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A differential operator approach to multidimensional optimal control

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We use recent improvements in the parametrizations of controllable linear multidimensional systems to show how to transform the study of a linear quadratic optimal problem into that of a variational problem without constraints. We give formal conditions on the differential module defined by the linear control system to pass from the Pontryagin approach to a purely Euler–Lagrange variational problem. This formal approach uses the cost function in order to link the locally exact sequence formed by the controllable system and its parametrizations with the sequence formed by their formal adjoint operators. In the case of partial differential equations, this scheme is typical for any problem of linear elasticity theory and electromagnetism.

1. Introduction

In this paper, we show how recent results on the parametrization of linear multidimensional control systems can be used to find new results on optimal control and variational calculus. In particular, we are interested in knowing how the structural properties of a linear multidimensional control system, described within the framework of module theory, can be useful in order to reduce a constrained variational problem to a free/unconstrained one.

We first recall the fact that a controllable linear system, in the sense that the system determines a torsion-free differential module (Pommaret and Quadrat 1999 a, Wood 2000, Zerz 2000, Pommaret 2001), is parametrizable and we show how to find effectively its parametrizations (Pommaret and Quadrat 1998, 1999 a). The problem investigated in this paper is to optimize a functional under the constraint given by a linear multidimensional control system. We prove that if the control system defines a torsion-free module and if the differential sequence formed by the system and one of its parametrizations is locally exact, then, by substitution, we are led to a simple variational problem without constraints. In particular, if the control system defines a projective differential module (e.g. controllable ordinary differential systems), one can always reduce our problem to this case. Moreover, if the system is defined by a surjective differential operator (i.e. by differentially independent equations) and the differential module associated with the system is projective, then the Lagrange multipliers can always be obtained explicitly without any integration. Many examples illustrate the formal approach developed in this paper and, in particular, we show how it can be used in order to study linear quadratic problems such as the ones arising in linear elasticity theory and electromagnetism.

Finally, we hope to convince the reader that these algebraic and geometric methods, developed for linear control theory (Pommaret and Quadrat 1998, 1999 a, b Pommaret 2001) and using, as main ingredients, the formal adjoints of differential operators, differential sequences and differential module theory, are in fact closely related to some physics principles such as the duality existing between geometry and physics in the sense of H. Poincaré.

2. Mathematical tools

Let us first briefly recall some results about the formal theory of differential operators (Spencer 1965, Pommaret 2001) and its dual approach in terms of differential modules. See Palamodov (1970), Bjork (1979), Pham (1980), Maisonobe and Sabbah (1993) and Pommaret and Quadrat (1999 b) for more details.

Let E and F be trivial vector bundles over a differential manifold X of dimension n with local coordinates $x = (x^1, \ldots, x^n)$ (Pommaret 2001). In what follows, we shall take $\mathbb{R}^n$ for X or open subsets. Let

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{D}} & F \\ (x, \xi^k(x)) & \mapsto & (x, \eta^\tau(x) = \displaystyle\sum_{0 \le |\mu| \le q,\, 1 \le k \le m} a_k^{\tau\mu}(x)\, \partial_\mu \xi^k, \\ & & 1 \le \tau \le l) \end{array} \tag{1}$$

be a *differential operator* from E to F, where the fibered dimension of E (resp. F) is equal to m (resp. equal to l), $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index of length $|\mu| = \mu_1 + \cdots + \mu_n$ and we adopt the notation $\partial_\mu = \partial_1^{\mu_1}, \ldots, \partial_n^{\mu_n}$. If we denote by Θ the kernel of the differential operator $\mathcal{D}$, then we have the following exact sequence

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F. \tag{2}$$

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Now, we associate with any differential operator $\mathcal{D}$ an algebraic object, namely a differential module M, in the following way (see Pommaret and Quadrat 1999 b, Pommaret 2001 for more details). For that, when K is a differential field containing $\mathbb{Q}$ with commuting derivations $\partial_1, \ldots, \partial_n$ (Ritt 1950, Kolchin 1973), let us introduce the ring $D = K[d_1, \ldots, d_n]$ of differential operators, i.e. the ring of elements of the form $P = \sum_{0 \le |\mu| < \infty} a^\mu \, d_\mu$, where the coefficients a^μ belong to K and where the derivatives d_i satisfy

$$d_i(a\, d_j) = a\, d_i\, d_j + \partial_i a\, d_j.$$

We associate with (1) the D-homomorphism $.\mathcal{D}$ defined as

$$\begin{array}{l} D^l \xrightarrow{.\mathcal{D}} D^m \\ (P_\tau) \mapsto \Big(\sum_{0 \le |\mu| \le q,\, 1 \le \tau \le l} P_\tau\, a_k^{\tau\mu}\, d_\mu,\ 1 \le k \le m \Big) \end{array} \qquad (3)$$

i.e. we let operate $\mathcal{D}$ on the right of a row vector of D^l to obtain a row vector of D^m. Now, we associate with (2) the *finitely presented* left D-module M defined by the exact sequence

$$D \otimes_K F^\star \xrightarrow{.\mathcal{D}} D \otimes_K E^\star \longrightarrow M \longrightarrow 0$$

where $E^\star$ and $F^\star$ denote the dual vector bundles, or simply, because the vector bundles are trivial

$$D^l \xrightarrow{.\mathcal{D}} D^m \longrightarrow M \longrightarrow 0 \qquad (4)$$

i.e. $M = D^m / D^l\, \mathcal{D}$. See Pham (1980), Maisonobe and Sabbah (1993), Pommaret and Quadrat (1999 b) and Pommaret (2001) for more details.

When $\mathcal{D}\colon \xi \mapsto \eta$ is a sufficiently regular differential operator (Pommaret 2001), the *compatibility conditions* of the inhomogeneous system

$$\mathcal{D}\xi = \eta \qquad (5)$$

are defined by a differential operator $\mathcal{D}_1\colon F \to F_1$. In other words, all the necessary conditions on η in order to have the local existence of ξ satisfying (5) are generated by $\mathcal{D}_1 \eta = 0$. If $\mathcal{D}_1$ denotes the compatibility conditions of the differential operator $\mathcal{D}$, then we have the *formally exact sequence* (Spencer 1965, Pommaret 2001)

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F \xrightarrow{\mathcal{D}_1} F_1. \qquad (6)$$

In the differential module language, the formally exact sequence (6) means that we have constructed the beginning of a *free resolution* (see, e.g. Rotman 1979) of the left D-module M associated with $\mathcal{D}$, i.e. we have the following exact sequence

$$D^{l_1} \xrightarrow{.\mathcal{D}_1} D^l \xrightarrow{.\mathcal{D}} D^m \longrightarrow M \longrightarrow 0. \qquad (7)$$

We can repeat the same procedure with $\mathcal{D}_1$ instead of $\mathcal{D}$ and we obtain a long *formally exact sequence* of compatibility conditions, in the operator language (Pommaret 2001), or a *free resolution* of left D-module M, in the algebraic one (Rotman 1979).

An historical problem was to construct effectively the operator $\mathcal{D}_1$ and it was investigated by C. Riquier and E. Cartan at the beginning of the century (Riquier 1910, Cartan 1945) but received a nice improvement with M. Janet's work in the 1920s (Janet 1929) and a final achievement with the work of D. C. Spencer in the 1970s. From the works of D. C. Spencer (Spencer 1965), we know that the operator $\mathcal{D}_1$ can be constructed by bringing the operator $\mathcal{D}$ to *involutiveness* (Spencer 1965, Pommaret 2001) and we can find a *projective resolution* of M (see, e.g. Definition 1 and Rotman 1979) of length equal to n, where n is the number of derivations ∂_i or, equivalently, the number of derivatives d_i in D. However, the Spencer resolution is generally very difficult to compute, and thus, it is much easier to compute the *Janet sequence* (see Pommaret (2001) for more details)

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_{n-1}} F_{n-1} \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0$$

giving rise to a free resolution of M of length equal to $n+1$

$$0 \longrightarrow D^{l_n} \xrightarrow{.\mathcal{D}_n} D^{l_{n-1}} \xrightarrow{.\mathcal{D}_{n-1}} \cdots \xrightarrow{.\mathcal{D}_2} D^{l_1} \xrightarrow{.\mathcal{D}_1} D^{l_0} \xrightarrow{.\mathcal{D}_0} D^m \longrightarrow M \longrightarrow 0 \qquad (8)$$

obtained by replacing $\mathcal{D}$ by an *involutive* operator $\mathcal{D}_0\colon E \to F_0$ with the same kernel Θ (Pommaret 2001) and where the $\mathcal{D}_i$ are involutive first-order operators. In this case, we know that the last operator $\mathcal{D}_n\colon F_{n-1} \to F_n$ defines a *projective* D-module (see, e.g. Definition 1 and Rotman 1979).

Applying the functor $\hom_D(\cdot, D)$ to (8) (Rotman 1979), we obtain the *dual sequence*

$$0 \longleftarrow D^{l_n} \xleftarrow{\mathcal{D}_n.} D^{l_{n-1}} \xleftarrow{\mathcal{D}_{n-1}.} \cdots \xleftarrow{\mathcal{D}_2.} D^{l_1} \xleftarrow{\mathcal{D}_1.} D^{l_0} \xleftarrow{\mathcal{D}_0.} D^m \longleftarrow \hom_D(M, D) \longleftarrow 0 \qquad (9)$$

where $\mathcal{D}_i.$ means that we make $\mathcal{D}_i$ operate on the left of a column vector of $D^{l_{i-1}}$ to obtain a column vector of D^{l_i}. The *defect of cohomology* at D^{l_i} is denoted by

$$\mathrm{ext}_D^i(M, D) = \ker(\mathcal{D}_{i+1}.)/\mathrm{im}(\mathcal{D}_i.).$$

The defects of cohomology $\mathrm{ext}_D^i(M, D)$ do only depend on M and not on its resolution (8), that is, if we have two different resolutions of the same left D-module M, then we obtain the same defect of cohomology (up to an isomorphism) from the two different dual sequences (see Rotman (1979)).

Now, we have to note that, using the fact that D is both a left and right D-module, we can endow the abelian group $\hom_D(M, D)$ of D-morphisms

(D-linear maps) from M to D with the structure of a right D-module by

$$\forall a \in D, \quad \forall \phi \in \hom_D(M, D) : (\phi a)(m) = \phi(m)\, a, \quad \forall m \in M.$$

The cokernel of $\mathcal{D}. : D^m \to D^l$ is the right D-module N_r defined by

$$0 \longleftarrow N_r \longleftarrow D^l \xleftarrow{\mathcal{D}.} D^m \longleftarrow \hom_D(M, D) \longleftarrow 0. \qquad (10)$$

It can be shown that N_r only depends on M up to a *projective equivalence* (Pommaret and Quadrat 200 b). If we want to give an interpretation of the *extension functor* derived from the functor $\hom_D(\cdot, D)$ in terms of differential operators, we have to use the concept of *formal adjoint* (Maisonobe and Sabbah 1993, Pommaret and Quadrat 1998, Pommaret 2001): if $T^\star$ denotes the cotangent bundle of X and $\mathcal{D}\colon E \to F$ is a differential operator, then its formal adjoint is the operator $\widetilde{\mathcal{D}} : \widetilde{F} = \bigwedge^n T^\star \otimes F^\star \to \widetilde{E} = \bigwedge^n T^\star \otimes E^\star$, defined by using the three following formal rules equivalent to integration by parts:

- the adjoint of a matrix with entries in K (zero-order operator) is the transposed matrix,
- the adjoint of ∂_i is $-\partial_i$,
- for two linear partial differential operators P, Q that can be composed, then: $\widetilde{P \circ Q} = \widetilde{Q} \circ \widetilde{P}$.

Moreover, we have the relation

$$\langle \lambda, \mathcal{D}_1\, \eta \rangle - \langle \widetilde{\mathcal{D}}_1\, \lambda, \eta \rangle = d(\cdot)$$

expressing a difference of n-forms and where d is the standard exterior derivative.

In homological algebra language, if we denote by $T = D_1/K$, where

$$D_1 = \left\{ P \in D \mid \operatorname{ord}(P) = \sup \left\{ |\mu| \mid P = \sum_{0 \le |\nu|} a^\nu d_\nu,\ a^\mu \neq 0 \right\} \le 1 \right\},$$

then $\bigwedge^n T \otimes_K \cdot$ is called the *side changing functor* (Pham 1980, Bjork 1993, Maisonobe and Sabbah 1993, Pommaret 2001) and it allows to pass from a right D-module N_r to a left D-module. Thus, the left D-module $N = \bigwedge^n T \otimes_K N_r$ is the module defined by the formal adjoint $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ and we have the following exact sequence:

$$0 \longleftarrow N \longleftarrow \bigwedge^n T \otimes_K D^l \xleftarrow{\widetilde{\mathcal{D}}} \bigwedge^n T \otimes_K D^m. \qquad (11)$$

Now, let us start with an involutive differential operator $\mathcal{D}_0\colon E \to F_0$ and let us denote by M the D-module associated with $\mathcal{D}_0$. We give an algorithm to check whether or not $\operatorname{ext}^i_D(M, D)$ is equal to zero (Pommaret and Quadrat 1999 b, Pommaret 2001).

