INTERNAL STABILIZATION OF COHERENT CONTROL SYSTEMS

Alban Quadrat

School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom, email: quadrat@amsta.leeds.ac.uk.

Abstract: We give a necessary and sufficient condition so that a system is internally stabilizable and we prove that any system − defined by a transfer matrix with entries in the field of fractions $K = Q(A)$ of A − is internally stabilizable if and only if A is a Prüfer domain. Hence, if A is a Prüfer domain which is not a Bézout domain, then there exist some internal stabilizable plants which have no doubly coprime factorizations. Moreover, we show that if the ring A is a Hermite ring, then it is possible to parametrize all the stabilizing controllers of an internally stabilizable plant by means of the Youla parametrization. Finally, our approach of synthesis problems, based on *algebraic analysis* [13], allows to recover in a unique framework the different results obtained in [9,10,15,16].

Keywords: Fractional representation of synthesis problems, internal stabilization, Youla parametrization, doubly coprime factorizations, Prüfer domains.

1. INTERNAL STABILITY

Let A be an algebra of SISO stable plants which forms a commutative integral domain and $K = Q(A)$ its field of fractions. Let us consider the closed-loop formed by a plant $P \in K^{q\times(p-q)}$ and a controller $C \in K^{(p-q)\times q}$ given in Figure 1. The equations of the closed-loop are:

$$\begin{cases} e_1 = u_1 + P\,e_2, \\ e_2 = u_2 + C\,e_1, \\ y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{cases} \quad (1)$$

Definition 1.
- The closed-loop (1) is *well-posed* if the matrix $\begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}$ is invertible.
- The plant P is *internally stablizable* if there exists a controller C such that the closed-loop is well-posed and:

$$\begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} \in A^{p\times p}. \quad (2)$$

Let us write $P = D_p{}^{-1} N_p$ and $C = D_c{}^{-1} N_c$, where D_p, N_p, D_c, N_c are matrices with entries in A (e.g. $D_p = d_p\, I_q$ and $D_c = d_c\, I_{p-q}$, where d_p (resp. d_c) is the product of all denominators of the entries of P (resp. of C)). We have:

$$(1) \Leftrightarrow \begin{cases} D_p\, e_1 - N_p\, e_2 - D_p\, u_1 = 0, \\ -N_c\, e_1 + D_c\, e_2 - D_c\, u_2 = 0, \\ y_1 - e_2 + u_2 = 0, \\ y_2 - e_1 + u_1 = 0. \end{cases} \quad (3)$$

Let us define $R_p = (D_p \;\; -N_p)$, $R_c = (-N_c \;\; D_c)$,

$$R = \begin{pmatrix} D_p & -N_p & -D_p & 0 \\ -N_c & D_c & 0 & -D_c \end{pmatrix},$$

$$R_s = \begin{pmatrix} D_p & -N_p & -D_p & 0 & 0 & 0 \\ -N_c & D_c & 0 & -D_c & 0 & 0 \\ 0 & -I_{p-q} & 0 & I_{p-q} & I_{p-q} & 0 \\ -I_q & 0 & I_q & 0 & 0 & I_q \end{pmatrix},$$

and the A-modules:

$$M_p = A^p/A^q\, R_p, \quad M_c = A^p/A^{p-q},\, R_c,$$
$$M = A^{2p}/A^p\, R, \quad M_s = A^{3p}/A^{2p}\, R_s.$$

Lemma 1. [13] We have $M_s = M \cong M_p \oplus M_c$.

Proposition 1. Let $P = D_p^{-1}\, N_p$ be a plant, $C = D_c^{-1}\, N_c$ a controller and the A-modules $M_p = A^p/A^q\,(D_p \ \ -N_p)$ and $M_c = A^p/A^{p-q}\,(-N_c \ \ D_c)$. If the plant P is internally stabilizable by the controller C, then $M_p/t(M_p)$ and $M_c/t(M_c)$ are projective A-modules.

