An introduction to internal stabilization of infinite-dimensional linear systems

A. Quadrat

***Abstract*— In these notes, we give a short introduction to the fractional representation approach to analysis and synthesis problems [12], [14], [17], [28], [29], [50], [71], [77], [78]. In particular, using *algebraic analysis* (commutative algebra, module theory, homological algebra, Banach algebras), we shall give necessary and sufficient conditions for a plant to be internally stabilizable or to admit (weakly) left/right/doubly coprime factorizations. Moreover, we shall explicitely characterize all the rings A of SISO stable plants such that every plant $-$ defined by means of a transfer matrix with entries in the quotient field $K = Q(A)$ of A $-$ satisfies one of the previous properties (e.g. internal stabilization, (weakly) doubly coprime factorizations). Using the previous results, we shall show how to parametrize all stabilizing controllers of an internally stabilizable plants which does not necessarily admits a doubly coprime factorization. Finally, we shall give some necessary and sufficient conditions so that a plant is strongly stabilizable (i.e. stabilizable by a stable controller) and prove that every internally stabilizable MIMO plant over $A = H_\infty(\mathbb{C}_+)$ is strongly stabilizable.**

***Index Terms*— Fractional representation approach to analysis and synthesis problems, internal stabilization, (weak) left/right/doubly coprime factorizations, parametrization of all stabilizing controllers, strong/simultaneous/robust stabilization, algebraic analysis, module theory, theory of fractional ideals, homological algebra, Banach algebras, stable range, $H_\infty(\mathbb{C}_+)$.**

I. A BRIEF INTRODUCTION

For the twentieth anniversary of the paper “Algebraic and topological aspects of feedback stabilization” by M. Vidyasagar, M. Schneider and H. Francis, published in IEEE Transactions on Automatic Control (August 1982) [77], we want to present in these notes certain of its main ideas as well as to give a personal overview of their recent progress.

The impacts of this paper, as well as the book [78], are difficult to evaluate in the present research [79], [83]. However, we can easily say that certain ideas of [77], [78] (fractional representation of systems, internal stabilization, Youla-Kučera parametrization of the stabilizing controllers, strong and simultaneous stabilizations, graph approach to plants, graph topology, margins of robustness…) have been at the core of the successful development of H_∞-control for finite-dimensional linear systems [20], [25], [29] in the nineties. We refer to [2], [41] for nice surveys about stabilization problems for finite-dimensional systems.

The question of the possibility to extend certain of the previous results to infinite-dimensional linear systems (e.g. delay systems, partial differential equations, convolution systems) was naturally asked in [17], [77] (see also the last chapter of [78]). However, the larger the class of systems becomes, the more difficult it is to give a general answer concerning these problems (internal stabilization, existence of doubly coprime factorizations, parametrization of the stabilizing controllers…). Hence, certain parts of the program developed in [17], [77] for infinite-dimensional linear systems are still in progress nowadays (see e.g. [14], [28], [44], [47], [56], [57], [58], [60], [69], [71], [73] and the references therein).

In these notes, we shall mainly focus on the following general questions [77], [78]:

1) Does it exist necessary and sufficient conditions to internal stabilization?
2) When can we parametrize all stabilizing controllers of a plant by means of the well-known Youla-Kučera parametrization?
3) Can we characterize all the rings A of single input single output (SISO) stable plants so that every transfer matrix $-$ defined by a matrix with entries in the quotient field $K = Q(A)$ of A $-$ is internally stabilizable?

For lack of space, we shall not have the possibility to develop certain results such as equivalences of external and internal closed-loop stability [14], [44], graph approach to plants (see [28], [83] and the references therein), graph topology, margins of robustness [14], H_2 or H_∞-optimal controllers [14], [29], [50]. Moreover, we shall only use an input-output approach to systems via transfer matrices as it is developed in [11], [12], [17], [50], [77], [78]. We refer to [14], [44] for the link between the frequency-domain approach and the state-space one (e.g. stabilizable and detectable state-space systems, Pritchard-Salamon class of systems). More generally, we refer the reader to the following nice references [14], [44], [50], [79], [83] for complementary information and bibliographies.

Throughout this paper, we shall denote by A a *commutative integral domain* [31], [66] (namely a ring with an identity which satisfies $\forall\, a,\, b \in A,\; a\, b = b\, a$ and $a\, b = 0,\; b \neq 0 \Rightarrow a = 0$), the *group of the units* of A by

$$\mathrm{U}(A) = \{a \in A \mid \exists\, b \in A :\; a\, b = 1\}$$

and the *quotient field* of A by:

$$K = Q(A) = \{a/b \mid 0 \neq b,\; a \in A\}.$$

By convention, every vector of A^n is a row vector. Moreover, $A^{q\times p}$ will denote the set of the $q \times p$ matrices with entries in A and

$$\mathrm{GL}_p(A) = \{U \in A^{p\times p} \mid \exists\, V \in A^{p\times p} :\; U\, V = V\, U = I_p\}$$

the group of invertible $p \times p$ matrices of $A^{p\times p}$ and I_p its identity. If $R \in A^{p\times p}$, then R^T will denote the transposed matrix and $(a_1, \ldots, a_n)$ the ideal $A\, a_1 + \ldots + A\, a_n$ of A.

INRIA Sophia Antipolis, CAFE project, 2004 route des lucioles, BP 93, 06902 Sophia Antipolis cedex, France. Alban.Quadrat@sophia.inria.fr

Finally, p, q and r will always be three positive integers satisfying $p = q + r$ and $\triangleq$ will mean 'by definition'.

II. The fractional representation approach to systems

"…As soon as I read this, my immediate reaction was 'What is so difficult about handling that case? All one has to do is to write the unstable part as a ratio of two stable rational functions!' Without exaggeration, I can say that the idea occured to me within no more than 10 min. So there it is $-$ the best idea I have had in my entire research career, and it took less than 10 min. All the thousands of hours I have spent thinking about problems in control theory since have not resulted in any ideas as good as this one. I don't think I know what the 'moral of this story' really is !", M. Vidyasagar [79].

The fractional representation approach to systems is an input-output theory based on the idea that the algebraic structure of a class of single input single output (SISO) plants needs to be a ring if we want to put two systems in connection ($\times$) and in parallel (+) [84]. Moreover, in the seventies, M. Vidyasagar [76], C. Desoer and coauthors [15] introduced the idea to consider an *integral domain A of SISO stable plants* in order to represent an unstable plant as a ratio of two stable plants, i.e. as an element of the *quotient field* of A, namely

$$K = Q(A) = \{n/d \mid 0 \neq d,\, n \in A\}$$

(see [79] for a historical survey). Examples of integral domains of SISO stable plants, usually encountered in the literature, are the following ones.

Example 2.1:

- The *ring of proper stable real rational functions* [41], [78]

$$RH_\infty = \{n/d \mid 0 \neq d,\, n \in \mathbb{R}[s],\, \deg n \leq \deg d,\quad d(s^\star) = 0 \Rightarrow \mathrm{Re}\,(s^\star) < 0\}. \tag{1}$$

  A transfer function p belongs to RH_∞ iff p is the transfer function of an exponentially stable time-invariant finite-dimensional SISO linear system.

- The *Hardy algebra* of bounded holomorphic functions on the open right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$, i.e.

$$H_\infty(\mathbb{C}_+) = \{f \in \mathcal{H}(\mathbb{C}_+) \mid \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}, \tag{2}$$

  where $\mathcal{H}(\mathbb{C}_+)$ denotes the ring of holomorphic functions in $\mathbb{C}_+$ [14], [84]. A transfer function p belongs to $H_\infty(\mathbb{C}_+)$ iff

$$\| \, p \, \|_\infty = \sup_{0 \neq u \in H_2(\mathbb{C}_+)} \frac{\| \, p\, u \, \|_2}{\| \, u \, \|_2} < +\infty,$$

  where $H_2(\mathbb{C}_+)$ is the Hilbert space of the holomorphic functions in $\mathbb{C}_+$ which are bounded w.r.t. the norm:

$$\| \, f \, \|_2^2 = \sup_{\mathrm{Re}\, x > 0} \int_{-\infty}^{+\infty} |f(x + i\, y)|^2 \, dy.$$

  Let us recall that $H_2(\mathbb{C}_+) = \mathcal{L}(L_2(\mathbb{R}_+))$, where $\mathcal{L}(\cdot)$ denotes the Laplace transform. Hence, p belongs to $H_\infty(\mathbb{C}_+)$ iff p is the transfer function of a $L_2(\mathbb{R}_+)$-stable time-invariant infinite-dimensional SISO system [14].

- The *Wiener algebra* defined by

$$\mathcal{A} = \{h(t) = f(t) + \textstyle\sum_{i=0}^{+\infty} a_i \, \delta_{t - t_i} \mid f \in L_1(\mathbb{R}_+),\quad (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\, 0 = t_0 < t_1 < t_2 < \ldots,\, t_i \in \mathbb{R}_+\} \tag{3}$$

  where h is bounded w.r.t. the norm:

$$\begin{aligned} \| \, h \, \|_{\mathcal{A}} &= \| \, f \, \|_{L_1(\mathbb{R}_+)} + \| \, (a_i)_{i \geq 0} \, \|_{l_1(\mathbb{Z}_+)} \\ &= \textstyle\int_0^{+\infty} |f(t)| \, dt + \sum_{i=0}^{+\infty} |a_i|. \end{aligned}$$

  Then, h belongs to $\mathcal{A}$ iff h is the impulse response of a $L_\infty(\mathbb{R}_+)$-stable time-invariant infinite-dimensional SISO linear system (BIBO stability) [11], [14]. Let us also consider the integral domain $\hat{\mathcal{A}} = \{\mathcal{L}(f) \mid f \in \mathcal{A}\}$ of transfer functions of BIBO stable time-invariant infinite-dimensional SISO linear systems [11], [14] .

- Let W_+ be the commutative integral domain of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ whose Taylor series converge absolutely:

$$W_+ = \{(a_i)_{i \geq 0},\ a_i \in k = \mathbb{R},\ \mathbb{C} \mid \sum_{i=0}^{+\infty} |a_i| < +\infty\}. \tag{4}$$

  Then, $p \in W_+$ iff p is the unit-pulse response of a BIBO-stable causal digital filter, i.e. W_+ is the *algebra of the bounded input bounded output (BIBO) causal digital filters* [78]:

- Let $M_{\mathbb{D}^n}$ be the *ring of structural stable multidimensional linear systems*, namely

$$\begin{aligned} M_{\mathbb{D}^n} = \{n/d \mid \quad & 0 \neq d,\, n \in \mathbb{R}[z_1, \ldots, z_n], \\ & d(z) = 0 \Rightarrow z \in \mathbb{C}^n \backslash \overline{\mathbb{D}}^n\}, \end{aligned} \tag{5}$$

  where $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n \mid |z_i| \leq 1,\, i = 1, \ldots, n\}$ is the closed unit polydisc of $\mathbb{C}^n$. See [43] and the references therein.

See [8], [9], [30], [34], [45], [82] for other examples of rings used in stabilization problems.

Example 2.2: Let us consider $A = RH_\infty$ and the transfer function $p = 1/(s-1)$. We easily check that $p \notin A$ because p has a pole in $1 \in \mathbb{C}_+$ (unstable pole). However, we have $p \in K = Q(A) = \mathbb{R}(s)$ because $p = n/d$, where:

$$\left\{ \begin{array}{l} n = 1/(s+1) \in A, \\ d = (s-1)/(s+1) \in A. \end{array} \right.$$

Testing the stability of a transfer function $p \in K = Q(A)$ becomes a membership problem: testing whether or not $p \in A$. By extension, a multi input multi output (MIMO) system is defined by means of a transfer matrix P whose entries belong to the quotient field $K = Q(A)$ of a certain integral domain A of SISO stable plants. Hence, if we have $P \in K^{q \times r}$, then we can always write P as

$$P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1},$$

where:

$$\left\{ \begin{array}{l} R = (D \,:\, -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}. \end{array} \right.$$

For instance, we can always take $D = d\, I_q$ and $\tilde{D} = d\, I_r$, where d is the product of the denominators of all the entries of P.

Example 2.3: Let us consider $A = H_\infty(\mathbb{C}_+)$, $K = Q(A)$ and the following transfer matrix with entries in K:

$$P = \left(\begin{array}{c} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{array}\right). \qquad (6)$$

We easily see that we have $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}$, where $R = (D\, :\, -N) \in A^{2\times 3}$ and $\tilde{R} = (\tilde{N}^T\, :\, \tilde{D}^T)^T \in A^{3\times 1}$ are for instance defined by:

$$\left\{\begin{array}{l} R = \left(\begin{array}{ccc} \frac{(s-1)^2}{(s+1)^2} & 0 & -\frac{(s-1)\, e^{-s}}{(s+1)^2} \\ 0 & \frac{(s-1)^2}{(s+1)^2} & -\frac{e^{-s}}{(s+1)^2} \end{array}\right), \\ \tilde{R} = \left(\begin{array}{c} \frac{(s-1)\, e^{-s}}{(s+1)^2} \\ \frac{e^{-s}}{(s+1)^2} \\ \frac{(s-1)^2}{(s+1)^2} \end{array}\right). \end{array}\right. \qquad (7)$$

In the fractional representation approach, instead of the transfer matrix $y = P\, u$, we usually prefer to study the system $D\, y - N\, u = 0$, i.e. $R\, z = 0$, where $P = D^{-1}\, N \in K^{q\times r}$, $R = (D\, :\, -N) \in A^{q\times p}$ and $z = (y^T\, :\, u^T)^T$. The idea to only consider the input and output variables together, i.e. without any separation between the inputs and the outputs, is similar to the module theory and the behavioural approaches to linear multidimensional systems (see [13], [22], [52], [53] and references therein). Hence, the structural properties of the plant, defined by P, can be studied by means of the linear system $R\, z = 0$ whose coefficients belong to a ring A. This can be achieved using *linear algebra over the ring* A (e.g. testing the existence of a right/left/doubly coprime factorization, invariant factors, equivalences…). However, linear algebra over a ring is a part of the *module theory* [5], [6], [7], [31], [66]. Therefore, it seems to be quite natural to introduce module theory into the study of linear systems. This idea is quite old and R. E. Kalman seems to be the first person who has used module theory in linear control theory during the sixties (see [38] and the references therein). Since this pioneering work, module theory has been more and more used in linear control theory (see [13], [22], [23], [24], [53] and the references therein). But, as surprising as it might be, module theory has only recently been introduced into fractional representation approach to analysis and synthesis problems in the pioneering work of V. R. Sule [73] (see also [69]) and, up to our knowledge, has only been developed since then in [47], [54], [55], [56], [58], [59], [61]. Let us recall the definition of an A-module (see [5], [31], [66] for more informations).

Definition 2.1: An *A-module* M over a ring A is a set M with two operations, namely an addition $+ : M \times M \longrightarrow M$, defined by

$$(m_1,\, m_2) \longmapsto m_1 + m_2,$$

and a scalar multiplication $A \times M \longrightarrow M$, defined by

$$(a,\, m) \longmapsto a\, m,$$

which satisfy

1) $m_1 + m_2 = m_2 + m_1$,
2) $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$,
3) $\exists\, 0 \in M,\ \forall\, m \in M :\ m + 0 = m$,
4) $\forall\, m \in M,\ \exists\, (-m) \in M :\ m + (-m) = 0$,
5) $a\,(m_1 + m_2) = a\, m_1 + a\, m_2$,
6) $(a + b)\, m = a\, m + b\, m$,
7) $(a\, b)\, m = a\,(b\, m)$,
8) $1\, m = m$,

for all $m,\, m_1,\, m_2,\, m_3 \in M$ and $a,\, b \in A$.

A *submodule* N of an A-module M is a subset N of M which also satisfies 1, 2, 3, 4 and:

$$\forall\, a \in A :\ a\, N = \{a\, n \mid n \in N\} \subseteq N.$$

Hence, an A-module shares the same definition as a k-vector space with the only distinction that the scalars belong to a ring A in the case of a module whereas they belong to a *field* k (i.e. a commutative ring such that every non-zero element has an inverse for the product) in the case of a vector space. This small difference implies huge ones in the respective theories (module theory and linear algebra over a field) that can be easily understood if we notice that an A-module has generally no basis. Indeed, if we want to obtain a basis of a k-vector space defined by a non minimal family of generators, we need to invert certain coefficients of k to obtain an independent subfamily of generators, i.e. a basis. But, if the scalars belong to a ring A instead of a field k, they generally do not admit inverses in A, and thus, we cannot generally obtain a basis from a family of generators.

Example 2.4: 1) If A is a commutative ring, then, for all $n \in \mathbb{Z}_+$, A^n is an A-module:

$$\forall\, \lambda_1,\, \lambda_2 \in A^n,\ \forall\, a_1,\, a_2 \in A :\ \ a_1\, \lambda_1 + a_2\, \lambda_2 \in A^n.$$

Let e_i be the vector of A^n defined by 1 in the i^{th} component and 0 for all the others. Then, $\{e_1, \ldots, e_n\}$ is a basis of A^n because every $\lambda = (\lambda_1 : \ldots : \lambda_n) \in A^n$ can be uniquely written as $\lambda = \sum_{i=1}^n \lambda_i\, e_i$. This basis is called the *canonical basis* of A^n.

2) If $f : M \longrightarrow N$ is an *A-morphism*, namely an A-linear application from the A-module M to the A-module N, i.e. $\forall\, \lambda_1,\, \lambda_2 \in M,\ \forall\, a_1,\, a_2 \in A$:

$$f(a_1\, \lambda_1 + a_2\, \lambda_2) = a_1\, f(\lambda_1) + a_2\, f(\lambda_2),$$

then

$$\left\{\begin{array}{l} \ker f = \{m \in M \mid f(m) = 0\}, \\ \operatorname{im} f = \{n \in N \mid \exists\, m \in M :\ n = f(m)\}, \\ \operatorname{coker} f = N/\operatorname{im} f, \end{array}\right.$$

$-$ where $N/\operatorname{im} f$ is the *quotient A-module* obtained by identifying two elements n_1 and n_2 of N if there exists $m \in M$ such that $n_1 - n_2 = f(m)$ $-$ are three A-modules [5], [31], [66].

3) Let $H_2(\mathbb{C}_+)$ be the *Hardy space* of holomorphic functions in the open right half plane $\mathbb{C}_+$ which are bounded with respect to the norm:

$$\parallel f \parallel_2 \triangleq \sup_{x\in\mathbb{R}_+} (\int_{-\infty}^{+\infty} |f(x + i\, y)|^2\, dy)^{1/2}.$$

It is well known that $H_2(\mathbb{C}_+)$ is a *Hilbert space* [14] and, by a theorem of Paley-Wiener, every function of $H_2(\mathbb{C}_+)$ is the Laplace transform of a unique function of $L_2(\mathbb{R}_+)$ [14]. Finally, $H_2(\mathbb{C}_+)$ has a natural structure of an $H_\infty(\mathbb{C}_+)$-module defined by:

$$\forall\, f,\, g \in H_2,\ \forall\, h,\, k \in H_\infty :\ h\, f + k\, g \in H_2.$$

Exercise 2.1: 1) Prove 2 of Example 2.4 (Hints for the structure of A-module of $\operatorname{coker} f$: if n_1 and n_2 are identified in $N/\operatorname{im} f$, i.e. there exists $m \in M$ such that $n_1 - n_2 = f(m)$, we say that n_1 and n_2 belong to the same *equivalence class* and we denote this class by $\pi(n_1) = \pi(n_2) \in N/\operatorname{im} f$. Then, we have an A-morphism $\pi : N \longrightarrow N/\operatorname{im} f$, defined by mapping any element $n \in N$ into its equivalence class $\pi(n)$, called the *quotient map*. The structure of A-module of $\operatorname{coker} f$ is defined by:

$$\forall\, a \in A,\ \forall\, n \in N :\ a\,\pi(n) \triangleq \pi(a\, n).$$

Check that $a\,\pi(n)$ does not depend on the choice of n, i.e. if $\pi(n_1) = \pi(n_2) = \pi(n)$, then $a\,\pi(n_1) = a\,\pi(n_2)$).