Algorithm 1: *Computation of* $\operatorname{ext}^i_D(M, D)$

Step 1. Start with $\mathcal{D}_0$.

Step 2. Find the sequence of the compatibility conditions operators $\mathcal{D}_r$ up to $\mathcal{D}_i$.

Step 3. Construct the adjoint sequence formed by the differential operators $\widetilde{\mathcal{D}}_i$ and $\widetilde{\mathcal{D}}_{i-1}$.

Step 4. Find the generating compatibility conditions $\widetilde{\mathcal{D}}'_{i-1}$ of $\widetilde{\mathcal{D}}_i$.

Step 5. Check whether or not $\widetilde{\mathcal{D}}_{i-1}$ generates all the compatibility conditions $\widetilde{\mathcal{D}}'_{i-1}$ of $\widetilde{\mathcal{D}}_i$. If yes, then $\operatorname{ext}^i_D(M, D) = 0$ else $\operatorname{ext}^i_D(M, D)$ is defined by all the compatibility conditions which are in $\widetilde{\mathcal{D}}'_{i-1}$ and not in $\widetilde{\mathcal{D}}_{i-1}$.

We can represent the above algorithm by the diagram

$$\begin{array}{llllll}
1 & E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} \cdots & \cdots \xrightarrow{\mathcal{D}_{i-2}} F_{i-2} & \xrightarrow{\mathcal{D}_{i-1}} F_{i-1} & \xrightarrow{\mathcal{D}_i} F_i & 2 \\
 & & 3 \quad \widetilde{F}_{i-2} & \xleftarrow{\widetilde{\mathcal{D}}_{i-1}} \widetilde{F}_{i-1} & \xleftarrow{\widetilde{\mathcal{D}}_i} \widetilde{F}_i & \\
 & & 4 \quad \widetilde{F}'_{i-2} & \xleftarrow{\widetilde{\mathcal{D}}'_{i-1}} & &
\end{array}$$

where the number indicates the step of the algorithm.

Example 1: Let us denote by $\partial_{ij} = \partial_i\, \partial_j$ and let $\mathcal{D}\colon \xi \mapsto \eta$ be defined by

$$\left.\begin{aligned} \partial_{12}\xi &= \eta^1 \\ \partial_{22}\xi &= \eta^2 \end{aligned}\right\}$$

$D = \mathbb{R}[d_1, d_2]$ and let $M = D/(D\, d_1 d_2 + D\, d_2^2)$ be the D-module associated with $\mathcal{D}$. Let us check whether or not $\operatorname{ext}^1_D(M, D)$ is equal to zero. We first find that the compatibility conditions $\mathcal{D}_1 : \eta \mapsto \zeta$ of $\mathcal{D}\xi = \eta$ are defined by $\partial_1 \eta^2 - \partial_2 \eta^1 = \zeta$. Its formal adjoint $\widetilde{\mathcal{D}}_1 : \lambda \mapsto \mu$, obtained by multiplying $\mathcal{D}_1$ on the left by a test function λ and integrating by parts, is defined by

$$\left.\begin{aligned} \partial_2 \lambda &= \mu_1 \\ -\partial_1 \lambda &= \mu_2. \end{aligned}\right\}$$

The compatibility conditions $\widetilde{\mathcal{D}}' : \mu \mapsto \nu$ of $\widetilde{\mathcal{D}}_1$ are generated by $\partial_1 \mu_1 + \partial_2 \mu_2 = \nu'$, whereas the formal adjoint $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is defined by $\partial_{12}\mu_1 + \partial_{22}\mu_2 = \nu$. Thus, $\widetilde{\mathcal{D}}$ does not generate all the compatibility conditions of $\widetilde{\mathcal{D}}_1$ as we have the relation $\partial_2 \nu' = \nu$. Finally, we obtain

$$\operatorname{ext}^1_D(M, D) = D\,(d_1 : d_2)/D\,(d_1\, d_2 : d_2^2) \neq 0.$$

From algebraic analysis (Palamodov 1970, Kashiwara 1995, Pommaret and Quadrat 1999 b, Pommaret 2001), we have the following first main theorem.

Theorem 1: *We can embed the left D-module M into the following exact sequence*

$$0 \longrightarrow M \longrightarrow D^{l_{-1}} \xrightarrow{.\mathcal{D}_{-2}} D^{l_{-2}} \xrightarrow{.\mathcal{D}_{-3}} \cdots \xrightarrow{.\mathcal{D}_{-r+1}} D^{l_{-r+1}} \xrightarrow{.\mathcal{D}_{-r}} D^{l_{-r}} \quad (12)$$

if and only if $\mathrm{ext}^i_D(N, D) = 0,\ i = 1, \ldots, r$, *where N is the left D-module associated with* $\widetilde{\mathcal{D}}$ *and which corresponds to the right D-module* N_r *defined by* (10).

Equivalently, within the differential operators framework, we have the following formally exact sequence

$$E_{-r} \xrightarrow{\mathcal{D}_{-r}} E_{-r+1} \xrightarrow{\mathcal{D}_{-r+1}} \cdots \xrightarrow{\mathcal{D}_{-2}} E_{-1} \xrightarrow{\mathcal{D}_{-1}} E_0 \xrightarrow{\mathcal{D}} F,$$

where $E_0 = E$ *and each differential operator generates all the compatibility conditions of the preceding one, if and only if* $\mathrm{ext}^i_D(N, D) = 0,\ i = 1, \ldots, r$.

Then, we say that the differential operator $\mathcal{D}$ *(resp.* $\mathcal{D}_{-i}$*) is parametrized by the differential operator* $\mathcal{D}_{-1}$ *(resp.* $\mathcal{D}_{-i-1}$*).*

Let us recall a few definitions coming from module theory (Rotman 1979).

Definition 1:

- A finitely generated D-module M is *free* if it is isomorphic to copies of D.
- A finitely generated D-module M is *projective* if there exist a free D-module F and a D-module N such as $F = M \oplus N$. Then, N is also a projective D-module.
- A finitely generated D-module M is *reflexive* if $M \cong \hom_D(\hom_D(M, D), D)$.
- A finitely generated D-module M is *torsion-free* if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D\colon\ P\, m = 0\} = 0.$$

We call $t(M)$ the *torsion submodule* of M and $m \in t(M)$ a *torsion element* of M.

We have the following standard results.

Theorem 2:

- *We have the following inclusions of D-modules*

  free $\subseteq$ *projective* $\subseteq$ *reflexive* $\subseteq$ *torsion* $-$ *free*. (13)

- *If D is a principal ideal domain (e.g.* $D = K[\mathrm{d}/\mathrm{d}t]$*), then every torsion-free D-module is free.*
- *If* $k = \{a \in K \mid \partial_i a = 0,\ i = 1, \ldots, n\}$ *is the field of constants of K, then every projective* $D = k[d_1, \ldots, d_n]$*-module is free.*

We refer to Rotman (1979) for the proofs. The third point is the famous non-trivial Quillen–Suslin theorem.

Now, we can state the second main theorem coming from algebraic analysis (Palamodov 1970, Kashiwara 1995, Pommaret and Quadrat 1999 b, Pommaret 2001).

Theorem 3: *Let M be a finitely presented left D-module associated with the differential operator* $\mathcal{D}$ *and N be the left D-module associated with* $\widetilde{\mathcal{D}}$*. We have the following propositions:*

- *M is a torsion-free left D-module* $\Leftrightarrow \mathrm{ext}^1_D(N, D) = 0$.
- *M is a reflexive left D-module* $\Leftrightarrow \mathrm{ext}^i_D(N, D) = 0$, $i = 1, 2$.
- *M is a projective left D-module* $\Leftrightarrow \mathrm{ext}^i_D(N, D) = 0$, $i = 1, \ldots, n$.

Let us note that, if $n = 1$, then it directly follows from Theorem 3 that a finitely presented torsion-free left D-module is projective.

If $D = k[d_1, \ldots, d_n]$ is a commutative polynomial ring over a field of constants k, then the concepts of torsion-freeness and projectiveness are the intrinsic formulations of the concepts of *minor left coprimeness* and *zero minor coprimeness* used in multidimensional systems theory (Youla and Gnavi 1979, Youla and Pickel 1984, Wood *et al.* 1998, Zerz 2000) for matrices with maximal generic rank. See Oberst (1990) and Pommaret and Quadrat (1999b) for more details.

Example 2: We let the reader check that the sequence of compatibility conditions of the gradient operator $\mathcal{D}\colon \xi \mapsto \eta$ in $\mathbb{R}^3$, defined by $\vec{\nabla}\xi = \eta$, is formed respectively by the curl operator $\mathcal{D}_1\colon \eta \mapsto \zeta$, defined by $\vec{\nabla} \wedge \eta = \zeta$, and the divergence operator $\mathcal{D}_2\colon \zeta \mapsto \theta$, defined by $\vec{\nabla}.\, \zeta = \theta$. Moreover, we can easily verify that the formal adjoint of the gradient (resp. the curl, the divergence) operator is minus the divergence (resp. the curl, minus the gradient) operator.

Now, let us start with the divergence operator and let us denote by $M = D^3/D(d_1\colon\ d_2\colon\ d_3)$ the $D = \mathbb{R}[d_1, d_2, d_3]$-module associated with $\mathcal{D}$. Then, we easily check that $\mathrm{ext}^1_D(N, D) = 0$ because the divergence operator is parametrized by the curl operator, namely, a generating set of compatibility conditions of $\vec{\nabla} \wedge \eta = \zeta$ is $\vec{\nabla}.\, \zeta = 0$. Moreover, we have $\mathrm{ext}^2_D(N, D) = 0$ because the curl operator is parametrized by the gradient operator, namely, a generating set of compatibility conditions of $\vec{\nabla}\xi = \eta$ is $\vec{\nabla} \wedge \eta = 0$. Finally, $\mathrm{ext}^3_D(N, D) = D/D^3(d_1\colon\ d_2\colon\ d_3)^{\mathrm{T}} \neq 0$ because the gradient operator is not formally injective, i.e. $\vec{\nabla}\xi = 0 \not\Rightarrow \xi = 0$. Hence, using Theorem 3, we obtain that the D-module M is reflexive but not projective, and thus, not free.

Similarly, we can prove that the D-module defined by the curl operator is only torsion-free and the gradient operator defines a torsion D-module.

Example 3: Let us consider the differential operator $\mathcal{D}\colon \xi \mapsto \eta$ defined by

$$\partial_1\xi^1 + \partial_2\xi^2 - x^2\xi^1 = \eta.$$

Let $D = \mathbb{R}(x^1, x^2)[d_1, d_2]$ and $M = D^2/D(d_1 - x^2 : d_2)$ be the left D-module defined by $\mathcal{D}$. Let us study the algebraic properties of M. First of all, we have to notice that $\mathcal{D}$ is *formally surjective*, i.e. $\mathcal{D}$ has no compatibility condition. The differential operator $\widetilde{\mathcal{D}} : \mu \mapsto \nu$ is defined by

$$\left.\begin{aligned} -\partial_1\mu - x^2\mu &= \nu_1 \\ -\partial_2\mu &= \nu_2. \end{aligned}\right\} \tag{14}$$

Then, from (14), we easily check that we have $\mu = \partial_1\nu_2 - \partial_2\nu_1 + x^2\nu_2$ which implies that $\widetilde{\mathcal{D}}$ is an injective operator. If we define the operator $\widetilde{\mathcal{P}} : \nu \mapsto \mu$ by $\partial_1\nu_2 - \partial_2\nu_1 + x^2\nu_2 = \mu$, then $\widetilde{\mathcal{P}} \circ \widetilde{\mathcal{D}} = id_{\widetilde{F}}$, i.e. $\widetilde{\mathcal{P}}$ is a *left-inverse* of $\widetilde{\mathcal{D}}$. Then, the $\bigwedge^2 T \otimes_K D$-morphism $.\widetilde{\mathcal{D}} : \bigwedge^2 T \otimes_K D^2 \to \bigwedge^2 T \otimes_K D$ is surjective because, for all $a \in \bigwedge^2 T \otimes_K D$, we can define $b = a\widetilde{\mathcal{P}}$ and we easily check that $a = b\widetilde{\mathcal{D}}$. Hence, the left D-module N, defined by (11), satisfies $N = \operatorname{coker}(.\widetilde{\mathcal{D}}) = 0$, which implies that $\operatorname{ext}^1_D(N, D) = 0$, $i = 1, 2$, and thus, by Theorem 3, M is a projective left D-module. Dualizing the operator $\widetilde{\mathcal{P}}$, we obtain a *right-inverse* $\mathcal{P}$ of $\mathcal{D}$, i.e. $\mathcal{D} \circ \mathcal{P} = id_F$. We refer the reader to Pommaret and Quadrat (1998) and Quadrat (1999) for the applications of left and right-inverses. Substituting in (14) the expression of μ in terms of ν_1 and ν_2, we obtain the operator $\widetilde{\mathcal{D}}_{-1} : \nu \mapsto \pi$ defined by

$$\left.\begin{aligned} &\partial_{11}\nu_2 - \partial_{12}\nu_1 + 2x^2\,\partial_1\nu_2 - x^2\,\partial_2\nu_1 + (x^2)^2\,\nu_2 + \nu_1 = \gamma_1 \\ &\partial_{12}\nu_2 - \partial_{22}\nu_1 + x^2\,\partial_2\,\nu_2 + 2\,\nu_2 = \gamma_2. \end{aligned}\right\}$$

Dualizing $\widetilde{\mathcal{D}}_{-1}$, we obtain the following differential operator $\mathcal{D}_{-1} : \theta \mapsto \xi$ defined by

$$\left.\begin{aligned} &-\partial_{22}\theta^2 - \partial_{12}\theta^1 + x^2\,\partial_2\theta^1 + 2\,\theta^1 = \xi^1 \\ &\partial_{12}\theta^2 + \partial_{11}\theta^1 - x^2\,\partial_2\theta^2 - 2x^2\,\partial_1\theta^1 + (x^2)^2\,\theta^1 + \theta^2 = \xi^2. \end{aligned}\right\}$$

We let the reader check by himself that the compatibility conditions of $\mathcal{D}_{-1}\,\theta = \xi$ exactly generate by $\mathcal{D}\xi = 0$. Hence, $\mathcal{D}$ is parametrized by $\mathcal{D}_{-1}$ in agreement with the fact that any projective module is torsion-free (see Theorem 2). Finally, let us point out that checking whether or not the left D-module M is free is a difficult problem. Indeed, the Quillen–Suslin theorem stated in the last point of Theorem 2 is no longer valid in this case as M is a left module defined over the non-commutative ring $D = \mathbb{R}(x^1, x^2)[d_1, d_2]$.

