{\pi_{n-h(M)}^\sigma}$. Chasing their denominators, we finally find the identities 1 and 2. Notice that $n - h(M)$ is given here by (12) □

Definition 6. A D-module has *pure dimension l* if M as well as any of its non-zero submodule have dimension l.

Theorem 7. *If M is a finitely generated $D = k[\chi_1, \ldots, \chi_n]$-module which satisfies $\mathrm{pd}_D(M) = i(N) + 1$, then:*

1. *$d(\mathrm{ext}_D^{i(N)+1}(M, D)) = n - h(M) - 1,$*

2. *$t(M) \cong \mathrm{ext}_D^{i(N)+1}(\mathrm{ext}_D^{i(N)+1}(M, D), D),$*

3. *if $t(M) \neq 0$, then $t(M)$ has pure dimension $n - i(N) - 1$.*

Proof. The fact that $\mathrm{pd}_D(M) = i(N) + 1$ means that there exists a projective resolution of M of the form:

$$0 \longrightarrow P_{i(N)+1} \xrightarrow{d_{i(N)+1}} P_{i(N)} \longrightarrow \ldots \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0.$$

By definition,

$$i(N) = \min_{i \geq 1} \{\, i - 1 \mid \mathrm{ext}_D^i(M, D) \neq 0 \},$$

which means that $\mathrm{ext}_D^i(M, D) = 0$ for $1 \leq i \leq i(N)$ and $\mathrm{ext}_D^{i(N)+1}(M, D) \neq 0$, i.e. we have the exact sequence:

$$0 \longleftarrow \mathrm{ext}_D^{i(N)+1}(M, D) \longleftarrow P_{i(N)+1}^\star \xleftarrow{d_{i(N)+1}^\star} \ldots \xleftarrow{d_2^\star} P_2^\star \longleftarrow N \longleftarrow 0.$$

1. We can apply Theorem 3 to the D-module $\mathrm{ext}_D^{i(N)+1}(M,D)$ to obtain:

$$\begin{aligned}\mathrm{cd}(\mathrm{ext}_D^{i(N)+1}(M,D)) &= j(\mathrm{ext}_D^{i(N)+1}(M,D)),\\ &= i(\mathrm{ext}_D^{i(N)+1}(M,D))+1,\\ &= i(N)+i(M)+1,\\ &= h(M)+1.\end{aligned}$$

2. We have $\mathrm{ext}_D^{i(N)+1}(N_{i(N)},D)\cong \mathrm{ext}_D^1(N,D)$ and, by Theorem 2, we have $t(M)\cong \mathrm{ext}_D^1(N,D)$, which shows that:

$$t(M)\cong \mathrm{ext}_D^{i(N)+1}(\mathrm{ext}_D^{i(N)+1}(M,D),D).$$

3. If $t(M)\neq 0$, then, by Theorem 7.10 of [1], we obtain that $t(M)$ has pure dimension $n-i(N)-1$. □

Example 5. In Example 3, we have seen that $M = D/(D\,\chi_1 + D\,\chi_2 + D\,\chi_3)$ satisfies that $i(N)=2$, and thus, $i(N)+1=3=\mathrm{pd}_D(M)$. Therefore, by Theorem 7, we obtain that $t(M)\cong \mathrm{ext}_D^3(\mathrm{ext}_D^3(M,D),D)$ has pure dimension 0, a fact that can be proved directly once noticing that $t(M)\cong M$.

Corollary 2. *If M is defined by the exact sequence (11) and $t(M)\neq 0$, then $t(M)$ has pure dimension $n-1$.*

Let us notice that Theorem 7 and Corollary 2 are also true if $D=K[d_1,\dots,d_n]$ where K is a differential field [7, 8].

4 Conclusion

Every results in this paper are effective by means of Gröbner basis: we can compute a finite free resolution of a finitely presented D-module and, by duality, $\mathrm{ext}_D^i(N,D)$ and $\mathrm{ext}_D^i(M,D)$ for $i\geq 1$ and determine $h(M)$. Moreover, the proves of Theorem 4 and 5 are totally constructive: we first compute $\mathrm{ext}_D^i(N,D)$ for $i\geq 1$ and their annihilators $\mathrm{ann}(\mathrm{ext}_D^i(N,D))$. Then, by means of techniques of elimination, we can determine explicitely $I_{n-h(M)}^{\sigma\, i} = \mathrm{ann}(\mathrm{ext}_D^i(N,D))\cap k[\chi_{\sigma(1)},\dots,\chi_{\sigma(n-h(M))}]$ to finally find $\pi_{n-h(M)}^{\sigma}$, $\forall\,\sigma\in S_n$. Moreover, extended Bézout identities, as well as generalized inverses obtained in [7], can be effectively obtained following the line of [7]. See also [12] for computational aspects.

By lack of space, we just give one application of the results obtained in this paper. In the case of differential delay systems, i.e. $D=k[\delta_1,\dots,\delta_{n-1},\frac{d}{dt}]$, Theorems 5 and 6 give an effective method to determine the polynomials π introduced in [4] to do motion planning. However, σ belongs to the subgroup S_{n-1} of permutations of the $n-1$ first variables of D. This remark and Theorem 5 show that a system satisfying $h(M)\geq 1$ is π-flat, where $\pi\in k[\delta_{\sigma(1)},\dots,\delta_{\sigma(n-h(M))}]$, $\sigma\in S_{n-1}$, and $n-h(M)=d(N)+1$ for a system defined by a full row rank matrix.

To finish, let us notice that Corollary 1 shows that R can be completed to a square matrix whose determinant divides a power of $\pi_{n-h(M)}^{\sigma}$ (if $n-h(M)=1$, then π_1^{σ} is the greatest common divisor of the $q\times q$ minors of R by Remark 4). See [3] for related questions.

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