Math. Control Signals Systems (2000) 13: 193–215

Mathematics of Control, Signals, and Systems

Formal Elimination for Multidimensional Systems and Applications to Control Theory*

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Abstract. Following Douglas's ideas on the inverse problem of the calculus of variations, the purpose of this article is to show that one can use formal integrability theory to develop a theory of elimination for systems of partial differential equations and apply it to control theory. In particular, we consider linear systems of partial differential equations with variable coefficients and we show that we can organize the integrability conditions on the coefficients to build an "intrinsic tree". Trees of integrability conditions naturally appear when we test the structural properties of linear multidimensional control systems with some variable or unknown coefficients (controllability, observability, invertibility, ...) or for generic linearization of nonlinear systems.

Key words. Elimination theory, Multidimensional linear control systems with variable or unknown coefficients, Generic linearization of nonlinear control systems, Structural properties of control systems, Controllability, Trees of integrability conditions, Formal integrability, Differential modules.

1. Introduction

Expansion into power series of analytic or formal solutions of a system of partial differential equations (PDEs) has been an early powerful tool in mathematics, physics and engineering sciences. In particular, the wish to have a theory which computes the dimension of the space of the analytic (formal) solutions of a system of PDEs, without integrating it explicitly, is not new, as Einstein explained in 1952 [8]: ... *we need a method which gives a measure of the strength of a system of equations ... The set of numbers of "free" coefficients* (*derivatives of the field variables at a point*) *for all degrees of differentiation is directly a measure of the "weakness" of the system of equations, and through this, indirectly, also of its "strength"*. The dimension of solutions of a system of PDEs has been particularly studied by Riquier [29], Cartan [3] and Janet [15] during the years 1900–1930. In particular, Janet has developed effective algorithms in order to compute it without

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integrating the system explicitly. His work has inspired Ritt while he was creating *differential algebra* (see the last two chapters of [30] for an exposition of the Riquier and Janet works). More recently, and independently of these precursors, the *theory of formal integrability* has been developed in an intrinsic way by Spencer, Quillen and Goldschmidt, using the modern techniques of differential geometry [14], [24], [33].

In this paper we are mainly interested in elimination problems encountered in control theory. We consider a system of PDEs with two sets of dependent variables and we search all the conditions (differential equalities and inequalities) that the first set of variables has to satisfy in order that the system has a solution. In [32], Seidenberg gives a complete answer to this problem, using a differential algebra approach [19], [30]. The purpose of this article is to investigate how the theory of formal integrability can be used to solve this problem. This approach seems to be more intrinsic and allows one to have a new viewpoint on elimination theory. In particular, following the Spencer–Goldschmidt criterion developed in [14] and [33], only three kinds of differential equalities and inequalities may appear in elimination problems: the first one appears for rank conditions where we check the conditions of fibred manifolds, the second one occurs where we test the surjectivity of the projection of prolongations of the system of PDEs, and the last one appears where we check a more technical property, namely the 2-acyclicity of the symbol of the system. In the case of a linear system of PDEs with variable coefficients, these differential relations on the coefficients can be arranged in order to build an "intrinsic tree". Each final leaf of the tree represents a solution of the system of PDEs which depends on the differential relations (knots of the branches) that the variable coefficients of the system verify. Such a point of view was first adopted in 1941 by Douglas in his study of the inverse problem of the calculus of variations [7], using Janet's ideas and techniques [15]. We propose in this paper to reconsider this approach to elimination problems within the framework of the modern theory of formal integrability. A nice problem would be to revisit the results obtained by Douglas on the inverse problem of calculus of variation with the theory of formal integrability.

Recently, the theory of differential modules (D-modules) has given new insight into the structural properties of multidimensional linear control systems. See [9]–[11], [20], [22], [24]–[27], [35] and the references within. Most of the intrinsic properties of linear multidimensional control systems such as controllability, observability, primeness, poles and zeros have been reformulated in terms of the algebraic properties of D-modules. Formal tests using only formal duality and formal integrability have been developed in [24] and [26]–[28] to check these properties of modules. We show that if we consider linear control systems with variable or unknown coefficients or generic linearization of nonlinear ones, trees of integrability conditions naturally appear when we test these properties.

Finally, notice that, to our knowledge, *elimination theory* was first introduced in control theory by Diop in [4]–[6]. He uses for that the effective methods of differential algebra developed by Ritt and Kolchin [19], [30] and Seidenberg's results on elimination theory [32]. In [1], both differential algebra and Groebner bases [2] have been used to make recent improvements in the theory of elimination. These algebraic methods are in general more effective than those of formal integrability theory but less intrinsic (dependence on the coordinate system through the ranking, choice of differential polynomials in the characteristic sets). (See [23] for more details.) Roughly speaking, we can say that "*effectivity always competes with intrinsicness*".

2. Formal Integrability Theory

2.1. Introduction

We introduce the main ideas of formal integrability theory, before presenting them by using more technical tools. In the course of this paper, we only consider differentiable manifolds and maps.

If we want to look for the formal solutions of a system of PDEs, we have to know the number of "arbitrary" ("free", "parametric",... depending on the author) derivatives at each order. The solutions $y^k = f^k(x)$ of a system of PDEs of order q satisfy a certain number of equations, say $\Phi^\tau(x, \partial_\mu f^k(x)) = 0$, where $\tau = 1, \ldots, l$, $k = 1, \ldots, m$ and $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index with length $|\mu| = \mu_1 + \cdots + \mu_n$. We may replace the derivative of the unknown functions $f^k(x)$ solutions by jet coordinates with the same indices ($\partial_\mu f^k(x) \to y^k_\mu$)), that is to say, we regard any derivative of the f^k as new unknowns. We say that a jet coordinate is of order q if the length of its indice is lower than or equal to q, and of strict order q if the length of its indice is equal to q. Thus, $\Phi^\tau(x, \partial_\mu f^k(x)) = 0$ is transformed into a pure equation relating the jet coordinates $\Phi^\tau(x, y^k_\mu) = 0$. We suppose that these equations define a fibred manifold $\mathscr{R}_q$ (no relation among the x only) in the space of jet coordinates of order q. Using the implicit function theorem, we can locally determine certain jet coordinates as a function of $\dim \mathscr{R}_q$ (the fibre dimension) other jet coordinates (we try to write the greatest number of jet coordinates of order strictly equal to q as functions of jet coordinates of lower order). We call the first ones "*principal*" jet coordinates and the second "*parametric*" jet coordinates. Thus, we have made a partition of the jets of order q into two sets, the principal and parametric ones, where the first one can be expressed in terms of the second.

Now, we notice that if we differentiate once the equations of $\Phi^\tau(x, \partial_\mu f^k(x)) = 0$ with respect to each x^i (prolongation ρ_1), and replace again the derivatives by the jet coordinates, we obtain

$$d_i\Phi^\tau \equiv \frac{\partial \Phi^\tau}{\partial x^i} + \sum_{0 \le |\mu| \le q, k=1,\ldots,m} \frac{\partial \Phi^\tau}{\partial y^k_\mu} y^k_{\mu+1_i} = 0, \qquad i = 1, \ldots, n, \quad \tau = 1, \ldots, l. \quad (1)$$

Thus, the terms of strict order $q+1$ appear linearly with coefficients defined on $\mathscr{R}_q$, i.e. with jets satisfying $\Phi^\tau(x, y^k_\mu) = 0$. This simple remark allows us to use linear algebra. We define $\mathscr{R}_{q+1} = \rho_1(\mathscr{R}_q)$ by

$$\Phi^\tau = 0, \qquad d_i\Phi^\tau = 0, \qquad i = 1, \ldots, n, \quad \tau = 1, \ldots, l. \quad (2)$$

Now, we call M_{q+1} the vector space defined by

$$\sum_{|\mu|=q,k=1,\ldots,m} \frac{\partial \Phi^\tau}{\partial y_\mu^k} v_{\mu+1_i}^k = 0, \qquad i = 1, \ldots, n, \quad \tau = 1, \ldots, l, \tag{3}$$

in the jet coordinates of order strictly equal to $q+1$. There are $m(q+n)!/((q+1)!\,(n-1)!)$ jet coordinates of order strictly equal to $q+1$ and if we denote by $\sigma_1(\Phi)$ the matrix in the left member of (3), then we have

$$\dim M_{q+1} = \frac{m\,(q+n)!}{(q+1)!\,(n-1)!} - \operatorname{rk} \sigma_1(\Phi)$$

parametric jet coordinates of strict order $q+1$. Indeed, we can find in (3), $\operatorname{rk} \sigma_1(\Phi)$ lineary independent equations and, by linear algebra in the upper part of (2) and substitution of the principal jet coordinates of order q by the parametric ones, we obtain $\operatorname{rk} \sigma_1(\Phi)$ principal jet coordinates of strict order $q+1$, which can be expressed with $\dim M_{q+1}$ parametric jet coordinates of strict order $q+1$ and with $\dim \mathscr{R}_q$ ones of order q.