3. Controllablity of linear multidimensional systems

In this section, we also briefly recall how the results of the preceding section can be used for analysing the structural properties of linear multidimensional systems. We refer the reader to Pommaret and Quadrat (1999 a,b) and Pommaret (2001) for more details.

In agreement with the concept of controllability used in multidimensional control theory, we have the following definition (Pommaret and Quadrat 1999 a, Wood 2000, Zerz 2000, Pommaret 2000).

Definition 2: A linear control system, defined by the differential operator $\mathcal{D}_1 : F_0 \to F_1$, is *controllable* if the left D-module $M = D^{l_0}/D^{l_1}\,\mathcal{D}_1$ associated with $\mathcal{D}_1$ is torsion-free, i.e. $t(M) = 0$.

By Theorem 3, a linear control system, defined by the operator $\mathcal{D}_1$, is controllable iff we have $\operatorname{ext}^1_D(N, D) = 0$, where $N = (\bigwedge^n T \otimes_K D^{l_1})/\ (\bigwedge^n T \otimes_K D^{l_0})\,\widetilde{\mathcal{D}}_1$ is the left D-module associated with $\widetilde{\mathcal{D}}_1$. In the case where the system is controllable, using Theorem 1, we know that $\mathcal{D}_1$ can be parametrized by a differential operator $\mathcal{D}_0$, i.e. $\mathcal{D}_1$ represents exactly all the compatibility conditions of $\mathcal{D}_0$. If we want to check whether or not a system is controllable, compute effectively the differential operator $\mathcal{D}_0$ or the *autonomous elements* of the system (i.e. the torsion elements of M), we have to proceed in the following way.

Algorithm 2: *Controllability test*

Step 1. Start with $\mathcal{D}_1$.

Step 2. Construct its formal adjoint $\widetilde{\mathcal{D}}_1$.

Step 3. Find the compatibility conditions of $\widetilde{\mathcal{D}}_1\,\lambda = \mu$ and denote this differential operator by $\widetilde{\mathcal{D}}_0$.

Step 4. Construct its formal adjoint $\mathcal{D}_0\ (=\widetilde{\widetilde{\mathcal{D}}}_0)$.

Step 5. Find the compatibility conditions of $\mathcal{D}_0\,\xi = \eta$ and denote this differential operator $\mathcal{D}_1'$.

This leads to two different cases:

- If $\mathcal{D}_1$ generates exactly the compatibility conditions $\mathcal{D}_1'$ of $\mathcal{D}_0$, then the linear system defined by $\mathcal{D}_1$ determines a torsion-free left D-module M and $\mathcal{D}_0$ is a parametrization of $\mathcal{D}_1$.
- Otherwise, the operator $\mathcal{D}_1$ is among, but not exactly, the compatibility conditions $\mathcal{D}_1'$ of $\mathcal{D}_0$. Then, the torsion elements of M are all the new compatibility conditions modulo the equations $\mathcal{D}_1\,\eta = 0$.

We refer the reader to Chyzak *et al.* (2003, 2004) for more details concerning effective computations of the extension functors $\operatorname{ext}^i_D(\cdot, D)$. Algorithms 1 and 2 have recently been implemented in the symbolic package *OreModules* using non-commutative Gröbner bases. See Chyzak *et al.* (2003, 2004) for more details and examples.

Remark 1: For a full row rank matrix $R \in D^{l\times m}$ with entries in a commutative polynomial ring $D = \mathbb{R}[\chi_1, \ldots, \chi_n]$, it is well-known that the D-module $M = D^m/D^l\,R$ is torsion-free iff 1 is the greatest common divisor on all the $l \times l$-minors of R (Oberst 1990, Wood *et al.* 1998, Pommaret and Quadrat 1999 b, Zerz 2000). However, Algorithm 2 can be used for more general systems (variable coefficients case, non-full row rank matrices). Moreover, if the left D-module M is torsion-free, Algorithm 2 effectively computes a parametrization of the system and, if the module is not torsion-free, it gives a family of generators of the torsion submodule $t(M)$. Finally, let us point out that the concept of parametrization developed in this paper generalizes the concept of *controller form* (Kailath 1980) to linear multidimensional systems defined by non-surjective differential operators (Pommaret and Quadrat 1999 a).

Example 4: In Example 3, we saw that, up to a change of notations, the system defined by the differential operator $\mathcal{D}_1 : \eta \mapsto \zeta$ as

$$\partial_1 \eta^1 + \partial_2 \eta^2 - x^2\, \eta^1 = \zeta$$

determines a projective left D-module, and thus, the linear multidimensional system

$$\partial_1 \eta^1 + \partial_2 \eta^2 - x^2\, \eta^1 = 0$$

is controllable. Moreover, we found a non-trivial parametrization $\mathcal{D}_0 : \xi \mapsto \zeta$ of $\mathcal{D}_1$, defined by

$$\left.\begin{array}{l} -\partial_{22}\xi^2 - \partial_{12}\xi^1 + x^2\,\partial_2\xi^1 + 2\,\xi^1 = \eta^1 \\ \partial_{12}\xi^2 + \partial_{11}\xi^1 - x^2\,\partial_2\xi^2 - 2\,x^2\,\partial_1\xi^1 + (x^2)^2\,\xi^1 + \xi^2 = \eta^2. \end{array}\right\}$$

Let us point out that there exist some parametrizations of $\mathcal{D}_1$ which have the minimal number of arbitrary parameters (also called *potentials*) ξ. These parametrizations are called *minimal parametrizations* (Pommaret and Quadrat 1999 b). We refer to (Pommaret and Quadrat (1999 b) for an effective algorithm which computes them (see also Chyzak *et al.* 2004). For instance, we have the following two minimal parametrizations of $\mathcal{D}_1$

$$\left\{\begin{array}{l} -\partial_{22}\,\xi = \eta^1 \\ \partial_{12}\,\xi - x^2\,\partial_2\,\xi + \xi = \eta^2 \end{array}\right. \qquad \left\{\begin{array}{l} -\partial_{12}\,\xi + x^2\,\partial_2\,\xi + 2\,\xi = \eta^1 \\ \partial_{11}\,\xi - 2\,x^2\,\partial_1\,\xi + (x^2)^2\,\xi = \eta^2. \end{array}\right.$$

We have the following corollary of Theorem 3.

Corollary 1: *A linear ordinary differential control system defined by a surjective differential operator $\mathcal{D}_1$, i.e. $\mathcal{D}_1$ has no compatibility condition, is controllable iff its formal adjoint $\widetilde{\mathcal{D}}_1$ is an injective differential operator, namely $\widetilde{\mathcal{D}}_1\lambda = 0 \Rightarrow \lambda = 0$.*

Proof: Let M be the left $D = K[\mathrm{d}/\mathrm{d}t]$-module defined by the surjective differential operator $\mathcal{D}_1$. Then, the left D-module N is defined by

$$0 \longleftarrow N \longleftarrow T \otimes_K D^{l_1} \overset{.\widetilde{\mathcal{D}}_1}{\longleftarrow} T \otimes_K D^{l_0}.$$

If $\widetilde{\mathcal{D}}_1$ is an injective differential operator, then there exists an operator $\widetilde{\mathcal{P}}_1 : \widetilde{F}_0 \to \widetilde{F}_1$ such that $\widetilde{\mathcal{P}}_1 \circ \widetilde{\mathcal{D}}_1 = id_{\widetilde{F}_1}$. This implies that the D-morphism $.\widetilde{\mathcal{D}}_1 : T \otimes_K D^{l_0} \to T \otimes_K D^{l_1}$ is surjective. Indeed, for all $a \in T \otimes_K D^{l_1}$, if we define $b = a\widetilde{\mathcal{P}}_1 \in T \otimes_K D^{l_0}$, then we have $a = b\widetilde{\mathcal{D}}_1$. Hence, we have $N = \operatorname{coker}(.\widetilde{\mathcal{D}}_1) = 0$, and thus, $\operatorname{ext}^1_D(N, D) = 0$, which implies that M is a torsion-free left D-module by Theorem 3 and the control system defined by $\mathcal{D}_1$ is controllable by Definition 2.

Conversely, let us suppose that M is a torsion-free left D-module. Since D is a principal left ideal domain, by Theorem 2, M is a projective left D-module. Thus, the following exact sequence

$$0 \longrightarrow D^{l_1} \overset{.\mathcal{D}_1}{\longrightarrow} D^{l_0} \longrightarrow M \longrightarrow 0$$

splits (see Rotman 1979 for more details), i.e., there exists a D-morphism $.\mathcal{P}_1 : D^{l_0} \to D^{l_1}$ such that $(.\mathcal{P}_1) \circ (.\mathcal{D}_1) = .id_{D^{l_1}}$, that is to say, $.(\mathcal{D}_1 \circ \mathcal{P}_1) = .id_{D^{l_1}}$, and thus, we have the following matrix equality $\mathcal{D}_1 \circ \mathcal{P}_1 = id_{D^{l_1}}$. By duality, we obtain that $\widetilde{\mathcal{P}}_1 \circ \widetilde{\mathcal{D}}_1 = id_{T \otimes_K D^{l_1}}$, which shows that $\widetilde{\mathcal{D}}_1$ is an injective differential operator with a left-inverse $\widetilde{\mathcal{P}}_1$. $\square$

Example 5: Let us consider the Kalman system $-\dot{x} + A(t)\,x + B(t)\,u = 0$, where A is a square $n \times n$ matrix and B is $n \times m$. By Corollary 1, the surjective ordinary differential operator $\mathcal{D}_1 : \eta \mapsto \zeta$, defined by $-\dot{\eta}^1 + A(t)\,\eta^1 + B(t)\,\eta^2 = \zeta$, defines a controllable system iff the formal adjoint operator $\widetilde{\mathcal{D}}_1 : \lambda \mapsto \mu$, defined by

$$\left.\begin{array}{l} \dot{\lambda} + \lambda\, A(t) = \mu_1 \\ \lambda\, B(t) = \mu_2 \end{array}\right\}$$

is injective. Differentiating the zero-order equation and using the first one, we obtain that

$$\lambda\,(AB - \dot{B}) = 0 \Rightarrow \lambda\,(A^2B - \dot{A}B - 2A\dot{B} + \ddot{B}) = 0 \ldots .$$

Therefore, the differential operator $\widetilde{\mathcal{D}}_1$ is injective iff the rank over K of the controllability matrix

$$(B\!:\; AB - \dot{B} : \ldots : A^{n-1}B + \cdots : \cdots)$$

is equal to n. Of course, we can proceed similarly if A and B do not depend on time, and we recover the classical Kalman test (Pommaret and Quadrat 1999 b).

4. Linear quadratic optimal problems

After having recalled in §§ 2 and 3 some results about differential operators, algebraic analysis and controllability of linear multidimensional systems, let us study the main problem of this paper, namely the linear quadratic optimal problem for multidimensional control systems.

In what follows, we shall use the following jet notation $\eta_q = (\eta_\mu, 0 \leq |\mu| \leq q)$. For example, if we take

$X = \mathbb{R}$, i.e. in the ordinary differential case, we have $\eta_q = (\eta, \dot{\eta}, \ddot{\eta}, \ldots, \eta^{(q)})$.