Proof. Let us note

$$T = \begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} = \begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} \begin{pmatrix} D_p & 0 \\ 0 & D_c \end{pmatrix}$$

$$= H\,(dI_p)^{-1},$$

where d is obtained by making all the entries of T under the same denominator after possible numerators/denominators simplifications. We have

$$\begin{pmatrix} R_p & -D_p & 0 \\ R_c & 0 & -D_c \end{pmatrix} d\, I_{2p} = \begin{pmatrix} R_p \\ R_c \end{pmatrix} \begin{pmatrix} dI_p & -H \end{pmatrix},$$

and the commutative diagram given in Figure 2 where $M_t = A^{2p}/A^p\,(dI_p \ \ -H)$ and $\phi : M \to M_t$ is defined by $\phi(m) = \pi'(u\, d)$, with $\pi(u) = m$. By the Snake lemma [2,14], we obtain the two following exact sequences:

$$0 \longrightarrow \ker\phi \longrightarrow M \stackrel{\phi}{\longrightarrow} M_t \longrightarrow \operatorname{coker}\phi \longrightarrow 0, \quad (4)$$

$$\begin{array}{l} 0 \longrightarrow \ker\phi \longrightarrow A^p/A^p(R_p^t \ \ R_c^t)^t \longrightarrow \\ \qquad A^{2p}/A^{2p}\, dI_{2p} \longrightarrow \operatorname{coker}\phi \longrightarrow 0. \end{array} \quad (5)$$

P is internally stabilizable if $d = 1$, i.e.:

$$\begin{cases} M_t \cong A^p, \\ A^{2p}/A^{2p}\, dI_{2p} = 0 \Rightarrow \operatorname{coker}\phi = 0, \\ \ker\phi \cong A^p/A^p\,(R_p^t \ \ R_c^t)^t. \end{cases}$$

Then, using (4), we obtain the exact sequence:

$$0 \longrightarrow A^p/A^p\,(R_p^t \ \ R_c^t)^t \longrightarrow M \stackrel{\phi}{\longrightarrow} M_t \longrightarrow 0. \quad (6)$$

The A-module $A^p/A^p\,(R_p^t \quad R_c^t)^t$ is a torsion module and $M_t \cong A^p$ is a torsion-free A-module. Thus, we have $t(M) \cong A^p/A^p\,(R_p^t \quad R_c^t)^t$ and $M/t(M) \cong M_t \cong A^p$. But, $t(M) \cong t(M_p) \oplus t(M_p)$, and thus, we have

$$M/t(M) \cong M_p/t(M_p) \oplus M_c/t(M_c) \cong A^p,$$

i.e. $M_p/t(M_p)$ is a projective A-module.

Theorem 1. The plant $P = D_p^{-1}\, N_p$ is internally stabilizable iff the A-module $M_p/t(M_p)$ is a projective A-module, where $M_p = A^p/A^q\, R_p$ with $R_p = (D_p \ \ -N_p)$.

Proof. $\Rightarrow$ It was proved in Proposition 1.

$\Leftarrow$ Let $M_p/t(M_p)$ be a projective A-module. We have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & t(M_p) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^q & \stackrel{.R_p}{\longrightarrow} & A^p & \stackrel{\pi}{\longrightarrow} & M_p & \longrightarrow & 0 \\
 & & \downarrow \kappa & & \| & & \downarrow p & & \\
0 & \longrightarrow & \ker\phi & \longrightarrow & A^p & \stackrel{\phi}{\longrightarrow} & M_p/t(M_p) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker}\kappa & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \quad (7)$$

where we have defined $\phi = p \circ \pi$ and $\kappa : A^q \longrightarrow \ker\phi$ is induced by id : $A^p \longrightarrow A^p$. Therefore, there exists an exact sequence of the form:

$$0 \longrightarrow \ker\phi \longrightarrow A^p \stackrel{\phi}{\longrightarrow} M_p/t(M_p) \longrightarrow 0. \quad (8)$$

The fact that $M_p/t(M_p)$ is a projective A-module implies that the exact sequence (8) splits [2,14], and thus, we have $A^p \cong M_p/t(M_p) \oplus \ker\phi$. Thus, $\ker\phi$ is also a projective A-module.