Now, the trouble begins if $\operatorname{rk} \sigma_1(\Phi) < l \times n$: we have certain equations of (3) which are linear combinations of $\operatorname{rk} \sigma_1(\Phi)$ others. Eliminating the jets of order $q+1$ in the corresponding equations of (2), we obtain equations of order q. Only two different cases may happen:

- Substituting the principal jets of order q in these new equations, we are led to 0, then we have no new equations relating the parametric jet coordinates up to the order q. Thus, we have determined for the moment the number of parametric jet coordinates of strict order $q+1$ and obtained $l \times n - \operatorname{rk} \sigma_1(\Phi)$ identities of the form

$$\sum_{i,\tau} A_\tau^{ip}(x, y_\mu^k)\, d_i \Phi^\tau + B_\tau^p(x, y_\mu^k) \Phi^\tau = 0, \qquad 0 \le |\mu| \le q.$$

  We notice that it leads to compatibility conditions in the linear case.
- Substituting the principal jet of order q in these new equations, we are led to some nonidentically zero equations $\Psi^\alpha(x, y_\mu^k) = 0$, $|\mu| \le q$, relating the parametric jet coordinates of order q. This contradicts the fact that they are parametric jet coordinates. Then we have to add these new equations to the system $\Phi^\tau(x, y_\mu^k) = 0$ and start anew with the following system:

$$\mathscr{R}_q^{(1)} \quad \begin{cases} \Phi^\tau(x, y_\mu^k) = 0, \\ \Psi^\alpha(x, y_\mu^k) = 0. \end{cases} \tag{4}$$

We have just shown how to compute the number of parametric jet coordinates of order $q+1$. It can be done similarly for each order. We have seen that the feedback of information on the lower-order derivatives (new equations $\Psi^\alpha(x, y_\mu^k) = 0$) modifies the calculus of the number of parametric jet coordinates and thus the calculus of the dimension of the space of solutions (the parametric jet coordinates determine the initial conditions that we have to fix to compute the power series of the solutions). Hence, certain systems of PDEs seem to be "nicer" than some others, that is, those in which no feedback of information on the lower-order derivatives appears when differentiating the equations of the system and projecting them onto lower-order jet spaces. Hence, we call a system of PDEs *formally integrable* whenever the formal power series of its solutions can be determined step by step by successive derivations without obtaining backward new information on lower-order derivatives. We may wonder how to recognize when a system of PDEs is formally integrable, as we have to verify that no new lower-order information appears at each order, that is, for an infinity of orders. So, we can ask: does there exist a finite algorithm testing whether a system of PDEs is formally integrable or not? In the case where the system is not formally integrable, we have seen that we have to add new equations. So, does there exist a procedure which adds enough equations to the system, in order to transform it into a formal integrable system, with the same solutions? During the years 1960–1975, Spencer and coworkers have given positive answers [14], [33] that we now present.

2.2. *Main Results*

We denote by X a manifold of dimension n with local coordinates $(x^1, \ldots, x^n)$, and by $T(X)$ and $T^\star(X)$, its tangent and cotangent bundles. Let $\mathscr{E}$ be a fibred manifold over X with fibre dimension m and local coordinates (x^i, y^k). We define the q-jet bundle $J_q(\mathscr{E})$ as a fibred manifold with local coordinates (x, y^k_μ), $\mu = (\mu_1, \ldots, \mu_n)$, $0 \leq |\mu| \leq q$, and a *nonlinear system of PDEs* of order q as a fibred submanifold $\mathscr{R}_q$ of $J_q(\mathscr{E})$, determined locally by $\Phi^\tau(x, y^k_\mu) = 0$. The *r-prolongation* of $\mathscr{R}_q$ is $\mathscr{R}_{q+r} = \rho_r(\mathscr{R}_q) = J_r(\mathscr{R}_q) \cap J_{r+q}(\mathscr{E})$, and is obtained by substituting the jet coordinates by the derivatives, differentiating r times and substituting again the derivatives by jet coordinates. The projection $\pi^{q+r+s}_{q+r} : J_{q+r+s}(\mathscr{E}) \to J_{q+r}(\mathscr{E})$ induces a projection of $\mathscr{R}_{q+r+s}$ on $\mathscr{R}_{q+r}$. We denote the image of this projection by $\mathscr{R}^{(s)}_{q+r}$. Notice that $\mathscr{R}_{q+r}$ and $\mathscr{R}^{(s)}_{q+r}$ are not in general fibred manifolds for any $r, s \geq 0$. The *linearized system* R_q of $\mathscr{R}_q$ is locally defined by

$$\sum_{0 \leq |\mu| \leq q, k=1,\ldots,m} \frac{\partial \Phi^\tau(x, y^k_\mu)}{\partial y^k_\mu} v^k_\mu = 0, \qquad \tau = 1, \ldots, l.$$

It is a linear system in v^k_μ with variable coefficients satisfying $\Phi^\tau(x, y^k_\mu) = 0$. We define the symbol M_q of $\mathscr{R}_q$, as the family of vector spaces over $\mathscr{R}_q$, by

$$\sum_{|\mu|=q, k=1,\ldots,m} \frac{\partial \Phi^\tau}{\partial y^k_\mu} v^k_\mu = 0, \qquad \tau = 1, \ldots, l, \tag{5}$$

and we denote by $\sigma(\Phi)$ the corresponding matrix. Then the symbol M_{q+r} of $\mathscr{R}_{q+r}$ is defined by

$$\sum_{|\mu|=q, |\nu|=r, k=1,\ldots,m} \frac{\partial \Phi^\tau}{\partial y^k_\mu} v^k_{\mu+\nu} = 0, \qquad \tau = 1, \ldots, l, \tag{6}$$

and only depends on M_q. We call $\sigma_r(\Phi) = \sigma(\rho_r(\Phi))$ the matrix in the left member of (6). We define the *δ-sequence* by

$$\Lambda^s T^\star \otimes M_{q+r+1} \stackrel{\delta}{\longrightarrow} \Lambda^{s+1} T^\star \otimes M_{q+r},$$

where

$$(\delta(\omega))^k_\nu = dx^i \wedge \omega^k_{\nu+1_i}, \qquad |\nu| = q + r,$$

where $\omega = (\omega^k_\mu = v^k_{\mu,I}\, dx^I) \in \Lambda^s T^\star \otimes M_{q+r+1}$, $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_s}$, $i_1 < \cdots < i_s$, $1 \leq k \leq m$ and $|\mu| = q + r + 1$. We verify that $\delta \circ \delta = 0$. The cohomology at $\Lambda^s T^\star \otimes M_{q+r}$ of the sequence

$$\Lambda^{s-1} T^\star \otimes M_{q+r+1} \xrightarrow{\delta} \Lambda^s T^\star \otimes M_{q+r} \xrightarrow{\delta} \Lambda^{s+1} T^\star \otimes M_{q+r-1}$$

is denoted by $H^s_{q+r}(M_q)$.

Definition 1. The symbol M_q of $\mathcal{R}_q$ is said to be *s-acyclic* if $H^1_{q+r}(M_q) = \cdots = H^s_{q+r}(M_q) = 0$, for all $r \geq 0$ and *involutive* if it is n-acyclic. In particular, every system $\mathcal{R}_q$ of ordinary differential equations (ODEs) has an involutive symbol. A symbol M_q is *of finite type* if there exists an integer r such that $M_{q+r} = 0$.