Let us consider the differential operator $\mathcal{D}_1 : \eta \mapsto \zeta$ of order q and the following Lagrangian function

$$L(\eta_q) = \tfrac{1}{2}\, \eta_q^{\mathrm{T}}\, R\, \eta_q,$$

where R is a symmetric matrix ($R_{k,l}^{\alpha,\beta} = R_{l,k}^{\beta,\alpha}$) with entries in K and $\eta = (\eta^k, 1 \leq k \leq m)^{\mathrm{T}}$. Let us consider the problem of minimizing

$$\int L(\eta_q)\,\mathrm{d}x \tag{15}$$

where $\mathrm{d}x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, under the *differential constraint*

$$\mathcal{D}_1\, \eta = 0. \tag{16}$$

The variation of the Lagrangian function is given by $\delta L(\eta_q) = \sum_{0 \leq |\alpha| \leq q,\, 1 \leq k \leq m} \pi_k^\alpha\, \delta\, \eta_\alpha^k$, where

$$\pi_k^\alpha = \frac{\partial L(\eta_q)}{\partial\, \eta_\alpha^k} = \sum_{1 \leq l \leq m,\ |\beta| \leq q} R_{k,l}^{\alpha,\beta}\, \eta_\beta^l$$

and $\delta\, \eta_\alpha^k$ denotes the variation of η_α^k. Let us define the differential operator $\mathcal{B} \colon \eta \mapsto \mu$ by

$$\mathcal{B}\,\eta = \sum_{0 \leq |\alpha| \leq q} (-1)^{|\alpha|}\, d_\alpha \pi_k^\alpha = \mu.$$

For any section η of F_0, $\mathcal{B}\,\eta$ belongs to $\widetilde{F}_0 = \bigwedge^n T^\star \otimes F_0^\star$ and we have the diagram

$$\begin{array}{l} F_0 \xrightarrow{\mathcal{D}_1} F_1 \\ \downarrow \mathcal{B} \\ \widetilde{F}_0. \end{array}$$

Proposition 1: *The operator* $\mathcal{B} \colon F_0 \to \widetilde{F}_0$ *is a self-adjoint operator, namely* $\widetilde{\mathcal{B}} = \mathcal{B}$.

Proof: If we multiply $\mathcal{B}\,\eta$ on the left by a vector $\theta \in F_0$ and integrate by parts, we obtain the following result once using implicit summation on the dumb indices

$$\begin{aligned} \langle \mathcal{B}\,\eta, \theta \rangle &= (-1)^{|\alpha|}\, (d_\alpha \pi_k^\alpha)\, \theta^k \\ &= \pi_k^\alpha\, d_\alpha\, \theta^k + d(\cdot) \\ &= R_{k,l}^{\alpha,\beta}\, \eta_\beta^l\, d_\alpha \theta^k + d(\cdot) \\ &= R_{k,l}^{\alpha,\beta} \theta_\alpha^k\, \eta_\beta^l + d(\cdot) \\ &= ((-1)^{|\beta|}\, d_\beta\, \pi_l^\beta)\, \eta^l + d(\cdot) \\ &= \langle \eta, \mathcal{B}\,\theta \rangle + d(\cdot). \end{aligned}$$

Finally, the result holds once we note that we have $\widetilde{\widetilde{F}}_0 = F_0$. □

Definition 3 (Pommaret 2001)**:** The sequence of differential operators $F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} F_2$ is said to be locally exact at F_1 if, for every $x \in X$ and every section ζ of F_1 in an open neighbourhood U of x satisfying $\mathcal{D}_2 \zeta = 0$, there exists an open neighbourhood V of x with $V \subset U$ and a section η of F_0 on V such that $\mathcal{D}_1 \eta = \zeta$ on V.

Let us give the formal optimal system which satisfies the linear quadratic optimal problem defined by (15) and (16).

Theorem 4 (Quadrat 1999)**:** *The optimal system of the variational problem* (15) *and* (16) *is defined by*

$$\left.\begin{aligned} \mathcal{D}_1\, \eta &= 0 \\ \mathcal{B}\,\eta - \widetilde{\mathcal{D}}_1\, \lambda &= 0 \end{aligned}\right\} \tag{17}$$

where λ *denotes the Lagrange multipliers.*

Moreover, if the compatibility conditions of $\widetilde{\mathcal{D}}_1$ *are written by means of the differential operator* $\widetilde{\mathcal{D}}_0$ *and if the sequence*

$$\widetilde{E} \xleftarrow{\widetilde{\mathcal{D}}_0} \widetilde{F}_0 \xleftarrow{\widetilde{\mathcal{D}}_1} \widetilde{F}_1 \tag{18}$$

is locally exact at $\widetilde{F}_0$, *then* (17) *is equivalent to the following system*

$$\left.\begin{aligned} \mathcal{D}_1\, \eta &= 0 \\ (\widetilde{\mathcal{D}}_0 \circ \mathcal{B})\, \eta &= 0. \end{aligned}\right\} \tag{19}$$

Moreover, we have the locally exact diagram

$$\begin{array}{ccccc} & & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\ & \swarrow & \downarrow \mathcal{B} & & \\ \widetilde{E} & \xleftarrow{\widetilde{\mathcal{D}}_0} & \widetilde{F}_0 & \xleftarrow{\widetilde{\mathcal{D}}_1} & \widetilde{F}_1. \end{array}$$

Proof: If we denote by $\lambda(x)$ the Lagrange multipliers, then we have

$$\delta \int (L(\eta_q) - \lambda\, \mathcal{D}_1 \eta)\,\mathrm{d}x = \int (\mathcal{B}\,\eta - \widetilde{\mathcal{D}}_1 \lambda)\, \delta\, \eta\ \mathrm{d}x + \cdots$$

where $\delta\,\eta$ denotes the variation of η. Accordingly, a necessary condition of optimality is given by

$$\mathcal{B}\,\eta - \widetilde{\mathcal{D}}_1 \lambda = 0 \tag{20}$$

and we obtain (17).

Let us suppose that (18) is a locally exact sequence at $\widetilde{F}_0$. Let us prove that (17) and (19) are equivalent. Let us suppose that we have (17). Then, eliminating the Lagrange multipliers by composing (20) on the left by $\widetilde{\mathcal{D}}_0$, we obtain $(\widetilde{\mathcal{D}}_0 \circ \mathcal{B})\, \eta = 0$, i.e. (19). Now, let us suppose that we have (19). By hypothesis, the sequence (18) is locally exact at $\widetilde{F}_0$, and thus, for every $x \in X$ and every section $\mathcal{B}\,\eta$ of $\widetilde{F}_0$ on an open neighbourhood U of x, there exist an open neighbourhood V of x with $V \subset U$ and a section λ of $\widetilde{F}_1$ on V such that we have $\widetilde{\mathcal{D}}_1\, \lambda = \mathcal{B}\,\eta$ on V. This proves the second part of the theorem. □

Let us illustrate Theorem 4 with the following example.

Example 6: Let us minimize $\int \frac{1}{2}((\eta^1)^2 + (\eta^2)^2)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ under the differential constraint

$$\partial_1 \eta^1 + \partial_2 \eta^2 - x^2 \eta^1 = 0.$$

The operator $\mathcal{B}\colon F_0 \to \widetilde{F}_0$ is defined by $\mathcal{B}\eta = \eta$ and the adjoint $\widetilde{\mathcal{D}}_1$ of $\mathcal{D}_1$ is (see Example 3)

$$\left.\begin{aligned} -\partial_1 \lambda - x^2 \lambda &= \mu_1 \\ -\partial_2 \lambda &= \mu_2. \end{aligned}\right\}$$

Therefore, the optimal system is defined by the equations

$$\left.\begin{aligned} &\partial_1 \eta^1 + \partial_2 \eta^2 - x^2 \eta^1 = 0 \\ &\eta^1 + \partial_1 \lambda + x^2 \lambda = 0 \\ &\eta^2 + \partial_2 \lambda = 0. \end{aligned}\right\}$$

In Example 3, we saw that the differential operator $\mathcal{D}_1 : \eta \mapsto \partial_1 \eta^1 + \partial_2 \eta^2 - x^2 \eta^1 = \zeta$ defined a projective left D-module. Thus, the following sequence $\widetilde{E} \xleftarrow{\widetilde{\mathcal{D}}_0} \widetilde{F}_0 \xleftarrow{\widetilde{\mathcal{D}}_1} \widetilde{F}_1$ is locally exact at $\widetilde{F}_0$ because this formally exact sequence splits (Rotman 1979). Hence, the optimal system is equivalently given by

$$\left.\begin{aligned} &\partial_1 \eta^1 + \partial_2 \eta^2 - x^2 \eta^1 = 0 \\ &\partial_{11} \eta^2 - \partial_{12} \eta^1 + 2x^2 \partial_1 \eta^2 - x^2 \partial_2 \eta^1 + (x^2)^2 \eta^2 + \eta^1 = 0 \\ &\partial_{12} \eta^2 - \partial_{22} \eta^1 + x^2 \partial_2 \eta^2 + 2\eta^2 = 0. \end{aligned}\right\}$$

We let the reader check by himself that the above system is not *formally integrable* (see Pommaret 2001 for more details) and that its solution space depends on two arbitrary functions of one variable.

Let us give the first main result of this paper.

Theorem 5 (Quadrat 1999)**:** *If the linear control system defined by the differential operator $\mathcal{D}_1$ is controllable, that is parametrizable by $\mathcal{D}_0 : E \to F_0$ in such a way that the sequences $\widetilde{E} \xleftarrow{\widetilde{\mathcal{D}}_0} \widetilde{F}_0 \xleftarrow{\widetilde{\mathcal{D}}_1} \widetilde{F}_1$ and $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ are locally exact at $\widetilde{F}_0$ and F_0, then* (17) *is equivalent to*

$$\left.\begin{aligned} \mathcal{A}\xi &= 0 \\ \eta &= \mathcal{D}_0 \xi \end{aligned}\right\} \tag{21}$$

where $\mathcal{A}$ is the differential operator defined by

$$\mathcal{A} = \widetilde{\mathcal{D}}_0 \circ \mathcal{B} \circ \mathcal{D}_0. \tag{22}$$

Moreover, we have the commutative locally exact diagram

$$\begin{array}{ccccc} E & \xrightarrow{\mathcal{D}_0} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\ \mathcal{A}\downarrow & & \downarrow \mathcal{B} & & \\ \widetilde{E} & \xleftarrow{\widetilde{\mathcal{D}}_0} & \widetilde{F}_0 & \xleftarrow{\widetilde{\mathcal{D}}_1} & \widetilde{F}_1. \end{array}$$

Proof: Using the fact that (18) is locally exact at $\widetilde{F}_0$, by Theorem 4, we know that (17) and (19) are equivalent. Therefore, let us prove the equivalence between (19) and (21).

Let us suppose that (19) is satisfied. Using the fact that $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ is a locally exact sequence at F_0, then there exists a local section ξ of E which locally satisfies $\mathcal{D}_1 \eta = 0 \Leftrightarrow \mathcal{D}_0 \xi = \eta$. Moreover, we have $(\widetilde{\mathcal{D}}_0 \circ \mathcal{B})\eta = (\widetilde{\mathcal{D}}_0 \circ \mathcal{B})(\mathcal{D}_0 \xi) = \mathcal{A}\xi = 0$, which proves (21).

Now, let us suppose that we have (21). Therefore, we have $\mathcal{D}_1 \eta = \mathcal{D}_1(\mathcal{D}_0 \xi) = 0$ because $\mathcal{D}_1$ generates the compatibility conditions of $\mathcal{D}_0$. Finally, we have $\mathcal{A}\xi = (\widetilde{\mathcal{D}}_0 \circ \mathcal{B})(\mathcal{D}_0 \xi) = (\widetilde{\mathcal{D}}_0 \circ \mathcal{B})\eta$, which proves (19). □

Remark 2: Theorem 5 gives the possibility to transform a variational problem for η with a differential constraint $\mathcal{D}_1 \eta = 0$ into a variational problem for ξ without any differential constraint. Let us note that a similar result was independently obtained in Pillai and Willems (2002) within the behavioural framework for differential operators with constant coefficients (in this case, the local exactness of the differential sequences defined in Theorem 5 hold for C^∞ or $\mathcal{D}'$-sections over open convex subsets of X. See Oberst (1990) for more details).

Corollary 2 (Quadrat 1999)**:** *If the linear control system defined by the differential operator $\mathcal{D}_1$ defines a projective left D-module $M = D^{l_0}/D^{l_1}\mathcal{D}_1$, then the optimal system* (17) *is equivalent to* (21)*, where $\mathcal{A}$ is defined by* (22)*. In particular, this result holds if the system is a controllable linear ordinary differential system.*

Proof: If $\mathcal{D}_1$ defines a projective left D-module $M = D^{l_0}/D^{l_1}\mathcal{D}_1$, then the differential operator $\mathcal{D}_1$ can be parametrized by a differential operator $\mathcal{D}_0$ such that the following two sequences $\widetilde{E} \xleftarrow{\widetilde{\mathcal{D}}_0} \widetilde{F}_0 \xleftarrow{\widetilde{\mathcal{D}}_1} \widetilde{F}_1$ and $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ split, and thus, are locally exact at $\widetilde{F}_0$ and F_0 respectively. Then, the first part of the result directly follows from Theorem 5.

Now, if the system is a controllable linear ordinary differential system, then $D = K[\mathrm{d}/\mathrm{d}t]$ is a principal left ideal domain (see Theorem 2) and, by Definition 2, the left D-module $M = D^{l_0}/D^{l_1}\mathcal{D}_1$ is torsion-free. Thus, by Theorem 2, the left D-module M is projective and the result follows from the previous point.