The fact that $\ker\phi$ is a projective A-module is equivalent to the existence of a family $\{a_1, \ldots, a_m\}$ of elements of A satisfying [2,14]:

(1) the ideal $(a_1, \ldots, a_m)$ is equal to A, i.e. $\exists\, x_i \in A : \sum_{i=1}^m x_i\, a_i = 1$,
(2) If $S_{a_i} = \{1, a_i, a_i^2, \ldots\}$ is the mutiplicative set defined by a_i, then $S_{a_i}^{-1}\, \ker\phi$ is a free $S_{a_i}^{-1}$ A-module.

$S_{a_i}^{-1}\, A$ is a flat A-module, and thus, we obtain the exact sequence of free $S_{a_i}^{-1}\, A$-modules:

$$\begin{array}{l} 0 \longrightarrow S_{a_i}^{-1}(\ker\phi) \longrightarrow (S_{a_i}^{-1}\, A)^p \\ \qquad \stackrel{S_{a_i}^{-1}\phi}{\longrightarrow} S_{a_i}^{-1}(M_p/t(M_p)) \longrightarrow 0. \end{array} \quad (9)$$

The fact that $t(M_p)$ is a torsion A-module implies that $\operatorname{rank}_A(t(M_p)) = 0$. Thus, we have $\operatorname{rank}_A(M_p/t(M_p)) = \operatorname{rank}_A(M_p)$ and:

$$\operatorname{rank}_A(\ker\phi) = p - \operatorname{rank}_A(M_p) = q. \quad (10)$$

If we note $S_{a_i}^{-1}\, A = A_i$, then $S_{a_i}^{-1}\, \ker\phi$ is a free A_i-module of rank q. Hence, taking a basis of $S_{a_i}^{-1}\, \ker\phi \cong A_i^q$, there exists a $q \times p$-matrix R_i with entries in A_i such that (9) becomes:

$$0 \longrightarrow A_i^q \stackrel{.R_i}{\longrightarrow} A_i^p \longrightarrow S_{a_i}^{-1}(M_p/t(M_p)) \longrightarrow 0.$$

By hypothesis, $M_p/t(M_p)$ is a projective A-module, and thus, $S_{a_i}^{-1}(M_p/t(M_p))$ is also a projective A_i-module [2,14]. Hence, the previous exact sequence splits [2,14], and thus, there exists a $p \times q$-matrix S_i with entries in A_i such that:

$$R_i\, S_i = I_q. \quad (11)$$

Let us note $R_p = (D_p \ \ -N_p)$, $R_i = (D_i \ \ -N_i)$. We first prove that we have $P = D_p^{-1}\, N_p = D_i^{-1}\, N_i$.

By localization of (7) by $S_{a_i}^{-1}$, we obtain the following commutative exact diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & S_{a_i}^{-1} t(M_p) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_i^q & \xrightarrow{.R_p} & A_i^p & \longrightarrow & S_{a_i}^{-1} M_p & \longrightarrow & 0 \\
 & & \downarrow .R_i' & & \| & & \downarrow & & \\
0 & \longrightarrow & A_i^q & \xrightarrow{.R_i} & A_i^p & \longrightarrow & S_{a_i}^{-1}(M_p/t(M_p)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_i^q/A_i^q\,R_i' & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Hence, we have $R_p = R_i'\,R_i$, i.e.

$$(D_p \quad -N_p) = R_i'\,(D_i \quad -N_i), \qquad (12)$$

where R_i' is a full rank $q \times q$ matrix and $S_{a_i}^{-1} t(M_p) \cong A_i^q/A_i^q\,R_i'$. Thus, we have:

$$P = D_p^{-1}\,N_p = (R_i'\,D_i)^{-1}(R_i'\,N_i) = D_i^{-1}\,N_i.$$

Chasing the denominators of each R_i and S_i, there exists $\alpha_i \in \mathbb{Z}_+$ such that all the entries of each matrix $a_i^{\alpha_i}\,S_i\,R_i$ are in A. If $\alpha = \max_{1 \leq i \leq m} \alpha_i$, then we have for $i = 1, \ldots, m$:

$$a_i^{\alpha}\,S_i\,R_i = a_i^{\alpha} \begin{pmatrix} X_i\,D_i & -X_i\,N_i \\ Y_i\,D_i & -Y_i\,N_i \end{pmatrix} \in A^{p \times p}. \quad (13)$$

