An introduction to internal stabilization of linear infinite-dimensional systems

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Introduction

Unstable plants

- **Finite-dimensional system:**

$$\dot{x}(t) = x(t) + u(t),\ x(0) = 0 \Rightarrow \widehat{x}(s) = \frac{1}{s-1}\,\widehat{u}(s).$$

- **Delay system:**

$$\begin{cases} \dot{x}(t) = x(t) + u(t),\ x(0) = 0, \\ y(t) = \begin{cases} 0, & 0 \le t \le 1, \\ x(t-1), & t \ge 1, \end{cases} \end{cases}$$

$$\Rightarrow \widehat{y}(s) = \frac{e^{-s}}{s-1}\,\widehat{u}(s).$$

- **System of partial differential equations:**

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ \frac{\partial z}{\partial x}(0,t) = 0,\ \frac{\partial z}{\partial x}(1,t) = u(t), \\ y(t) = \frac{\partial z}{\partial t}(1,t), \end{cases}$$

$$\Rightarrow \widehat{y}(s) = \frac{1+e^{-2\,s}}{1-e^{-2\,s}}\,\widehat{u}(s).$$

- The **poles of the transfer functions** ($1, 1, k\,\pi\,i, k \in \mathbb{Z}$)

$$h_1(s) = \tfrac{1}{s-1},\ h_2(s) = \tfrac{e^{-s}}{s-1},\ h_3(s) = \tfrac{1+e^{-2\,s}}{1-e^{-2\,s}}$$

belong to $\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \ge 0\} \Rightarrow$ **unstability.**

Stabilization by feedback

- $\mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 0\}$,

$H_\infty(\mathbb{C}_+) = \{$holomorphic functions f in $\mathbb{C}_+ \,|$
$\qquad \| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}$,

$H_2 \; = \{$holomorphic functions f in $\mathbb{C}_+ \,|$
$\| f \|_2 = \sup_{x \in \mathbb{R}_+} (\int_{-\infty}^{+\infty} |f(x+iy)|^2 dy)^{1/2} < +\infty\}$.

$\quad = \mathcal{L}(L_2(\mathbb{R}_+)),\; \mathcal{L}(\cdot)$ Laplace transform.

- The transfer functions h_i **do not belong to** $H_\infty(\mathbb{C}_+)$:

$$h_1(s) = \frac{1}{s-1},\; h_2(s) = \frac{e^{-s}}{s-1},\; h_3(s) = \frac{1+e^{-2\,s}}{1-e^{-2\,s}}$$

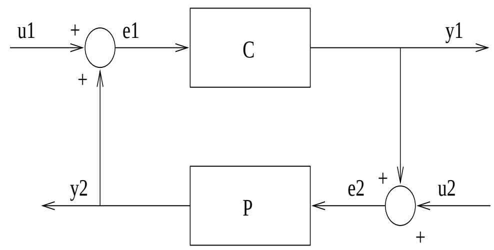
$\Rightarrow$ we have the **linear unbounded operator**

$$\begin{array}{rcl} T_{h_i} : H_2 & \longrightarrow & H_2, \\ \widehat{u} & \longrightarrow & \widehat{y} = h_i\,\widehat{u}, \end{array}$$

$\Rightarrow \mathrm{dom}(T_{h_i}) = \{\widehat{u} \in H_2 \,|\, \widehat{y} = h_i\,\widehat{u} \in H_2\} \subsetneq H_2$

$\Rightarrow \exists\, \widehat{u} \in H_2 :\; \widehat{y} = h_i\,\widehat{u} \notin H_2$.

- Is it possible to find a (robust/optimal) controller C such that the closed-loop is stable $\forall\, \widehat{u}_i \in H_2^{n_i}$?



Importance of coprime factorizations

• Let P the transfer matrix of an **unstable plant**.

• **The problem of finding all the internal stabilizing controllers C of P,** i.e. all the transfer matrices C such that

$$\left\{\begin{array}{l} (I-P\,C)^{-1}, \\ (I-P\,C)^{-1}\,P, \\ C\,(I-P\,C)^{-1}, \\ I+C\,(I-P\,C)^{-1}\,P, \end{array}\right. \quad \text{are stable,}$$

is **a non-linear problem**.

• If P **admits a doubly coprime factorization**

$$P=D^{-1}\,N=\tilde{N}\,\tilde{D}^{-1}, \left(\begin{array}{cc} D & -N \\ -\tilde{Y} & \tilde{X} \end{array}\right)\left(\begin{array}{cc} X & \tilde{N} \\ Y & \tilde{D} \end{array}\right)=I,$$

then, all the stabilizing controllers of P are parametrized by the **affine Youla-Kučera parametrization**:

$$\boxed{C(Q)=(\tilde{Y}-Q\,N)^{-1}\,(\tilde{Y}-Q\,D)=(Y-\tilde{D}\,Q)\,(X-\tilde{N}\,Q)^{-1}.}$$

• **The problems of finding the stabilizing optimal controllers** (e.g. $\inf_C \parallel W_1\,(I-P\,C)^{-1}\,W_2 \parallel_\infty$) **are no more a non-linear problem but an affine one** (e.g. $\inf_Q \parallel W_1\,(X-\tilde{N}\,Q)\,D\,W_2 \parallel_\infty$).