2) Prove that $L_p(\mathbb{R}_+)$ is an $\mathcal{A}$-module for $1 \leq p \leq +\infty$, $H_2(\mathbb{C}_+)$ is an $\hat{\mathcal{A}}$-module (see (3) for the definitions of $\mathcal{A}$ and $\hat{\mathcal{A}}$) and H_2 is an RH_∞-module (see (1) for the definition of RH_∞) (Hints: show that if $f \in \mathcal{A}$, $g \in L_p(\mathbb{R}_+)$, $k \in \hat{\mathcal{A}}$, $l \in RH_\infty$ and $h \in H_2(\mathbb{C}_+)$, then $f \star g \in L_p(\mathbb{R}_+)$, $k\, h \in H_2(\mathbb{C}_+)$ and $l\, h \in H_2(\mathbb{C}_+)$. See [16] for informations and details).

III. Weakly doubly coprime factorizations

A. Definitions

A useful tool for time-invariant finite-dimensional linear systems ($A = RH_\infty$ or $k[s]$, $k = \mathbb{R}, \mathbb{C}$) is the concept of *coprime factorization*. The coprime factorization of a rational matrix goes back to the work of H. H. Rosenbrock [65] and has played since then a major role in analysis and synthesis problems (controllability, observability, stabilizability, detectability, Youla-Kučera parametrization of all stabilizing controllers, graph topology, equivalences…). This technique has been popularized by the book of M. Vidyasagar [78]. However, contrary to finite-dimensional systems, the transfer matrix of more general systems (delays systems, systems of partial differential equations, convolution equations…) generally does not admit a coprime factorization [12], [14], [17], [44], [77], [78], [82]. Intuitively, this comes from the fact that the algebraic properties of the rings $H_\infty(\mathbb{C}_+)$, $\mathcal{A}$ and $\hat{\mathcal{A}}$… are more complex than the ones of RH_∞. For finite-dimensional systems ($A = RH_\infty$ or $k[s]$, $k = \mathbb{R}, \mathbb{C}$), one can prove that there exists only one concept of primeness, but, for more sophisticated rings as $H_\infty(\mathbb{C}_+)$ or $\hat{\mathcal{A}}$, this fact is no longer true. We are going to introduce the concept of *weak primeness* which plays a major role in these notes. This concept generalizes the one introduced by M. C. Smith for $H_\infty(\mathbb{C}_+)$ in the important contribution [71].

Definition 3.1:

- [56] A matrix $R \in A^{q \times p}$ is *weakly left-prime* if we have

$$K^q\, R \cap A^p \triangleq \{\lambda \in A^p \mid \exists\, \mu \in K^q :\ \lambda = \mu\, R\} = A^q\, R \triangleq \{\lambda \in A^p \mid \exists\, \nu \in A^q :\ \lambda = \nu\, R\},$$

i.e. if a row vector $\mu \in K^q$ is such that $\mu\, R \in A^p$, then there exists $\nu \in A^q$ satisfying:

$$\mu\, R = \nu\, R. \tag{8}$$

- R is *weakly right-prime* if R^T is weakly left-prime.

Exercise 3.1: Show that, if R has a *full row rank*, namely the q rows of R are A-linear independent, then R is weakly left-prime iff, if there exists $\mu \in K^q$ such that $\mu\, R \in A^q$, then $\mu \in A^q$ (Hints: factorize (8) by R and use the fact that R has full row rank to obtain $\mu = \nu \in A^q$).

Example 3.1: Let us consider the matrix R defined by (7). The matrix R is not weakly left-prime because we have

$$\left(\frac{s+1}{s-1}\ :\ 0\right) \left(\begin{array}{ccc} \frac{(s-1)^2}{(s+1)^2} & 0 & -\frac{(s-1)\, e^{-s}}{(s+1)^2} \\ 0 & \frac{(s-1)^2}{(s+1)^2} & -\frac{e^{-s}}{(s+1)^2} \end{array}\right) = \left(\frac{s-1}{s+1}\ :\ 0\ :\ -\frac{e^{-s}}{s+1}\right) \in A^3$$

and the vector $(\frac{s+1}{s-1}\ :\ 0)$ belongs to K^2 but not to A^2.

Definition 3.2:

- A couple $(a,\, b)$ of elements of A has a *common divisor* $c \in A$ if there exist $a',\, b' \in A$ such that:

$$\begin{cases} a = a'\, c, \\ b = b'\, c. \end{cases}$$

If there exists a common divisor c of a and b which satisfies that, for every other divisor c' of a and b, there exists $d \in A$ such that $c = d\, c'$, then c is called the *greatest common divisor* of a and b and is denoted by $[a,\, b]$. A greatest common divisor is defined up to an invertible element, i.e. up to an element of $\mathrm{U}(A)$.

- A ring A is *a greatest common divisor domain* (gcdd) if every couple $(a,\, b)$ of elements of A has a greatest common divisor $[a,\, b]$.

Proposition 3.1: [71] If A is a greatest common divisor domain, then a full row rank matrix $R \in A^{q \times p}$ ($\Rightarrow 0 < q \leq p$) is weakly left-prime iff 1 is a greatest common divisor of all the $q \times q$ minors of R.

Exercise 3.2: 1) We shall see in Theorem 3.4 that $H_\infty(\mathbb{C}_+)$ is a greatest common divisor domain. Prove that the matrix R defined by (7) is not weakly left-prime because $\frac{(s-1)^2}{(s+1)^2}$ is a common divisor of the 2×2 minors of R.

2) Check that 1 is a greatest common divisor of $\frac{1}{s+1}$ and $e^{-s} \in H_\infty(\mathbb{C}_+)$. Similar problem for $\frac{s-1}{s+1}$ and e^{-s}.

3) Find a common divisor of the two elements $\frac{1-e^{-s}}{s+1}$ and $\frac{s}{s+1}$ of $A = H_\infty(\mathbb{C}_+)$ (Hint: $\frac{1}{s+1}$ is not a common divisor of the two elements because $s \notin A$ but use the fact that $\frac{1-e^{-s}}{s} \in A$).

Definition 3.3:

- A transfer matrix $P \in K^{q \times r}$ admits a *weakly left-coprime factorization* if there exists a weakly left-prime matrix $R = (D \,:\, -N) \in A^{q \times r}$ such that $D \in A^{q \times q}$ has full rank (i.e. $\det D \neq 0$) and:

$$P = D^{-1}\,N.$$

- Dually, $P \in K^{q \times r}$ admits a *weakly right-coprime factorization* if there exists a weakly right-prime matrix $\tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}$ such that the matrix $\tilde{D} \in A^{r \times r}$ has full rank (i.e. $\det \tilde{D} \neq 0$) and:

$$P = \tilde{N}\,\tilde{D}^{-1}.$$

- $P \in K^{q \times r}$ admits a *weakly doubly coprime factorization* if P has a weakly left-coprime factorization as well as a weakly right-coprime factorization.

Let us note that the matrix R defined by (7) was obtained by removing all the denominators of P. In Example 3.1, we saw that R was not weakly left-prime. Hence, the procedure which consists in writing all the entries of a transfer matrix over a common denominator generally leads to matrices which are not weakly left/right-prime. Moreover, we shall show that this concept of weak primeness is the weakest existing concept of primeness, and thus, if $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is not a weakly doubly coprime factorization, then it is also not a doubly coprime factorization. Hence, if we want to compute effectively a doubly coprime factorization of a transfer matrix (when it exists), then we first need to have an algorithm which computes a weakly doubly coprime factorization of a transfer matrix (if such a factorization also exists).

B. Transfer matrices & Torsion-freeness

In order to understand when a transfer matrix $P \in K^{q \times r}$ admits a weakly doubly coprime factorization, we need to introduce the concepts of a *torsion element* of an A-module and of a *torsion/torsion-free* A-module. All the A-modules which will be considered in the rest of this paper are *finitely generated*, namely are defined by means of a finite family of generators [7], [31], [66].

Definition 3.4:
- An A-module M is *free* if it admits a basis, or equivalently, if $M \cong A^r$ with $r \in \mathbb{Z}_+$. Then, r is called the *rank* of the A-module M.
- The *torsion submodule* $t(M)$ of an A-module M is defined:

$$t(M) = \{m \in M \mid \exists\, 0 \neq a \in A,\ a\,m = 0\}.$$

  An element of $t(M)$ is called a *torsion element*. An A-module M is *torsion-free* if $t(M) = 0$, or equivalently, $M/t(M) = M$ and M is a *torsion* module if $t(M) = M$.
- If N is a submodule of an A-module M, then we call the A-*closure* of N in M the A-module defined by:

$$\overline{N} = \{m \in M \mid \exists\, 0 \neq a \in A,\ a\,m \in N\}.$$

Remark 3.1: Let us notice that the concept of a torsion element of a k-vector space (k is a field) is trivial: if m is a torsion element of k-vector space, then there exists $0 \neq a \in k$ such that $a\,m = 0$. But, using the fact that $0 \neq a \in k$ and k is a field, then $a^{-1} \in k$, and thus:

$$a\,m = 0 \Rightarrow a^{-1}\,(a\,m) = m = 0.$$

Hence, every k-vector space is torsion-free, i.e. this concept is only interesting for A-modules. More generally, every k-vector space is a free k-module.

If $R \in A^{q \times p}$, then we define the A-morphism $.R$ (see 2 of Example 2.4) by:

$$\begin{array}{ccc} A^q & \xrightarrow{.R} & A^p \\ (\lambda_1 : \ldots : \lambda_q) & \longmapsto & (\lambda_1 : \ldots : \lambda_q)\,R. \end{array}$$

From 2 of Example 2.4, we know that $\operatorname{im} .R = A^q\,R$ and $\operatorname{coker} .R = A^p/A^q\,R$ are two A-modules. These two A-modules will be of very common use in all these notes. The A-module $A^q\,R \triangleq \{\lambda \in A^p \mid \exists\, \nu \in A^q :\ \lambda = \nu\,R\}$ is defined by the A-linear combination of the rows of R. Let us give an interpretation of the A-module $A^p/A^q\,R$. From Example 2.4, we know that A^q (resp. A^p) is a free A-module having a canonical basis denoted by $\{e_1, \ldots, e_q\}$ (resp. $\{f_1, \ldots, f_p\}$). Let us denote by $z_i = \pi(f_i)$ the equivalence class of f_i in $A^p/A^q\,R$ (see 1 of Exercise 2.1). For $i = 1, \ldots, q$, we have:

$$e_i\,R = (R_{i1} : \ldots : R_{ip}) = \sum_{j=1}^{p} R_{ij}\,f_j \in A^q\,R \Rightarrow \pi(e_i\,R) = 0.$$

Using the structure of A-module of $M = A^p/A^q\,R$ and the A-morphism $\pi : A^p \longrightarrow M$ (see 1 of Exercise 2.1), then, in $A^p/A^q\,R$, for $i = 1, \ldots, q$, we have:

$$\pi(e_i\,R) = \pi(\sum_{j=1}^{p} R_{ij}\,f_j) = \sum_{j=1}^{p} R_{ij}\,\pi(f_j) = \sum_{j=1}^{p} R_{ij}\,z_j = 0. \tag{9}$$

Thus, M is defined by the system $R\,z = 0$ and all the A-linear combinations of its equations, where $z = (z_1 : \ldots : z_p)$ is a vector of *formal variables* which correspond to the generators of M (they do not belong to any "functional space").

Example 3.2: Let us reconsider the matrix $R \in A^{2 \times 3}$ defined by (7) ($A = H_\infty(\mathbb{C}_+)$). Let us call y_1 (resp. y_2) the class of f_1 (resp. f_2) and u the class of f_3 in $M = A^3/A^2\,R$. We find that the A-module M is defined by the system

$$\left\{\begin{array}{l} \frac{(s-1)^2}{(s+1)^2}\,y_1 - \frac{(s-1)\,e^{-s}}{(s+1)^2}\,u = 0, \\ \frac{(s-1)^2}{(s+1)^2}\,y_2 - \frac{e^{-s}}{(s+1)^2}\,u = 0, \end{array}\right.$$

as well as the A-linear combinations of these two equations. We can check that the element $z = \frac{(s-1)}{(s+1)}\,y_1 - \frac{e^{-s}}{(s+1)}\,u$ of M (i.e. class of $\left(\frac{(s-1)}{(s+1)} \,:\, 0 \,:\, -\frac{e^{-s}}{(s+1)}\right) \in A^3$ in M) satisfies the equation $\frac{(s-1)}{(s+1)}\,z = 0$, i.e. m is a torsion element of M.

Lemma 3.1: [56] Let us consider $R \in A^{q \times p}$ and the A-module $M = A^p/A^q\,R$. Then, we have:

1) *The A-closure of the A-module $A^q\,R$ in A^p is:*

$$\overline{A^q\,R} = K^q\,R \cap A^p.$$

2) $t(M) = (K^q\,R \cap A^p)/A^p\,R = \overline{A^q\,R}/A^q\,R$.
3) $M/t(M) = A^p/(K^q\,R \cap A^p) = A^p/\overline{A^q\,R}$.
4) *$M = A^p/A^q\,R$ is a torsion-free A-module ($t(M) = 0$) iff R is weakly left-prime.*

Exercise 3.3: Prove Lemma 3.1. See [56] for the answers.

A transfer matrix $P \in K^{q \times r}$ has lots of different fractional representations of the form $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$, where:

$$\begin{cases} R = (D \,:\, -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}. \end{cases}$$

In the next proposition, we show that the concepts of A-closure and torsion submodule allow us to capture the intrinsic informations of these different representations.

Proposition 3.2: [56] If a transfer matrix $P \in K^{q \times r}$ can be written as

$$P = D_1^{-1} N_1 = D_2^{-1} N_2, \quad P = \tilde{N}_1 \tilde{D}_1^{-1} = \tilde{N}_2 \tilde{D}_2^{-1},$$

with

$$\begin{cases} R_i = (D_i \,:\, -N_i) \in A^{q \times p}, \\ \tilde{R}_i = (\tilde{N}_i^T \,:\, \tilde{D}_i^T)^T \in A^{p \times r}, \end{cases} \quad i = 1, 2,$$

then we have:

1) $\overline{A^q R_1} = \overline{A^q R_2}$,
2) $\overline{A^r \tilde{R}_1^T} = \overline{A^r \tilde{R}_2^T}$,
3) $A^p R_1^T \cong A^p R_2^T \cong \tilde{N}_i / t(\tilde{N}_i) = A^p / \overline{A^r \tilde{R}_i^T}$
4) $A^p \tilde{R}_1 \cong A^p \tilde{R}_2 \cong M_i / t(M_i) = A^p / A^r \tilde{R}_i^T$,

with the notations:

$$\begin{cases} M_i = A^p / A^q R_i, \\ \tilde{N}_i = A^p / A^r \tilde{R}_i^T, \end{cases} \quad i = 1, 2.$$

Example 3.3: Let us consider $A = \hat{\mathcal{A}}$ and:

$$p = e^{-s}/(s-1) \in K = Q(A).$$

There are different ways to obtain a fractional representation of p: for instance, we have $p = n_1/d_1 = n_2/d_2$ with:

$$\begin{cases} n_1 = e^{-s}/(s+1) \in A, \\ d_1 = (s-1)/(s+1) \in A, \\ n_2 = (e^{-s}\,(s-1))/(s+1)^2 \in A, \\ d_2 = (s-1)^2/(s+1)^2 \in A. \end{cases}$$

If we denote by

$$\begin{cases} R_1 = (d_1 \,:\, n_1) \in A^{1 \times 2}, \\ R_2 = (d_2 \,:\, n_2) \in A^{1 \times 2}, \end{cases}$$

then we have:

$$\begin{cases} I_1 = A^2 R_1^T & = (d_1,\, n_1), \\ I_2 = A^2 R_2^T & = (d_2,\, n_2) \\ & = ([d_2,\, n_2]\, d_1,\ [d_2,\, n_2]\, n_1) = \left(\frac{s-1}{s+1}\right) I_1. \end{cases}$$

If we define the following A-morphisms

$$\begin{cases} \phi : I_1 \longrightarrow I_2, & \phi(a) = \frac{(s-1)}{(s+1)}\, a,\ \forall\, a \in I_1, \\ \psi : I_2 \longrightarrow I_1 & \psi(b) = \frac{(s+1)}{(s-1)}\, b = c_1\, d_1 + c_2\, n_1, \\ & \forall\, b = \frac{(s-1)}{(s+1)}\, (c_1\, d_1 + c_2\, n_1) \in I_2, \end{cases}$$

and we easily check that $\phi \circ \psi = \mathrm{id}_{I_2}$ and $\psi \circ \phi = \mathrm{id}_{I_1}$, which proves that $I_1 \cong I_2$.

Corollary 3.1: [56] The structural (intrinsic) properties of a transfer matrix

$$P = D^{-1} N = \tilde{N} \tilde{D}^{-1} \in K^{q \times r},$$

where

$$\begin{cases} R = (D \,:\, -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}, \end{cases}$$

only depend on the A-modules $\overline{A^q R}$ and $\overline{A^r \tilde{R}^T}$ and, up to an isomorphism, on the A-modules $A^p R^T$ and $A^p \tilde{R}$.

C. Algorithm

The next theorem gives necessary and sufficient conditions for a transfer matrix to admit a weakly left/right-coprime factorization.

Theorem 3.1: [56] $P = D^{-1} N = \tilde{N} \tilde{D}^{-1} \in K^{q \times r}$, where

$$\begin{cases} R = (D \,:\, -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}, \end{cases}$$

admits a weakly left (resp. right) coprime factorization iff $\overline{A^q R}$ (resp. $\overline{A^r \tilde{R}^T}$) is a free A-module of rank q (resp. r), or equivalently, iff there exists a full row rank matrix $R' \in A^{q \times p}$ (resp. a full column rank matrix $\tilde{R}' \in A^{p \times r}$) such that $\overline{A^q R} = A^q R'$ (resp. $\overline{A^r \tilde{R}^T} = A^r \tilde{R}'^T$).

Exercise 3.4:

1) [58], [59] Prove that $P \in K^{q \times r}$ admits a weakly left-coprime factorization iff there exists a non-singular matrix $D \in A^{q \times q}$ such that:

$$\{\lambda \in A^q \mid \lambda\, P \in A^r\} = A^q\, D.$$

Deduce that $P = D^{-1} N$ is a weakly left-coprime of P.

2) [58], [59] Prove that $P \in K^{q \times r}$ admits a weakly right-coprime factorization iff there exists a non-singular matrix $\tilde{D} \in A^{r \times r}$ such that:

$$\{\lambda \in A^r \mid \lambda\, P^T \in A^q\} = A^r\, \tilde{D}^T.$$

Prove that $P = \tilde{N}\, \tilde{D}^{-1}$ is a weakly right-coprime of P.

Corollary 3.2: [56] $P = D^{-1} N = \tilde{N} \tilde{D}^{-1} \in K^{q \times r}$, where

$$\begin{cases} R = (D \,:\, -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}, \end{cases}$$

admits a weakly doubly coprime factorization iff the $\overline{A^q R}$ and $\overline{A^r \tilde{R}^T}$ are two free A-modules of rank q and r.