Theorem 1. *Let M_q be the symbol of the system $\mathcal{R}_q$, then there exists an integer r large enough such that M_{q+r} is involutive.*

A test checking the 2-acyclicity of the symbol is still lacking. Indeed, we have to verify $H^2_{q+r}(M_q) = 0$ for any $r \geq 0$, and thus for an infinity of orders. Only the case of a finite type symbol can be checked as we only have to verify $H^2_q(M_q) = \cdots = H^2_{q+r-1}(M_q) = 0$ when $M_{q+r} = 0$ and $M_{q+r-1} \neq 0$. Nevertheless, we can test whether a symbol is involutive or not. However, it must only be done in "sufficently generic coordinates": the *δ-regular coordinates*. Roughly speaking, the δ-regular coordinates are not the most generic coordinates but "generic enough" to give the right dimension of various spaces. We say that v^k_μ is of class $i > 1$ if $\mu_1 = \cdots = \mu_{i-1} = 0$ and $\mu_i > 0$ and of class 1 if $\mu_1 > 0$. Now, using the equations defining M_q, we try to express the maximum number of v^k_μ of class n, in function of the other v^l_ν. Next, we substitute these v^k_μ in the other equations to make the v^k_μ of class n disappear. We respectively do the same for the v^k_μ of class $n-1, \ldots, 1$. We usually say that M_q is in *the solved form*. We associate a system of "dots" to these equations, as follows:

equations of class n	1	$\cdots$	$\cdots$	$\cdots$	n
equations of class $n-1$	1	$\cdots$	$\cdots$	$n-1$	$\bullet$
$\vdots$					
equations of class i	1	$\cdots$	i	$\bullet$	$\bullet$
$\vdots$					
equations of class 1	1	$\bullet$	$\cdots$	$\cdots$	$\bullet$

Though this classification looks like Janet's original one, it is in fact quite different. For a detailed study, we refer the reader to [12] and [13]. Moreover, let M^i_q be the vector space locally defined by $\sigma(\Phi)$ where we have the v^k_μ of class strictly lower than i equal to zero. We call $\sigma(\Phi)^i$ the matrix of the equations defining M^i_q.

We have

$$\dim M_q^i = \frac{m(q+n-i-1)!}{(q-1)!\,(n-i)!} - \operatorname{rk}\sigma(\Phi)^i, \qquad i = 0, \ldots, n.$$

We set

$$\alpha_q^i = \dim M_q^{i-1} - \dim M_q^i, \qquad i = 1, \ldots, n. \tag{7}$$

Theorem 2. *The symbol M_q is involutive if there exists a system of coordinates, called δ-regular coordinates, in which one of the following properties is satisfied*:

1. $\dim M_{q+1} = \alpha_q^1 + 2\alpha_q^2 + \cdots + n\alpha_q^n$.
2. *Prolongations with respect to the dots do not bring new equations.*

Then we have

$$\dim M_{q+r} = \sum_{i=1}^{n} \frac{(r+i-1)!}{r!\,(i-1)!} \alpha_q^i, \qquad \forall r \geq 0.$$

We have seen that a "good system" $\mathscr{R}_q$ of PDEs was a system in which no lower-order information appeared when projecting its prolongations $\mathscr{R}_{q+r+s} = \rho_{r+s}(\mathscr{R}_q)$ on lower-order jet space $J_{q+r}(\mathscr{E})$. Using the previous notation, it leads to the following definition.

Definition 2. A system $\mathscr{R}_q$ is called *formally integrable* if $\mathscr{R}_{q+r}$ is a fibred manifold and the projection $\pi_{q+r}^{q+r+s} : \mathscr{R}_{q+r+s} \to \mathscr{R}_{q+r}$ is surjective, for all $r, s \geq 0$, i.e. $\mathscr{R}_{q+r}^{(s)} = \mathscr{R}_{q+r}$.

A system $\mathscr{R}_q$ is called *involutive* if $\mathscr{R}_q$ is formally integrable with an involutive symbol M_q.

Spencer–Goldschmidt Criterion. If M_q is 2-acyclic and $\mathscr{R}_{q+1}$ is a fibred manifold such that $\mathscr{R}_q^{(1)} = \mathscr{R}_q$, then $\mathscr{R}_q$ is formally integrable.

The reader has to keep in mind that the previous criterion gives sufficient but not necessary conditions in order to have a formally integrable system, as is shown in the following example.

Example 1. The system $\partial_i \xi^j + \partial_j \xi^i = (2/n)\omega_{ij}\,\partial_r \xi^r$, $i, j, r = 1, \ldots, n$, is neither 2-acyclic nor involutive but the first prolongation becomes 2-acyclic when $n \geq 4$ and the system is formally integrable. More generally, any homogeneous system is formally integrable even if the criterion is not satisfied.

We have the following corollary. See [24] for a proof.

Corollary 1. *Let $\mathscr{R}_q$ be an involutive system of PDEs and let $\mathscr{R}_{q-1}$ be the projection of $\mathscr{R}_q$ on $J_{q-1}(\mathscr{E})$, then*

$$\dim \mathscr{R}_{q+r} = \dim \mathscr{R}_{q-1} + \sum_{i=1}^{n} \frac{(r+i)!}{r!\,i!} \alpha_q^i.$$

Therefore, we have to fix α_q^1 functions in x^1, α_q^2 functions in $(x^1, x^2), \ldots,$ and α_q^n functions in $(x^1, \ldots, x^n)$ to determine a formal solution of $\mathcal{R}_q$.

Definition 3. A system $\mathcal{R}_q$ is called *sufficiently regular* if:

1. $\mathcal{R}_{q+r}^{(s)}$ is a fibred manifold, for all $r, s \geq 0$.
2. The symbol $M_{q+r}^{(s)}$ is induced from a vector bundle over X, for all $r, s \geq 0$.

If $\mathcal{R}_q$ is not formally integrable, then there exists a finite procedure which gives a formally integrable system with the same solutions as $\mathcal{R}_q$ [14], [24], [33].

Theorem 3. *If $\mathcal{R}_q$ is a sufficiently regular system, we can find two integers, r, $s \geq 0$, such that $\mathcal{R}_{q+r}^{(s)}$ is formally integrable (involutive) with the same solutions as $\mathcal{R}_q$.*

Algorithm ("up and down"). We start with $\mathcal{R}_q$. Find $r \geq 0$ such that $\mathcal{R}_{q+r}$ is 2-acyclic (involutive). Test whether $\mathcal{R}_{q+r}^{(1)} = \mathcal{R}_{q+r}$. If it is the case, then the algorithm stops, else, start anew with $\mathcal{R}_{q+r}^{(1)}$. We stop whenever we have found two integers r, s such that $\mathcal{R}_{q+r}^{(s)}$ is a formally integrable system (involutive).

2.3. *Formal Elimination Theory*

We consider the following system of PDEs of order q,

$$\Phi^\tau(x, y_\mu^k, z_\nu^l) = 0, \qquad 0 \leq |\mu|, \quad |\nu| \leq q, \quad \tau = 1, \ldots, t, \tag{8}$$

where $y = (y^1, \ldots, y^r)$ and $z = (z^1, \ldots, z^s)$ are two sets of unknowns. We would like to determine the differential relations that z has to satisfy in order that (8) admits formal solutions. For that, we look at (8) as a system of PDEs in the unknowns y, with coefficients in z_ν, $0 \leq |\nu| \leq q$:

$$\Psi^\tau(x, y_\mu^k) = 0, \qquad 0 \leq |\mu| \leq q, \quad \tau = 1, \ldots, t. \tag{9}$$

Suppose that z is given, we can locally find the formal solutions of (9) by bringing it to formal integrability. However, in doing this, we have to check certain rank conditions which give differential relations (equalities and inequalities) that z has to satisfy in order that (9) admits formal solutions. These differential relations belong to three different classes depending on whether they are obtained

(1) in checking fibred manifold conditions,
(2) in projecting prolongations of the system on lower-order jets bundles,
(3) in testing the 2-acyclicity (or involutivity) of certain symbols.