Using the fact that $(a_1, \ldots, a_m) = A$, then there exists a family $\{b_1, \ldots, b_m\}$ of elements of A such that $\sum_{i=1}^m b_i\,a_i^{\alpha} = 1$. From (12), we obtain:

$$D_p = \sum_{i=1}^m b_i\,a_i^{\alpha}\,D_p = \sum_{i=1}^m b_i\,a_i^{\alpha}\,R_i'\,D_i, \qquad (14)$$

$$N_p = \sum_{i=1}^m b_i\,a_i^{\alpha}\,N_p = \sum_{i=1}^m b_i\,a_i^{\alpha}\,R_i'\,N_i. \qquad (15)$$

Let us define:

$$S = \sum_{i=1}^m b_i\;a_i^{\alpha}\;S_i\,D_i. \qquad (16)$$

If we note $S_i = (X_i^t \;\; Y_i^t)^t$, then we have:

$$S = \left(\left(\sum_{i=1}^m b_i\;a_i^{\alpha}\;X_i\,D_i \right)^t \quad \left(\sum_{i=1^m} b_i\;a_i^{\alpha}\;Y_i\,D_i \right)^t \right)^t .$$

We claim that the following controller

$$C = -\left(\sum_{i=1}^m b_i\;a_i^{\alpha}\;Y_i\,D_i \right) \left(\sum_{i=1}^m b_i\;a_i^{\alpha}\;X_i\,D_i \right)^{-1}$$

internally stabilizes the plant P, i.e. all the entries of the following matrix belong to A:

$$\begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - PC)^{-1} & (I_q - PC)^{-1}\,P \\ C(I_q - PC)^{-1} & I_{p-q} + C(I_q - PC)^{-1}\,P \end{pmatrix}.$$

We check the four identities given in Figure 3.

Remark 1. Similarly as Theorem 1, one can prove that $P = \tilde{N}_p\,\tilde{D}_p^{-1}$ is internally stabilizable by $C = \tilde{X}^{-1}\,\tilde{Y}$ iff $\tilde{M}_p = A^{p-q}/A^p\,(\tilde{N}_p^t \;\; \tilde{D}_p^t)^t$ is such that $\tilde{M}_p/t(\tilde{M}_p)$ is a projective A-module.

Corollary 1. [13] If $M_p = A^p/A^q\,(D_p \quad -N_p)$ is a torsion-free A-module, then $P = D_p^{-1}\,N_p$ is internally stabilizable iff M_p is a projective A-module. Moreover, a stabilizing controller has the form $C = -Y\,X^{-1}$, where $S = (X^t \;\; Y^t)^t$ is any right inverse of $R_p = (D_p \quad -N_p)$, i.e. any matrix $S \in A^{p \times q}$ satisfying $R_p\;S = I_q$.

Proposition 2. [13] Let $M = A^p/A^q\,R$ be an A-module defined by a matrix $R \in A^{q \times p}$. Then, $M/t(M)$ is a projective A-module iff $R\,A^q$ is a projective A-module.

Corollary 2. The plant $P = D^{-1}\,N$ is internally stabilizable iff $(D \quad -N)\,A^p$ is a projective A-module.

This corollary was first proved in [9,16].

Definition 2. [2,14] A domain A is a *Prüfer domain* if any finitely generated ideal is projective, or equivalently, if any finitely generated torsion-free A-module is projective. A *Dedekind domain* is a noetherian Prüfer domain.

Theorem 2. The propositions are equivalent:

(1) A is a Prüfer domain,
(2) any MIMO system defined by a transfer matrix T with entries in $K = Q(A)$ is internally stabilizable,
(3) any SISO system defined by a transfer function $T = n/d$, with $(n,\,d) \in A \times A \backslash 0$, is internally stabilizable.