We give an algorithm which computes the A-closure $\overline{A^q R}$ of an A-module of the form $A^q R$ if a certain hypothesis on the ring A is satisfied, namely A is a *coherent ring* (see next section). This hypothesis allows us to certify that, for every matrix $R \in A^{q \times p}$, the A-modules $\ker .R^T$ and $\ker .R_{-1}$ that we need to compute are finitely generated.

Algorithm 1: Input: A a coherent ring and $R \in A^{q \times p}$. Output: $R' \in A^{q' \times p}$ such that $\overline{A^q R} = A^{q'} R'$.

1) Start with $R \in A^{q \times p}$.
2) Transpose R to obtain $R^T \in A^{p \times q}$.
3) Find a family of generators of:

$$\ker .R^T = \{\lambda \in A^p \mid \lambda\, R^T = 0\}.$$

If $\{\lambda_1, \ldots, \lambda_m\}$ is a family of generators of $\ker .R^T$, then denote by $R_{-1}^T \in A^{m \times p}$ the matrix whose i^{th} row is λ_i.

4) Tranpose R_{-1}^T in order to obtain $R_{-1} \in A^{p\times m}$.
5) Find a family of generators of

$$\ker .R_{-1} = \{\eta \in A^p \mid \eta\, R_{-1} = 0\}.$$

If $\{\eta_1, \ldots, \eta_{q'}\}$ is a family of generators of $\ker .R_{-1}$, then denote by $R' \in A^{q'\times p}$ the matrix whose i^{th} row is η_i. Then, we have:

$$\overline{A^q\, R} = A^{q'}\, R'.$$

Remark 3.2: Let us notice that the previous algorithm was obtained using a concept of homological algebra called *extension functor* [7], [31], [66]. More precisely, in [56], we proved that we have $t(M) \cong \text{ext}^1_A(A^q/A^p\, R^T, A)$ and gave a proof of the previous algorithm. This result generalizes to a more general situation certain results obtained in [13], [53].

Example 3.4: In Example 3.1, we saw that the matrix R defined by (7) is not weakly left-prime, and thus, that the following fractional representation

$$P = \left(\begin{array}{cc} \frac{(s-1)^2}{(s+1)^2} & 0 \\ 0 & \frac{(s-1)^2}{(s+1)^2} \end{array}\right)^{-1} \left(\begin{array}{c} \frac{(s-1)\, e^{-s}}{(s+1)^2} \\ \frac{e^{-s}}{(s+1)^2} \end{array}\right)$$

is not a weakly coprime factorization of the transfer matrix P defined by (6). Let us check whether or not P admits a weakly left-coprime factorization using the previous algorithm (in Theorem 3.3, we shall see that $A = H_\infty(\mathbb{C}_+)$ is a coherent ring).

1) We first start with $R \in A^{2\times 3}$ defined by (7).
2) We compute $R^T \in A^{3\times 2}$.
3) Let us compute

$$\ker .R^T = \{\lambda = (\lambda_1 : \lambda_2 : \lambda_3) \in A^3 \mid \lambda\, R^T = 0\}.$$

Let $\lambda \in \ker .R^T$, i.e.:

$$\left\{\begin{array}{l} \frac{(s-1)^2}{(s+1)^2}\, \lambda_1 - \frac{(s-1)\, e^{-s}}{(s+1)^2}\, \lambda_3 = 0, \\ \frac{(s-1)^2}{(s+1)^2}\, \lambda_2 - \frac{e^{-s}}{(s+1)^2}\, \lambda_3 = 0. \end{array}\right. \quad (10)$$

From the first equation, we obtain

$$\frac{(s-1)}{(s+1)} \left(\frac{(s-1)}{(s+1)}\, \lambda_1 - \frac{e^{-s}}{(s+1)}\, \lambda_3\right) = 0$$
$$\Leftrightarrow \frac{(s-1)}{(s+1)}\, \lambda_1 - \frac{e^{-s}}{(s+1)}\, \lambda_3 = 0,$$

because A is an integral domain and $\lambda_i \in A$. Using the fact $\left[\frac{s-1}{s+1}, \frac{e^{-s}}{s+1}\right] = 1$, we obtain:

$$\left\{\begin{array}{l} \lambda_1 = \frac{e^{-s}}{s+1}\, \mu, \\ \lambda_3 = \frac{s-1}{s+1}\, \mu, \\ \mu \in A. \end{array}\right.$$

Substituting λ_3 in the second equation of (10), we obtain:

$$\frac{(s-1)}{(s+1)} \left(\frac{(s-1)}{(s+1)}\, \lambda_2 - \frac{e^{-s}}{(s+1)^2}\, \mu\right) = 0$$
$$\Leftrightarrow \frac{(s-1)}{(s+1)}\, \lambda_2 - \frac{e^{-s}}{(s+1)^2}\, \mu = 0.$$

Finally, using the fact that $\left[\frac{s-1}{s+1}, \frac{e^{-s}}{(s+1)^2}\right] = 1$, we obtain

$$\left\{\begin{array}{l} \lambda_2 = \frac{e^{-s}}{(s+1)^2}\, \mu', \\ \mu = \frac{(s-1)}{(s+1)}\, \mu', \\ \mu' \in A \end{array}\right. \Rightarrow \left\{\begin{array}{l} \lambda_1 = \frac{(s-1)\, e^{-s}}{(s+1)^2}\, \mu', \\ \lambda_2 = \frac{e^{-s}}{(s+1)^2}\, \mu', \\ \lambda_3 = \frac{(s-1)^2}{(s+1)^2}\, \mu'. \end{array}\right.$$

Therefore, we have $\lambda = \mu'\, R_{-1}^T$, where:

$$R_{-1}^T = \left(\frac{(s-1)\, e^{-s}}{(s+1)^2} : \frac{e^{-s}}{(s+1)^2} : \frac{(s-1)^2}{(s+1)^2}\right) \in A^{1\times 3}.$$

4) We transpose R_{-1}^T in order to obtain $R_{-1} \in A^{3\times 1}$.
5) Let us compute

$$\ker .R_{-1} = \{\eta = (\eta_1 : \eta_2 : \eta_3) \in A^3 \mid \eta\, R_{-1} = 0\}.$$

Let us consider $\eta = (\eta_1 : \eta_2 : \eta_3) \in \ker .R_{-1}$, i.e.:

$$\frac{(s-1)\, e^{-s}}{(s+1)^2}\, \eta_1 + \frac{e^{-s}}{(s+1)^2}\, \eta_2 + \frac{(s-1)^2}{(s+1)^2}\, \eta_3 = 0$$
$$\Leftrightarrow \frac{(s-1)}{(s+1)} \left(\frac{e^{-s}}{s+1}\, \eta_1 + \frac{(s-1)}{(s+1)}\, \eta_3\right) = -\frac{e^{-s}}{(s+1)^2}\, \eta_2.$$

Using the fact that $\left[\frac{s-1}{s+1}, \frac{e^{-s}}{(s+1)^2}\right] = 1$, we obtain:

$$\left\{\begin{array}{l} \frac{e^{-s}}{(s+1)}\, \eta_1 + \frac{(s-1)}{(s+1)}\, \eta_3 = \frac{e^{-s}}{(s+1)^2}\, \zeta_1, \\ \eta_2 = -\frac{(s-1)}{(s+1)}\, \zeta_1, \\ \zeta_1 \in A. \end{array}\right. \quad (11)$$

From the first equation of (11), we deduce that

$$\frac{e^{-s}}{(s+1)} \left(\eta_1 - \frac{1}{(s+1)}\, \zeta_1\right) = -\frac{(s-1)}{(s+1)}\, \eta_3,$$

and using the fact that $\left[\frac{e^{-s}}{s+1}, \frac{s-1}{s+1}\right] = 1$, we obtain

$$\left\{\begin{array}{l} \eta_1 = \frac{1}{(s+1)}\, \zeta_1 + \frac{(s-1)}{(s+1)}\, \zeta_2, \\ \eta_2 = -\frac{(s-1)}{(s+1)}\, \zeta_1, \\ \eta_3 = -\frac{e^{-s}}{s+1}\, \zeta_2, \\ \zeta_1,\ \zeta_2 \in A, \end{array}\right. \Leftrightarrow \eta = (\zeta_1 : \zeta_2)\, R',$$

where:

$$R' = \left(\begin{array}{ccc} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} & 0 \\ \frac{(s-1)}{(s+1)} & 0 & -\frac{e^{-s}}{s+1} \end{array}\right) \in A^{2\times 3}. \quad (12)$$

6) We have $\overline{A^2\, R} = A^2\, R'$ and R' is a full row rank matrix. Thus, $A^2\, R'$, i.e. $\overline{A^2\, R}$, is a free A-module of rank 2. Hence, by Theorem 3.1, we know that the following fractional representation of P

$$P = \left(\begin{array}{cc} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} \\ \frac{(s-1)}{(s+1)} & 0 \end{array}\right)^{-1} \left(\begin{array}{c} 0 \\ \frac{e^{-s}}{s+1} \end{array}\right) \quad (13)$$

is a weakly left-coprime factorization of P (check that there is no common factor to all the 2×2 minors of the matrix R').

Finally, by Lemma 3.1, we know that:

$$\left\{\begin{array}{l} \overline{A^2\, R} = A^2\, R' = K^2\, R \cap A^2, \\ t(M) = \overline{A^2\, R}/A^2\, R = A^2\, R'/A^2\, R, \\ M/t(M) = A^2/A^2\, R'. \end{array}\right.$$

Let us compute a family of generators of the torsion elements of M. We know that the torsion submodule of M is defined by $t(M) = A^2\, R'/A^2\, R$, and thus, the class of the first row of R' in $t(M)$ corresponds to the element

$$z_1 = \frac{1}{(s+1)}\, y_1 - \frac{(s-1)}{(s+1)}\, y_2$$

whereas

$$z_2 = \frac{(s-1)}{(s+1)}\, y_1 - \frac{e^{-s}}{(s+1)}\, u$$

corresponds to the class of the second row of R' in $t(M)$. We easily check that we have $\frac{(s-1)^2}{(s+1)^2}\, z_1 = 0$ and $\frac{(s-1)}{(s+1)}\, z_2 = 0$ in $t(M)$, and thus, z_1 and z_2 constitute a family of generators of $t(M)$. Finally, $M/t(M) = A^2/A^2\, R'$ is defined by

$$\begin{cases} \frac{1}{(s+1)}\, y_1 - \frac{(s-1)}{(s+1)}\, y_2 = 0, \\ \frac{(s-1)}{(s+1)}\, y_1 - \frac{e^{-s}}{(s+1)}\, u = 0, \end{cases}$$

as well as all the A-linear combinations of these two equations (see (9) for more details).

Exercise 3.5: Let $A = H_\infty(\mathbb{C}_+)$, $K = Q(A)$ and let us consider the following transfer matrix:

$$P = \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s-1} \end{pmatrix} \in K^{2\times 2}.$$

1) Show that we have $P = D^{-1}\, N$, where:

$$D = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-1}{s+1} \end{pmatrix} \in A^{2\times 2},$$

$$N = \begin{pmatrix} \frac{(s-1)\, e^{-s}}{(s+1)^2} & \frac{(s-1)^2}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{pmatrix} \in A^{2\times 2}.$$

2) Check that $P = D^{-1}\, N$ is not a weakly left-coprime factorization of P. Can you exhibit a torsion element of the A-module $M = A^4/A^2\, R$, $R = (D\, :\, -N) \in A^{2\times 4}$?
3) Doing similarly as Example 3.4, show that we have the following weakly left-coprime factorization of P:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{pmatrix}^{-1} \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s+1} \end{pmatrix}.$$

4) Give a family of generators of $t(M)$ of M and the equations which generate $M/t(M)$.
5) Dually, find a weakly right-coprime factorization of P.

We can check your computations looking at [56].

D. Sylvester coherent domains

Recall that an *ideal* I of A is just an A-submodule of A [31], [66], i.e. $\forall\, a_1,\, a_2 \in I, \forall\, b_1,\, b_2 \in A$, we have $b_1\, a_1 + b_2\, a_2 \in I$.

Definition 3.5: A ring is *noetherian* if every ideal I of A is *finitely generated*, namely there exists a finite family $\{a_1, \ldots, a_n\}$ of elements of A such that:

$$I = (a_1, \ldots, a_n) \triangleq \left\{ \sum_{i=1}^{n} b_i\, a_i \mid b_i \in A \right\}.$$

Example 3.5: The ring $A = RH_\infty$ of proper stable real rational functions is a *principal ideal domain* [78], namely every ideal of A is generated by means of a single element of A. In particular, A is a noetherian ring. Similarly for $A = k[s]$ with $k = \mathbb{R}$, $\mathbb{C}$.

Definition 3.6: A *Banach algebra* A is a k-algebra (with $k = \mathbb{R}, \mathbb{C}$) (namely a ring A which has the structure of a k-module) with a *norm* $\|\cdot\|_A$ (namely an application $\|\cdot\|_A$: $A \longrightarrow \mathbb{R}_+$ which satisfies

- $\| a \|_A = 0 \Leftrightarrow a = 0$, $\forall\, a \in A$,
- $\| \alpha\, a \|_A = |\, \alpha\, |_k \, \| a \|_A$, $\forall\, \alpha \in k$, $\forall\, a \in A$,
- $\| a + b \|_A \leq \| a \|_A + \| b \|_A$, $\forall\, a,\ b \in A$)

which satisfies the following properties:

- $\| a\, b \|_A \leq \| a \|_A \, \| b \|_A$, $\forall\, a,\ b \in A$,
- $\| 1 \|_A = 1$,
- A is *complete* as a k-vector space, namely every *Cauchy sequence* $(a_n)_{n\geq 0}$ of elements of A (i.e. a sequence $(a_n)_{n\geq 0}$ satisfying:

$$\forall\, \epsilon > 0, \exists\, N \in \mathbb{Z}_+, \forall\, n,\, m > N : \| a_n - a_m \|_A < \epsilon)$$

converges (namely,

$$\exists\, l \in A,\ \forall\, \epsilon > 0, \exists\, N \in \mathbb{Z}_+, \forall\, n > N : \| a_n - l \|_A < \epsilon).$$

Example 3.6: The following four examples

- $(H_\infty(\mathbb{C}_+),\ \| f \|_\infty = \sup_{s\in\mathbb{C}_+} |f(s)|)$,
- $(\mathcal{A},\ \| g \|_{\mathcal{A}} = \| f \|_{L_1(\mathbb{R}_+)} + \sum_{n=0}^{+\infty} |a_n|)$,
- $(\hat{\mathcal{A}},\ \| \hat{g} \|_{\hat{\mathcal{A}}} = \| g \|_{\mathcal{A}})$,
- $(W_+,\ \| (a_n)_{n\geq 0} \|_{W_+} = \sum_{n=0}^{+\infty} |a_n|)$,

are Banach algebras (see [11], [12], [14], [78] for more details).

Theorem 3.2: [68] A noetherian Banach algebra is finitely dimensional (as a k-vector space).

Therefore, $H_\infty(\mathbb{C}_+)$, $\mathcal{A}$, $\hat{\mathcal{A}}$ and W_+ are not noetherian rings, and thus, certain of their ideals are not finitely generated. Hence, it seems that we cannot use the main part of commutative algebra which was developed for noetherian rings in order to study the algebraic properties of these rings. In the fifties, H. Cartan and J. P. Serre developed the concept of a *coherent sheaf* in order to study analytic and algebraic geometries. This concept is closely related to the concept of a *coherent ring* which was introduced in commutative algebra by S. U. Chase in 1960. This concept plays a crucial role in these notes.

Definition 3.7:

- [5], [7], [26], [66] A ring A is *coherent* if the A-module of the relations (*syzygy A-module*) of every finitely generated ideal $I = (a_1, \ldots, a_n)$ of A, namely

$$S(I) = \{r = (r_1 : \ldots : r_n) \in A^n \mid \sum_{i=1}^{n} r_i\, a_i = 0\},$$

is finitely generated, i.e. there exist $m \in \mathbb{Z}_+$ and a matrix $R \in A^{m\times n}$ such that,

$$\forall\, r \in S(I),\ \ \exists\, b = (b_1 : \ldots : b_m) \in A^m : \ \ r = b\, R,$$

or, equivalently, $S(I) = A^m\, R$.
- [5], [7], [26], [66] A finitely generated ideal I of A which satisfies that the A-module of the relations $S(I)$ is finitely generated is called a *finitely presented* ideal of A.

The class of modules over a coherent ring enjoys very nice algebraic properties (e.g. it is closed by respect to (direct) sums, intersections, quotients, tensor products, morphisms...) which makes every computation of a *finitely presented* module (i.e. an A-module of the form $A^p/A^q\,R$ for a certain matrix $R \in A^{q\times p}$ and $p,\, q \in \mathbb{Z}_+$) very tractable (as in the case of a noetherian ring).

Example 3.7:
- Any noetherian ring is coherent [7], [26], [66]. In particular, RH_∞ and $k[s]$ ($k = \mathbb{R},\, \mathbb{C}$) are two coherent integral domains.
- A coherent ring is not necessarily a noetherian ring. For instance, the ring $k[x_i,\, i \geq 1]$ of polynomials in an infinite number of independent variables x_i with coefficients in the field $k = \mathbb{R}, \mathbb{C}$ is not a noetherian ring but a coherent one [66].
- A *Bézout domain*, namely an integral domain such that every finitely generated ideal of A is generated by a single element of A, is a coherent ring. For instance, the ring of entire functions in $\mathbb{C}$ with coefficients in $k = \mathbb{R}, \mathbb{C}$, namely

$$E(k) = \{f(s) = \textstyle\sum_{n=0}^{+\infty} a_n\, s^n \mid \quad a_n \in k, \\ \lim_{n \to +\infty} |a_n|^{1/n} = 0\},$$

  and $\mathcal{E} = E(\mathbb{R}) \cap \mathbb{R}(s)[e^{-s}]$ are two Bézout domains [30], [33], [45], and thus, coherent rings.

Exercise 3.6: Show that $k[x_i,\, i \geq 1]$, with $k = \mathbb{R}, \mathbb{C}$, is not a noetherian ring (Hint: consider the ideal $\sum_{i \geq 1} A\, x_i$ and prove that this ideal is not finitely generated).

Theorem 3.3: [46] $H_\infty(\mathbb{D})$, $H_\infty(\mathbb{C}_+)$, $L_\infty(\mathbb{T})$ and $L_\infty(\mathbb{R})$ are coherent rings, where:

$$\left\{ \begin{array}{l} \mathbb{D} = \{s \in \mathbb{C} \,|\, |s| < 1\}, \\ \mathbb{T} = \{s \in \mathbb{C} \,|\, |s| = 1\}. \end{array} \right.$$

For all these rings, the algorithm given in section III-C finishes because we can prove that if A is a coherent ring and $R \in A^{q\times p}$, then $\ker .R = \{\lambda \in A^q \mid \lambda\, R = 0\}$ is a finitely generated A-module, i.e. is defined by means a finite family of generators. Let us introduce another concept which will play an important role in the rest of these notes.

Definition 3.8: [18] An integral domain A is a *coherent Sylvester domain* if, for every $q \in \mathbb{Z}_+$ and every column vector $R^T \in A^q$, the A-module $\ker .R^T = \{\lambda \in A^q \mid \lambda\, R = 0\}$ is a free A-module.