However, the third kind is a rather "technical one" because the definition of formal integrability does not need the 2-acyclicity of the symbol but only fibred manifolds and projections conditions. However, most of the time, we have to use the Spencer–Goldschmidt criterion for which the 2-acyclicity (or the involutivity) has to be tested.

The following example illustrates the first two kinds of differential relations. This example is taken from [6] where the elimination has been done by differential algebra techniques. Difficulties may arise even if the equations are linear in y, though not in y, z, the z playing the role of arbitrary coefficients.

Example 2. We consider the system defined by

$$R_1 \quad \begin{cases} \dot{z}^1 - uz^2 = 0, \\ \dot{z}^2 - z^1 - uz^2 = 0, \\ z^1 - y = 0. \end{cases} \tag{10}$$

In the control framework u is the input, z is the state and y is the ouput and we look for input–output relations by eliminating the state z. The system R_1 is not formally integrable in $z = (z^1, z^2)$. As this system is a system of ODEs, we know that the symbol $M_1 = 0$ is trivially involutive and we have only to saturate the system by lower-order equations. We have

$$R_1^{(1)} \quad \begin{cases} \dot{z}^1 - uz^2 = 0, \\ \dot{z}^2 - z^1 - uz^2 = 0, \\ z^1 - y = 0, \\ uz^2 - \dot{y} = 0. \end{cases}$$

1. If $u = 0$, then $R_1^{(1)}$ is defined by

$$\begin{cases} \dot{z}^1 = 0, \\ \dot{z}^2 - z^1 = 0, \\ z^1 - y = 0, \\ \dot{y} = 0, \end{cases}$$

   and $R_1^{(1)}$ is a fibred manifold. In this case, we have $R_1^{(2)} = R_1^{(1)}$ and $R_1^{(1)}$ is an involutive system. Moreover, $\dim R_1^{(1)} = \dim M_1^{(1)} + \dim R_0^{(1)} = 0 + (2 - 1) = 1$, where $R_0^{(1)}$ is the projection of $R_1^{(1)}$ on $J_0(E)$ (i.e., the zero-order equations of the system $R_1^{(1)}$).
2. If $u \neq 0$, then $R_1^{(2)}$ is a strict subset of $R_1^{(1)}$, defined by

$$R_1^{(2)} \quad \begin{cases} \dot{z}^1 - uz^2 = 0, \\ \dot{z}^2 - z^1 - uz^2 = 0, \\ z^1 - y = 0, \\ uz^2 - \dot{y} = 0, \\ u\ddot{y} - (\dot{u} + u^2)\dot{y} - u^2 y = 0. \end{cases}$$

   $R_1^{(2)}$ is a fibred manifold iff $u\ddot{y} - (\dot{u} + u^2)\dot{y} - u^2 y = 0$ and, in this case, $R_1^{(2)}$ is an involutive system. Moreover, $\dim R_1^{(2)} = \dim M_1^{(2)} + \dim R_0^{(2)} = 0 + (2 - 2) = 0$.

We can notice that the dimension of the fibre is generically equal to zero and the dimension jumps to one in the differentially algebraic set $\{u = 0, \dot{y} = 0\}$.

Finally, the input–output behaviour [34] is the disjoint union of the two following systems:

$$\begin{cases} u = 0, \\ \dot{y} = 0, \end{cases} \qquad \begin{cases} u \neq 0, \\ u\ddot{y} - (\dot{u} + u^2)\dot{y} - u^2 y = 0. \end{cases}$$

2.3.1. *Trees of Integrability Conditions*

The dimension of the formal solutions of a system of linear PDEs with variable coefficients highly depends on differential relations that these coefficients satisfy. In the case of linear PDEs with variable coefficients, we can organize these integrability conditions in order to build a tree. Each final leaf of the tree represents a solution of the system of PDEs which depends on the differential relations (knots of the branches) that the variable coefficients of the system verify. We use in all the examples the well-known notation of derivatives $\partial_i \partial_j y = y_{ij}$.

Example 3. We define the following system:

$$R_2 \quad \begin{cases} y_{33} - a y_{11} = 0, \\ y_{23} = 0, \\ y_{22} - b y_{11} = 0, \\ y_{13} = 0, \\ y_{12} = 0, \end{cases} \tag{11}$$

where a and $b \in \mathbb{R}$. We have the following multiplicative variables:

$$M_2 \quad \begin{cases} v_{33} - a v_{11} = 0, \\ v_{23} = 0, \\ v_{22} - b v_{11} = 0, \\ v_{13} = 0, \\ v_{12} = 0. \end{cases} \qquad \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ 1 & \bullet & \bullet \\ \hline \end{array} \tag{12}$$

If we prolong with respect to the dots, we find two new equations: $a v_{111} = 0$ and $b v_{111} = 0$. Thus M_2 is involutive if $a = b = 0$. Else, if we prolong once the symbol M_2, we obtain $M_3 = 0$, i.e. M_2 is a finite type and M_3 is a trivial involutive symbol. In that case, we can easily check whether the symbol M_2 is 2-acyclic or not: we have to compute the cohomology $H_2^2(M_2)$ of the following sequence:

$$0 \longrightarrow \Lambda^2 T^\star \otimes M_2 \xrightarrow{\delta} \Lambda^3 T^\star \otimes T^\star.$$

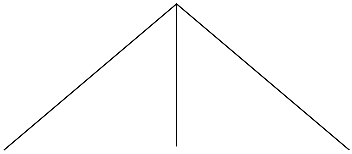
Thus, we only have to check under what conditions on a and b, δ is injective, with

$$\forall \omega = v_{kl,ij}\, dx^i \wedge dx^j \in \Lambda^2 T^\star \otimes M_2,$$

$$\delta(\omega)_k = (v_{k3,12} + v_{k1,23} + v_{k2,31})\, dx^1 \wedge dx^2 \wedge dx^3.$$

Thus $\delta(\omega) = 0$ with $v_{kl} \in M_2 \Rightarrow v_{11,23} = v_{22,31} = v_{33,12} = 0 \Rightarrow a v_{11,12} = 0,\ b v_{11,31} = 0$ and δ is injective iff $a \neq 0$ and $b \neq 0$. In this case, M_2 is 2-acyclic but not involutive otherwise we would have the exact sequence $\cdots \longrightarrow \Lambda^2 T^\star \otimes M_3 \xrightarrow{\delta} \Lambda^3 T^\star \otimes M_2 \longrightarrow 0$ and thus $M_3 = 0 \Rightarrow M_2 = 0$, which is obviously not true.

We obtain the following tree of integrability conditions:



$$\begin{cases} a = 0, \\ b = 0. \end{cases} \qquad \begin{cases} \max(|a|,|b|) \neq 0, \\ \min(|a|,|b|) = 0. \end{cases} \qquad \begin{cases} a \neq 0, \\ b \neq 0. \end{cases}$$

M_2 involutive, $M_3 \neq 0$ — M_2 not 2-acyclic, $M_3 = 0$ involutive — M_2 2-acyclic, M_2 not involutive, $M_3 = 0$ involutive

1. If $a = 0$, $b = 0$, then M_2 is involutive and we easily see that $R_2^{(1)} = R_2$. Thus, R_2 is an involutive system. Moreover, $\dim M_2^0 = 1, \dim M_2^1 = 0$, $\dim M_2^2 = 0 \Rightarrow \alpha_2^1 = 1$, $\alpha_2^2 = 0$, $\alpha_2^3 = 0$. Thus $\dim M_{2+r} = \dim R_{2+r} = 1$, $\forall r \geq 0$. We find the compatibility conditions of

$$\begin{cases} y_{33} = z^1, \\ y_{23} = z^2, \\ y_{22} = z^3, \\ y_{13} = z^4, \\ y_{12} = z^5, \end{cases} \qquad \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ 1 & \bullet & \bullet \end{matrix}} \tag{13}$$

by derivating the equations with respect to the dots and projecting on the) system R_2. This computation naturally leads to the following inhomogeneous *first-order* system:

$$\begin{cases} z_3^2 - z_2^1 = t^1, \\ z_3^3 - z_2^2 = t^2, \\ z_3^5 - z_1^2 = t^3, \\ z_3^4 - z_1^1 = t^4, \\ z_2^4 - z_1^2 = t^5, \\ z_2^5 - z_1^3 = t^6. \end{cases} \qquad \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \end{matrix}} \tag{14}$$