Proof. $1 \Rightarrow 2$ Let $K = Q(A)$ and $T = D^{-1}\,N \in K^{q \times (p-q)}$ be the transfer matrix defining the plant (take for $D = d\,I_q$ the product of all the denominators of the entries of T). Therefore, any system is defined by a finitely presented A-module $M_p = A^p/A^q\,R$, where $R = (D \quad -N) \in A^{q \times p}$. If A is a Prüfer domain, then $M_p/t(M_p)$ is a torsion-free, i.e. projective, A-module, and by Theorem 1, the plant is internally stabilizable.

$2 \Rightarrow 3$ trivial.

$3 \Rightarrow 1$ Any system SISO defined by $T = n/d$ is internally stabilizable. If $R = (d \quad -n)$ is the full rank matrix with entries in A then, by Theorem 1, the A-module $M_p = A^2/A\,R$ satisfies that $M_p/t(M_p)$ is a projective A-module. The tranposed A-module $T(M_p) = A/A^2\,R$ [12] is defined by $0 \longleftarrow T(M_p) \longleftarrow A \xleftarrow{R.} A^2$, and we have the following exact sequence

$$0 \longleftarrow I \xleftarrow{R.} A^2 \longleftarrow M_p^\star \longleftarrow 0,$$

where $I = (n, d) = R\, A^2$ is the ideal of A defined by n and d. By Proposition 2, $M_p/t(M_p)$ is a projective A-module iff $I = (n, d)$ is a projective one. Hence, any ideal I generated by two elements n and d of A is a projective A-module, a fact which is equivalent to A is a Prüfer domain [6].

Example 1. For instance, the ring $A = \mathbb{Z}[i\sqrt{5}]$ used in [1] is a Dedekind domain [14], and thus, a Prüfer domain. Hence, any MIMO plant over $K = Q(A)$ is internally stabilizable. In particular, any SISO plant defined by a tranfer function over $K = Q(A)$ is internally stabilizable [10].

Example 2.
- The integral closure of $\mathbb{Z}$ into a finite extension of $\mathbb{Q}$ is a Dedekind domain. More generally, if A is a one-dimensional noetherian domain, K its field of fractions and L a finite algebraic extension field of K, then the integral closure of A in L is a Dedekind domain.
- Any non-singular algebraic surface defines a Dedekind affine domain. More generally, a one-dimensional noetherian normal domain is a Dedekind domain.
- The domain of entire functions $\mathbb{C} < s >$, i.e functions $f(s) = \sum_{n=0}^{+\infty} a_n s^n$, $a_n \in \mathbb{C}$ and $\lim_{n \to +\infty} |a_n|^{1/n} = 0$, is a Bézout domain [5], i.e. a Prüfer domain. The ring of *meromorphic bounded Nash functions* on a Nash submanifold of $\mathbb{R}^n$ is a Prüfer domain [7].

Lemma 2. Let $I = (n, d)$ be an ideal of a ring A with $d \neq 0$ and $I^{-1} = \{z \in K = Q(A) \mid z\, I \subset A\}$ its *fractional ideal* [14], then we have:

$$I\, I^{-1} = (d : n) + (n : d),$$

where $(a : b) = \{c \in A \mid c\, b \in (a)\}$ for $a, b \in A$.

Proof. Let us prove $(d : n) + (n : d) \subseteq I\, I^{-1}$. Let $a \in (d : n) = \{a \in A \mid \exists\, p \in A : a\, n = p\, d\}$, then we have:

$$\begin{cases} \left(\dfrac{a}{d}\right) n = p \in A, \\ \left(\dfrac{a}{d}\right) d = a \in A, \end{cases} \Rightarrow \left(\dfrac{a}{d}\right) \in I^{-1} \Rightarrow a \in I\, I^{-1}.$$

Similary, we can prove that $\left(\dfrac{b}{n}\right) \in I^{-1}$, and thus, $b \in I\, I^{-1}$, which proves the inclusion.