Remark 3.3: The previous definition of a coherent Sylvester domain is the simplest one that we know. A more useful but abstract definition (by means of homological algebra) of a coherent Sylvester domain is a *projective-free coherent integral domain of weak global dimension* $\mathrm{w.gl.dim}(A) \leq 2$. See VII for more details. For instance, the next examples of coherent Sylvester domains are obtained using this last definition.

Example 3.8:
- A *Bézout domain*, namely an integral domain such that every finitely generated ideal I of A has the form $I = (a)$ for a certain element of A, is a coherent Sylvester domain. Since, RH_∞ and $\mathcal{E}$ are two Bézout domains [30], [45], [78], and thus, they are two coherent Sylvester domains.
- In [19], it is shown that $A = B[x]$ is a coherent Sylvester domain iff B is a Bézout domain. In particular, if B is a *principal ideal domain*, namely an integral domain such that every ideal of B has the form $I = (a)$ for a certain element of A (e.g. $B = \mathbb{Z},\, k[s],\; k = \mathbb{R}, \mathbb{C}, \quad RH_\infty$), then $A = B[x]$ is a coherent Sylvester domain. Therefore, $A = \mathbb{Z}[x]$ and $A = k[s][z] = k[s,z]$ are two examples of coherent Sylvester domains.

Theorem 3.4: [56] $H_\infty(\mathbb{C}_+)$ is a coherent Sylvester domain.

Proposition 3.3: [19] Every coherent Sylvester domain is a greatest common divisor domain.

Corollary 3.3: $H_\infty(\mathbb{C}_+)$ is a greatest common divisor domain (see [63], [71] for direct proofs).

The next result links the existence of a weakly doubly coprime factorization of any transfer matrix $-$ with entries in $K = Q(A)$ $-$ to a coherent Sylvester domain A.

Theorem 3.5: [56] We have the following equivalences:
- *Every transfer matrix $-$ with entries in $K = Q(A)$ $-$ admits a weakly doubly coprime factorization,*
- *A is a coherent Sylvester domain.*

Corollary 3.4: [56] Every transfer matrix with entries in $K = Q(H_\infty(\mathbb{C}_+))$ admits a weakly doubly coprime factorization (see [71] for a direct proof).

Hence, Theorem 3.5 generalizes a result on $H_\infty(\mathbb{C}_+)$ obtained by M. C. Smith [71] to a large class of rings (namely coherent Sylvester domains).

Exercise 3.7: Let us consider the ring $A = \mathbb{C}[x_1, x_2, x_3]$ of polynomials in $x_1,\, x_2,\, x_3$ whose coefficients belong to $\mathbb{C}$ and the following vector $R = (x_1\, :\, x_2\, :\, x_3)^T \in A^3$ (gradient operator).

1) Prove that $\ker .R = A^3\, R_1$, where the matrix R_1 is defined by (curl operator):

$$R_1 = \left(\begin{array}{ccc} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{array} \right) \in A^{3\times 3}.$$

2) Prove that $\ker .R_1 = A\, R^T$.
3) If $f : M \longrightarrow N$ is any A-morphism, then show that $M/\ker\, f \cong \mathrm{im}\, f$. Deduce that

$$A^3/\ker .R_1 \cong A^3\, R_1 = \ker .R,$$

   and thus, $\ker .R \cong A^3/A\, R^T$.
4) Using the fact that $A^3/A\, R^T$ is defined by the single equation $x_1\, z_1 + x_2\, z_2 + x_3\, z_3 = 0$ (divergent operator) and its A-linear combinations, prove that $A^3/A\, R^T$, and thus, $\ker .R$ is not a free A-module (show that $A^3/A\, R^T$ has no basis). Deduce that A is not a coherent Sylvester domain.
5) Deduce that the multidimensional linear system defined by $P = \left(\frac{x_1}{x_3} : \frac{x_2}{x_3}\right)^T \in K^{2\times 1}$ has no weakly left-coprime factorization ($K = \mathbb{C}(x_1, x_2, x_3)$ is the ring of rational functions in $x_1,\, x_2$ and x_3).

IV. Left/right/doubly coprime factorizations

Let us recall the well-known concepts of left/right/doubly coprime factorizations [12], [14], [77], [78].

Definition 4.1:

- A matrix $R = (D \,:\, -N) \in A^{q\times p}$ is *left-prime* if R has a *right-inverse*, namely a matrix
$$S = (X^T \,:\, Y^T)^T \in A^{p\times q}$$
which satisfies $R\,S = D\,X - N\,Y = I_q$.
- A transfer matrix $P \in K^{q\times r}$ admits a *left-coprime factorization* if there exists a left-prime matrix
$$R = (D \,:\, -N) \in A^{q\times p}$$
such that $D \in A^{q\times q}$ has full rank (i.e. $\det D \neq 0$) and:
$$P = D^{-1}\,N.$$
- A matrix $\tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p\times r}$ is *right-prime* if $\tilde{R}$ has a *left-inverse*, namely a matrix
$$\tilde{S} = (-\tilde{Y} \,:\, \tilde{X}) \in A^{r\times p}$$
which satisfies $\tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r$.
- A transfer matrix $P \in K^{q\times r}$ admits a *right-coprime factorization* if there exists a right-prime matrix
$$\tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p\times r}$$
such that $\tilde{D} \in A^{r\times r}$ has full rank (i.e. $\det \tilde{D} \neq 0$) and:
$$P = \tilde{N}\,\tilde{D}^{-1}.$$
- A transfer matrix $P \in K^{q\times r}$ admits a *doubly coprime factorization* if P admits a left and right-coprime factorization.

In order to give necessary and sufficient conditions of the existence of a left/right/doubly coprime factorization, we need to introduce the following definitions.

Definition 4.2: [5], [7], [26], [66] If M is a finitely generated A-module (i.e. M is defined by means of a finite family of generators), then, we have:

- M is a *stably free* A-module if there exist $r,\, s \in \mathbb{Z}_+$ such that $M \oplus A^s \cong A^r$ ($\oplus$ denotes the direct sum).
- M is a *projective* A-module if there exist an A-module P and $r \in \mathbb{Z}_+$ such that $M \oplus P \cong A^r$, i.e. M is a direct summand of a free A-module. Let us note that, in this case, P is also a projective A-module.

Proposition 4.1: [7], [66] We have the following implications of A-modules:

$$\textit{free} \;\Rightarrow\; \textit{stably free} \;\Rightarrow\; \textit{projective} \;\Rightarrow\; \textit{torsion-free}.$$

Definition 4.3: [42], [66] We have the following definitions:

- A ring A is a *projective-free ring* if every finitely generated projective A-module is free.
- A ring A is a *Hermite ring* if every finitely generated stably free A-module is free.

Let us introduce the *Fitting ideals* of a finitely presented A-module (namely an A-module of the form $A^p/A^q\,R$, for a certain matrix $R \in A^{q\times p}$). In the next proposition, this concept will give a tractable characterization of the finitely presented projective A-module $M = A^p/A^q\,R$ in terms of the minors of the matrix R.

Definition 4.4:

- If $R \in A^{q\times p}$, then we denote by $I_i(R)$ the ideal of A generated by:
  - all the $i \times i$ minors of R, if $1 \leq i \leq \min\{p,\, q\}$,
  - $I_i(R) = 0$, if $i > \min\{p,\, q\}$,
  - $I_i(R) = A$, if $i \leq 0$.
- [31] If $R \in A^{q\times p}$ and $M = A^p/A^q\,R$, then $I_i(R)$ only depends on M and not on R (the same module M can be defined by means of different matrices). Then, we call the *Fitting ideals of* M the ideals defined by:
$$\mathrm{Fitt}_i(M) = I_{p-i}(R), \;\; \forall\, i \in \mathbb{Z}.$$

Proposition 4.2: [31] The A-module $M = A^p/A^q\,R$ is projective iff there exists $r \in \mathbb{Z}_+$ such that:
$$\left\{\begin{array}{l} \mathrm{Fitt}_r(M) = 0, \\ \mathrm{Fitt}_{r+1}(M) = A \Leftrightarrow 1 \in \mathrm{Fitt}_{r+1}(M). \end{array}\right.$$

Example 4.1: Let us consider the matrix $R' \in A^{2\times 3}$ defined by (12) and the A-module $M' = A^3/A^2\,R'$ where $A = H_\infty(\mathbb{C}_+)$. We have $\mathrm{Fitt}_0(M') = 0$ and:
$$\mathrm{Fitt}_1(M') = \left(\tfrac{e^{-s}}{s+1},\; \tfrac{(s-1)^2}{(s+1)^2},\; \tfrac{(s-1)\,e^{-s}}{(s+1)^2}\right) \subseteq A.$$
We have the following *Bézout identity*
$$\tfrac{e^{-s}}{(s+1)}\,a + \tfrac{(s-1)^2}{(s+1)^2}\,b = 1 \Rightarrow \mathrm{Fitt}_1(M') = A,$$
where
$$\left\{\begin{array}{ll} a & = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \\ b & = \frac{(s+25)}{(s+1)} + \frac{4\,(5\,s-3)}{(s+1)}\,\frac{(2-s-e^{-(s-1)})}{(s-1)^2} \in A, \\ & = \frac{(s+1)^3-4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2}, \end{array}\right. \tag{14}$$
and thus, $M' = A^3/A^2\,R'$ is a projective A-module.

Exercise 4.1: Let $A = H_\infty(\mathbb{C}_+)$ and let us consider the matrix $R' \in A^{2\times 4}$ defined by
$$R' = \left(\begin{array}{cccc} 1 & 0 & -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{array}\right),$$
which corresponds to the weakly left-coprime factorization of Exercise 3.5. Prove that the finitely presented A-module $M' = A^4/A^2\,R'$ is a projective A-module (Hint: consider the two elements $\frac{s-1}{s+1}$ and $\frac{1}{s+1}$ of $\mathrm{Fitt}_2(M')$ and prove that 1 is an A-linear combination of them).

The following theorem gives necessary and sufficient conditions for a transfer matrix to admit left/right/doubly coprime factorizations.

Theorem 4.1: [56] Let $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ be any fractional representation of the transfer matrix $P \in K^{q\times r}$, where:
$$\left\{\begin{array}{l} R = (D \,:\, -N) \in A^{q\times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p\times r}. \end{array}\right.$$
Then, we have:

- *P admits a left-coprime factorization iff the A-module $\overline{A^q\,R}$ is a free A-module of rank q and $A^p/\overline{A^q\,R}$ is a stably free A-module.*
- *P admits a right-coprime factorization iff the A-module $\overline{A^r\,\tilde{R}^T}$ is a free A-module of rank r and $A^p/\overline{A^r\,\tilde{R}^T}$ is a stably free A-module.*
- *P admits a doubly coprime factorization iff $\overline{A^q\,R}$ and $\overline{A^r\,\tilde{R}^T}$ are two free A-modules of rank respectively q*

and r and $A^p/\overline{A^q\,R}$ and $A^p/\overline{A^r\,\tilde{R}^T}$ are two stably free A-modules.

Remark 4.1: If a transfer matrix P admits a left (resp. right or doubly) coprime factorization, then P also admits a weakly left (resp. right or doubly) coprime factorization (see Theorems 3.1 and 4.1). Thus, every left (resp. right or doubly) coprime factorization is a weakly left (resp. right or doubly) coprime factorization.

Exercise 4.2:

- [58], [59] Prove that $P \in K^{q\times r}$ admits a right-coprime factorization iff there exists a non-singular matrix $\tilde{D} \in A^{r\times r}$ such that

$$\begin{aligned} A^p\,(P^T\,:\,I_r)^T &= \{\lambda_1\,P + \lambda_2 \mid \lambda_1 \in A^q,\, \lambda_2 \in A^r\} \\ &= A^p\,\tilde{D}^{-1}. \end{aligned}$$

Deduce that $P = \tilde{N}\,\tilde{D}^{-1}$, where $\tilde{N} \triangleq P\,\tilde{D} \in A^{q\times r}$, is a right-coprime factorization of P.

- [58], [59] Prove that $P \in K^{q\times r}$ admits the left-coprime factorization iff there exists a non-singular matrix $D \in A^{q\times q}$ such that

$$\begin{aligned} A^p\,(I_q\,:\,-P)^T &= \left\{\lambda_1 - \lambda_2\,P^T \mid \lambda_1 \in A^q,\, \lambda_2 \in A^r\right\} \\ &= A^q\,(D^{-1})^T. \end{aligned}$$

Deduce that $P = D^{-1}\,N$, where $N \triangleq D\,P \in A^{q\times r}$, is a left-coprime factorization.

Proposition 4.3: [56] If $R \in A^{q\times p}$ is a full row rank matrix, then the A-module $M = A^p/A^q\,R$ is stably free iff the A-module $N = A^q/A^p\,R^T = 0$, i.e. iff there exists $S \in A^{p\times q}$ such that:

$$R\,S = I_q.$$

Example 4.2: Let us determine whether or not the transfer matrix P defined by (6) admits a left-coprime factorization. In Example 3.4, we proved that $\overline{A^2\,R} = A^2\,R'$, where $R' \in A^{2\times 3}$ is defined by (12). Hence, the A-module $\overline{A^2\,R}$ is a free A-module of rank 2. By Proposition 4.3, $A^3/\overline{A^2\,R} = A^3/A^2\,R'$ is a stably free A-module iff $A^2/A^3\,R'^T = 0$. The A-module $A^2/A^3\,R'^T$ is defined by the following equations

$$\left\{\begin{array}{l} \frac{1}{(s+1)}\,\lambda_1 + \frac{(s-1)}{(s+1)}\,\lambda_2 = 0, \\ -\frac{(s-1)}{(s+1)}\,\lambda_1 = 0, \\ -\frac{e^{-s}}{(s+1)}\,\lambda_2 = 0, \end{array}\right. \tag{15}$$

as well as their A-linear combinations. If we put a second member $\mu = (\mu_1\,:\,\mu_2\,:\,\mu_3)^T$ to the equations (15), combining the first two equations, we obtain:

$$\frac{(s-1)^2}{(s+1)^2}\,\lambda_2 = \frac{(s-1)}{(s+1)}\,\mu_1 + \frac{1}{(s+1)}\,\mu_2.$$

Combining this new equation with the last one of (15), we obtain

$$\lambda_2 = b\,\frac{(s-1)}{(s+1)}\,\mu_1 + b\,\frac{1}{(s+1)}\,\mu_2 - a\,\frac{1}{(s+1)}\,\mu_3, \tag{16}$$

where a and b are defined by (14). From the first two equations of (15), we also obtain:

$$\lambda_1 + 2\,\frac{(s-1)}{(s+1)}\,\lambda_2 = 2\,\mu_1 - \mu_2$$

Using this new equation and (16), we obtain:

$$\begin{aligned} \lambda_1 = \;& 2\,(-b\,\tfrac{(s-1)^2}{(s+1)^2} + 1)\,\mu_1 - (2\,b\,\tfrac{(s-1)}{(s+1)^2} + 1)\,\mu_2 \\ & +2\,a\,\tfrac{(s-1)}{(s+1)^2}\,\mu_3, \end{aligned} \tag{17}$$

Hence, if $\mu_1 = \mu_2 = \mu_3 = 0$, then, from (16) and (17), we obtain $\lambda_1 = \lambda_2 = 0$, i.e. we have $A^2/A^3\,R'^T = 0$, and thus, $A^3/\overline{A^2\,R} = A^3/A^2\,R'$ is a stably free A-module. By Theorem 4.1, P admits a left-coprime factorization. We have already done all the computations for such a left-coprime factorization: from (16) and (17), we obtain

$$(\lambda_1\,:\,\lambda_2) = (\mu_1\,:\,\mu_2\,:\,\mu_3)\,S,$$

where

$$S = \left(\begin{array}{cc} -2\,b\,\frac{(s-1)^2}{(s+1)^2} + 2 & b\,\frac{(s-1)}{(s+1)} \\ -2\,b\,\frac{(s-1)}{(s+1)^2} - 1 & b\,\frac{1}{(s+1)} \\ 2\,a\,\frac{(s-1)}{(s+1)^2} & -a\,\frac{1}{(s+1)} \end{array}\right) \in A^{3\times 2},$$

and thus, $R\,S = I_2$. Therefore, (13) is a left-coprime factorization of P because we have:

$$\begin{aligned} &\left(\begin{array}{cc} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} \\ \frac{(s-1)}{(s+1)} & 0 \end{array}\right) \left(\begin{array}{cc} -2\,b\,\frac{(s-1)^2}{(s+1)^2} + 2 & b\,\frac{(s-1)}{(s+1)} \\ -2\,b\,\frac{(s-1)}{(s+1)^2} - 1 & b\,\frac{1}{(s+1)} \end{array}\right) \\ &- \left(\begin{array}{c} 0 \\ \frac{e^{-s}}{(s+1)} \end{array}\right) \left(\begin{array}{cc} 2\,a\,\frac{(s-1)}{(s+1)^2}\,: & -a\,\frac{1}{(s+1)} \end{array}\right) = I_2. \end{aligned} \tag{18}$$

Exercise 4.3: Doing as in the previous example, show that

$$P = \left(\begin{array}{cc} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{array}\right)^{-1} \left(\begin{array}{cc} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s+1} \end{array}\right) \in K^{2\times 2}$$

is a left-coprime factorization of the transfer matrix P defined in Exercise 3.5 ($K = Q(H_\infty(\mathbb{C}_+))$).

Equivalent necessary and sufficient conditions of the existence of left/right/doubly coprime factorizations can be obtained.

Theorem 4.2: [56] Let $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ be any fractional representation of the transfer matrix $P \in K^{q\times r}$, where:

$$\left\{\begin{array}{l} R = (D\,:\,-N) \in A^{q\times p}, \\ \tilde{R} = (\tilde{N}^T\,:\,\tilde{D}^T)^T \in A^{p\times r}. \end{array}\right.$$

Then, we have:

- *P admits a left-coprime factorization iff $A^p/\overline{A^r\,\tilde{R}^T}$ is a free A-module of rank q.*
- *P admits a right-coprime factorization iff $A^p/\overline{A^q\,R}$ is a free A-module of rank r.*
- *P admits a doubly coprime factorization iff $A^p/\overline{A^r\,\tilde{R}^T}$ and $A^p/\overline{A^q\,R}$ are two free A-modules of rank respectively q and r.*

A direct consequence of the last point of Theorem 4.2 is the following corollary first obtained by V. R. Sule in [73].

Corollary 4.1: Let $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ be any fractional representation of the transfer matrix $P \in K^{q\times r}$, where:

$$\left\{\begin{array}{l} R = (D\,:\,-N) \in A^{q\times p}, \\ \tilde{R} = (\tilde{N}^T\,:\,\tilde{D}^T)^T \in A^{p\times r}. \end{array}\right.$$

Then, P admits a doubly coprime factorization iff the A-modules $A^p\, R^T$ and $A^p\, \tilde{R}$ are two free A-modules of rank respectively q and r.

Exercise 4.4: Using the last point of Theorem 4.2 and 3 and 4 of Proposition 3.2, prove Corollary 4.1.

Corollary 4.2: A SISO plant, defined by a transfer function $p = n/d \in K = Q(A)$, where $0 \neq d, n \in A$, admits a coprime factorization iff the ideal $I = (d, n)$ of A is a free A-module, i.e. I is a principal ideal of A (namely $I = (d, n)$ is defined by a single element of A).

This result was already proved by M. Vidyasagar in [78].

Exercise 4.5: Let us consider $R = (d : \, -n) \in A^{1\times 2}$. Show that the A-module $A^2\, R^T$ is the ideal $I = (d, n)$ of A defined by d and n. Then, using Theorem 4.2 and the result that an ideal I of an integral domain A is free iff I is a principal ideal (prove this result), prove Corollary 4.2.