We let the reader check that the corresponding homogeneous system is involutive (it is a general property of involutive systems [23]). Differentiating the equations with respect to the dots, we obtain two compatibility conditions:

$$\begin{cases} t_3^5 - t_2^4 + t_1^1 = s^1, \\ t_3^6 - t_2^2 + t_1^2 = s^2. \end{cases} \qquad \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{matrix}} \tag{15}$$

This system does not have compatibility conditions. We have just built the *Janet sequence* of the operator $\mathscr{D}_0 : y \to z$ defined by (13). We have

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathscr{D}_0} F_1 \xrightarrow{\mathscr{D}_1} F_2 \xrightarrow{\mathscr{D}_2} F_3 \longrightarrow 0,$$

where Θ is the kernel of $\mathscr{D}_0$ and the operators $\mathscr{D}_1 : z \to t$ and $\mathscr{D}_2 : t \to s$ are defined by (14) and (15).

2. If $a \neq 0$, $b \neq 0$ (for example $a = b = 1$), then M_2 is 2-acyclic and $R_2^{(1)} = R_2$. Hence, the system is formally integrable. In this case, we can compute the compatibility conditions of

$$\begin{cases} y_{33} - a y_{11} = z^1, \\ y_{23} = z^2, \\ y_{22} - b y_{11} = z^3, \\ y_{13} = z^4, \\ y_{12} = z^5, \end{cases} \tag{16}$$

by computing $R_2^{(1)} = R_2$. We find only five homogeneous *first-order* compatibility conditions, defined by

$$\begin{cases} z_3^2 - z_2^1 - a z_1^5 = 0, \\ z_3^3 - z_2^2 + b z_3^4 = 0, \\ z_3^5 - z_1^2 = 0, \\ b z_3^4 - a z_2^5 - b z_1^1 + a z_1^3 = 0, \\ z_2^4 - z_1^2 = 0. \end{cases}$$

3. Finally, if $\max(|a|, |b|) \neq 0$ and $\min(|a|, |b|) = 0$, then M_2 is not 2-acyclic and we have to prolong the system and see whether or not $R_3^{(1)} = R_3$, as we already know that $M_3 = 0$ is a trivial involutive symbol. We let the reader check that it is the case and R_3 is an involutive system. We suppose that $a \neq 0$ and $b = 0$. Computing the compatibility conditions by differentiating with respect to the dots of M_3 and projecting on R_3, we find six homogeneous *first- and second-order* compatibility conditions:

$$\begin{cases} z_{33}^4 - z_{13}^1 - a z_{11}^4 = 0, \\ z_3^5 - z_1^2 = 0, \\ z_2^5 - z_1^3 = 0, \\ z_3^3 - z_2^2 = 0, \\ z_3^2 - z_2^1 - a z_1^5 = 0, \\ z_4^2 - z_1^2 = 0. \end{cases}$$

Notice that R_2 is formally integrable ($\mathscr{R}_{2+r+s}$ is a fibred manifold and $\mathscr{R}_{2+r+s} \to \mathscr{R}_{2+r}$ is surjective for all $r, s \geq 0$) even if the Spencer–Goldschmidt criterion is not satisfied (see Example 1).

Notice that a simple change of the parameters a and b has totally changed the compatibility conditions of the system R_2 (the number and the orders). Moreover, in that example, the 2-acyclicity of M_2 is a generic property. Obviously, we can find examples combining the three kinds of inequations. *Such examples are extremely rare*.

3. Applications to Control Theory

3.1. Controllability

Let $\mathscr{D} : E \to F$ be a linear differential operator, where E and F are two vector bundles over X of fibre dimensions m and l. The differential operator $\mathscr{D}$ is called *injective* if $\mathscr{D}\eta = 0 \Rightarrow \eta = 0$ and *surjective* if the equations $\mathscr{D}\eta = 0$ are linearly differential independent [19], [30], or, equivalently, if $\mathscr{D}\eta = \zeta$ has no *compatibility conditions*, i.e. there is no nontrivial operator $\mathscr{D}_1$ such that $\mathscr{D}\eta = \zeta \Rightarrow \mathscr{D}_1\zeta = 0$ [14], [24], [33]. The sequence $E \xrightarrow{\mathscr{D}} F_0 \xrightarrow{\mathscr{D}_1} F_1$ is *formally exact* if $\mathscr{D}_1$ generates all the compatibility conditions of $\mathscr{D}$. In this case, $\mathscr{D}_1$ is said to be *parametrizable* by $\mathscr{D}$ and the arbitrary functions ξ such that $\mathscr{D}\xi = \eta \Rightarrow \mathscr{D}_1\eta = 0$ are called *potentials*.

Let K be a differential field with n commuting derivations $\partial_1, \ldots, \partial_n$, i.e. $\partial_i\partial_j = \partial_j\partial_i$, and containing $\mathbb{Q}$. We denote by $D = K[d_1, \ldots, d_n]$ the ring of scalar differential operators with coefficients in K, satisfying

$$d_i(bd_j) = bd_id_j + (\partial_i b)d_j, \qquad \forall b \in K.$$

In general, the ring D is a noncommutative integral domain, but if K is a *field of constants*, i.e. $\forall a \in K$, $\partial_i a = 0$, $\forall i = 1, \ldots, n$, then D is commutative. Moreover, D possesses the left (resp. right) *Ore property*, namely, $\forall P, Q \in D$, there exists $R, S \in D\backslash 0$ (resp. U, V) such that $RP = SQ$ (resp. $PU = QV$) [27]. Let $\eta = \{\eta^1, \ldots, \eta^m\}$ and we form the free left D-module $D\eta^1 + \cdots + D\eta^m$, denoted by $D\eta \simeq D^m$. Every element of $D\eta$ has the form $\sum_{0 \le |\mu| < \infty, k=1,\ldots,m} a_k^\mu d_\mu \eta^k$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index.

A fundamental idea is to associate to any differential operator $\mathscr{D} : E \to F$ the left D-module

$$M = D\eta / D(\mathscr{D}\eta),$$

and we say that $\mathscr{D}$ determines the left D-module M.

We recall that an element z of M is called a *torsion element* of M if there exists a nonzero element P of D such that $Pz = 0$. We denote by $t(M)$ the D-submodule formed by all the torsion elements of M which is called the *torsion submodule* of M. We say that a D-module M is *torsion-free* if $t(M) = 0$ [31]. Therefore, the D-module $M/t(M)$ is always torsion-free.

Controllability is one of the key concepts of control theory which goes back to Kalman's work [17]. We recall certain recent improvements in this direction. We call *observable* any function of the system variables (inputs and outputs) and their derivatives. Only two possibilities may happen for an observable: it may or may not verify a PDE by itself. An observable which does not satisfy any PDE is called *free* or *unconstrained* [24]. On the contrary, an observable which satisfies at least a PDE is called *autonomous* or *constrained* [24], [34]. In [24] we can find the following definition.

Definition 4. A control system is said to be *controllable* if every observable of the system is free.

A characterization of controllability is given in [24] in terms of differential closure. In [20], the same definition has been found independently for linear multidimensional control systems into the differential modules framework. See also [9], [10], and [34]. Accordingly, a linear multidimensional control system is controllable if it determines a torsion-free D-module M.

Hence, we need an algorithm which allows us to check effectively if a D-module M determined by a differential operator $\mathscr{D} : E \to F$ is torsion-free or not. For this, we defined the formal duality of differential operators. We denote the dual bundle of E by $E^\star$ and its adjoint bundle by $\tilde{E} = \bigwedge^n T^\star \otimes E^\star$. If $\mathscr{D} : E \to F$ is a linear differential operator, then its *formal adjoint* $\tilde{\mathscr{D}} : \tilde{F} \to \tilde{E}$ is defined by the following formal rules:

- The adjoint of a matrix (zero-order operator) is the transposed matrix.
- The adjoint of ∂_i is $-\partial_i$.
- For two linear PD operators P, Q that can be composed: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

We can easily check that the following identity $(\tilde{\tilde{\mathscr{D}}}) = \mathscr{D}$, and one can prove that

$$\langle \mu, \mathscr{D}\xi \rangle = \langle \tilde{\mathscr{D}}\mu, \xi \rangle + d(\cdot),$$

where d is the exterior derivative. We can directly compute the adjoint of an operator by multiplying it by a row vector of test functions and integrating the result by parts.