Remark 3: Corollary 2 explains why the concept of controllability plays a central role (though not really emphasized) in the study of optimal control problems for ordinary differential control systems. For partial differential systems, let us point out that controllability is generally a necessary but not a sufficient condition in order to have the local exact sequences $\widetilde{E} \xleftarrow{\widetilde{\mathcal{D}}_0} \widetilde{F}_0 \xleftarrow{\widetilde{\mathcal{D}}_1} \widetilde{F}_1$ and $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$. However, the fact that the left

D-module M is projective is generally a sufficient but not a necessary condition in order to have these sequences locally exact. For instance, the Poincaré sequence, induced by the exterior derivative, is a locally exact sequence (Pommaret 2001) but none of its operators defines a projective D-module (see Example 2).

Let us illustrate Theorem 5 and Corollary 2 with an example of a standard linear quadratic problem for an ordinary differential control system.

Example 7: Let us minimize $\int_0^{\mathrm{T}} \frac{1}{2}\,(x(t)^2 + u(t)^2)\,\mathrm{d}t$, where x and u satisfy

$$\left.\begin{array}{l} \dot{x}(t) + x(t) - u(t) = 0 \\ x(0) = x_0. \end{array}\right\} \tag{23}$$

We easily check that (23) is controllable, and thus, parametrizable and a parametrization of (23) is defined by

$$\xi \mapsto \left\{\begin{array}{l} \xi(t) = x(t) \\ \dot{\xi}(t) + \xi(t) = u(t). \end{array}\right.$$

Hence, the previous optimization problem is equivalent to minimize the Lagrangian

$$I = \int_0^T \tfrac{1}{2}\,(\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\,\mathrm{d}t$$

with the only algebraic constraint $\xi(0) = x_0$. We easily check that we have

$$\delta I = \int_0^T \left(-\ddot{\xi}(t) + 2\,\xi(t)\right)\delta\xi(t)\,\mathrm{d}t + \left[(\dot{\xi}(t) + \xi(t))\,\delta\xi(t)\right]_0^{\mathrm{T}}$$

and thus, the optimal system (17) is equivalent to the system

$$\left.\begin{array}{l} \ddot{\xi}(t) - 2\,\xi(t) = 0 \\ \xi(0) = x_0 \\ \dot{\xi}(T) + \xi(T) = 0 \\ \xi(t) = x(t) \\ \dot{\xi}(t) + \xi(t) = u(t). \end{array}\right\} \tag{24}$$

After integrating (24) and eliminating x_0 between x and u, we finally obtain that the optimal controller is defined by

$$u(t) = \frac{-\mathrm{e}^{\sqrt{2}(t-T)} + \mathrm{e}^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\,\mathrm{e}^{\sqrt{2}(t-T)} - (1+\sqrt{2})\,\mathrm{e}^{-\sqrt{2}(t-T)}}\;x(t).$$

In the particular case of ordinary differential equations, let us show how to find again some well-known results developed in the literature (Kwakernaak and Sivan 1972, Kailath 1980, Anderson and Moore 1990).

Example 8: Let us write $s = (\mathrm{d}/\mathrm{d}t)$ and let us consider the system be defined by $D(s)\,y + N(s)\,u = 0$, where $\det D(s) \neq 0$ and the polynomial matrix $(D\colon N)$ is left-coprime, i.e. controllable (Kailath 1980). Thus, differential operator $\mathcal{D}_1 : (y : u) \mapsto \zeta$, defined by

$$D(s)\,y + N(s)\,u = \zeta$$

has a right-inverse and the $D = \mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module $M = D^{l_0}/D^{l_1}\,(D\colon N)$ associated with $\mathcal{D}_1$ is a projective D-module (see Example 3 and Pommaret and Quadrat (1998) for more details). But, $D = \mathbb{R}[\mathrm{d}/\mathrm{dt}]$ is a principal ideal domain, and thus M is a free D-module, i.e. $\mathcal{D}_1$ admits an injective parametrization $\mathcal{D}_0$, which is the *controller form*

$$\left.\begin{array}{l} \overline{N}(s)\,\xi = y \\ \overline{D}(s)\,\xi = u \end{array}\right\}$$

where the basis ξ of M is called the *partial state* of the system $D(s)\,y + N(s)\,u = 0$ (Kailath 1980). See Pommaret and Quadrat (1998, 1999 a) for more details. The formal adjoint operator $\widetilde{\mathcal{D}}_0 : (\mu_1 : \mu_2) \mapsto \nu$ of $\mathcal{D}_0$ is defined by

$$\overline{N}^{\mathrm{T}}(-s)\,\mu_1 + \overline{D}^{\mathrm{T}}(-s)\,\mu_2 = \nu.$$

Now, let us find the optimal system which corresponds to the minimization of the cost

$$\int_0^{+\infty} \tfrac{1}{2}\,\eta^{\mathrm{T}}(t)\,R\,\eta(t)\,\mathrm{d}t$$

where $\eta = (y : u)^{\mathrm{T}}$ and R is a symmetric matrix with entries in $\mathbb{R}$. We easily check that $\mathcal{B} = R$, and thus, we have

$$\widetilde{\mathcal{D}}_0 \circ \mathcal{B} = \left(\overline{N}^{\mathrm{T}}(-s) : \overline{D}^{\mathrm{T}}(-s)\right) \circ R$$

and the differential operator $\mathcal{A}\colon \xi \mapsto \nu$ is defined by

$$\left(\left(\overline{N}^{\mathrm{T}}(-s) : \overline{D}^{\mathrm{T}}(-s)\right) \circ R \circ \begin{pmatrix} \overline{N}(s) \\ \overline{D}(s) \end{pmatrix}\right)\xi = \nu. \tag{25}$$

In particular, if we take

$$R = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$$

where S is a positive definite symmetric matrix acting on the inputs, we find that the dynamic of the optimal system is given by (Kwakernaak and Sivan 1992, Kailath 1980, Anderson and Moore 1990).

$$\left(\overline{N}^{\mathrm{T}}(-s) \circ \overline{N}(s) + \overline{D}^{\mathrm{T}}(-s) \circ S \circ \overline{D}(s)\right)\xi = 0.$$

Corollary 3 (Quadrat 1999)**:** *The differential operator* $\mathcal{A} : E \to \widetilde{E}$ *defined by* (21) *is self-adjoint, i.e.* $\widetilde{\mathcal{A}} = \mathcal{A}$.

Proof: We have

$$\widetilde{\mathcal{A}} = \widetilde{(\widetilde{\mathcal{D}}_0 \circ \mathcal{B} \circ \mathcal{D}_0)} = \widetilde{\mathcal{D}}_0 \circ \widetilde{\mathcal{B}} \circ \widetilde{\widetilde{\mathcal{D}}}_0 = \mathcal{A}$$

because, from Proposition 1, we know that $\mathcal{B}$ is a self-adjoint operator, i.e. $\widetilde{\mathcal{B}} = \mathcal{B}$, and $\widetilde{\widetilde{\mathcal{D}}}_0 = \mathcal{D}_0$. $\square$

Example 9: Let us consider again Example 6. The operator $\mathcal{D}_1$ defines a projective left D-module $M = D^2/D\,\mathcal{D}_1$, and thus, the sequence $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ is locally exact at F_0. The optimal system is then equivalent to $\mathcal{A}\xi = 0$ and $\eta = \mathcal{D}_0\,\xi$. But, as we have $\mathcal{B} = id$, with a slight abuse of language, it follows that $\mathcal{A} = \widetilde{\mathcal{D}}_0 \circ \mathcal{D}_0$ and the fourth-order square operator $\mathcal{A}$ is trivially self-adjoint.

Example 10: We consider again Example 8. We saw that the dynamic of the optimal system was given by $\mathcal{A}\xi = 0$, where $\mathcal{A}$ is defined by (25). We easily check that $\mathcal{A}$ is a self-adjoint differential operator. Now, if we denote by

$$\Delta(s) = \left(\overline{N}^{\mathrm{T}}(-s) : \overline{D}^{\mathrm{T}}(-s)\right) \circ R \circ \begin{pmatrix} \overline{N}(s) \\ \overline{D}(s) \end{pmatrix}$$

and $\delta(s) = \det \Delta(s)$. Thus, we have

$$\begin{aligned} \delta(s) &= \det(\Delta(s)^{\mathrm{T}}) \\ &= \det(\Delta(-s)) \\ &= \delta(-s). \end{aligned}$$

Hence, if there exists $s_0 \in \mathbb{C}$ such that $\delta(s_0) = 0$, then $\delta(-s_0) = 0$, showing that the eigenvalues of the dynamic $\mathcal{A}\xi = 0$ are symmetric with respect to the real axis (Kwakernaak and Sivan 1992, Kailath 1980, Anderson and Moore 1990).

Let us give the second main result of this paper.

Proposition 2 (Quadrat 1999): *If the surjective differential operator $\mathcal{D}_1 : \eta \mapsto \zeta$ defines a projective left D-module $M = D^{l_0}/D^{l_1}\,\mathcal{D}_1$, then we can express the Lagrangian multipliers λ as differential linear combinations of η. More precisely, we then have $\lambda = (\widetilde{\mathcal{P}}_1 \circ \mathcal{B})\,\eta$, where $\widetilde{\mathcal{P}}_1$ is a left-inverse of the injective operator $\widetilde{\mathcal{D}}_1$. Therefore, the differential operator $\widetilde{\mathcal{P}}_1 \circ \mathcal{B} : F_0 \to \widetilde{F}_1$ allows us to observe λ and we have the exact diagram*

$$\begin{array}{ccc} F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \longrightarrow 0 \\ \mathcal{B}\downarrow & \searrow & \\ \widetilde{F}_0 & \xleftarrow{\widetilde{\mathcal{D}}_1} & \widetilde{F}_1 \longleftarrow 0. \\ & \xrightarrow{\widetilde{\mathcal{P}}_1} & \end{array}$$

Proof: The facts that the differential operator $\mathcal{D}_1$ is surjective and the left D-module $M = D^{l_0}/D^{l_1}\,\mathcal{D}_1$ associated with $\mathcal{D}_1$ is projective imply that $\widetilde{\mathcal{D}}_1$ is an injective differential operator, and thus $\widetilde{\mathcal{D}}_1$ admits a left-inverse $\widetilde{\mathcal{P}}_1$, i.e. we have $\widetilde{\mathcal{P}}_1 \circ \widetilde{\mathcal{D}}_1 = id_{\widetilde{F}_1}$ (see the end of the proof of Corollary 1). Accordingly, we have $\widetilde{\mathcal{D}}_1\,\lambda = \mathcal{B}\,\eta \Rightarrow \lambda = (\widetilde{\mathcal{P}}_1 \circ \mathcal{B})\,\eta$, which proves the result. □

Remark 4: By Corollary 1, the linear ordinary differential control system defined by a surjective differential operator $\mathcal{D}_1$ is controllable iff $\widetilde{\mathcal{D}}_1$ is an injective differential operator. Therefore, Proposition 2 always holds in this case.

Example 11: We consider again Example 6. In Example 3, we saw that $\widetilde{\mathcal{D}}_1$ was an injective differential operator and that $\widetilde{\mathcal{D}}_1$ admitted a left-inverse $\widetilde{\mathcal{P}}_1 : \mu \mapsto \lambda$ defined by

$$-\partial_2\mu_1 + \partial_1\mu_2 + x^2\,\mu_2 = \lambda.$$

Thus, the differential operator $\widetilde{\mathcal{P}}_1 \circ \mathcal{B}: \eta \mapsto \lambda$ is defined by

$$-\partial_2\eta^1 + \partial_1\eta^2 + x^2\,\eta^2 = \lambda.$$

Hence, we explicitly obtain the Lagrangian multiplier $\lambda = -\partial_2\eta^1 + \partial_1\eta^2 + x^2\,\eta^2$ in terms of the system variables η^1 and η^2 and their derivatives.

The third main result of this paper is motivated by linear elasticity theory as we shall later see in § 5 and by the fact that we want to close the diagram of Theorem 5 on the right.

Proposition 3 (Quadrat 1999): *If the differential operator $\mathcal{B} : F_0 \to \widetilde{F}_0$ is invertible, then the optimal system* (17) *is equivalent to the system*

$$\left.\begin{aligned} &\mathcal{C}\,\lambda = 0 \\ &\eta = (\mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1)\,\lambda \end{aligned}\right\} \tag{26}$$

where the differential operator $\mathcal{C} : \widetilde{F}_1 \to F_1$ is defined by

$$\mathcal{C} = \mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1. \tag{27}$$

Moreover, we have the commutative diagram

$$\begin{array}{ccc} F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\ \mathcal{B}\downarrow\uparrow\mathcal{B}^{-1} & & \uparrow\mathcal{C} \\ \widetilde{F}_0 & \xleftarrow{\widetilde{\mathcal{D}}_1} & \widetilde{F}_1 \end{array}$$

In particular, this result holds if $\mathcal{D}_1$ is a first-order differential operator (e.g. a Kalman system) and $\mathcal{B} = R$ is a positive definite symmetric matrix with constant entries.