Let us prove $I\, I^{-1} \subseteq (d : n) + (n : d)$. Any element of $c \in I\, I^{-1}$ can be written as

$$c = \left(\sum_{i=1}^{n} a_i\, x_i\right) n + \left(\sum_{j=1}^{m} b_j\, x_j\right) d,$$

where $a_i, b_j \in A$ and $x_i \in K$ is such that $x_i\, n \in A$ and $x_i\, d \in A$. We have $d\,(\sum_{i=1}^n a_i\, x_i\, n) = (\sum_{i=1}^n a_i\, x_i\, d)\, n \in A\, n$ because $\sum_{i=1}^n a_i\, x_i\, d \in A$. Similarly, $n\,(\sum_{j=1}^m b_j\, x_j\, d) = (\sum_{j=1}^m b_j\, x_j\, n)\, d \in A\, d$, and thus, $c \in (d : n) + (n : d)$.

Proposition 3. If A is an algebra of SISO stable systems, then a SISO plant defined by a transfer function $p = n/d$, with $(n, d) \in A \times A \backslash 0$, is internally stabilizable iff:

$$I\, I^{-1} = (d : n) + (n : d) = A.$$

Proof. Following proof $3 \Rightarrow 1$ of Theorem 2 and using Proposition 2, we obtain that $p = n/d$ is internally stabilizable iff $I = (n, d)$ is a projective A-module. Now, using the fact that $I \neq 0$, it is easy to prove that I is a projective A-module iff I is an *invertible ideal*, i.e. $I\, I^{-1} = A$ [14]. Using Lemma 2, we finally obtain that $p = n/d$ is internally stabilizable iff $I\, I^{-1} = A$, i.e. iff $(d : n) + (n : d) = A$ by Lemma 2.

The condition $(d : n) + (n : d) = A$ of internal stabilization first appeared in [15].

2. PARAMETRIZATION OF ALL STABILIZING CONTROLLERS

Theorem 3. [13] If $M = A^p/A^q\, R$ is an A-module defined by a full row rank matrix $R \in A^{q \times p}$ $(0 \leq q \leq p)$, then M is a free A-module iff there exist matrices R_{-1}, S and S_{-1} with entries in A such that we have the splitting exact sequence:

$$0 \longrightarrow A^q \underset{\xleftarrow{.S}}{\xrightarrow{.R}} A^p \underset{\xleftarrow{.S_{-1}}}{\xrightarrow{.R_{-1}}} A^{p-q} \longrightarrow 0.$$

Moreover, we have the splitting exact sequence

$$0 \longrightarrow A^q \underset{\xleftarrow{.S(Q)}}{\xrightarrow{.R}} A^p \underset{\xleftarrow{.S_{-1}(Q)}}{\xrightarrow{.R_{-1}}} A^{p-q} \longrightarrow 0, \qquad (17)$$

with $S_{-1}(Q) = S_{-1} + Q\, R$, $\;S(Q) = S - R_{-1}\, Q$ and Q is any matrix which belongs to $A^{(p-q) \times q}$. (17) is equivalent to the following generalized Bézout identities:

$$(1)\ \ \begin{pmatrix} S(Q) & R_{-1} \end{pmatrix} \begin{pmatrix} R \\ S_{-1}(Q) \end{pmatrix} = I_p,$$

$$(2)\ \ \begin{pmatrix} R \\ S_{-1}(Q) \end{pmatrix} \begin{pmatrix} S(Q) & R_{-1} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix}.$$

The map

$$Q \in A^{(p-q) \times q} \rightarrow M_c(Q) = A^p/A^{p-q}\, S_{-1}(Q)$$

is the Youla parametrization of all the stabilizing controllers of $M = A^p/A^q\, R$.

Theorem 4. If A is a Hermite domain, then any internally stabilizable plant defined by a transfer matrix with entries in $K = Q(A)$ has doubly coprime factorizations and all the stabilizing controllers of a stabilizable plant are parametrized by the Youla parametrization.