Example 4. Let the differential operator $\mathscr{D} : \xi \to \eta$ defined by $\partial_{33}\xi^1 - x^2\partial_{11}\xi^1 - \partial_{22}\xi^2 = \eta$. Multiplying the system by the row function λ and integrating the result by parts, we find that the adjoint operator $\tilde{\mathscr{D}} : \mu \to \nu$ of $\mathscr{D}$ is defined by

$$\begin{cases} \partial_{33}\mu - x^2\partial_{11}\mu = \nu_1, \\ \partial_{22}\mu = \nu_2. \end{cases}$$

We let the reader check by himself that the compatibility condition of $\tilde{\mathscr{D}}$ is formed by two equations of order 3 and 6 [24]. This operator $\tilde{\mathscr{D}}$ is a famous example of Janet [15].

We have the following theorem. See [24] and [27] for proof and examples (see [31] for a definition of the ext functor). See also [18] and [21] for more general results.

Theorem 4. *The following assertions are equivalent*:

- *A control system, defined by an operator* $\mathscr{D}_1$, *is controllable.*
- *The operator* $\mathscr{D}_1$ *determines a torsion-free D-module M.*
- *The operator* $\mathscr{D}_1$ *is parametrizable by an operator* $\mathscr{D}_0$.
- $\mathrm{ext}^1_D(N, D) = 0$, *where N is the left D-module defined by* $\tilde{\mathscr{D}}_1$.

Hence, we have the following test to check whether an operator $\mathscr{D}_1$ determines or not a torsion-free D-module M:

1. Start with $\mathscr{D}_1$.
2. Construct its adjoint $\tilde{\mathscr{D}}_1$.
3. Find the compatibility conditions of $\tilde{\mathscr{D}}_1\lambda = \mu$ and denote this operator by $\tilde{\mathscr{D}}_0$.
4. Construct its adjoint $\mathscr{D}_0 = \widetilde{(\tilde{\mathscr{D}}_0)}$.
5. Find the compatibility conditions of $\mathscr{D}_0\xi = \eta$ and denote this operator by $\mathscr{D}_1'$.

We are led to two different cases:

- If $\mathscr{D}_1$ generates all the compatibility conditions of $\mathscr{D}_0$, then the system $\mathscr{D}_1$ determines a torsion-free D-module M and $\mathscr{D}_0$ is a parametrization of $\mathscr{D}_1$.
- Otherwise, the operator $\mathscr{D}_1$ does not describe all the compatibility conditions of $\mathscr{D}_0$. The torsion elements of M are formed by all the new single compatibility conditions in $\mathscr{D}_1'$ modulo the equations $\mathscr{D}_1\eta = 0$.

See [24], [26], and [28] for a proof of this test. We can represent the test by the following differential sequences where the number indicates the different stages:

$$
\begin{array}{ccccccc}
 & & & & \xrightarrow{\mathscr{D}_1'} & F_1' & 5 \\
4 & E & \xrightarrow{\mathscr{D}_0} & F_0 & \xrightarrow{\mathscr{D}_1} & F_1 & 1 \\
3 & \tilde{E} & \xleftarrow{\tilde{\mathscr{D}}_0} & \tilde{F}_0 & \xleftarrow{\tilde{\mathscr{D}}_1} & \tilde{F}_1 & 2
\end{array}
$$

Corollary 2. *The controllability of a linear multidimensional control system with variable coefficients depends at most on two trees of integrability conditions* (*the first one in computing* $\tilde{\mathscr{D}}_0$ *and the other in determining* $\mathscr{D}_1'$).

Notice that linear multidimensional control systems with variable coefficients naturally appear when we linearize nonlinear control systems with respect to generic solutions. One can easily prove that the controllability of the (generically) linearized system implies the controllability of the nonlinear one. Thus, this gives a sufficient condition of controllability for nonlinear multidimensional systems.

Example 5. We consider the finite transformation $y = f(x)$ satisfying the Pfaffian system:

$$dy^3 - a(y^2)\,dy^1 = \rho(x)(dx^3 - a(x^2)\,dx^1).$$

Linearizing such a transformation around the identity by setting $y = x + t\xi(x) + \cdots$ and making $t \to 0$, after eliminating $\rho(x)$, we discover that infinitesimal transformations are defined, through the use of a correct geometric object, by the kernel of the differential system $\mathscr{D}_0\xi = \eta$ as follows:

$$\begin{cases} -a(x^2)\partial_1\xi^1 + \partial_1\xi^3 + \frac{1}{2}a(x^2)(\partial_1\xi^1 + \partial_2\xi^2 + \partial_3\xi^3) - \xi^2\partial_2 a(x^2) = \eta^1, \\ -a(x^2)\partial_2\xi^1 + \partial_2\xi^3 = \eta^2, \\ -a(x^2)\partial_3\xi^1 + \partial_3\xi^3 - \frac{1}{2}(\partial_1\xi^1 + \partial_2\xi^2 + \partial_3\xi^3) = \eta^3. \end{cases}$$

See p. 237 of [24] for more details. From the theory of Lie pseudogroups [24], we can prove that the PD system $\mathscr{D}_0\xi = 0$ is formally integrable if and only if $\partial_2 a(x^2) = c = \text{cst}$, the "classical case" of *contact transformations* corresponding to $a(x^2) = x^2$ ($\Rightarrow c = 1$). It follows that the only compatibility condition $\mathscr{D}_1\eta = 0$ is

$$-a(x^2)(\partial_2\eta^3 - \partial_3\eta^2) + \partial_1\eta^2 - \partial_2\eta^1 + \partial_2 a(x^2)\eta^3 = 0,$$

and the operator $\mathscr{D}_1$ is surjective. The adjoint operator $\tilde{\mathscr{D}}_1$ is then defined by

$$\begin{cases} \partial_2\lambda = \mu_1, \\ -a(x^2)\partial_3\lambda - \partial_1\lambda = \mu_2, \\ a(x^2)\partial_2\lambda + 2c\lambda = \mu_3. \end{cases}$$

As $\mu_3 - a(x^2)\mu_1 = 2c\lambda$, the operator $\tilde{\mathscr{D}}_1$ is injective if and only if $c \neq 0$. In that case, the two independent compatibility conditions can be written by

$$\begin{cases} \partial_2\mu_3 - a(x^2)\partial_2\mu_1 - 3c\mu_1 = 2\nu_2, \\ -a(x^2)\partial_3(\mu_3 - a(x^2)\mu_1) - \partial_1(\mu_3 - a(x^2)\mu_1) - 2c\mu_2 = -2(\nu_1 + a(x^2)\nu_3), \end{cases}$$

after introducting the adjoint $\tilde{\mathscr{D}}_0$ of $\mathscr{D}_0$ as follows:

$$\begin{cases} \frac{1}{2}a(x^2)\partial_1\mu_1 + \frac{1}{2}\partial_1\mu_3 + a(x^2)\partial_3\mu_3 + a(x^2)\partial_2\mu_2 + \partial_2 a(x^2)\mu_2 = \nu_1, \\ -\frac{1}{2}a(x^2)\partial_2\mu_1 + \frac{1}{2}\partial_2\mu_3 - \frac{3}{2}\partial_2 a(x^2)\mu_1 = \nu_2, \\ -\partial_1\mu_1 - \frac{1}{2}a(x^2)\partial_3\mu_1 - \partial_2\mu_2 - \frac{1}{2}\partial_3\mu_3 = \nu_3. \end{cases}$$