Corollary 4.3: If A is a Hermite ring, namely a ring such that every finitely generated stably free A-module is free (see Definition 4.3), then a transfer matrix $P \in K^{q\times r}$ admits a doubly coprime factorization iff P admits a left-coprime factorization or a right-coprime factorization.

This result was firstly proved by M. Vidyasagar in [78].

Exercise 4.6: In this exercise, we prove Corollary 4.3.

1) Suppose that the transfer matrix P admits a left-coprime factorization $P = D^{-1}\, N$, $R = (D : \, -N) \in A^{q\times p}$. Using the first point of Theorem 4.1, deduce that the A-module $\overline{A^q\, R} = A^q\, R$ is free of rank q and the A-module $A^p/\overline{A^q\, R} = A^p/A^q\, R$ is stably free of rank r.
2) Using the definition of a Hermite ring (see Definition 4.3), deduce that $A^p/\overline{A^q\, R}$ is a free A-module.
3) Using the second point of Theorem 4.2, deduce that P admits a right-coprime factorization, i.e. P admits a doubly coprime factorization.
4) Do the same by admitting that P admits now a right-coprime factorization.

Finally, we have the following theorem which characterizes the class of rings A of SISO stable plants over which every transfer matrix admits a doubly coprime factorization.

Theorem 4.3: [78] We have the following equivalences:

1) *Every transfer function with entries in $K = Q(A)$ admits a coprime factorization.*
2) *Every transfer matrix with entries in $K = Q(A)$ admits a doubly coprime factorization.*
3) *A is a Bézout domain.*

Exercise 4.7: 1) Prove that $2 \Rightarrow 1 \Rightarrow 3$ (use Corollary 4.2 for the last implication).
2) Use the following result that A is a Bézout domain iff every finitely generated torsion-free A-module is free [26], Theorem 4.2 and Lemma 3.1 to prove $3 \Rightarrow 2$.

Example 4.3: For instance, if $A = RH_\infty$ or $A = \mathcal{E}$ (two Bézout domains), then every transfer matrix whose entries belong to $K = Q(A)$ admits a doubly coprime factorization. Recall that in a Bézout domain, two elements $a, b \in A$ generate an ideal $I = (a, b)$ which satisfies $I = ([a, b])$ (a Bézout domain is a gcdd by Example 3.8 and Proposition 3.3). Let us recall that we have [14], [16], [78]

$$\begin{cases} \forall\, a, b \in A = H_\infty(\mathbb{C}_+), \ (a, b) = A \\ \qquad\qquad \Leftrightarrow \inf_{s\in\mathbb{C}_+} (|a(s)| + |b(s)|) > 0, \\ \forall\, a, b \in A = \hat{\mathcal{A}}, \ (a, b) = A \\ \qquad\qquad \Leftrightarrow \inf_{s\in\overline{\mathbb{C}_+}} (|a(s)| + |b(s)|) > 0, \end{cases} \tag{19}$$

where $\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) \geq 0\}$ is the *closed right half plane*. Therefore, if we take $A = H_\infty(\mathbb{C}_+)$ or $A = \hat{\mathcal{A}}$, then $[\frac{1}{s+1}, e^{-s}] = 1$ (see Exercise 3.2) but the ideal

$$I = \left(\tfrac{1}{s+1},\, e^{-s}\right) \subsetneq (1) = A$$

because we have:

$$\inf_{s\in\mathbb{C}_+} \left(\left|\tfrac{1}{s+1}\right| + |e^{-s}|\right) = 0.$$

Indeed, if we take a sequence $(x_n)_{n\geq 0}$, with $x_n \in \mathbb{R}_+$ and $\lim_{n\to+\infty} x_n = +\infty$, then we have:

$$\lim_{n\to+\infty} \left|\tfrac{1}{x_n+1}\right| = \lim_{n\to+\infty} |e^{-x_n}| = 0.$$

Therefore, $A = H_\infty(\mathbb{C}_+)$ and $A = \hat{\mathcal{A}}$ are not Bézout domains.

Exercise 4.8: 1) Let us consider the plant defined by the transfer function $p = \frac{e^{-s}}{s-1}$. Show that p belongs to $K = Q(A)$, where $A = H_\infty(\mathbb{C}_+)$ or $A = \hat{\mathcal{A}}$, because we have:

$$\begin{cases} n = \frac{e^{-s}}{s+1} \in A, \\ d = \frac{s-1}{s+1} \in A. \end{cases}$$

2) Using (19), show that the two elements $d = \frac{s-1}{s+1}$ and $n = \frac{e^{-s}}{s+1}$ of A satisfy that the ideal $I = (d, n)$ is equal to A, and thus, that p admits a coprime factorization.
3) Show that $p = n/d$ is a coprime factorization of p with:

$$\tfrac{(s-1)}{(s+1)}\, \left(1 + 2\left(\tfrac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(\tfrac{e^{-s}}{s+1}\right)\, 2\, e = 1.$$

The effective computation of a doubly coprime factorization is generally a difficult task. See [8], [9], [78] for the explicit forms of coprime factorizations for some classes of SISO systems.

V. The fractional representation approach to synthesis problems

A. Introduction

> "The central idea that is used repeatedly in the book is that "of factoring" the transfer matrix of a (not necessarily stable) system as the "ratio" of two *stable* rational matrices. This idea was first used in a paper published in 1972 (see [76]), but the emphasis there was on analyzing the stability of a *given* system rather than on the *synthesis* of control systems as is the case here. It turns out that this seemingly simple stratagem leads to conceptually simple and computationally tractable solutions to many important and interesting problems...", M. Vidyasagar [78].

In the eighties, the fractional representation approach to synthesis problems was created in order to study in a unique mathematical framework some synthesis problems (e.g.

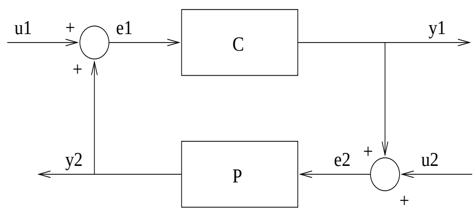

Fig. 1. Closed-loop system

internal/strong/simultaneous/robust stabilization, parametrization of all stabilizing controllers, robustness, H_2/H_∞-optimal controllers) for different classes of time-invariant linear systems (continuous-time, discrete, finite or infinite-dimensional systems) [2], [14], [17], [41], [77], [78]. The main idea of this approach was to give general formulations of different synthesis problems so that a wide variety of classes of systems (e.g. lumped or delay systems, systems of partial differential equations) could be studied using the same concepts and tools. In this approach, synthesis problems are reformulated independently to the classes of systems which are considered so that general conditions on the solvability of a specific synthesis problem can be obtained. Hence, the verification of the solvability of a synthesis problem for a particular system of a certain class is brought back to the verification of an abstract condition for which the parameters are specified. This allows to separate as much as possible the problems coming from the specific synthesis problem to the difficulties arriving from the class of systems which is considered. It is not surprising that the fractional representation approach to synthesis problems is then a ring-theoretic approach: algebra develops general (universal) concepts which can be used in very different situations. Therefore, it is not surprising to use module theory and homological algebra in the studies of the fractional representation approach to synthesis problems. Indeed, these two algebraic theories have been developed to understand general features of algebraic stuctures without specifying a particular ring. Hence, we could easily say that the fractional representation approach to synthesis problems is *a homological algebra approach to stabilization problems*.

B. Internal stabilization

Let us consider the closed-loop system defined in Figure 1 where u_2 (resp. u_1) is the consign (resp. a perturbation), y_1 and y_2 the outputs and e_1 and e_2 the internal inputs. We have the following equations of the closed-loop system:

$$\left\{\begin{array}{l} \left(\begin{array}{cc} I_q & -P \\ -C & I_r \end{array}\right) \left(\begin{array}{c} e_1 \\ e_2 \end{array}\right) = \left(\begin{array}{c} u_1, \\ u_2 \end{array}\right), \\ y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{array}\right.$$

The following definition plays a crucial role in all the rest of the paper.

Definition 5.1: [17], [41], [44], [77], [78] Let A be an integral domain of SISO stable plants and $K = Q(A)$ its quotient field. Let $P \in K^{q \times r}$ be a transfer matrix of a plant and $C \in K^{r \times q}$ a transfer matrix of a controller. Then, C is called *an internal stabilizing controller of* P if

$$H(P, C) = \left(\begin{array}{cc} I_q & -P \\ -C & I_r \end{array}\right)^{-1}$$

$$= \left(\begin{array}{cc} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{array}\right) \in A^{p \times p},$$

i.e. all the entries of the transfer matrix from $(u_1 : u_2)^T$ to $(e_1 : e_2)^T$ are A-stable.

Example 5.1: Let us consider $p = \frac{1}{(s-1)} \in K = \mathbb{R}(s)$ given in [37] and $A = RH_\infty$. The controller $c = -\frac{(s-1)}{(s+1)}$ proposed in [37] is not a stabilizing controller of p because we have

$$\left\{\begin{array}{l} e_1 = \frac{(s+1)}{(s+2)}\,u_1 + \frac{(s+1)}{(s+2)\,(s-1)}\,u_2, \\ e_2 = -\frac{(s-1)}{(s+2)}\,u_1 + \frac{(s+1)}{(s+2)}\,u_2, \end{array}\right.$$

and the transfer function from u_2 to e_1 is not stable (unstable pole at $s = 1$). Hence, unstable pole-zero cancellations between the plant p and the controller c lead to an instability in the closed-loop, i.e. c is not a stabilizing controller of p.

Proposition 5.1: We have the following equivalences:

- *If $A = H_\infty(\mathbb{C}_+)$, then internal stabilizability is equivalent to the fact that the linear operator $T_{H(P,\,C)}$, defined by*

$$\begin{array}{rcl} H_2(\mathbb{C}_+)^p & \longrightarrow & H_2(\mathbb{C}_+)^p, \\ u = (u_1 : u_2)^T & \longmapsto & (e_1 : e_2)^T = H(P, C)\,u, \end{array}$$

  is bounded [14], [28], namely:

$$\mathrm{dom}(T_{H(P,\,C)}) = \{u \in H_2^p \mid H(P, C)\,u \in H_2^p\} = H_2^p.$$

  This means that there is no input u with a finite energy, i.e. $u \in H_2^p$, so that the corresponding internal input $e = (e_1 : e_2)^T$ has an infinite energy, i.e. $e \notin H_2^p$.
- *If $A = RH_\infty(\mathbb{C}_+)$ or $A = \hat{\mathcal{A}}$, then internal stabilizability implies that the linear operator $T_{H(P,\,C)}$, defined by*

$$\begin{array}{rcl} H_2(\mathbb{C}_+)^p & \longrightarrow & H_2(\mathbb{C}_+)^p, \\ u = (u_1 : u_2)^T & \longmapsto & (e_1 : e_2)^T = H(P, C)\,u, \end{array}$$

  is bounded [12], [15], [78], namely:

$$\mathrm{dom}(T_{H(P,\,C)}) = \{u \in H_2^p \mid H(P, C)\,u \in H_2^p\} = H_2^p.$$

- *If $A = \mathcal{A}$, then internal stabilization implies that the operator $T_{H(P,\,C)}$, defined by*

$$\begin{array}{rcl} L_q(\mathbb{R}_+)^p & \longrightarrow & L_q(\mathbb{R}_+)^p, \\ u = (u_1 : u_2)^T & \longmapsto & (e_1 : e_2)^T = H(P, C) \star u, \end{array}$$

  is bounded for $1 \leq q \leq +\infty$, namely

$$\begin{aligned} &\mathrm{dom}(T_{H(P,\,C)}) \\ &\quad = \{u \in L_q(\mathbb{R}_+)^p \mid H(P, C)\,u \in L_q(\mathbb{R}_+)^p\} \\ &\quad = L_q(\mathbb{R}_+)^p. \end{aligned}$$

  Moreover, if the convolution kernel $H(P, C)$ has a vanishing singular part, then internal stabilization is equivalent to BIBO stability*, i.e. to the fact that the* *previous linear operator is bounded for $q = +\infty$ [12], [14], [15].*

The following theorem characterizes internal stabilization in terms of module theory.

Theorem 5.1: [54], [56] A plant defined by a transfer matrix $P = D^{-1} N = \tilde{N}\, \tilde{D}^{-1} \in K^{q\times r}$, where

$$\begin{cases} R = (D \,:\, -N) \in A^{q\times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p\times r}, \end{cases}$$

is internally stabilized by a controller of the form $C = Y\, X^{-1}$ (resp. $C = \tilde{X}^{-1}\, \tilde{Y}$) iff $A^p/\overline{A^q\, R}$ (resp. $A^p/\overline{A^r\, \tilde{R}^T}$) is a projective A-module.

From Theorem 5.1, we obtain the following algorithm:

Algorithm 2: Input: A coherent domain A and a matrix $R = (D \,:\, -N) \in A^{q\times p}$.
Ouput: Stabilizability or not of $P = D^{-1}\, N \in K^{q\times r}$.

1) Using Algorithm 1, compute $\overline{A^q\, R}$: we obtain $q' \in \mathbb{Z}_+$ and $R' \in A^{q'\times p}$ such that $\overline{A^q\, R} = A^{q'}\, R'$.
2) For increasing i, check whether or not:
$$1 \in \mathrm{Fitt}_i(A^p/A^{q'}\, R').$$
If there exists i such that $1 \in \mathrm{Fitt}_i(A^p/A^{q'}\, R')$, then P is internally stabilizable, else not.

Remark 5.1: In order to be able to check effectively internal stabilizability, we need to be able to:

- compute the kernel of matrices with entries in A,
- test whether or not 1 belongs to a finitely generated ideal of A.

Example 5.2: Let us reconsider Example 4.2. We proved that the A-module $A^3/A^2\, R'$ was projective ($A = H_\infty(\mathbb{C}_+)$), where the matrix $R' \in A^{2\times 3}$ is defined by (12). Moreover, in Example 3.4, we proved that $\overline{A^2\, R} = A^2\, R'$, where R is defined by (7). Thus, the A-module $A^3/\overline{A^2\, R} = A^3/A^2\, R'$ is projective and, by Theorem 5.1, the plant defined by the transfer matrix P (6) is internally stabilized by a certain controller of the form $C = Y\, X^{-1}$.

Exercise 5.1: Using Exercises 3.5 and 4.1, prove that the transfer matrix P defined in Exercise 3.5 is internally stabilizable.

Corollary 5.1: [56] If a transfer matrix $P \in K^{q\times r}$ admits a weakly left (resp. right) coprime factorization of the form $P = D^{-1}\, N$ (resp. $P = \tilde{N}\, \tilde{D}^{-1}$), where

$$R = (D \,:\, -N) \in A^{q\times p}$$

(resp. $\tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p\times r}$), then P is internally stabilizable iff $P = D^{-1}\, N$ (resp. $P = \tilde{N}\, \tilde{D}^{-1}$) is a left (resp. right) coprime factorization of P. Moreover, if we have

$$\begin{cases} D\, X - N\, Y = I_q, \\ S = (X^T \,:\, Y^T)^T \in A^{p\times q}, \end{cases} \tag{20}$$

(resp.

$$\begin{cases} \tilde{Y}\, \tilde{N} - \tilde{X}\, \tilde{D} = I_r, \\ \tilde{S} = (\tilde{Y} \,:\, \tilde{X}) \in A^{r\times p}\,), \end{cases} \tag{21}$$

then, the controller $C = Y\, X^{-1}$ (resp. $C = \tilde{X}^{-1}\, \tilde{Y}$) internally stabilizes P.

Exercise 5.2: 1) If P admits a left-coprime factorization (resp. a right-coprime factorization) of the form (20) (resp. (21)), then prove that P is internally stabilized by $C = Y\, X^{-1}$ (resp. $C = \tilde{X}^{-1}\, \tilde{Y}$) (Hints: for instance, if P admits the left-coprime factorization (20), then prove we have $I_q - P\, C = (X\, D)^{-1}$, and thus,
$$\begin{cases} (I_q - P\, C)^{-1} = X\, D \in A^{q\times q}, \\ (I_q - P\, C)^{-1}\, P = X\, N \in A^{q\times r}, \\ C\, (I_q - P\, C)^{-1} = Y\, D \in A^{r\times q}, \\ C\, (I_q - P\, C)^{-1}\, P = Y\, N \in A^{r\times r}, \end{cases}$$
i.e. C internally stabilizes P. See [60] for the explicit computations).

2) Prove the converse of Corollary 5.1 using the following result "if P admits a weakly left-coprime factorization $P = D^{-1}\, N$, with $R = (D \,:\, -N) \in A^{q\times p}$, then $A^p/A^q\, R$ is a projective A-module iff $A^p/A^q\, R$ is a stably free A-module" (see [56] for a proof of this result).

Example 5.3: In Example 4.2, we gave a left-coprime factorization (18) of the transfer matrix P defined by (6). Thus, by Corollary 5.1, the controller defined by

$$\begin{aligned} C &= Y\, X^{-1} \\ &= \left(2\, a\, \tfrac{(s-1)}{(s+1)^2} \,:\, -a\, \tfrac{1}{(s+1)}\right) \begin{pmatrix} -2\, b\, \frac{(s-1)^2}{(s+1)^2} + 2 & b\, \frac{(s-1)}{(s+1)} \\ -2\, b\, \frac{(s-1)}{(s+1)^2} - 1 & b\, \frac{1}{(s+1)} \end{pmatrix}^{-1} \\ &= -\frac{4\, (5\, s-3)\, e\, (s-1)^2}{(s+1)\, ((s+1)^3 - 4\, (5\, s-3)\, e^{-(s-1)})}\, (1 \,:\, 2), \end{aligned}$$

internally stabilizes P.

Example 5.4: Let us consider the following transfer function $p = e^{-\sqrt{s}}/(s-1)$ arising in the theory of transmission lines [9]. Let $A = H_\infty(\mathbb{C}_+)$ and let us denote by:

$$\begin{cases} n = e^{-\sqrt{s}}/(s+1) \in A, \\ d = (s-1)/(s+1) \in A. \end{cases}$$

Then, we have $p = n/d$ and $[d,\, n] = 1$ which shows that $p = n/d$ is a weakly coprime factorization of p. Hence, p is internally stabilizable iff p admits a coprime factorization, i.e. there exists $x,\, y \in A$ such that $d\, x - n\, y = 1$. Hence, the existence of a coprime factorization for p is equivalent to the existence of $y \in A$ such that:

$$x = \frac{1 + y\, e^{-\sqrt{s}}/(s+1)}{(s-1)/(s+1)} = \frac{(s+1) + y\, e^{-\sqrt{s}}}{(s-1)} \in A.$$

Therefore, we must try to remove the unstable pole 1 by choosing an appropriate y, i.e. $y \in A$ such that:

$$((s+1) + y\, e^{-\sqrt{s}})(1) = 2 + y(1)\, e^{-1} = 0.$$

If we choose $y = y(1) = -2\, e \in A$, then we have:

$$x = \frac{(s+1) - 2\, e^{1-\sqrt{s}}}{(s-1)} \in A.$$

Therefore, $c = y/x$ is a stabilizing controller of p.

We refer the reader to [8], [9] for explicit coprime factorizations for some classes of infinite-dimensional linear SISO systems (e.g differential time-delay or fractional differential systems).

Corollary 5.2: [56] If A is a projective-free integral domain, then every plant, defined by a transfer matrix with entries in $K = Q(A)$, is internally stabilizable iff it admits a doubly coprime factorization.

In particular, Corollary 5.2 is true for coherent Sylvester domains (e.g. $H_\infty(\mathbb{C}_+)$ [71], RH_∞ [78]).