Proof. From Theorem 1 and the exact sequence (8), we obtain that $M_p/t(M_p)$ and $\ker \phi$ are two projective A-modules. Using the fact that A is a Hermite ring, we obtain that $M_p/t(M_p)$ and $\ker \phi$ are two free A-modules. From (10), we obtain that $\ker \phi \cong A^q$, and thus, we have the exact sequence

$$0 \longrightarrow A^q \xrightarrow{.R''} A^p \longrightarrow M_p/t(M_p) \longrightarrow 0,$$

where R'' is a certain matrix of $A^{q \times p}$. Using (7), we obtain that there exists $R' \in A^{q \times q}$ such that $R = R' R''$, and thus, $P = D^{-1} N = (R' D'')^{-1} (R' N'') = (D'')^{-1} N''$. Therefore, by Theorem 3, the plant P has doubly coprime factorizations, and thus, all stabilizing controllers of P are parametrized by means of the Youla parametrization.

Proposition 4. [17] Any plant defined by a transfer matrix with entries in $K = Q(A)$ has doubly coprime factorizations iff A is a Bézout domain.

Hence, if A is a Prüfer domain which is not a Bézout domain, then there exist some internal stabilizable plants which have no doubly coprime factorizations, i.e. the stabilizing controllers of certain plants cannot be parametrized by means of the Youla parametrization. In particular, this is true for $A = \mathbb{Z}[i\sqrt{5}]$ [1,10,13].

3. CONCLUSION

We hope to have convinced the reader that algebraic analysis allowed to unify the different results on internal stabilization obtained in [9,15,16]. This approach gives a necessary and sufficient condition so that a plant is internally stabilizable. This condition, given in terms of modules, cannot be obtained into the usual matricial framework [4,17] and justifies the introduction of algebraic analysis in the fractional representation approach of synthesis problems. We proved that any plant defined by a transfer matrix with entries in the field of fractions $K = Q(A)$ was internally stabilizable iff A was a Prüfer domain. Vidyasagar proved in [17] that any plant defined by a transfer matrix with entries in the field of fractions $K = Q(A)$ had doubly coprime factorizations iff A was a Bézout domain. Hence, if A is a Prüfer domain which is not a Bézout domain, then there exist some internally stabilizable plants which fail to have doubly coprime factorizations. Finally, we proved that over a Hermite ring, any internally stabilizable system has doubly coprime factorizations.

4. REFERENCES

[1] Anantharam, V. (1985). "On stabilization and existence of coprime factorizations", *IEEE Trans. Automat. Contr.*, vol. 30, pp. 1030-1031.

[2] Bourbaki, N. (1980). *Algèbre Commutative*, Chap. X, Masson.

[3] Curtain, R. F. and Zwart, H. J. (1991). *An Introduction to Infinite-Dimensional Linear Systems Theory*, TAM 21, Springer-Verlag.

[4] Desoer, C. A., Liu, R.-W., Murray, J. and Saeks, R. (1980). "Feedback system design: the fractional representation approach to analysis and synthesis", *IEEE Trans. Automat. Contr.*, vol. 25, pp. 399-412.

[5] Helmer, O. (1940). "Divisibility properties of integral functions", *Duke Math. J.*, pp. 345-356.

[6] Jensen, C. U. (1966). "Arithmetical rings", *Acta Mathematica Academiae Scientiarum Hungaricae Tomus*, vol. 17, pp. 115-123.

[7] Kucharz, W. and Rusek, K. (1996). "On the ring of locally bounded Nash meromorphic functions", *Bull. Autral. Math. Soc.*, vol. 54, pp. 503-507.

[8] Loiseau, J. J. (2000). "Algebraic tools for the control and stabilization of time-delay systems", *IFAC Reviews*, Annual Reviews in Control, vol. 24, pp. 135-149.

[9] Mori, K. and Abe, K. (1999). "Feedback stabilization over commutative rings: further study of coordinate-free approach", to appear in *SIAM J. Contr. and Optim.*.

[10] Mori, K. (1999). "Feedback stabilization over commutative rings with no right-/left coprime factorizations", *CDC 99*, pp. 973-975.

[11] Pommaret, J. F. and Quadrat, A. (1999). "Algebraic analysis of linear multidimensional control systems", *IMA J. Math. Contr. Infor.*, vol. 16, pp. 275-297.

[12] Quadrat, A. (2001). "Coherent $H_\infty(D)$-modules in control theory", to appear in the proceedings of *First IFAC symposium on system structure and control, Prague 2001.*

[13] Quadrat, A. (2001). "A fractional representation approach of synthesis problems: an algebraic analysis point of view", presented for publication to *SIAM J. Contr. Optim.*.