Now, we start with the operator $\tilde{\mathscr{D}}_0$ depending on the arbitrary function $a(x^2)$ and we ask about the algebraic property of $M = (D\mu)/D(\tilde{\mathscr{D}}_0\mu)$. According to the test, we must construct the adjoint of $\tilde{\mathscr{D}}_0$ which is $\mathscr{D}_0$ and look for its compatibility conditions $\mathscr{D}_1$, a result bringing out the condition $\partial_2 a(x^2) = c$, where c is an arbitary constant. When $c = 0$, we should find the zero-order compatibility condition $\mu_3 - a(x^2)\mu_1 = 0$ which is not a consequence of $\tilde{\mathscr{D}}_0$ and thus the D-module M has torsion elements. Indeed, we can easy verify that the nonzero element $z = \mu_3 - a(x^2)\mu_1 \in M$ satisfies $\partial_2 z = 0$. When $c \neq 0$, the adjoint $\tilde{\mathscr{D}}_1$ admits the compatibility condition expressed by $\tilde{\mathscr{D}}_0$ because we have in that case

$$a(x^2)\partial_3\nu_2 - \partial_2\nu_1 + \partial_1\nu_2 - a(x^2)\partial_2\nu_3 - 2c\nu_3 = 0,$$

which gives $c\nu_3 = \frac{1}{2}((\partial_1 + a(x^2)\partial_3)\nu_2 - \partial_2(\nu_1 + a(x^2)\nu_3))$ and M is a torsion-free D-module. Such an example with two trees of integrability conditions is *very rare* and was first obtained in [25].

We recall that a D-module M is *free* if there exists a basis of M, i.e. a set of elements which forms a generative and a D-independent family of M. The following proposition is just a rephrasing of the definition of free modules in the differential operators context.

Proposition 1. *An operator $\mathscr{D}_1$ determines a free D-module M iff there exists an injective parametrization $\mathscr{D}_0$ of $\mathscr{D}_1$, i.e. a formally exact sequence* $0 \longrightarrow E \xrightarrow{\mathscr{D}_0} F_0 \xrightarrow{\mathscr{D}_1} F_1$.

Recall that a module M is a *projective* if there exist a module N and a free module F such that $F \cong M \oplus N$ [31]. Notice that N is in his turn a projective module. We have the following proposition. We refer to [26] for a proof of the equivalences, a generalization of this theorem to nonsurjective operators and for applications of projective modules to control theory.

Theorem 5. *The following assertions are equivalent*:

- *A surjective operator $\mathscr{D}_1 : F_0 \to F_1$ determines a projective D-module M.*
- *The adjoint operator $\tilde{\mathscr{D}}_1$ is injective.*
- *There exists an operator $P_1 : F_1 \to F_0$ such that $\mathscr{D}_1 \circ \mathscr{P}_1 = \mathrm{id}_{F_1}$, where id_{F_1} is the identity operator of F_1.*

In particular, projectiveness depends at least on a single tree of integrability conditions.

We let the reader check by himself that every free module is projective and every projective module is torsion-free, which can be summed up by the following inclusions:

$$\text{free} \subseteq \text{projective} \subseteq \text{torsion-free}.$$

For a principal ideal domain (for example, $D = K[d/dt]$), every torsion-free module is a free module. In 1976 Quillen and Suslin independently showed the 1950 Serre conjecture, claiming that every projective module over a polynomial ring $k[\chi_1, \ldots, \chi_n]$ (k a field) is free. See [31] and [36] for more details. In particular, this result is true for the ring $D = k[d_1, \ldots, d_n]$ if k is a field of constants. Thus, a surjective differential operator $\mathscr{D}_1 : F_0 \to F_1$ with constant coefficients determines a free D-module M iff $\tilde{\mathscr{D}}_1$ is injective. We notice that it is quite often supposed that the inputs of a control system are linearly differentially independent [19], [30], a fact that leads to the surjectivity of the corresponding operator. Then we have the following useful corollary:

Corollary 3. *The following assertions are equivalent*:

- *An OD control system, defined by a surjective operator $\mathscr{D}_1$, is controllable.*
- *The adjoint operator $\tilde{\mathscr{D}}_1$ is injective.*
- *The D-module M determined by $\mathscr{D}_1$ is free.*

In particular, the controllability of a linear OD control system depends on a single tree of integrability conditions.

Example 6. We study the controllability of the system defined by

$$\begin{cases} \ddot{y}^1 + y^1 - y^2 + \alpha u = 0, \\ \ddot{y}^2 + y^2 - y^1 - u = 0, \end{cases} \tag{17}$$

where α is a real parameter. Dualizing the surjective operator $\mathscr{D}_1$, we obtain that the operator $\tilde{\mathscr{D}}_1$ is defined by

$$\begin{cases} \ddot{\lambda}_1 + \lambda_1 - \lambda_2 = \mu_1, \\ \ddot{\lambda}_2 + \lambda_2 - \lambda_1 = \mu_2, \\ -\lambda_2 + \alpha\lambda_1 = \mu_3. \end{cases} \tag{18}$$

We put $\mu_1 = \mu_2 = \mu_3 = 0$ and bring (18) to formal integrability to obtain the new zero-order equation:

$$(\alpha + 1)(\alpha - 1)\lambda_1 = 0.$$

Thus $\tilde{\mathscr{D}}_1$ is injective and the system (17) is controllable iff $\alpha \neq -1$ and $\alpha \neq 1$. For example, if $\alpha = -1$, then we get a torsion element $z = y^1 - y^2$ which satisfies $(d^2/dt^2 + 2)z = 0$. A tree with more branches of integrability conditions has been exhibited in [25] for an ordinary time-varying control system. See [27] for the link between torsion elements and first integrals of motion.

We give an example showing that the algebraic properties of a D-module, determined by a linear PD system with variable or unknown coefficients, depends on integrability conditions.

Example 7. Let $\mathscr{D}_1 : \eta \to \zeta$ be the operator defined by

$$\partial_2\eta^1 - \alpha\partial_1\eta^1 - \partial_2\eta^2 + a(x)\eta^2 = \zeta, \tag{19}$$

where α is a real parameter. We determine how the algebraic properties of the D-module M, determined by $\mathscr{D}_1$, depend on the coefficients α and $a(x)$. Dualizing $\mathscr{D}_1$, we obtain the operator $\tilde{\mathscr{D}}_1 : \lambda \to \mu$ defined by

$$\begin{cases} -\partial_2\lambda + \alpha\partial_1\lambda = \mu_1, \\ \partial_2\lambda + a(x)\lambda = \mu_2, \end{cases}$$

which can be rearranged under the following form:

$$\begin{cases} \partial_2\lambda + a(x)\lambda = \mu_2, \\ \alpha\partial_1\lambda + a(x)\lambda = \mu_1 + \mu_2. \end{cases}$$

We put $\mu_1 = \mu_2 = 0$ and call R_1 the corresponding system:

$$R_1 \quad \begin{cases} \partial_2\lambda + a(x)\lambda = 0, \\ \alpha\partial_1\lambda + a(x)\lambda = 0. \end{cases} \tag{20}$$

We study the formal integrability of the system R_1. First, M_1 is an involutive symbol for any α, but its dimension depends on whether $\alpha = 0$ or not.

1. If $\alpha = 0$, then $\dim M_1 = 2 - 1 = 1$ and M_1 is involutive. The dimension of R_1 depends on whether or not $a = 0$.
   (a) If $a = 0$, then $\dim R_1 = 1$ and R_1 is formally integrable and we easily find that M has the torsion element $z = \eta^1 - \eta^2$ which satisfies $\partial_2 z = 0$.
   (b) If $a \neq 0$, then $\tilde{\mathscr{D}}_1$ is an injective operator. Thus $\mathscr{D}_1$ determines a projective D-module and we have $\tilde{\mathscr{P}}_1 : \mu \to \lambda$ defined by $(\mu_1 + \mu_2)/a(x) = \lambda$. Dualizing, we obtain that the right-inverse $\mathscr{P}_1 : \zeta \to \eta$ of $\mathscr{D}_1$ is defined by

$$\begin{cases} \zeta/a(x) = \eta^1, \\ \zeta/a(x) = \eta^2. \end{cases}$$

   We let the reader check that a parametrization $\mathscr{D}_0 : \xi \to \eta$ of $\mathscr{D}_1$ is defined by

$$\begin{cases} -a(x)\partial_2\xi + (a(x)^2 - 2\partial_2 a(x))\xi = \eta^1, \\ -a(x)\partial_2\xi - 2\partial_2 a(x)\xi = \eta^2. \end{cases}$$

   We easily see that $\mathscr{D}_0$ is injective with $\xi = (\eta^1 - \eta^2)/a(x)^2$, and thus $M \cong D\xi$.