Corollary 5.3: The integral domain $M_{\mathbb{D}^n}$, defined in (5), is projective-free [10], [39], and thus, every internally stabilizable admits a doubly coprime factorization [59].

Corollary 5.3 answers to a conjecture of Z. Lin. See [43] and the references therein. See [59] for more details.

Proposition 5.2: [56] We have the following equivalences:

- *The A-module $A^p / \overline{A^q\, R}$ is projective ($R \in A^{q \times p}$).*
- *The A-module $A^p\, R^T$ is projective.*

Hence, we have the following corollary of Theorem 5.1 and Proposition 5.2 which was firstly proved by V. R. Sule in [73].

Corollary 5.4: $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times r}$, where

$$\begin{cases} R = (D \,:\, -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}, \end{cases}$$

is internally stabilizable by a controller $C = Y\, X^{-1}$ (resp. $C = \tilde{X}^{-1}\, \tilde{Y}$) iff $A^p\, R^T$ (resp. $A^p\, \tilde{R}$) is a projective A-module.

In [47], K. Mori developed an algorithm in order to check whether or not an A-module of the form $A^p\, R^T$ is projective. Alternatively, using the approach developed in these notes, we can first compute the A-closure $\overline{A^q\, R}$ of the A-module $A^q\, R$ (see Algorithm 1 of section III-C) and use Proposition 4.2 to check whether or not $A^p / \overline{A^q\, R}$ is a projective A-module, i.e. whether or not P is internally stabilizable (see Algorithm 2). In the next corollary, we give two characterizations of internal stabilizability only using matrices.

Corollary 5.5:

1) *[55], [56] $P = D^{-1}\, N \in K^{q \times r}$, where $R = (D \,:\, -N) \in A^{q \times p}$, is internally stabilizable iff there exists $S = (X^T \,:\, Y^T)^T \in K^{p \times q}$, with $\det X \neq 0$, such that:*
   - $S\, R = \begin{pmatrix} X\, D & -X\, N \\ Y\, D & -Y\, N \end{pmatrix} \in A^{p \times p}$,
   - $R\, S = D\, X - N\, Y = I_q$.

   Then, the controller $C = Y\, X^{-1}$ internally stabilizes P.
2) *[55], [56] $P = \tilde{N}\, \tilde{D}^{-1}$, where $\tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times r}$, is internally stabilizable iff there exists a matrix $T = (-\tilde{Y} \,:\, \tilde{X}) \in K^{r \times p}$, with $\det \tilde{X} \neq 0$, such that:*
   - $S\, R = \begin{pmatrix} -\tilde{N}\, \tilde{Y} & \tilde{N}\, \tilde{X} \\ -\tilde{D}\, \tilde{Y} & \tilde{D}\, \tilde{X} \end{pmatrix} \in A^{p \times p}$,
   - $T\, R = -\tilde{Y}\, \tilde{N} + \tilde{X}\, \tilde{D} = I_r$.

   Then, the controller $C = \tilde{X}^{-1}\, \tilde{Y}$ internally stabilizes P.

Exercise 5.3: Give the proofs of 1 and 2 of Corollary 5.5 using only matrices. Compare your proofs with [58], [59].

Exercise 5.4: Check that $S = (X^T \,:\, Y^T)^T \in K^{3 \times 2}$ defined by

$$S = \begin{pmatrix} b\, \frac{(s-1)}{(s+1)} + 2\, \frac{(s+1)}{(s-1)^2} & 2\, b\, \frac{(s-1)}{(s+1)} - 2\, \frac{(s-1)}{s+1} \\ \frac{b}{(s+1)} - \frac{(s+1)}{(s-1)^2} & \frac{2\, b}{(s+1)} + \frac{(s+1)}{(s-1)} \\ -\frac{a}{(s+1)} & -\frac{2\, a}{(s+1)} \end{pmatrix},$$

where a and b are defined by (14), satisfies:

$$\begin{cases} S\, R \in A^{3 \times 3}, \\ R\, S = D\, X - N\, Y = I_2. \end{cases}$$

Deduce that P is internally stabilized by the controller:

$$\begin{aligned} C &= Y\, X^{-1} = \left(-\tfrac{a}{(s+1)} \,:\, -\tfrac{2\, a}{(s+1)}\right) \\ &\quad \begin{pmatrix} b\, \frac{(s-1)}{(s+1)} + 2\, \frac{(s+1)}{(s-1)^2} & 2\, b\, \frac{(s-1)}{(s+1)} - 2\, \frac{(s-1)}{s+1} \\ \frac{b}{(s+1)} - \frac{(s+1)}{(s-1)^2} & \frac{2\, b}{(s+1)} + \frac{(s+1)}{(s-1)} \end{pmatrix}^{-1}, \\ &= -\frac{4\, (5\, s-3)\, e\, (s-1)^2}{(s+1)\, ((s+1)^3 - 4\, (5\, s-3)\, e^{-(s-1)})}\, (1 \,:\, 2). \end{aligned}$$

Corollary 5.6: A SISO plant, defined by a transfer function $p = n/d \in K = Q(A)$, where $0 \neq d,\, n \in A$, is internally stabilizable iff the ideal $I = (d,\, n)$ of A is a projective A-module, i.e. there exist $x,\, y \in K$ such that:

$$\begin{cases} d\, x - n\, y = 1, \\ d\, x,\ d\, y,\ n\, x \in A. \end{cases} \tag{22}$$

If $x \neq 0$ (resp. $x = 0$), then the controller $c = y/x \in K$ (resp. $c = 1 - d\, y \in A$) internally stabilizes p.

Exercise 5.5: The main purpose of the exercise is to prove Corollary 5.6. See [57] for the proofs.

1) Let us consider the matrix $R = (d \,:\, -n) \in A^{1 \times 2}$. Show that $A^2\, R^T$ is the ideal $I = (d,\, n)$ of A defined by d and n.
2) Using Theorem 5.1 and Corollary 5.4, prove that the plant $p = n/d$ is internally stabilizable iff the ideal $I = (d,\, n)$ of A is a projective A-module.
3) Using Corollary 5.5, prove that $p = n/d$ is internally stabilizable iff (22) is satisfied for a certain couple $(x,\, y) \in K^2$.
4) If $x \neq 0$ (resp. $x = 0$), prove directly that $c = y/x$ (resp. $c = 1 - d\, y$), where $x,\, y \in K$ satisfy (22), is a stabilizing controller of p by showing that we have:

$$\begin{aligned} H(p,\, c) &= \begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{1-p\, c} & \frac{p}{1-p\, c} \\ \frac{c}{1-p\, c} & \frac{1}{1-p\, c} \end{pmatrix} \in A^{2 \times 2}. \end{aligned} \tag{23}$$

5) One can show that $I = (d,\, n)$ is a projective A-module iff I is an *invertible ideal* of A, namely I is such that the product $I\, (A : I) \triangleq \{\sum_{i=1}^n a_i\, b_i \mid a_i \in I,\, b_i \in A : I\}$ of I by $A : I = \{k \in K = Q(A) \mid k\, d,\ k\, n \in A\}$ equals A [54], [56]. Recover point 3 using the fact that $p = n/d$ is internally stabilizable iff $I = (d,\, n)$ is an invertible ideal of A.
6) Prove that $c = s/r$ internally stabilizes $p = n/d$ iff we have the following equality of ideals of A:

$$(d,\, n)\, (r,\, s) = (d\, r - n\, s).$$

7) Prove that:

$$I\, (A : I) = \{a \in A \mid a\, n \in (d)\} + \{a \in A \mid a\, d \in (n)\}.$$

Deduce that p is internally stabilizable iff we have

$$\{a \in A \mid a\, n \in (d)\} + \{a \in A \mid a\, d \in (n)\} = A$$

(see [54], [56] for a proof). This last result was firstly proved by S. Shankar and V. R. Sule in [69].

8) Prove that $p = n/d$ admits a weakly coprime factorization iff $A : I$ is a principal *fractional ideal* of A (see Exercise 5.8 for the definition of fractional ideals).
9) Prove $p = n/d$ admits a coprime factorization iff I is a principal ideal of A.
10) Prove that $p = n/d$ is is *strongly* (resp. *bistably*) *stabilizable*, namely p is internally stabilizable by means of a stable controller $c \in A$ (resp. by a stable controller c whose inverse is stable [4], [20], [78], i.e. $c \in \mathrm{U}(A)$) iff there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that:

$$I = (d - n\, c).$$

Exercise 5.6: [57] Let us consider the wave equation:

$$\left\{ \begin{array}{l} \frac{\partial^2 z}{\partial t^2}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ \frac{\partial z}{\partial x}(0,t) = 0, \\ \frac{\partial z}{\partial x}(1,t) = u(t), \\ y(t) = \frac{\partial z}{\partial t}(1,t), \end{array} \right. \tag{24}$$

1) Prove that the transfer function of (24) is given by:

$$p = (e^s + e^{-s})/(e^s - e^{-s}).$$

2) Prove that $p \in K = Q(H_\infty(\mathbb{C}_+))$.
3) Using the fact that $A = H_\infty(\mathbb{C}_+)$ is a gcdd (see Corollary 3.3), compute a weakly coprime factorization of p.
4) Prove that p is internally stabilizable and compute a stabilizing controller of p.
5) Determine a coprime factorization of p.
6) Prove that p is bistably stabilizable.

The next theorem gives some explicit characterizations of internal stabilizability only using the transfer matrix P of the system, i.e. without using any fractional representation of P.

Theorem 5.2: [58], [59] $P \in K^{q \times r}$ is internally stabilizable iff one of the following conditions is satisfied:

1) There exists $S = (U^T \,:\, V^T)^T \in A^{p \times q}$ such that:

$$\left\{ \begin{array}{ll} S\, P & = \left(\begin{array}{c} U\, P \\ V\, P \end{array} \right) \in A^{p \times r}, \\ (I_q \,:\, -P)\, S & = U - P\, V = I_q. \end{array} \right.$$

Then, $C = V\, U^{-1}$ is a stabilizing controller of P.

2) There exists $T = (-X \,:\, Y) \in A^{r \times p}$ such that:

$$\left\{ \begin{array}{ll} P\, T & = (P\, X \,:\, P\, Y) \in A^{q \times p}, \\ T \left(\begin{array}{c} P \\ I_r \end{array} \right) & = -X\, P + Y = I_r. \end{array} \right.$$

Then, $C' = Y^{-1}\, X$ is a stabilizing controller of P.

If P is internally stabilizable, then there exist $S \in A^{p \times q}$, $T \in A^{r \times p}$ satisfying 1 and 2 and such that

$$T\, S = -X\, U + Y\, V = 0,$$

i.e. there exists a stabilizing controller of P of the form:

$$C = V\, U^{-1} = Y^{-1}\, X.$$

Exercise 5.7: Check that $S = (U^T \,:\, V^T)^T \in A^{3 \times 2}$ defined by

$$S = \left(\begin{array}{cc} \frac{2}{s+1} + b \left(\frac{s-1}{s+1}\right)^3 & 2\, b \left(\frac{s-1}{s+1}\right)^3 - 2\, \frac{(s-1)}{(s+1)} \\ b\, \frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\, b\, \frac{(s-1)}{(s+1)^3} \\ -a\, \frac{(s-1)^2}{(s+1)^3} & -2\, a\, \frac{(s-1)^2}{(s+1)^3} \end{array} \right),$$

where a and b are defined by (14), satisfies:

$$\left\{ \begin{array}{l} S\, (I_2 \,:\, -P) \in A^{3 \times 3}, \\ (I_2 \,:\, -P)\, S = U - P\, V = I_2. \end{array} \right.$$

Deduce that P is internally stabilized by the controller:

$$\begin{array}{ll} C & = V\, U^{-1} = \left(\begin{array}{cc} -a\, \frac{(s-1)^2}{(s+1)^3} \,: & -2\, a\, \frac{(s-1)^2}{(s+1)^3} \end{array} \right) \\ & \left(\begin{array}{cc} \frac{2}{s+1} + b \left(\frac{s-1}{s+1}\right)^3 & 2\, b \left(\frac{s-1}{s+1}\right)^3 - 2\, \frac{(s-1)}{(s+1)} \\ b\, \frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\, b\, \frac{(s-1)}{(s+1)^3} \end{array} \right)^{-1}, \\ & = -\frac{4\,(5\,s-3)\, e\, (s-1)^2}{(s+1)\,((s+1)^3 - 4\,(5\,s-3)\, e^{-(s-1)})}\, (1 \,:\, 2). \end{array}$$

Corollary 5.7: [59] $P \in K^{q \times r}$ is internally stabilized by the controller $C \in K^{r \times q}$ iff one of the following conditions is satisfied:

- The matrix

$$\Pi_1 = \left(\begin{array}{cc} (I_q - P\, C)^{-1} & -(I_q - P\, C)^{-1}\, P \\ C\, (I_q - P\, C)^{-1} & -C\, (I_q - P\, C)^{-1}\, P \end{array} \right)$$

is a *projector of* $A^{p \times p}$, namely $\Pi_1^2 = \Pi_1 \in A^{p \times p}$.

- The matrix

$$\Pi_2 = \left(\begin{array}{cc} -P\, (I_r - C\, P)^{-1}\, C & P\, (I_r - C\, P)^{-1} \\ -(I_r - C\, P)^{-1}\, C & (I_r - C\, P)^{-1} \end{array} \right)$$

is a *projector of* $A^{p \times p}$, namely $\Pi_2^2 = \Pi_2 \in A^{p \times p}$.

Moreover, we have:

$$\Pi_1 + \Pi_2 = I_p.$$

Corollary 5.7 was already proved for $H_\infty(\mathbb{C}_+)$ [28].

Remark 5.2: First of all, let us notice that we can prove that Corollary 5.7 is equivalent to the fact that $P \in K^{q \times r}$ is internally stabilizable iff one of the following conditions is satisfied [58], [59]:

- $A^p\, (P^T \,:\, I_r)^T$ is a *projective lattice* of K^r, namely a projective A-submodule of K^r of rank r,
- $A^p\, (I_q \,:\, -P)^T$ is a *projective lattice* of K^q, namely a projective A-submodule of K^q of rank q.

Secondly, in the *loop-shaping procedure* [20], [29], let us notice that the *robustness radius* is defined by [20], [25], [29]:

$$b_{P,C} \triangleq \|\, \Pi_1 \,\|_\infty^{-1} = \|\, \Pi_2 \,\|_\infty^{-1}.$$

Corollary 5.8:

- If $P \in K^{q \times r}$ admits a left-coprime factorization $P = D^{-1}\, N$, $D\, X - N\, Y = I_q$, then $S = ((X\, D)^T \,:\, (Y\, D)^T)^T$ satisfies 1 of Theorem 5.2, and thus, $C = (Y\, D)\, (X\, D)^{-1} = Y\, X^{-1}$ is a stabilizing controller of P.

- Similarly, if $P \in K^{q\times r}$ admits a right-coprime factorization $P = \tilde{N}\,\tilde{D}^{-1}$, $-\tilde{Y}\,X + \tilde{X}\,\tilde{D} = I_r$, then $T = (-\tilde{D}\,\tilde{Y} : \tilde{D}\,\tilde{X})$ satisfies 2 of Theorem 5.2, and thus, $C = (\tilde{D}\,\tilde{X})^{-1}\,(\tilde{D}\,\tilde{Y}) = \tilde{X}^{-1}\,\tilde{Y}$ is a stabilizing controller of P.

Exercise 5.8: This exercise is based on certain results obtained in [55], [57], [62]. We refer the reader to these papers for more details and the solutions.

1) The lattices of K are called the *fractional ideals* of A. A fractional ideal J of A is an A-submodule of the quotient field $K = Q(A)$ which satisfies that there exists $0 \neq a \in A$ such that $a\,J \subseteq A$. Let $p \in K$ be a transfer function. Prove that $J = (1,\,p) \triangleq A + A\,p$ is a fractional ideal of A.
2) Prove that p admits a weakly coprime factorization iff the ideal $J = (1,\,p)$ satisfies that
$$A : J \triangleq \{k \in K \mid k,\, k\,p \in A\} = \{d \in A \mid d\,p \in A\}$$
is a *principal integral ideal of* A, namely has the form $A : J = (d)$, with $0 \neq d \in A$.
$A : J$ is called the *ideal of the denominators* of p whereas $(p)\,(A : J)$ is the *ideal of the numerators* of p.
3) Prove that p admits a coprime factorization iff the fractional ideal $J = (1,\,p)$ is principal.
4) $c \in K$ is said to *externally stabilizes* $p \in K$ if the transfer function $(p\,c)/(1 - p\,c) \in A$. Prove that $c \in K$ externally stabilizes p iff we have $(1,\,p\,c) = (1 - p\,c)$.
5) Prove p is internally stabilizable iff the fractional ideal $J = (1,\,p)$ is *invertible*, namely satisfies $J\,(A : J) = A$, where the product $J\,(A : J)$ is defined by:
$$J\,(A : J) = \{a + b\,p \mid a,\, b \in A :\ a\,p,\ b\,p \in A\}.$$
If J is an invertible fractional ideal of A, then $A : J$ is called *the inverse* of J and is denoted by J^{-1}. Deduce that p is internal stabilizable iff there exist $a,\, b \in A$ which satisfy [1]:
$$\begin{cases} a - b\,p = 1, \\ a\,p \in A. \end{cases} \tag{25}$$
Then, prove that if $a \neq 0$ (resp. $a = 0$), $c = b/a \in K$ (resp. $c = 1 - b \in A$) is a stabilizing controller of p and $J^{-1} = (a,\, b)$. Finally, if $a \neq 0$, then prove that we have:
$$\begin{cases} a = 1/(1 - p\,c) \quad (\textit{sensitivity transfer function}), \\ b = c/(1 - p\,c). \end{cases}$$
6) Prove directly that $c = b/a \in K$, where $0 \neq a,\, b \in A$ satisfy (25), is an internally stabilizing controller of p by showing that we then have (23).
7) Prove that $c \in K$ internally stabilizes $p \in K$ iff we have the following equality of fractional ideals of A:
$$(1,\,p)\,(1,\,c) = (1 - p\,c). \tag{26}$$
8) Consider the transfer function p defined in Example 4.8. Prove that p is internally stabilizable and p admits a coprime factorization.
9) Prove that $c = -(s-1)/(s+1) \in A$ cannot internally stabilize the plant $p = 1/(s-1)$ (see Example 5.1) using only (26) and the fact that $1 - p\,c \in \mathrm{U}(A)$.
10) Prove that if p admits a weakly coprime factorization and is internal stabilizable, then p admits a coprime factorization.
11) Let $c \in K$ be a stabilizing controller of p. Using 3 and (26), prove that c admits a coprime factorization iff p admits a coprime factorization.

The next theorem gives a general parametrization of all stabilizing controllers of an internal stabilizable plant which does not necessarily admit a doubly coprime factorization.

Theorem 5.3: [58], [59] Let $P \in K^{q\times r}$ be an internally stabilizable plant. Then, all stabilizing controllers of P have the form
$$\begin{aligned} C(Q) &= (V + Q)\,(U + P\,Q)^{-1} \\ &= (Y - Q\,P)^{-1}\,(X - Q), \end{aligned} \tag{27}$$
where $C = V\,U^{-1} = Y^{-1}\,X$ is a particular stabilizing controller of P, i.e. we have
$$\begin{cases} U - P\,V = I_q, \\ Y - X\,P = I_r, \\ \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{p\times r}, \\ (-P\,X :\ P\,Y) \in A^{q\times p}, \end{cases}$$
and Q is any matrix which belongs
$$\Omega = \{L \in A^{r\times q} \mid \quad \begin{array}{l} L\,P \in A^{r\times r},\ P\,L \in A^{q\times q}, \\ P\,L\,P \in A^{q\times r}\} \end{array} \tag{28}$$
such that $\det(U + P\,Q) \neq 0$ and $\det(Y - Q\,P) \neq 0$.