Proof: In Theorem 4, we saw that the optimal system was given by (17), namely

$$\left.\begin{aligned} &\mathcal{D}_1\,\eta = 0 \\ &\mathcal{B}\,\eta - \widetilde{\mathcal{D}}_1\lambda = 0. \end{aligned}\right\}$$

By inverting $\mathcal{B}$ in the second equation, we obtain $\eta = (\mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1)\,\lambda$, and thus, by substituting this result in the first equation, we obtain $\mathcal{D}_1\,\eta = (\mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1)\,\lambda = 0$, i.e. (26). Conversely, if we have (26), then, from the second equation of (26), we obtain $\widetilde{\mathcal{D}}_1\lambda = \mathcal{B}\,\eta$ and thus

$$\mathcal{C}\,\lambda = (\mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1)\,\lambda = \mathcal{D}_1\,((\mathcal{B}^{-1} \circ \mathcal{B})\,\eta) = \mathcal{D}_1\,\eta$$

which concludes the proof. □

Example 12: We consider again Example 6. The operator $\mathcal{B} : F_0 \to \widetilde{F}_0$ is invertible and $\mathcal{B}^{-1}\mu = \eta$. Therefore, the differential operator $\mathcal{C} : \widetilde{F}_1 \to F_1$ is defined by $-\Delta\lambda + (x^2)^2\lambda = \zeta$. Therefore, by Proposition 3, the optimal system is governed by the equations

$$\left.\begin{array}{l}\Delta\lambda - (x^2)^2\lambda = 0\\ \eta^1 = -\partial_1\lambda - x^2\lambda\\ \eta^2 = -\partial_2\lambda.\end{array}\right\}$$

We find again that the optimal system only depends on two arbitrary functions of one variable, which are needed for integrating the first equation above.

Example 13: Let us find the solutions which minimize the following cost

$$I = \int_{t_0}^{t_1} \tfrac{1}{2}\,(x : u)^{\mathrm{T}} \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \mathrm{d}t + \tfrac{1}{2}\, x(t_1)^{\mathrm{T}}\, S\, x(t_1)$$

where R (resp. Q, S) is a positive definite (resp. semi-definite) symmetric matrix with entries in $\mathbb{R}$, while x and u satisfy the time-invariant Kalman system $\dot{x} - A\,x - B\,u = 0$ and $x(t_0) = x_0$.

By Theorem 4, we obtain that the optimal system (17) is defined by

$$\left.\begin{array}{l}\dot{x} - A\,x - B\,u = 0\\ \dot{\lambda} + A^{\mathrm{T}}\lambda + Qx = 0\\ R\,u + B^{\mathrm{T}}\lambda = 0\\ x(t_0) = x_0\\ \lambda(t_1) = S\,x(t_1).\end{array}\right\} \qquad (28)$$

In particular, the operator $\mathcal{B} : F_0 \to \widetilde{F}_0$ is defined by

$$\mathcal{B}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} Q\,x \\ R\,u \end{pmatrix}.$$

If we suppose that Q is a positive definite matrix, then $\mathcal{B}$ is invertible and $\mathcal{B}^{-1}$ is defined by:

$$\mathcal{B}^{-1}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} Q^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} Q^{-1}\,\mu_1 \\ R^{-1}\,\mu_2 \end{pmatrix}.$$

Then, by Proposition 3, the optimal system (28) is equivalent to the system

$$\left.\begin{array}{l}-Q^{-1}\ddot{\lambda} + (A\,Q^{-1} - Q^{-1}A^{\mathrm{T}})\dot{\lambda} + (A\,Q^{-1}A^{\mathrm{T}} + B\,R^{-1}B^{\mathrm{T}})\lambda = 0\\ x = -Q^{-1}(\dot{\lambda} + A^{\mathrm{T}}\lambda)\\ u = -R^{-1}B^{\mathrm{T}}\lambda\\ S\,Q^{-1}(\dot{\lambda}(t_1) + A^{\mathrm{T}}\lambda(t_1)) + \lambda(t_1) = 0\\ \dot{\lambda}(t_0) + A^{\mathrm{T}}\lambda(t_0) + Q\,x_0 = 0.\end{array}\right\}$$

To our knowledge, the previous system is new. Let us finish by interpreting the standard *Riccati equation* (Kwakernaak and Sivan 1972, Kailath 1980, Anderson and Moore 1990) as an *integrability condition* (Pommaret and Quadrat 2000a, Pommaret 2001). Let us suppose that R (resp. Q) is a positive definite (resp. semi-definite) symmetric matrix. Using the fact that R is invertible, equation (28) is equivalent to the system

$$\left.\begin{array}{l}\dot{x} - A\,x + B\,R^{-1}B^{\mathrm{T}}\lambda = 0\\ \dot{\lambda} + A^{\mathrm{T}}\lambda + Q\,x = 0\\ u = -R^{-1}\,B^{\mathrm{T}}\lambda.\end{array}\right\}$$

This new system in x and λ is determined, and thus, if we add to it the new equation $\lambda - Px = 0$ as a feedback law for the total system, *it becomes non-formally integrable*, i.e. we cannot find step by step the solution of the system as a formal power series (see Spencer 1965 and Pommaret 2001 for more information). Indeed, if we differentiate the zero-order equation and take into account the other equations, we find the new zero-order equation

$$(\dot{P} + A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q)\,x = 0.$$

Hence, the system

$$\left.\begin{array}{l}\dot{x} - A\,x + B\,R^{-1}B^{\mathrm{T}}\lambda = 0\\ \dot{\lambda} + A^{\mathrm{T}}\lambda + Q\,x = 0\\ u = -R^{-1}\,B^{\mathrm{T}}\lambda\\ \lambda - P\,x = 0\end{array}\right\}$$

has a solution different from zero iff the following integrability condition on P is satisfied

$$\dot{P} + A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q = 0$$

that is, iff the previous Riccati equation for P is satisfied. In this case, we can rewrite the system as

$$\left.\begin{array}{l}\dot{P} + A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q = 0\\ \dot{x} - (A - B\,R^{-1}\,B^{\mathrm{T}}\,P)\,x = 0\\ u = -R^{-1}\,B^{\mathrm{T}}\,P\,x\\ \lambda = P\,x.\end{array}\right\}$$

We have recently shown in Pommaret and Quadrat (2000 a) that the controllability of a linear system with unspecified coefficients depends on trees of integrability conditions on the coefficients. The same thing may happen in a linear optimal control problem. To finish, we provide an illustrative example.

Example 14: Let us find the solutions that optimize

$$\int \tfrac{1}{2}(y^2 - u^2)\,\mathrm{d}t$$

where y and u satisfy the system $\dot{y} + a\,y - \dot{u} - u = 0$, in which a is a constant coefficient. System (17) is given by

$$\left.\begin{array}{l}\dot{\lambda} - a\lambda + y = 0\\ -\dot{\lambda} - \lambda - u = 0\\ \dot{y} + a\,y - \dot{u} - u = 0.\end{array}\right\}$$

Let us eliminate λ in order to find (19): summing the first two equations, we obtain the new zero-order equation

$(1-a)\lambda = u - y$. Thus, two cases may happen depending on the value of a:

1. If $a=1$, then $y-u=0$ and the optimal system is given by
$$\left.\begin{array}{l}\dot{y}+y-\dot{u}-u=0\\ y-u=0\end{array} \quad \Leftrightarrow \quad y-u=0.\right\}$$
2. If $a\neq 1$, then $\lambda = (y-u)/(a-1)$ and, after substituting, we get:
$$\left\{\begin{array}{l}\dot{y}+a\,y-\dot{u}-u=0\\ \dot{y}-y-\dot{u}+a\,u=0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\dot{y}+a\,y-\dot{u}-u=0\\ (a+1)\,(y-u)=0.\end{array}\right.$$
We are led to new integrability conditions:
(a) If $a=-1$, then the optimal system is given by the only equation $\dot{y}-\ y-\dot{u}-u=0$. In fact, we can note that a parametrization $\mathcal{D}_0 : \xi \mapsto (\,y\,:\,u)$ of this system is given by
$$\xi \mapsto \left\{\begin{array}{l}\dot{\xi}+\xi=y\\ \dot{\xi}-\xi=u\end{array}\right.$$
and thus, $\frac{1}{2}(y^2-u^2)=2\,\xi\,\dot{\xi}=(\mathrm{d}/\mathrm{d}t)\,(\xi^2)$.
(b) If $a\neq -1$, then the only solution is $y=u=0$.

We note that the condition $a\neq 1$ is in fact the condition on a for the system to be controllable (if $a=1$, then the element $z=y-u$ satisfies $(\mathrm{d}/\mathrm{d}t+1)\,z=0$, i.e. z is a torsion element of the $D=\mathbb{R}(a)[\mathrm{d}/\mathrm{d}t]$-module $M=D^2/D\,((\mathrm{d}/\mathrm{d}t)+1:-(\mathrm{d}/\mathrm{d}t)-1)$.

5. Applications to mathematical physics

In this section, we show how all the results obtained in the preceding sections can be applied to linear multidimensional systems appearing in some applications and specially in linear elasticity theory (Landau and Lifschitz 1990) and electromagnetism (Bok and Hulin-Jung 1979, Landau and Lifschitz 1989).

5.1. *Linear elasticity theory*

Let us denote the displacement in $\mathbb{R}^n$ by $\xi=(\xi^i)_{1\le i\le n}$ and contract the index of ξ^i by the euclidean metric $\omega_{ij}=\omega_{ji}=\delta_{ij}$, $1\le i,j\le n$, of $\mathbb{R}^n$ in order to lower the index with $\xi_i=\omega_{ij}\,\xi^j$. Then, the so-called small strain tensor is defined by the differential operator
$$\begin{array}{rcl}\mathcal{L}(\cdot)\,\omega : T & \longrightarrow & S_2T^\star\\ \xi & \mapsto & (\epsilon_{ij}=\frac{1}{2}\,(\mathcal{L}(\xi)\,\omega)_{ij}=\frac{1}{2}(\partial_i\,\xi_j+\partial_j\,\xi_i))_{1\le i,j\le n}\end{array}$$
where $\mathcal{L}$ is the Lie derivative of the euclidean metric. In what follows, we only consider the case $n=2$. Thus, the small strain tensor is given by
$$\left.\begin{array}{l}\epsilon_{11}=\partial_1\xi_1\\ \epsilon_{12}=\epsilon_{21}=\frac{1}{2}\,(\partial_1\xi_2+\partial_2\xi_1)\\ \epsilon_{22}=\partial_2\xi_2.\end{array}\right\}$$
See Landau and Lifschitz (1990).

This system has only one compatibility condition of order two, namely
$$\partial_{11}\epsilon_{22}+\partial_{22}\epsilon_{11}-2\,\partial_{12}\epsilon_{12}=0 \tag{29}$$
and we have the following formally exact sequence of differential operators
$$0\longrightarrow\Theta\longrightarrow E\xrightarrow{\mathcal{D}}F_0\xrightarrow{\mathcal{D}_1}F_1\longrightarrow 0 \tag{30}$$
where $E=T, F_0=S_2T^\star, \Theta$ is the field of small rigid displacements and $\mathcal{D}\,\xi=\frac{1}{2}(\mathcal{L}(\xi)\,\omega)$. In the spirit of H. Poincaré, this sequence is only based on geometry whereas the adjoint sequence, i.e. the sequence formed by the formal adjoint operators, and the constitutive law give the physics. Indeed, the formal adjoint $\widetilde{\mathcal{D}}:\sigma\mapsto f$ of the differential operator $\mathcal{D}$ is obtained by multiplying ϵ by σ and integrating by parts, i.e.
$$\begin{aligned}&\sigma^{11}\,\epsilon_{11}+\sigma^{12}\,\epsilon_{12}+\sigma^{21}\,\epsilon_{21}+\sigma^{22}\,\epsilon_{22}\\ &\quad=\sigma^{11}\,\epsilon_{11}+2\,\sigma^{12}\,\epsilon_{12}+\sigma^{22}\,\epsilon_{22}\\ &\quad=-(\partial_1\sigma^{11}+\partial_2\sigma^{12})\,\xi_1-(\partial_1\sigma^{12}+\partial_2\sigma^{22})\,\xi_2+\cdots\end{aligned}$$
where we have supposed that $\sigma^{12}=\sigma^{21}$. Thus, $-\widetilde{\mathcal{D}}:\sigma\mapsto f$ is given by
$$\left\{\begin{array}{l}\partial_1\sigma^{11}+\partial_2\sigma^{12}=f^1\\ \partial_1\sigma^{12}+\partial_2\sigma^{22}=f^2\end{array}\right. \tag{31}$$
where σ is the stress tensor and f is a density of forces. Similarly, the formal adjoint $\widetilde{\mathcal{D}}_1$ of the differential operator $\mathcal{D}_1$ is obtained by multiplying (29) by λ and integrating by parts
$$\begin{aligned}&\lambda\,(\partial_{11}\epsilon_{22}+\partial_{22}\epsilon_{11}-2\,\partial_{12}\epsilon_{12})\\ &\quad=\partial_{11}\lambda\;\epsilon_{22}+\partial_{22}\lambda\;\epsilon_{11}-2\,\partial_{12}\lambda\;\epsilon_{12}+\cdots\end{aligned}$$
and thus, $\widetilde{\mathcal{D}}_1:\lambda\mapsto\sigma$ is given by
$$\left.\begin{array}{r}\partial_{22}\lambda=\sigma^{11}\\ -\partial_{12}\lambda=\sigma^{12}\\ \partial_{11}\lambda=\sigma^{22}.\end{array}\right\} \tag{32}$$
We easily check that all the compatibility conditions of $\widetilde{\mathcal{D}}_1$ are generated by $\widetilde{\mathcal{D}}$ or by $-\widetilde{\mathcal{D}}$. Meanwhile, we find again the well-known parametrization of the stress tensor by the Airy function λ. Finally, we have the formally exact sequence
$$0\longleftarrow\widetilde{E}\xleftarrow{-\widetilde{\mathcal{D}}}\widetilde{F}_0\xleftarrow{\widetilde{\mathcal{D}}_1}\widetilde{F}_1. \tag{33}$$
In fact, it can be shown that the sequence (30) is locally equivalent to the Poincaré sequence
$$\bigwedge^0 T^\star\xrightarrow{d}\bigwedge^1 T^\star\xrightarrow{d}\bigwedge^2 T^\star\longrightarrow 0$$
(see Pommaret (2001) for more details), which is a locally exact and self-adjoint sequence. Hence, the sequences (30) and (33) are locally exact. Moreover,

the Poincaré sequence being a self-adjoint one, this is the reason why the kernel Θ of the differential operator $\mathcal{D}$ and the kernel Ω of $\widetilde{\mathcal{D}}_1$ both depend on three arbitrary constants.