2. If $\alpha \neq 0$, then $\dim M_1 = 2 - 2 = 0$ and $M_1 = 0$ is a trivial involutive symbol. Therefore, we just have to study the projection $\pi_1^2 : R_2 \to R_1$, i.e.

$$R_1^{(1)} \quad \begin{cases} \lambda_2 + a(x)\lambda = 0, \\ \alpha\lambda_1 + a(x)\lambda = 0, \\ (\partial_2 a(x) - \alpha\partial_1 a(x))\lambda = 0. \end{cases}$$

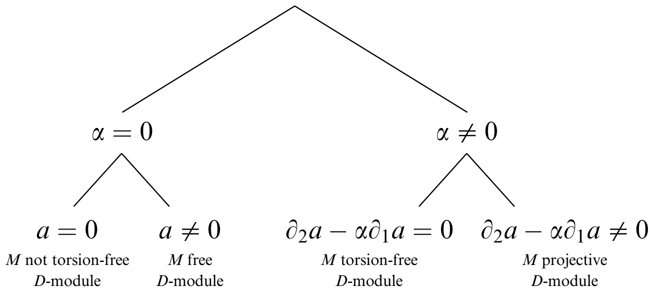
   The dimension of $R_0^{(1)}$ depends on whether $\partial_2 a(x) - \alpha\partial_1 a(x) = 0$ or not.
   (a) If $\partial_2 a(x) - \alpha\partial_1 a(x) = 0$, then $\mathscr{D}_1$ does not determine a projective D-module M. However, we easily find a parametrization $\mathscr{D} : \xi \to \eta$ defined by

$$\begin{cases} \partial_2\xi - a(x)\xi = \eta^1, \\ \partial_2\xi - \alpha\partial_1\xi = \eta^2. \end{cases}$$

   Thus $\mathscr{D}_1$ determines a torsion-free but not a projective D-module M.
   (b) If $\partial_2 a(x) - \alpha\partial_1 a(x) \neq 0$, then $\tilde{\mathscr{D}}_1$ is an injective operator and $\mathscr{D}_1$ determines a projective D-module M. We let the reader check that $\lambda = (\partial_2\mu_1 + \partial_2\mu_2 - \alpha\partial_1\mu_2 + a(x)\mu_1)/(\partial_2 a(x) - \alpha\partial_1 a(x))$. We denote $\varphi = \zeta/(\partial_2 a(x) - \alpha\partial_1 a(x))$, then the right-inverse $\mathscr{P}_1 : \zeta \to \eta$ of $\mathscr{D}_1$ is defined by

$$\begin{cases} -\partial_2\varphi + a(x)\varphi = \eta^1, \\ -\partial_2\varphi + \alpha\partial_1\varphi = \eta^2. \end{cases}$$

We can sum up the previous study by the following tree of integrability conditions:



Each of the previously defined properties of modules is obtained and illustrated in the above tree. *Such an example has never been provided before*.

3.2. *Observability*

Another key concept in control theory is *observability* [16]. It has recently been reformulated into the differential operators and the D-module frameworks in [5], [20], and [24] as follows.

Definition 5. Let a control system be defined by a differential operator $\mathscr{D}_1 : \eta \to \zeta$ and let $\eta = (\eta', \eta'')$. Then the system is called *observable with respect* to η' if every component of η'' can be expressed as a linear combination of the components of η' and their derivatives.

Proposition 2. *Let $\mathscr{D}_1 : F_0 \to F_1$ be a differential operator and let M be the D-module determined by $\mathscr{D}_1$. We denote by F_0' (resp. F_0'') the subbundle with sections $(\eta', 0)$ (resp. $(0, \eta'')$) and by $\mathscr{D}_1' : F_0' \to F_1$ and $\mathscr{D}_1'' : F_0'' \to F_1$ the induced operators. Hence, we have $\mathscr{D}_1 \eta = \mathscr{D}_1' \eta' + \mathscr{D}_1'' \eta''$. The control system defined by $\mathscr{D}_1$ is observable with respect to η' iff we have one of the following equivalent assertions*:

- *The operator $\mathscr{D}_1'' : F_0'' \to F_1$ is injective. In this case we have $\eta'' = -(\mathscr{P}_1'' \circ \mathscr{D}_1')\eta'$, where $\mathscr{P}_1'' : \zeta'' \to (0, \eta'')$ is a right-inverse of $\mathscr{D}_1''$.*
- *$M = (D\eta' + D\eta'')/D(\mathscr{D}_1 \eta) \cong (D\eta')/D((\mathscr{D}_2'' \circ \mathscr{D}_1')\eta')$, where $\mathscr{D}_2''$ denotes all the compatibility conditions of $\mathscr{D}_1''$.*

From the above proposition, we only have to study the formal integrability of a single system of PDEs and thus we have the following corollary:

Corollary 4. *The observability of a control system with variable or unknown coefficients depends at most on a single tree of integrability conditions.*

Example 8. We consider the following control system:

$$\begin{cases} \ddot{x}^1 + x^1 - x^2 - u^1 = 0, \\ \ddot{x}^2 + x^2 - x^1 - u^2 = 0, \\ y = x^2 - \alpha x^1, \end{cases}$$

where $x = (x^1, x^2)$ is the state, $u = (u^1, u^2)$ is the input, y is the output and $\alpha \in \mathbb{R}$. This system is observable with respect to (y, u) if x^1 and x^2 are linear combinations of y and u and their derivatives, i.e. iff the operator $\mathscr{D}_1'' : x \to \zeta$ defined by

$$\begin{cases} \ddot{x}^1 + x^1 - x^2 = \zeta^1, \\ \ddot{x}^2 + x^2 - x^1 = \zeta^2, \\ -x^2 + \alpha x^1 = \zeta^3, \end{cases}$$

is injective. We recognize that this operator is the dual of the operator $\mathscr{D}_1$ in Example 6 and thus the system is observable iff $\alpha \neq -1$ and $\alpha \neq 1$. If $\alpha \neq -1$ and $\alpha \neq 1$, we have

$$\begin{cases} x^1 = \dfrac{1}{(\alpha+1)(\alpha-1)}(\ddot{y} + (\alpha+1)y + u^2 - \alpha u^1), \\ x^2 = \dfrac{\alpha}{(\alpha+1)(\alpha-1)}(\ddot{y} + (\alpha+1)y + u^2 - \alpha u^1) - y. \end{cases}$$

Many others properties of control systems have been reformulated into the *D*-modules framework. For example:

- Computation of the *differential transcendence degree*: we bring the system to involutiveness and compute the last character $\alpha_q^n \Rightarrow$ computation of *output rank* $\Rightarrow$ invertibility [5].
- *State elimination*: we bring the system in (z, y, u) to formal integrability in z only to obtain the input–output behaviour of the system (see Example 2).
- *Structure at infinity*: we bring the system in (y, u) to formal integrability in u [24].

Thus, these properties depend on trees of integrability conditions if the system has variable coefficients or parameters. We also refer the reader to [5] for other properties of control systems which depend on trees of integrability conditions.

4. Conclusion

In this article we have developed a theory of differential elimination based on formal integrability theory and we have applied it to control theory. We hope that we have convinced the reader that this approach seems to be natural for many control problems and in particular for studying the structural properties of linear multidimensional control systems with variable coefficients. As we have already noticed, the effective character always competes with the intrinsic character, and thus we think that this approach of the theory of elimination will give more intrinsic results than the purely differential algebraic methods developed in [32]. We also think that an interesting problem should be to revisit Douglas's classification of the inverse problem of the calculus of variations [7] in this modern framework.

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