Let us notice that some attempts in order to parametrize all stabilizing controllers of an internally stabilizable plant which does not necessarily admit a doubly coprime factorization have been done in [48], [73]. Unfortunately, these parametrizations are either not explicit in the free parameters or the set of free parameters is not characterized.

Remark 5.3: The number of free parameters in the parametrization (27) is completely characterized by the projective A-module Ω of rank $r \times q$ defined by (28). Let us notice that determining the cardinal $\mu(\Omega)$ of a minimal generating system of an A-module is a well-known and difficult problem in algebra. Some bounds on $\mu(\Omega)$ have been given in [59] for different cases of systems but the general case is still open. However, for SISO systems, a complete answer is given in the next corollary.

Corollary 5.9: [57] Let $p = n/d \in K = Q(A)$ be an internally stabilizable plant.

[1]While we were completing the paper at the beginning of 2004, we have found that a similar characterization of internal stabilizability was obtained in the paper "Feedback, minimax sensitivity, and optimal robustness", G. Zames, B. A. Francis, IEEE Trans. Autom. Contr., 28 (1983), pp. 585-601, under the form: p is internally stabilizable iff there exists a stable q such that $a = 1 - p\,q$ and $a\,p = (1 - p\,q)\,p$ are both stable. This characterization corresponds to $b = -q$, up to the sign convention in the closed-loop system (see Figure 1).

- All stabilizing controllers of p have the form
$$c(q_1, q_2) = \frac{y + q_1\, d\, x^2 + q_2\, d\, y^2}{x + q_1\, n\, x^2 + q_2\, n\, y^2},$$
where $c = y/x$ is a stabilizing controller of p, namely
$$\begin{cases} d\, x - n\, y = 1, \\ d\, x,\ d\, y,\ n\, x \in A. \end{cases} \tag{29}$$
(see (22)) and q_1, q_2 are any element of A satisfying:
$$x + q_1\, n\, x^2 + q_2\, n\, y^2 \neq 0.$$
- All stabilizing controllers of p have the form
$$c(q_1, q_2) = \frac{b + q_1\, a^2 + q_2\, b^2}{a + q_1\, a^2\, p + q_2\, b^2\, p},$$
where $c = b/a$ is a stabilizing controller of p, namely
$$\begin{cases} a - b\, p = 1, \\ a,\ b,\ a\, p \in A, \end{cases} \tag{30}$$
(see (25)), and q_1, q_2 are any element of A satisfying:
$$a + q_1\, a^2\, p + q_2\, b^2\, p \neq 0.$$

The parametrizations (29) and (30) have only one free parameter iff p^2 admits a coprime factorization. If $p^2 = s/r$ is a coprime factorization of p, then:

- The parametrization (29) becomes the following one
$$c(q) = \frac{d\, y + q\, r}{d\, x + q\, r\, p},$$
where q is any element of A such that $d\, x + q\, p\, r \neq 0$.
- The parametrization (30) becomes the following one
$$c(q) = \frac{b + q\, r}{a + q\, r\, p},$$
where q is any element of A such that $a + q\, p\, r \neq 0$.

Exercise 5.9: Let $A = \mathbb{R}[x^2, x^3]$ be the polynomial ring in x^2 and x^3. Using the fact that every integer $n \geq 2$ is of the form $n = 2\, i + 3\, j$, we obtain that $x^n = (x^2)^i\, (x^3)^j \in A$ for $n > 1$ and $x \notin A$, which proves that:
$$A = \{p = \sum_{i=0}^{n} a_i\, x^i \in \mathbb{R}[x] \mid a_1 = 0\}.$$
In [47], the ring A has been used in order to modelize the set of discrete finite-time delay systems which do not contain the unit time-delay x. For instance, such a system appears in high-speed electronic circuits (see [47] for more details).

1) Let us consider $p = (1 - x^3)/(1 - x^2) \in K = Q(A)$. Using the identity
$$(1 - x^3)\,(1 + x^3) = (1 - x^2)\,(1 + x^2 + x^4),$$
prove that p does not admit a weakly coprime factorization, and thus, does not admit a coprime factorization.
2) Show that $c = (-1 + x^2)/(1 + x^3)$ is a stabilizing controller of p. Conclude that there is no Youla-Kučera parametrization of all stabilizing controllers of p.
3) Compute the parametrization of all stabilizing controllers of p. Prove that this parametrization of all stabilizing controllers of p admits two parameters and there does not exist a parametrization of all stabilizing controllers of p with only one free parameter.

Reconsider the exercise with $p = (1 + i\,\sqrt{5})/2 \in Q(A)$ and $A = \mathbb{Z}[i\,\sqrt{5}]$ [1]. For both of them, see [57] for the results.

Corollary 5.10:
- [59] If $P \in K^{q \times r}$ admits a left-coprime factorization $P = D^{-1}\, N$, then:
$$\Omega = \{L \in A^{r \times q} \mid \quad P\, L \in A^{q \times q}\}\, D.$$
- [59] If $P \in K^{q \times r}$ admits a right-coprime factorization $P = \tilde{N}\, \tilde{D}^{-1}$, then:
$$\Omega = \tilde{D}\, \{L \in A^{r \times q} \mid \quad L\, P \in A^{r \times r}\}.$$

Corollary 5.11: [58], [59] Let $P \in K^{q \times r}$ be a plant which admits a doubly coprime factorization:
$$\begin{cases} P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}, \\ \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I_p. \end{cases}$$
Then, the A-module Ω of free parameters defined by (28) is the free A-module of rank $r \times q$ defined by:
$$\begin{aligned} \Omega\ &= \tilde{D}\, A^{r \times q}\, D \\ &= \{L \in A^{r \times q} \mid L = \tilde{D}\, R\, D,\ \forall\, R \in A^{r \times q}\}. \end{aligned}$$
Therefore, all stabilizing controllers of P have the form
$$C(Q) = (Y + \tilde{D}\, Q)\, (X + \tilde{N}\, Q)^{-1} = (\tilde{X} - Q\, N)^{-1}\, (\tilde{Y} - Q\, D),$$
where $Q \in A^{r \times q}$ is any matrix such that:
$$\det(X + \tilde{N}\, Q) \neq 0, \quad \det(\tilde{X} - Q\, N) \neq 0.$$
We recover the well-known *Youla-Kučera parametrization of all stabilizing controllers of P* [17], [40], [78], [80], [81].

Example 5.5: Let us consider the transfer function
$$p = p_0\, e^{-\tau\, s},$$
where $p_0 \in RH_\infty$ is a proper and stable rational transfer function and $\tau \geq 0$. Hence, we have $p \in A = H_\infty(\mathbb{C}_+)$, and thus, p admits the coprime factorization $p = n/d$ with $n = p_0\, e^{-\tau\, s}$ and $d = 1$. Thus, we have the following Youla-Kučera parametrization of the stabilizing controllers of p
$$c(q) = \frac{q}{1 + q\, p_0\, e^{-\tau\, s}},$$
where $q \in A$ is a free parameter.

Let $c_0 \in RH_\infty$ be a stabilizing controller of $p_0 \in RH_\infty$ achieving some prescribed performances. Then, we have:
$$\tilde{q} \triangleq \frac{c_0}{(1 - p_0\, c_0)} \in RH_\infty \subseteq A.$$
Therefore, we obtain the stabilizing controller of p [50]
$$c(\tilde{q}) = \frac{c_0}{1 + p_0\, c_0\, (e^{-\tau\, s} - 1)} = \frac{c_0}{1 - c_0\, (p_0 - p)}$$
which is called the *Smith predictor* [49], [51]. Let us notice that the complementary sensitivity transfer function has the following form
$$\frac{p\, c(\tilde{q})}{1 - p\, c(\tilde{q})} = \left(\frac{p_0\, c_0}{1 - p_0\, c_0}\right)\, e^{-\tau\, s},$$

showing that the Smith predictor allows us to reject the time-delay $e^{-\tau s}$ outside the closed-loop formed by p_0 and c_0. See [24] for recent results on the Smith predictor.

Exercise 5.10:
- Following Example 5.4, prove that the unstable transfer function $p = e^{-s}/(s-1)$ is internally stabilized by the following controller:

$$c = -\frac{2\,e}{1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)} = -\frac{2\,e\,(s-1)}{s+1-2\,e^{-(s-1)}}.$$

Let us notice that $(1-e^{-(s-1)})/(s-1) \in A = H_\infty(\mathbb{C}_+)$ is called a *distributed delay*. See [8], [49] for more details.
- Compute the Youla-Kučera parametrization of all stabilizing controllers of p.

We refer the reader to [2], [40], [41], [78] for applications of the Youla-Kučera parametrization to synthesis problems.

Corollary 5.12: [59] Let A be a Banach algebra (e.g. $\hat{\mathcal{A}}$, W_+, $H_\infty(\mathbb{C}_+)$), $K = Q(A)$, $P \in K^{q\times r}$ a stabilizable plant and $W_1,\, W_2 \in A^{q\times q}$ two weighted transfer matrices. Let us denote by $\mathrm{Stab}(P)$ the set of all stabilizing controllers of P. Then, we have:

$$\begin{array}{c} \Xi \triangleq \inf_{C\in\mathrm{Stab}(P)} \parallel W_1\,(I_q - P\,C)^{-1}\,W_2 \parallel_A \\ = \\ \inf_{Q\in\Omega} \parallel W_1\,(U + P\,Q)\,W_2 \parallel_A, \end{array} \tag{31}$$

where $(U^T : V^T)^T \in A^{p\times q}$ satisfy

$$\left\{\begin{array}{l} U - P\,V = I_q, \\ \left(\begin{array}{c} U\,P \\ V\,P \end{array}\right) \in A^{p\times r}, \end{array}\right.$$

and $\underline{C} = V\,U^{-1}$ is a particular stabilizing controller of P.

Exercise 5.11:
1) [59] Let $P \in K^{q\times r}$ be a plant which admits the doubly coprime factorization:

$$\left\{\begin{array}{l} P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \\ \left(\begin{array}{cc} D & -N \\ -\tilde{Y} & \tilde{X} \end{array}\right)\left(\begin{array}{cc} X & \tilde{N} \\ Y & \tilde{D} \end{array}\right) = I_p. \end{array}\right.$$

Prove that $U + P\,Q = (X + \tilde{N}\,R)\,D$, and thus:

$$\Xi = \inf_{R\in A^{r\times q}} \parallel W_1\,(X + \tilde{N}\,R)\,D\,W_2 \parallel_A .$$

2) [61] Let $p \in K = Q(A)$ be a stabilizable plant and $w \in A$ a weighted transfer function.
   a) Using Corollary 5.9, prove that we have:

$$\inf_{c\in\mathrm{Stab}(p)} \parallel w/(1-p\,c) \parallel_A \tag{32}$$

$$= \inf_{q_1,\,q_2\in A} \parallel w\,(a + a^2\,p\,q_1 + b^2\,p\,q_2) \parallel_A \tag{33}$$

   where $a,\, b \in A$ satisfy $a - b\,p = 1,\; a\,p \in A$, and $\underline{c} = b/a$ is a stabilizing controller of p. Conclude that we have transformed the non-linear problem (32) into an affine, and thus, convex one (33).
   b) If $p = n/d$ is a coprime factorization of p

$$d\,x - n\,y = 1, \quad x,\, y \in A,$$

   then prove that we have $a = 1/(1-p\,c) = d\,x$ and $b = c/(1-p\,c) = d\,y$. Deduce that we have

$$a + a^2\,p\,q_1 + b^2\,p\,q_2 = d\,(x + q\,n),$$

   where $q = x^2\,q_1 + y^2\,q_2 \in A$.
   c) Using the following identity

$$(d^2\,(1 - 2\,n\,y))\,x^2 + (n^2\,(1 + 2\,d\,x))\,y^2 = 1,$$

   show that, for any $q \in A$,

$$\left\{\begin{array}{l} q_1 = d^2\,(1 - 2\,n\,y)\,q, \\ q_2 = n^2\,(1 + 2\,d\,x)\,q. \end{array}\right.$$

   are such that $q = x^2\,q_1 + y^2\,q_2$.
   d) Finally, deduce that we have:

$$\inf_{c\in\mathrm{Stab}(p)} \parallel w/(1-p\,c) \parallel_A = \inf_{q\in A} \parallel w\,d\,(x + n\,q) \parallel_A .$$

VI. Strong and simultaneous stabilizations

Definition 6.1: We have the following definitions [4], [78]:
- A plant $P \in K^{q\times r}$ is *strongly stabilizable* if there exists a stable stabilizing controller $C \in A^{r\times q}$ of P.
- Two plants $P_1,\, P_2 \in K^{q\times r}$ are *simultaneously stabilizable* if there exists a controller $C \in K^{r\times q}$ which internally stabilizes P_1 and P_2.

The strong and simultaneous stabilization problems have largely been investigated in the literature (see [4], [78] and the references therein). This can be explained by the fact that strongly stabilizable plants have a good ability to track reference inputs [78]. Moreover, in practice, engineers are usually reluctant to use unstable controllers specially when the plant is stable. Finally, simultaneous stabilization plays an important role in the study of *reliable stabilization*, i.e. when we want to design a controller which stabilizes a finite family of plants which describes a given system during normal operating conditions and various failed modes (e.g. loss of sensors or actuators, changes in operating points). We refer the reader to [4], [78] for more details and references.

Let us introduce some definitions [3], [27], [75].

Definition 6.2:
- $a = (a_1 : \ldots : a_n) \in A^n$ is *unimodular* if there exists a vector $b = (b_1 : \ \ldots : \ b_n) \in A^n$ such that $a\,b^T = \sum_{i=1}^n a_i\,b_i = 1$. We denote the set of all the unimodular vectors of A^n by $\mathrm{U}_n(A)$.
- A matrix $R \in A^{q\times p}$ is *unimodular* if there exists a matrix $S \in A^{p\times q}$ such that $R\,S = I_q$.
- A unimodular matrix $R = \mathrm{col}(R_1, \ldots, R_p) \in A^{q\times p}$ is called k-*stable* ($1 \le k \le r = p - q$) if there exists a $(p-k)$-tuple $(c_i)_{1\le i\le p-k}$ belonging to the A-module

$$R_{p-k+1}\,A + \ldots + R_p\,A \triangleq \left\{\sum_{i=1}^{k} R_{p-k+i}\,b_i \mid b_i \in A\right\}$$

  such that the matrix

$$\mathrm{col}(R_1 + c_1 : R_2 + c_2 : \ldots : R_{p-k} + c_{p-k}) \in A^{q\times(p-k)}$$

  is a unimodular matrix, where

$$\mathrm{col}(R_1 : \ldots : R_{p-k})$$

  denotes the matrix formed by the $(p-k)$ first columns of R.

Remark 6.1: A unimodular matrix $R \in A^{q\times p}$ is k-stable iff there exists a matrix $T_k \in A^{k\times(p-k)}$ such that

$$R_k = \mathrm{col}(R_1 : \ldots : R_{p-k}) + \mathrm{col}(R_{p-k+1} : \ldots : R_p)\, T_k$$

is a unimodular $q \times (p-k)$-matrix.

Definition 6.3: [3], [27], [75] $a = (a_1 : \ldots : a_n) \in \mathrm{U}_n(A)$ *is called* stable *(or* reductible*) if there exists a* $(n-1)$*-tuple* $b = (b_1 : \ \ldots : b_{n-1}) \in A^{n-1}$ *such that*

$$(a_1 + a_n\, b_1 : \ \ldots : a_{n-1} + a_n\, b_{n-1}) \in \mathrm{U}_{n-1}(A),$$

i.e. there exists $(c_1 : \ldots : c_{n-1}) \in A^{n-1}$ *such that we have:*

$$\sum_{i=1}^{n-1} (a_i + a_n\, b_i)\, c_i = 1.$$

Definition 6.4: [64], [74], [75] The stable range $\mathrm{sr}(A)$ *of* A *is the smallest* $n \in \mathbb{N} \cup \{+\infty\}$ *such that every vector of* $\mathrm{U}_{n+1}(A)$ *is stable.*

Remark 6.2: Let us notice that the stable range $\mathrm{sr}(A)$ is also called the *stable rank* of A in the literature of algebra.

Theorem 6.1:
- [74] $\mathrm{sr}(H_\infty(\mathbb{C}_+)) = 1$.
- [60], [78] $\mathrm{sr}(RH_\infty) = 2$.
- [36] $\mathrm{sr}(A(\mathbb{D})) = 1$.
- [67] $\mathrm{sr}(W_+) = 1$.
- [35] $\mathrm{sr}(E(k)) = 1$ if $k = \mathbb{C}$, and 2 if $k = \mathbb{R}$.
- [32] $\mathrm{sr}(L_\infty(i\,\mathbb{R})) = 1$.
- [75] $\mathrm{sr}(\mathbb{R}[x_1, \ldots, x_n]) = n + 1$.

Remark 6.3: Let us notice that $\mathrm{sr}(H_\infty(\mathbb{C}_+)) = 1$ does not contradict the fact that $\mathrm{sr}(RH_\infty) = 2$. Indeed, the functions of $H_\infty(\mathbb{C}_+)$ can have some complex coefficients whereas a function of RH_∞ can only have real coefficients. It seems that the ring $\{f \in H_\infty(\mathbb{C}_+) \,|\, \overline{f(\overline{s})} = f(s)\}$ has stable range 2 but, up to now, there is no proof of it.

The following proposition explains the link between strong stabilizability and k-stability.

Proposition 6.1: [60] The transfer matrix $P \in K^{q\times r}$ is strongly stabilizable iff P admits a doubly coprime factorization $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}$ such that $R = (D : \ -N) \in A^{q\times p}$ and $(\tilde{D}^T : \ \tilde{N}^T) \in A^{r\times p}$ are respectively r and q-stable.

Remark 6.4: Let us notice that if $P = D_1^{-1}\, N_1 = D_2^{-1}\, N_2$ are two left-coprime factorizations of P, then, we can prove that there exists a matrix $U \in \mathrm{GL}_q(A)$ such that:

$$(D_2 : \ -N_2) = U\,(D_1 : \ -N_1).$$

Hence, we can easily show that R_1 is k-stable iff R_2 is also k-stable. Similar results also hold for right-coprime factorizations. Therefore, Proposition 6.1 does not depend on a particular choice of a doubly coprime factorization of P.

Secondly, let us notice that strong stabilizability implies the existence of a doubly coprime factorization for the plant.