We can link the two differential sequences (30) and (33) with the constitutive law, namely the Hooke law $\mathcal{B} : \epsilon \mapsto \sigma$, defined by

$$\left.\begin{aligned} \sigma^{11} &= (\alpha + 2\,\beta)\,\epsilon_{11} + \alpha\,\epsilon_{22} \\ \sigma^{12} &= \sigma^{21} = 2\,\beta\,\epsilon_{12} \\ \sigma^{22} &= \alpha\,\epsilon_{11} + (\alpha + 2\,\beta)\,\epsilon_{22} \end{aligned}\right\}$$

where (α, β) are the Lamé constants and we obtain the locally exact diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Theta & \longrightarrow & E & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \longrightarrow & 0 \\ & & & & & & \downarrow \mathcal{B} & & & & \\ 0 & \longleftarrow & \widetilde{E} & \xleftarrow{-\tilde{\mathcal{D}}} & \widetilde{F}_0 & \xleftarrow{\mathcal{D}_1} & \widetilde{F}_1 & \longleftarrow & \Omega & \longleftarrow & 0. \end{array} \tag{34}$$

With such a diagram, we naturally want to define the operator $\mathcal{A} = -\widetilde{\mathcal{D}} \circ \mathcal{B} \circ \mathcal{D} : E \to \widetilde{E}$. Now, let us note that $\mathcal{B}$ is a symmetric matrix, i.e. $\widetilde{\mathcal{B}} = \mathcal{B}$, and thus, $\mathcal{A}$ is a self-adjoint differential operator. This fact can be easily verified on the direct expression of the differential operator $\mathcal{A}$

$$\left.\begin{aligned} (\alpha + 2\,\beta)\,\partial_{11}\xi_1 + \beta\,\partial_{22}\xi_1 + (\alpha + \beta)\,\partial_{12}\xi_2 &= f^1 \\ (\alpha + \beta)\,\partial_{12}\xi_1 + \beta\,\partial_{11}\xi_2 + (\alpha + 2\,\beta)\,\partial_{22}\xi_2 &= f^2 \end{aligned}\right\}$$

or, equivalently, on the so-called Navier equations

$$\left.\begin{aligned} (\alpha + \beta)\,\partial_1(\partial_1\xi_1 + \partial_2\xi_2) + \beta\,\Delta\,\xi_1 &= f^1 \\ (\alpha + \beta)\,\partial_2(\partial_1\xi_1 + \partial_2\xi_2) + \beta\,\Delta\,\xi_2 &= f^2. \end{aligned}\right\}$$

The Hooke law is in fact invertible and the operator $\mathcal{B}^{-1}$ is given by

$$\left.\begin{aligned} \epsilon_{11} &= \frac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\sigma^{11} - \frac{\alpha}{4\,\beta\,(\alpha + \beta)}\sigma^{22} \\ \epsilon_{12} &= \frac{1}{2\,\beta}\sigma^{12} \\ \epsilon_{22} &= -\frac{\alpha}{4\,\beta\,(\alpha + \beta)}\sigma^{11} + \frac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\sigma^{22} \end{aligned}\right\}$$

and thus, we obtain the differential operator $\mathcal{C} = \mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1 : \widetilde{F}_1 \to F_1$ defined by

$$\frac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\Delta\Delta\,\lambda = \zeta.$$

Finally, we can sum up the different differential operators by the locally exact diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Theta & \longrightarrow & E & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \longrightarrow & 0 \\ & & & & \downarrow \mathcal{A} & & \downarrow \mathcal{B} & & \uparrow \mathcal{C} & & \\ 0 & \longleftarrow & \widetilde{E} & \xleftarrow{-\tilde{\mathcal{D}}} & \widetilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \widetilde{F}_1 & \longleftarrow & \Omega & \longleftarrow & 0. \end{array}$$

To finish this section, let us connect the above results to parametrizability of multidimensional systems. From the controllability test, we can conclude from what precedes, that the operator $\mathcal{D}_1$ determines a torsion-free D-module M, with $\mathcal{D}_0$ as a parametrization (it is not surprising because, by definition, $\mathcal{D}_1$ is the compatibility condition of $\mathcal{D}_0$). More surprisingly, we proved that the D-module defined by the differential operator $\widetilde{\mathcal{D}}_0$ is also torsion-free with the parametrization given by the differential operator (32).

5.1.1. *Case without forces*. In the case where there is no force, let us minimize the energy of deformation defined by

$$\int \tfrac{1}{2}\,\epsilon^{\mathrm{T}}\,\mathcal{B}\,\epsilon\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

under the differential constraint $\mathcal{D}_1\epsilon = 0$. Introducing new unknowns λ as Lagrange multipliers, it is equivalent to optimize the new integral

$$\int \left(\tfrac{1}{2}\epsilon^{\mathrm{T}}\,\mathcal{B}\,\epsilon - \lambda\,\mathcal{D}_1\,\epsilon\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

where the ϵ are now considered as independent unknowns. Thus, by Theorem 4, we have to solve the system

$$\left.\begin{aligned} \mathcal{B}\,\epsilon - \widetilde{\mathcal{D}}_1\lambda &= 0 \\ \mathcal{D}_1\,\epsilon &= 0 \end{aligned}\right\} \tag{35}$$

or, in other words, the system defined by

$$\left.\begin{aligned} &(\alpha + 2\,\beta)\,\epsilon_{11} + \alpha\,\epsilon_{22} - \partial_{22}\lambda = 0 \\ &2\,\beta\,\epsilon_{12} + \partial_{12}\lambda = 0 \\ &\alpha\,\epsilon_{11} + (\alpha + 2\,\beta)\,\epsilon_{22} - \partial_{11}\lambda = 0 \\ &\partial_{11}\epsilon_{22} + \partial_{22}\epsilon_{11} - 2\,\partial_{12}\epsilon_{12} = 0. \end{aligned}\right\}$$

We can solve $\epsilon = (\mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1)\,\lambda$ in the first equation of (35), and, substituting it in the second, we obtain $\mathcal{C}\,\lambda = (\mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \widetilde{\mathcal{D}}_1)\,\lambda = 0$. Finally, we have to solve the following system

$$\left.\begin{aligned} &\Delta\,\Delta\,\lambda = 0 \\ &\epsilon_{11} = \frac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\partial_{22}\lambda - \frac{\alpha}{4\,\beta\,(\alpha + \beta)}\partial_{11}\lambda \\ &\epsilon_{12} = -\frac{1}{2\,\beta}\partial_{12}\lambda \\ &\epsilon_{22} = -\frac{\alpha}{4\,\beta\,(\alpha + \beta)}\partial_{22}\lambda + \frac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\partial_{11}\lambda \end{aligned}\right\}$$

and, from the first equation of the previous system, we deduce that λ is a biharmonic function.

Also, the differential sequence (30) is locally exact, and thus, as we saw in Theorem 5, the solution of the equivalent unconstrained optimization problem

$$\int \frac{1}{2}\left(\xi^{\mathrm{T}}\,(\widetilde{\mathcal{D}}\circ\mathcal{B}\circ\mathcal{D})\,\xi\right)\mathrm{d}x^1\wedge\mathrm{d}x^2$$

is defined by

$$\left.\begin{array}{l}\mathcal{A}\,\xi=0\\ \mathcal{D}\,\xi=\epsilon\end{array}\right\}$$

or, in other words, we have to solve the following system of partial differential equations only in the displacements ξ

$$\left.\begin{array}{l}(\alpha+\beta)\,\partial_1(\partial_1\xi_1+\partial_2\xi_2)+\beta\,\Delta\,\xi_1=0\\ (\alpha+\beta)\,\partial_2(\partial_1\xi_1+\partial_2\xi_2)+\beta\,\Delta\,\xi_2=0\\ \partial_1\xi_1=\epsilon_{11}\\ \frac{1}{2}(\partial_1\xi_2+\partial_2\xi_1)=\epsilon_{12}\\ \partial_2\xi_2=\epsilon_{22}.\end{array}\right\}$$

5.1.2. *Forces coming from a potential.* If the force f comes from a potential ψ, namely we have

$$\left.\begin{array}{l}f^1=\partial_1\psi\\ f^2=\partial_2\psi\end{array}\right\}$$

then, from (31), we have the system

$$\left.\begin{array}{l}\partial_1\sigma^{11}+\partial_2\sigma^{12}-\partial_1\psi=0\\ \partial_1\sigma^{12}+\partial_2\sigma^{22}-\partial_2\psi=0\end{array}\right\} \tag{36}$$

and, if we introduce by $\overline{\sigma}^{11}=\sigma^{11}-\psi$, $\overline{\sigma}^{12}=\sigma^{12}$ and $\overline{\sigma}^{22}=\sigma^{22}-\psi$, we find that the new system without forces is defined by

$$\left.\begin{array}{l}\partial_1\overline{\sigma}^{11}+\partial_2\overline{\sigma}^{12}=0\\ \partial_1\overline{\sigma}^{12}+\partial_2\overline{\sigma}^{22}=0.\end{array}\right\} \tag{37}$$

Moreover, we have $\widetilde{\mathcal{D}}\,\overline{\sigma}=0\Leftrightarrow\overline{\sigma}=\widetilde{\mathcal{D}}_1\lambda$ because the differential sequence $\widetilde{F}_1\xrightarrow{\widetilde{\mathcal{D}}_1}\widetilde{F}_0\xrightarrow{\widetilde{\mathcal{D}}}\widetilde{E}$ is locally exact at $\widetilde{F}_0$. Hence, (37) is equivalent to the system

$$\left.\begin{array}{l}\sigma^{11}=\partial_{22}\lambda+\psi\\ \sigma^{12}=-\partial_{12}\lambda\\ \sigma^{22}=\partial_{11}\lambda+\psi.\end{array}\right\} \tag{38}$$

Therefore, solving system (36), where σ satisfies $\epsilon=\mathcal{B}^{-1}\sigma$ and $\mathcal{D}_1\,\epsilon=0$, is the same as solving system (38) with $\epsilon=\mathcal{B}^{-1}\sigma$ and $\mathcal{D}_1\,\epsilon=0$. Hence, substituting (38) into $\mathcal{D}_1\,(\mathcal{B}^{-1}\sigma)=0$, we have to solve the following partial differential equation

$$\Delta\Delta\,\lambda+\frac{2\,\beta}{(\alpha+2\,\beta)}\,\Delta\,\psi=0$$

and substitute the result in (38) to obtain the corresponding stress tensor σ. As a matter of fact, when the only forces involved are of gravitational type, then $\Delta\,\psi=0$, and we are brought back to the preceding situation. In particular, see the introduction of Pommaret (2001) for the construction of a dam.