[14] Rotman, J. J. (1979). *An Introduction to Homological Algebra*, Academic Press.

[15] Shankar, S. and Sule, V. R. (1992). "Algebraic geometric aspects of feedback stabilization", *SIAM J. Contr. Optim.*, vol. 30, pp. 11-30.

[16] Sule, V. R. (1994). "Feedback stabilization over commutative rings: the matrix case", *SIAM J. Contr. Optim.*, vol. 32, pp. 1675-1695, and (1998). "Corrigendum", vol. 36, pp. 2194-2195.

[17] Vidyasagar, M. (1985). *Control System Synthesis*, MIT Press.

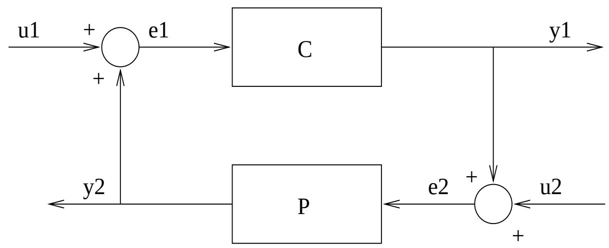

Fig. 1. Closed loop

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & 0 & & & & \\
 & & & \cdot\begin{pmatrix} R_p & -R_p' & 0 \\ R_c & 0 & -R_c' \end{pmatrix} & \downarrow & & & & \\
0 & \longrightarrow & A^p & \xrightarrow{\hspace{6em}} & A^{2p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & \cdot\begin{pmatrix} R_p \\ R_c \end{pmatrix} & \downarrow & & .\mathrm{d}\,\mathrm{I}_{2\mathrm{p}}\downarrow & & \phi\downarrow & & \\
0 & \longrightarrow & A^p & \xrightarrow{\cdot\left(\,d\,I_p \;\; -N\,\right)} & A^{2p} & \xrightarrow{\pi'} & M_t & \longrightarrow & 0,
\end{array}
$$

Fig. 2. Commutative exact diagram

$$
\begin{aligned}
I_q - PC &= I_q - D_p^{-1}\,N_p\left(\sum_{i=1}^m b_i\;a_i^\alpha\;Y_i\,D_i\right)\left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right)^{-1} \\
&= D_p^{-1}\left[D_p\left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right) - N_p\left(\sum_{i=1}^m b_i\;a_i^\alpha\;Y_i\,D_i\right)\right]\left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right)^{-1} \\
&= D_p^{-1}\left[\sum_{i=1}^m b_i\;a_i^\alpha\;(D_p\,X_i - N_p\,Y_i)\,D_i\right]\left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right)^{-1} \\
&= D_p^{-1}\left[\sum_{i=1}^m b_i\;a_i^\alpha\;R_i'\,(D_i\,X_i - N_i\,Y_i)\,D_i\right]\left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right)^{-1} \quad \text{(by (12))} \\
&= D_p^{-1}\left[\sum_{i=1}^m b_i\;a_i^\alpha\;R_i'\,D_i\right]\left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right)^{-1} \quad \text{(by (11))} \\
&= D_p^{-1}\,D_p\left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right)^{-1} \quad \text{(by (14))} \\
&= \left(\sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i\right)^{-1} \\
\Rightarrow (I_q - PC)^{-1} &= \sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,D_i \in A^{q\times q}, \\
\Rightarrow C(I_q - PC)^{-1} &= -\sum_{i=1}^m b_i\;a_i^\alpha\;Y_i\,D_i \in A^{(p-q)\times q}, \\
\Rightarrow (I_q - PC)^{-1}P &= \sum_{i=1}^m b_i\;a_i^\alpha\;X_i\,N_i \in A^{q\times(p-q)}, \\
\Rightarrow I_{p-q} + C(I - PC)^{-1}P &= I_{p-q} + \sum_{i=1}^m b_i\;a_i^\alpha\;Y_i\,N_i \in A^{(p-q)\times(p-q)}.
\end{aligned}
$$

Fig. 3. Four blocks of T