Theorem 6.2: [60] Let $P = D^{-1}\, N$ be a left-coprime factorization of P with $R = (D : \ -N) \in A^{q\times p}$. If R is k-stable and $s \triangleq r - k \geq 0$, then there exist two stable matrices $T_1 \in A^{k\times q}$ and $T_2 \in A^{k\times s}$ such that the matrix

$$R_k = (D - \Lambda\, T_1 : \ -(N_s + \Lambda\, T_2)) \in A^{q\times(p-k)}$$

admits a right-inverse with entries in A, with the notations:

$$R = (D : \ -N) = (\underset{\underset{q}{\leftrightarrow}}{D} \quad : \underset{\underset{r}{\leftrightarrow}}{-N_s} \quad : \underset{\underset{k}{\leftrightarrow}}{-\Lambda}) \quad \in A^{q\times p}.$$

Let us define by $S_k = (U^T : \ V^T)^T \in A^{(p-k)\times q}$, $U \in A^{q\times q}$, $V \in A^{s\times q}$, any right-inverse of R_k such that $\det U \neq 0$. Then, the controller $C \in K^{r\times q}$, defined by

$$C = \begin{pmatrix} V\,U^{-1} \\ T_1 + T_2\,(V\,U^{-1}) \end{pmatrix}, \quad \begin{array}{l} \updownarrow s = r - k \\ \updownarrow k \end{array}$$

internally stabilizes P. Moreover, if $\det(D - \Lambda\, T_1) \neq 0$, then the controller $C_s = V\,U^{-1} \in K^{s\times q}$ internally stabilizes

$$P_s = (D - \Lambda\, T_1)^{-1}\,(N_r + \Lambda\, T_2) \in K^{q\times s}.$$

The unstable part of C is only contained in the transfer matrix $C_s = V\,U^{-1}$ and its dimension is less or equal to $s \times q$.

Similar results also hold for a transfer matrix P admitting a right-coprime factorization.

Up to our knowledge, there is no general algorithm checking whether or not a matrix R is k-stable. However, we can prove that any matrix $R \in A^{q\times p}$ such that $r \geq \mathrm{sr}(A)$ is $r - \mathrm{sr}(A) + 1$-stable [60]. Therefore, we obtain the following corollary which only depends on $\mathrm{sr}(A)$, i.e. on the integral domain A.

Corollary 6.1: [60] Let $P = D^{-1}\, N$ be a left-coprime factorization $P \in K^{q\times r}$ such that $r \geq \mathrm{sr}(A)$. Then, there exist two stable matrices

$$\begin{cases} T_1 \in A^{(r-\mathrm{sr}(A)+1)\times q}, \\ T_2 \in A^{(r-\mathrm{sr}(A)+1)\times(\mathrm{sr}(A)-1)}, \end{cases}$$

such that the following $q \times (q + \mathrm{sr}(A) - 1)$-matrix

$$R_{r-\mathrm{sr}(A)+1} \triangleq (D - \Lambda\, T_1 : \ -(N_{\mathrm{sr}(A)-1} + \Lambda\, T_2))$$

admits a right-inverse, with the notations:

$$R = (D : \ -N) = (\underset{\underset{q}{\longleftrightarrow}}{D} \quad : \underset{\underset{\mathrm{sr}(A)-1}{\longleftrightarrow}}{-N_{\mathrm{sr}(A)-1}} \quad : \underset{\underset{r-\mathrm{sr}(A)+1}{\longleftrightarrow}}{-\Lambda}).$$

If $S_{r-\mathrm{sr}(A)+1} = (U^T : \ V^T)^T \in A^{(q+\mathrm{sr}(A)-1)\times q}$ is any right-inverse of $R_{r-\mathrm{sr}(A)+1}$ such that $\det U \neq 0$, then the controller C defined by

$$C = \begin{pmatrix} V\,U^{-1} \\ T_1 + T_2\,(V\,U^{-1}) \end{pmatrix} \quad \begin{array}{l} \updownarrow \mathrm{sr}(A) - 1 \\ \updownarrow r - \mathrm{sr}(A) + 1 \end{array}$$

internally stabilizes the plant $P = D^{-1}\, N$. Moreover, if $\det(D - \Lambda\, T_1) \neq 0$, then the controller $C_{\mathrm{sr}(A)-1} = V\,U^{-1}$ internally stabilizes the plant

$$P_{\mathrm{sr}(A)-1} = (D - \Lambda\, T_1)^{-1}\,(N_{\mathrm{sr}(A)-1} + \Lambda\, T_2).$$

Finally, the unstable part of the controller C is only contained in $C_{\mathrm{sr}(A)-1} = V\,U^{-1}$ and its dimension is less or equal to $(\mathrm{sr}(A) - 1) \times q$.

Corollary 6.2: [60] If $\mathrm{sr}(A) = 1$, then every transfer matrix which admits a left or a right-coprime factorization is strongly stabilizable (i.e. is internally stabilized by a stable controller). In particular, this result holds for $A = W_+$ or $A(\mathbb{D})$.

Moreover, every internally stabilizable plant, defined by a transfer matrix P with entries in the quotient field of $H_\infty(\mathbb{C}_+)$ is strongly stabilizable.

Let us notice that Corollary 6.4 solves a question asked by A. Feintuch in [21] on the generalization of S. Treil's result [74] for MIMO systems defined over $H_\infty(\mathbb{C}_+)$.

Corollary 6.3: [62] If $\mathrm{sr}(A) = 1$, then A is a Hermite ring. In particular, this is the case for the rings $H_\infty(\mathbb{C}_+)$, $A(\mathbb{D})$, W_+, $E(\mathbb{C})$ and $L_\infty(i\,\mathbb{R})$. Moreover, if $K = Q(A)$ and the transfer matrix $P \in K^{q\times r}$ admits a left or a right-coprime factorization, then P admits a doubly coprime factorization.

Let us state the link between strong and simultaneous stabilizabilities.

Proposition 6.2: [78] Let $P_1,\, P_2 \in K^{q\times r}$ be two transfer matrices which admit the following doubly coprime factorizations $P_i = D_i^{-1}\, N_i = \tilde{N}_i\, \tilde{D}_i^{-1}$ and:

$$\left(\begin{array}{cc} D_i & -N_i \\ -\tilde{Y}_i & \tilde{X}_i \end{array}\right) \left(\begin{array}{cc} X_i & \tilde{N}_i \\ Y_i & \tilde{D}_i \end{array}\right) = I_p, \quad i = 1,\, 2.$$

Then, P_1 and P_2 are simultaneously stabilized by a controller C iff there exists $T \in A$ such that $U + V\, T \in \mathrm{GL}_q(A)$, where:

$$\left\{\begin{array}{l} U = D_1\, X_0 - N_1\, Y_0, \\ V = -D_1\, \tilde{N}_0 + N_1\, \tilde{D}_0. \end{array}\right.$$

Remark 6.5: Let us notice that if P_1 and P_2 are two stabilizable plants which do not admit doubly coprime factorizations, then the simultaneous stabilization problem for two plants is no more equivalent to a strong stabilization problem. The relationships between these two problems seem to be highly open for stabilizable plants which do not admit doubly coprime factorizations.

Corollary 6.4: [60] If $\mathrm{sr}(A) = 1$, then every couple of plants, defined by two transfer matrices P_0 and P_1 with entries in $K = Q(A)$, having the same dimensions, and admitting doubly coprime factorizations, is simultaneously stabilized by a controller (simultaneous stabilization). In particular, this result holds for $A = W_+$ or $A(\mathbb{D})$.

Moreover, if $A = H_\infty(\mathbb{C}_+)$ and P_0, P_1 are two internally stabilizable plants with entries in $K = Q(A)$, then P_0 and P_1 are simultaneously stabilized by a controller C.

We refer to [70] for a promising work on the simultaneous stabilization problem for multidimensional systems, i.e. for the ring $M_{\mathbb{D}^n}$ defined in Example 2.1.

Exercise 6.1: [62] Using Exercise 5.8, prove the results:

1) Prove that $p \in K = Q(A)$ is strongly (resp. bistably) stabilizable iff there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that $J = (1 - p\, c)$. Deduce that p is strongly stabilizable iff there exists $c \in A$ such that $p/(1 - p\, c) \in A$.
2) Using (26), prove that $c \in K$ internally stabilizes 0 iff $c \in A$.
3) Let $p_1 = n_1/d_1$, $p_2 = n_2/d_2 \in K$ be two coprime factorizations with $d_1\, x_1 - n_1\, y_1 = 1$. Prove that p_1 and p_2 are simultaneously stabilizable iff
$$p_3 \triangleq \frac{(d_1\, n_2 - n_1\, d_2)}{(d_2\, x_1 - n_2\, y_1)}$$
is strongly stabilizable [4], [78].
4) Let $p_1 = n_1/d_1, \ldots, p_k = n_k/d_k \in K$ be k coprime factorizations with $d_1\, x_1 - n_1\, y_1 = 1$. Prove that $p_1, \ldots, p_k$ are simultaneously stabilizable iff the plants $p_{k+1}, \ldots, p_{2\,k-1}$, defined by
$$p_{k+i-1} \triangleq \frac{d_i\, n_1 - n_i\, d_1}{d_i\, x_1 - n_i\, y_1}, \quad i = 2, \ldots, k,$$
are simultaneously stabilized by a stable controller [4].
5) Let $p_1 \in A$ and $p_2 \in K$. Using (26), prove that c simultaneously stabilizes p_1 and p_2 iff $c/(1 - p_1\, c)$ strongly stabilizes $p_2 - p_1$.
6) Let $p_1 \in A$ and $p_2, \ldots, p_k \in K$. Prove that c simultaneously stabilizes $p_1, \ldots, p_k$ iff $c/(1 - p_1\, c) \in A$ simultaneously stabilizes the plants $p_2 - p_1, \ldots, p_k - p_1$.
7) Let $p,\, c \in A$. Using (26), prove that c internally stabilizes p iff $1/(1 - p\, c) \in A$. Hence, deduce that c internally stabilizes p iff c externally stabilizes p.
Let us recall that if A is a Banach algebra, then:
$$\parallel 1 - a \parallel_A < 1 \Rightarrow a \in \mathrm{U}(A). \tag{34}$$
Let A be a Banach algebra and:
$$\parallel c \parallel_A < 1/ \parallel p \parallel_A .$$
Prove that $c \in A$ internally stabilizes p. This result is generally called the *small gain theorem* [14], [84].
8) Using (26), prove that $0 \neq c \in K$ internally stabilizes $0 \neq p \in K$ iff $1/c$ internally stabilizes $1/p$.
9) Let $\delta \in A$. Using (26), prove that c internally stabilizes $p \in K$ iff $c/(1 + \delta\, c)$ internally stabilizes $p + \delta$. Similarly, prove that c internally stabilizes $p \in K$ iff $c + \delta$ internally stabilizes $p/(1 + \delta\, p)$.
10) Let $\delta \in A$ and c be a stabilizing controller of $p \in K$. Using (26), prove that $p + \delta$ (resp. $p/(1 + \delta\, p)$) is internally stabilized by c iff:
$$\begin{array}{c} 1 - (\delta\, c/(1 - p\, c)) \in \mathrm{U}(A) \\ (\text{resp. } 1 + (\delta\, p/(1 - p\, c)) \in \mathrm{U}(A)). \end{array}$$
If A is a Banach algebra, using (34), deduce that
$$\begin{array}{c} \forall\, \delta \in A : \parallel \delta \parallel_A < \frac{1}{\parallel c/(1 - p\, c) \parallel_A} \\ (\text{resp. } \forall\ \delta \in A : \parallel \delta \parallel_A < \ 1/(\parallel p/(1 - p\, c) \parallel_A)), \end{array}$$
c internally stabilizes $p + \delta$ (resp. $p/(1 + \delta\, p)$). Let us notice that $p + \delta$ is generally called an *additive pertubation* of p whereas $p/(1 + \delta\, p)$ is called a *multiplicative perturbation* of p [20].

To finish, let us introduce the concept of *topological stable range* of a Banach algebra.

Definition 6.5: [64] If A is a Banach algebra, then the *topological stable range* $\mathrm{tsr}(A)$ of A is the smallest $n \in \mathbb{N} \cup \{+\infty\}$ such that $\mathrm{U}_n(A)$ is dense in A^n for the product topology.

Remark 6.6: As for the stable range, the topological stable range $\mathrm{tsr}(A)$ is also called the *topological stable rank* of A.

Theorem 6.3: We have the following results:

- [72] $\mathrm{tsr}(H_\infty(\mathbb{D})) = 2$,
- [64] $\mathrm{tsr}(A(\mathbb{D})) = 2$.

Proposition 6.3: [64] If A is a Banach algebra, then we have $\mathrm{sr}(A) \leq \mathrm{tsr}(A)$.

Let us notice that we can have $\mathrm{sr}(A) < \mathrm{tsr}(A)$ as we can easily see it in Theorems 6.1 and 6.3.

Proposition 6.4: [60] If A is a Banach algebra such that $\mathrm{tsr}(A) = 2$, then every SISO plant, defined by the transfer function $p = n/d$ ($0 \neq d, n \in A$), satisfies:

$$\forall\, \epsilon > 0,\ \exists\, (d_\epsilon\ :\ n_\epsilon) \in \mathrm{U}_2(A) : \quad \left\{ \begin{array}{l} \| \, n - n_\epsilon \, \|_A \leq \epsilon, \\ \| \, d - d_\epsilon \, \|_A \leq \epsilon. \end{array} \right.$$

If $d_\epsilon \neq 0$, then, in the product topology, p is as close as we want to a transfer function $p_\epsilon = n_\epsilon / d_\epsilon$ which admits a coprime factorization. In particular, this result holds for $A = H_\infty(\mathbb{D})$ or $A(\mathbb{D})$.

Remark 6.7: From Proposition 6.4, we obtain that if p is not internally stabilizable, then there exists a stabilizable plant p_ϵ as close as we want to p in the product topology.

VII. Classification of the rings of SISO stable plants

> " ...The foregoing results about rational functions are so elegant that one can hardly resist the temptation to try to generalize them to non-rational functions. But to what class of functions? Much attention has been devoted in the engineering literature to the identification of a class that is wide enough to encompass all the functions of physical interest and yet enjoys the structural properties that allow analysis of the robust stabilisation problem", N. Young [83].

To finish these notes, we shall give few results of commutative algebra and homological algebra which allow to start a classification of rings of SISO stable plants by respect to certain system properties (e.g. existence of (weakly) doubly coprime factorizations, internal stabilization).

Definition 7.1: [6], [26], [66] A *Prüfer domain* A is an integral domain which satisfies one of the following equivalent assertions:

- Every finitely generated torsion-free A-module is projective.
- Every ideal of the form $I = (d,\, n)$, $0 \neq d,\, n \in A$, is a projective A-module, i.e. there exist $x,\, y \in K$ such that:

$$\left\{ \begin{array}{l} d\, x - n\, y = 1, \\ d\, x,\ d\, y,\ n\, x \in A. \end{array} \right.$$

- For every $p \in K = Q(A)$, the fractional ideal $J = (1,\, p)$ of A is invertible (see Exercise 5.8).

Prüfer domains were named after H. Prüfer who initiated their study in 1923.

Example 7.1: We have the following examples:

- Every integral closure of $\mathbb{Z}$ into a finite extension of $\mathbb{Q}$ is a *Dedekind domain*, namely a noetherian Prüfer domain. For example, the integral closure of $\mathbb{Z}$ into $\mathbb{Q}(i\,\sqrt{5})$ is the Dedekind domain $\mathbb{Z}[i\,\sqrt{5}]$, and thus, a Prüfer domain [26], [66]. This fact allowed us in [56], [57] to explain the counter-example given in [1]
- Every non-singular algebraic surface defines a Dedekind affine domain. For instance, the ring $\mathbb{R}[t_1, t_2]/(t_1^2 + t_2^2 - 1)$ is a Dedekind domain, and thus, a Prüfer domain [66].
- Every Bézout domain is a Prüfer domain. Thus, the ring of entire functions $E(k)$, with $k = \mathbb{R}, \mathbb{C}$, and $\mathcal{E} = E(\mathbb{R}) \cap \mathbb{R}(s)[e^{-s}]$ are Prüfer domains [26], [66].
- The ring of $\mathbb{Z}$-valued polynomials in $\mathbb{Q}[x]$, namely
$$A = \{p \in \mathbb{Q}[x] \mid p(\mathbb{Z}) \subset \mathbb{Z}\},$$
is a Prüfer domain [26].

The next theorem gives a complete characterization of the rings A of SISO stable plants over which every plant is internally stabilizable.

Theorem 7.1: [56] We have the following equivalences:

1) *Every SISO plant, defined by a transfer function with entries in $K = Q(A)$, is internally stabilizable.*
2) *Every MIMO plant, defined by a transfer matrix with entries in $K = Q(A)$, is internally stabilizable.*
3) *A is a Prüfer domain.*

Let us notice that Theorem 7.1 has a similar form as Theorem 4.3.

Exercise 7.1: Using Definition 7.1, Theorem 5.1, Lemma 3.1 and Exercises 5.5 and 5.8, prove Theorem 7.1.

Remark 7.1: Let us notice the fact that the integral domains over which

- every transfer matrix admits a weakly doubly coprime factorization, i.e. coherent Sylvester domains (see Theorem 3.5),
- every plant, defined by a transfer matrix, is internally stabilizable, i.e. Prüfer domains (see Theorem 7.1),
- every transfer matrix admits a doubly coprime factorization, i.e. Bézout domains (see Theorem 4.3),

are all coherent rings (see Definition 3.7) and *integrally closed* [26] (namely, every element k of $K = Q(A)$ satisfying a monic polynomial, i.e. $\sum_{i=0}^{n} a_i\, k^i = 0$, with $a_n = 1$ and $a_i \in A$, belongs to A). In terms of homological algebra, a coherent Sylvester domain A is a projective-free coherent integral domain (see Definition 4.3) of weak global dimension $\mathrm{w.gl.dim}(A) \leq 2$, a Prüfer domain is an integral domain of weak global dimension $\mathrm{w.gl.dim}(A) \leq 1$ and a Bézout domain is a projective-free domain of weak global dimension $\mathrm{w.gl.dim}(A) \leq 1$ (see [54], [56], [66] for more details). Roughly speaking, the concept of *weak global dimension* [7], [66] measures the number of different concepts of primeness: a ring A with $\mathrm{w.gl.dim}(A) \leq 1$ has only one concept of primeness (the standard one) whereas a ring A with $\mathrm{w.gl.dim}(A) \leq 2$ has two concepts of primeness (the same standard one as well as the concept of weak primeness). Over a ring A with $\mathrm{w.gl.dim}(A) \geq 3$ (see e.g. Exercise 3.7), not every transfer matrix with entries in the quotient field $K = Q(A)$ admits a weakly doubly coprime factorization, and thus, the fractional representation approach seems to fail to be interesting. Finally, let us notice that the problem to recognize whether or not a finitely generated projective/stably free A-module is free (i.e. whether or not a stabilizing plant admits coprime factorizations) is an important issue in algebra and a theory, so called *algebraic K-theory*, was developed in the seventies in order to study these problems (as well as others). We refer the interested reader to [60], [57], [58] for an introduction to basic concepts of K-theory as well as their applications to synthesis problems.

For lack of space, in these notes, we were not able to show how to use the algebraic analysis approach developed in this paper in order to recover the operator-theoretic approach developed in [28] (see [83] for a nice introduction to this approach). Indeed, a nearly complete characterization of the functional spaces (e.g. H_2, $L_p(\mathbb{R}_+)$) so that internal stabilization is equivalent to the existence of the bounded inverse of the linear operator from e to u (see Proposition 5.1) is obtained in [61]. This result can also be used in order to model rings of SISO stable plants with prescribed stabilization properties (for instance, find a ring of SISO stable plants over which internal stabilization is equivalent to the existence of a bounded inverse of the linear operator from e to u, where e and u belong to a certain functional space [61]).

VIII. Conclusion

We hope to have convinced the reader that the algebraic analysis (commutative algebra, module theory, homological algebra, Banach algebras) develops powerful concepts and tools which allow, on the one hand, to recover different results of the classical literature on the fractional representation approach to analysis and synthesis problems and, on the other hand, to develop new ones. For lack of space, we were not able to treat in these notes certain other results that can also be obtained using this mathematical framework. We refer to [54], [55], [56], [57], [58], [59], [60], [61], [62] for more details.

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