5.2. *Electromagnetism*

Let us consider the formally exact sequence

$$0\longrightarrow\Theta\longrightarrow E\xrightarrow{\mathcal{D}_0}F_0\xrightarrow{\mathcal{D}_1}F_1\xrightarrow{\mathcal{D}_2}F_2\xrightarrow{\mathcal{D}_3}F_3\longrightarrow 0 \tag{39}$$

where Θ is the kernel of the differential operator $\mathcal{D}_0\colon E\to F_0$ defined by

$$\xi\mapsto\left\{\begin{array}{l}\vec{\nabla}\xi=\vec{A}\\ -\dfrac{\partial\xi}{\partial t}=V.\end{array}\right.$$

The differential operator $\mathcal{D}_1:F_0\to F_1$ gives the electromagnetism field $(\vec{B},\vec{E})$ from the potential $(\vec{A},V)$, i.e.

$$(\vec{A},V)\mapsto\left\{\begin{array}{l}\vec{\nabla}\wedge\vec{A}=\vec{B}\\ -\vec{\nabla}V-\dfrac{\partial\vec{A}}{\partial t}=\vec{E}.\end{array}\right. \tag{40}$$

The differential operator $\mathcal{D}_2:F_1\to F_2$ is the first set of Maxwell equations, namely

$$(\vec{B},\vec{E})\mapsto\left\{\begin{array}{l}\vec{\nabla}.\,\vec{B}=\kappa^1\\ \vec{\nabla}\wedge\vec{E}+\dfrac{\partial\vec{B}}{\partial t}=\vec{\kappa}^2\end{array}\right.$$

and the differential operator $\mathcal{D}_3:F_2\to F_3$ defines the compatibility conditions of $\mathcal{D}_2$, i.e.

$$\frac{\partial\kappa^1}{\partial t}-\vec{\nabla}.\,\vec{\kappa}^2=0.$$

The use of space-time formalism with $x^4=c\,t$ and special relativity amounts to rewrite the formally exact sequence (39) in the intrinsic form of the locally exact and self-adjoint Poincaré sequence for the exterior derivative (Pommaret 2001)

$$T^\star\xrightarrow{d}\wedge^2\,T^\star\xrightarrow{d}\wedge^3\,T^\star\xrightarrow{d}\wedge^4\,T^\star\longrightarrow 0.$$

Moreover, the differential operator $\widetilde{\mathcal{D}}_1:\widetilde{F}_1\to\widetilde{F}_0$ defined by

$$(\vec{H},\,-\vec{D})\mapsto\left\{\begin{array}{l}\vec{\nabla}\wedge\vec{H}-\dfrac{\partial\vec{D}}{\partial t}=\vec{j}\\ -\vec{\nabla}.\,\vec{D}=-\rho\end{array}\right.$$

is the second set of Maxwell equations, where $\vec{H}$ is the magnetic induction, $\vec{D}$ the electric induction, $\vec{j}$ the density of current and ρ is the density of electric charge. The compatibility conditions of $\widetilde{\mathcal{D}}_1(\vec{H}:-\vec{D})^{\mathrm{T}}=(\vec{j}:-\rho)^{\mathrm{T}}$ is the conservation law defined by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{j} = 0. \qquad (41)$$

Again, in space-time formalism, the differential sequence formed by the formal adjoints of the differential operators of (39) is isomorphic to the Poincaré sequence, and thus, it is locally exact.

Now, let us consider the problem of minimizing the electromagnetism Lagrangian defined by

$$\int \left(\frac{1}{2\,\mu_0} \, \|\vec{B}\|^2 - \frac{\epsilon_0}{2} \, \|\vec{E}\|^2 \right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4$$

where ϵ_0 is the dielectric constant and μ_0 is the magnetic constant, under the differential constraint formed by the first set of Maxwell equations

$$\left. \begin{array}{l} \vec{\nabla}.\,\vec{B} = 0 \\ \vec{\nabla} \wedge \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0. \end{array} \right\} \qquad (42)$$

From the Lagrangian, we obtain the operator (Minkowski law) $\mathcal{B} : F_1 \to \widetilde{F}_1$ is defined by

$$(\vec{B},\, \vec{E}) \mapsto \left\{ \begin{array}{l} \dfrac{1}{\mu_0} \vec{B} = \vec{H} \\ -\epsilon_0 \, \vec{E} = -\vec{D}. \end{array} \right.$$

Therefore, we obtain the commutative locally exact diagram

$$\begin{array}{ccccc} F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \xrightarrow{\mathcal{D}_2} & F_2 \\ \downarrow \mathcal{A} & & \downarrow \mathcal{B} & & \\ \widetilde{F}_0 & \xleftarrow{\widetilde{\mathcal{D}}_1} & \widetilde{F}_1 & \xleftarrow{\widetilde{\mathcal{D}}_2} & \widetilde{F}_2 \end{array}$$

where, using the relation $\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \Delta \vec{A}$, the differential operator $\mathcal{A} = \widetilde{\mathcal{D}}_1 \circ \mathcal{B} \circ \mathcal{D}_1$ is defined by

$$\left. \begin{array}{l} \dfrac{1}{\mu_0} \left(\left(\dfrac{1}{c^2} \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta \, \vec{A} \right) + \vec{\nabla} \left(\vec{\nabla}.\vec{A} + \dfrac{1}{c^2} \dfrac{\partial V}{\partial t} \right) \right) = \vec{j} \\ \epsilon_0 \left(\Delta \, V + \dfrac{\partial \vec{\nabla}.\vec{A}}{\partial t} \right) = -\rho \end{array} \right\}$$

and $c_0 = 1/\sqrt{\epsilon_0 \, \mu_0}$ is the speed of light in the vacuum. See Bok and Hulin-Jung (1979) for more details. Hence, we obtain that the optimal system (17) is equivalent to (21), i.e.

$$\left. \begin{array}{l} \dfrac{1}{c^2} \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta \, \vec{A} + \vec{\nabla} \left(\vec{\nabla}.\vec{A} + \dfrac{1}{c^2} \dfrac{\partial V}{\partial t} \right) = 0 \\ \dfrac{1}{c^2} \dfrac{\partial^2 V}{\partial t^2} - \Delta \, V - \dfrac{\partial}{\partial t} \left(\vec{\nabla}.\vec{A} + \dfrac{1}{c^2} \dfrac{\partial V}{\partial t} \right) = 0 \\ \vec{\nabla} \wedge \vec{A} = \vec{B} \\ -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} = \vec{E}. \end{array} \right\} \qquad (43)$$

Finally, let us note that the first set of Maxwell equations (42) is parametrized by (40), where the potential $(\vec{A},\, V)$ is not uniquely determined. Using the fact that $\vec{A}$ (resp. V) is defined up to a gradient $\vec{\nabla}\xi$ (resp. $\partial\xi/\partial t$) of a function ξ, we can choose the potential $(\vec{A},\, V)$ in such a way that it satisfies the following equation (*gauge condition*)

$$\vec{\nabla}.\vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0.$$

Therefore, system (43) becomes

$$\left. \begin{array}{l} \dfrac{1}{c^2} \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta \, \vec{A} = 0 \\ \dfrac{1}{c^2} \dfrac{\partial^2 V}{\partial t^2} - \Delta \, V = 0 \\ \vec{\nabla} \wedge \vec{A} = \vec{B} \\ -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} = \vec{E}. \end{array} \right\} \qquad (44)$$

From the first two equations of (44), we deduce that $\vec{A}$ and V satisfy a wave equation with a speed of propagation equal to c_0 (electromagnetic waves). We refer to Pommaret (2001) for a more intrinsic formulation of the Maxwell equations.

6. Conclusion

We hope to have convinced the reader about the possibility to extend optimal control theory from ordinary differential systems to multidimensional systems described by partial differential equations. In particular, using the parametrizations of controllable linear multidimensional systems, we showed how to transform variational problems with partial differential equations constraints into variational problems without differential constraints. Moreover, different equivalent forms for the optimal systems were given which depend on the structural properties of the multidimensional systems. Finally, within the differential operators and algebraic analysis frameworks, we illustrated the main new results either on classical optimal control problems or on variational problems coming from mathematical physics (e.g. linear elasticity, electromagnetism).

Let us stress that the different equivalent forms for the optimal systems obtained in this paper can be explicitly computed using the algorithms developed in Pommaret and Quadrat (1998, 1999 a,b), Pommaret (2001) and Chyzak *et al.* (2004). We refer the reader to Chyzak *et al.* (2003, 2004) for computational issues as well as for a description of the package *OreModules* used to perform such computations.

In forthcoming publications, we shall study the extension of the results obtained in this paper to more general classes of linear systems and, in particular, to systems defined over *Ore algebras* (see Chyzak *et al.* (2003, 2004) for more details). Such a class of linear systems includes differential time-delay systems, multidimensional discrete systems, etc.

Moreover, the problem of spectral factorization of the differential operator $\mathcal{A}$ needs to be studied in the future (see Quadrat 1999). In the particular case of partial differential operators with constant coefficients, the independent work (Pillai and Willems 2002) has shown how such a problem was related to Hilbert's 17th problem.

Finally, using an extension of the concept of parametrization, some generalizations of this paper have been recently obtained in Quadrat and Robertz (2004) for uncontrollable multidimensional systems.

References

Anderson, B. D. O., and Moore, J. B., 1990, *Optimal Control: Linear Quadratic Methods* (Englewood Cliffs, NJ: Prentice Hall).

Bjork, J. E., 1979, *Rings of Differential Operators* (Amsterdam; Oxford; New York: North Holland).

Bjork, J. E., 1993, *Analytic $\mathcal{D}$-modules and Applications* (Amsterdam; Oxford; New York: Kluwer).

Bok, J., and Hulin-Jung, N., 1979, *Ondes électromagnétiques Relativité* (Paris: Hermann).

Cartan, E., 1945, *Les systémes différentiels extérieurs et leurs applications géométriques* (Paris: Hermann).

Chyzak, F., Quadrat, A., and Robertz, D., 2003, *OreModules* project http://wwwb.math.rwth-aachen.de/OreModules/.

Chyzak, F., Quadrat, A., and Robertz, D., 2004, Effective algorithms for parametrizing linear control systems over Ore algebras. INRIA report 5181, available at http://www.inria.fr/rrrt/index.fr.html, submitted to *Applicable Algebra in Engineering, Communications and Computing*.

Janet, M., 1929, *Leçons sur les systémes d'équations aux dérivés partielles*, Cahiers Scientifiques IV (Paris: Gauthier-Villars).

Kailath, T., 1980, *Linear Systems* (Englewood Cliffs, NJ: Prentice-Hall).

Kashiwara, M., 1995, *Algebraic Study of Systems of Partial Differential Equations*. Mémoires de la Société Mathématiques de France 63.

Kolchin, E. R., 1973, *Differential Algebra and Algebraic Groups* (New York; London: Academic Press).

Kwakernaak, H., and Sivan, R., 1972, *Linear Optimal Control Systems* (New York: Wiley).

Landau, L., and Lifschitz, E., 1989, *Physique théorique, Tome 2: Théorie des champs* (Moscow: MIR), fourth edition.

Landau, L., and Lifschitz, E., 1990, *Physique théorique, Tome 7: Elasticité* (Moscow: MIR), second edition.

Oberst, U., 1990, Multidimensional constant linear systems. *Acta Applicandae Mathematicae*, **20**, 1–175.

Maisonobe, P., and Sabbah, C., 1993, *$\mathcal{D}$-modules cohérents et holonomes*, Travaux en cours 45 (Paris: Hermann).

Palamodov, V. P., 1970, *Linear Differential Operators with Constant Coefficients*, Grundlehren der mathematischen Wissenschaften 168 (Berlin; Heidelberg; New York: Springer-Verlag).

Pham, F., 1980, *Singularités des systèmes différentiels de Gauss-Manin*, Progress in Mathematics, 2 (Boston: Birkhäuser).

Pillai, H., and Willems, J. C., 2002, Lossless and dissipative distributed systems. *SIAM Journal of Control and Optimization*, **40**, 1406–1430.

Pinch, E. R., 1993, *Optimal Control and the Calculus of Variations* (Oxford: Oxford University Press).

Pommaret, J.-F., 2001, *Partial Differential Control Theory* (Dordrecht; Boston; London: Kluwer).

Pommaret, J.-F., and Quadrat, A., 1998, Generalized Bezout identity. *Applicable Algebra in Engineering, Communication and Computing*, **9**, 91–116.

Pommaret, J.-F., and Quadrat, A., 1999a, Localization and parametrization of linear multidimensional control systems. *Systems and Control Letters*, **37**, 247–260.

Pommaret, J.-F., and Quadrat, A., 1999b, Algebraic analysis of linear multidimensional control systems. *IMA Jounal of Control and Information*, **16**, 275–297.

Pommaret, J.-F., and Quadrat, A., 2000a, Formal elimination for multidimensional systems and applications to control theory. *Mathematics of Control, Signals, and Systems*, **13**, 193–215.

Pommaret, J.-F., and Quadrat, A., 2000b, Equivalences of linear control systems. *MTNS 2000*, Perpignan (France), CDRom.

Quadrat, A., 1999, Analyse algébrique des systèmes de contrôle linéaires multidimensionnels. PhD thesis, Ecole Nationale des Ponts et Chaussées (France), 23/09.

Quadrat, A., and Robertz, D., 2004, On Monge problem for uncontrollable linear systems, in preparation.

Spencer, D. C., 1965, Overdetermined systems of partial differential equations. *Bulletin of the American Mathematics Society*, **75**, 1–114.

Ritt, J. F., 1950, *Differential Algebra*, AMS Colloq. Publ. 33.

Riquier, Ch., 1910, *La méthode des fonctions majorantes et les systèmes d'équations aux dérivées partielles*, Mémorial des Sciences Mathématiques XXXII (Paris: Gauthier-Villars).

Rotman, J. J., 1979, *An Introduction to Homological Algebra* (Orlando, FL: Academic Press).

Youla, D. C., and Gnavi, G., 1979, Notes on *n*-dimensional system theory. *IEEE Transactions on Circuits and Systems*, **26**, 105–111.

Youla, D. C., and Pickel, P. F., 1984, The Quillen–Suslin theorem and the structure of *n*-dimensional elementary polynomial matrices. *IEEE Transactions on Circuits and Systems*, **31**, 513–518.

Wood, J., 2000, Modules and behaviours in *n*D systems theory. *Multidimensional Dimensional Systems and Signal Processing*, **11**, 11–48.

Wood, J., Rogers, E., and Owens, H., 1998, Formal theory of matrix primeness. *Mathematics of Control, Signals, and Systems*, **11**, 40–78.

Zerz, E., 2000, *Topics in Multidimensional Linear Systems Theory*, Lecture Notes in Control and Information Sciences (London: Springer).