$\boxed{\text{Open questions}}$

• **For finite-dimensional systems** (ODE), **robust control solved completely the following problems**:

1. Internal/simultaneous/strong/robust stabilization,

2. Parametrization of all the stabilizing controllers,

3. Computations of the stabilizing H_2/H_∞-controllers …

• **For infinite-dimensional systems (PDE, delay systems), the same problems are generally open.**

• We are interesting here in the following questions:

1. Does it exist necessary and sufficient conditions to internal stabilization?

2. What are the links between internal stabilization and the existence of coprime factorizations?

3. Does it exist necessary and sufficient conditions to the existence of the Youla-Kučera parametrization of the stabilizing controllers?

The fractional representation approach to synthesis problems

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The Fractional Representation of Plants

• (Zames) **The set of transfer functions of SISO systems has the structure of an algebra** (parallel +, serie $\circ$, proportional feedback . by scalar in $\mathbb{R}$).

• (Vidyasagar) **Let A be an algebra of transfer functions of SISO stable systems** with a structure of an intregral domain ($a\,b = 0, a \neq 0 \Rightarrow b = 0$) and its **the field of fractions**:

$$K = Q(A) = \left\{ p = \frac{n}{d} \mid 0 \neq d, n \in A \right\}.$$

K represents the class of (unstable) systems

$\Rightarrow$ Any unstable plant is defined by a transfer matrix $T \in K^{p\times q}$ with entries in $K = Q(A)$.

• (Zames) **The algebra A of SISO stable systems has to be a normed algebra** in order to take into account the errors in the modelization & approximation of the real plant by a mathematical model.

• We usually ask **the normed algebra A of SISO stable systems to be complete** (robustness problems), i.e. A is a **Banach algebra**.

Examples of stable algebras A of SISO systems

1. $RH_\infty = \left\{ \frac{n(s)}{d(s)} \in \mathbb{R}(s) \mid \deg n(s) \leq \deg d(s), \; d(s) = 0 \Rightarrow \mathrm{Re}(s) < 0 \right\}$

- $h_1(s) = \frac{1}{s-1} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{1}{s+1}, \; \frac{s-1}{s+1} \in RH_\infty$

$$\Rightarrow h_1 \in Q(RH_\infty) = \mathbb{R}(s).$$

2. $\mathcal{A} = \{ f(t) + \sum_{i=0}^{+\infty} a_i \, \delta_{t-t_i} \mid f \in L_1(\mathbb{R}_+), \; (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+), \; 0 = t_0 \leq t_1 \leq t_2 ... \}$,

and $\widehat{\mathcal{A}} = \{ \widehat{g} \mid g \in \mathcal{A} \}$, the **Wiener algebras**.

- $h_2(s) = \frac{e^{-s}}{s-1} = \frac{\left(\frac{e^{-s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{e^{-s}}{s+1}, \; \frac{s-1}{s+1} \in \widehat{\mathcal{A}}.$

$$\Rightarrow h_2 \in Q(\widehat{\mathcal{A}}).$$

3. $\mathbb{C}_+ = \{ s \in \mathbb{C} \mid \mathrm{Re}\, s > 0 \}$. The **Hardy algebra**

$H_\infty(\mathbb{C}_+) = \{$ holomorphic functions f in $\mathbb{C}_+ \mid \| f \|_\infty = \sup_{s \in \mathbb{C}_+} | f(s) | < +\infty \}$.

- $h_3(s) = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})}, 1+e^{-2\,s}, 1-e^{-2\,s} \in H_\infty(\mathbb{C}_+)$

$$\Rightarrow h_3 \in Q(H_\infty(\mathbb{C}_+)).$$

A module approach of synthesis problems

• **Methodology:**

1. An integral domain A of **SISO stable systems** is chosen (e.g. $A = RH_\infty,\ H_\infty(\mathbb{C}_+)\ldots$).

2. **The plant is defined by a transfer matrix**:

$$P \in K^{q\times(p-q)},\ K = Q(A) = \left\{\frac{n}{d} \mid 0 \neq d,\, n \in A\right\}.$$

3. **We write** P as:

$$P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \quad \begin{cases} (D\,:\,-N) \in A^{q\times p}, \\ (\tilde{N}^T\,:\,\tilde{D}^T)^T \in A^{p\times(p-q)}. \end{cases}$$

(e.g. $D = d\,I_q,\ \ N = d\,P,\ \ \tilde{D} = d\,I_{p-q},\ \ \tilde{N} = \ d\,P$).

3. **We have** $y = P\,u \Leftrightarrow \begin{cases} (D\,:\,-N)\begin{pmatrix} y \\ u \end{pmatrix} = 0, \\ \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} z. \end{cases} \quad (\star)$

4. **Analysis & synthesis problems are reformulated in terms of the properties of** $(\star)$.

• **Linear algebra over rings is the module theory**

$\Rightarrow$ a module approach to analysis & synthesis problems of linear infinite dimensional systems.

Example

- Let us consider the following transfer matrix:

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix}.$$

- Let us consider $A = H_\infty(\mathbb{C}_+)$ and $K = Q(A)$.

- We have:

$$\begin{cases} y_1 = \frac{e^{-s}}{(s-1)}\, u, \\ y_2 = \frac{e^{-s}}{(s-1)^2}\, u \end{cases} \Rightarrow \begin{cases} \frac{(s-1)}{(s+1)}\, y_1 - \frac{e^{-s}}{(s+1)}\, u = 0, \\ \left(\frac{s-1}{s+1}\right)^2 \, y_2 - \frac{e^{-s}}{(s+1)^2}\, u = 0, \end{cases}$$

$$\Rightarrow R \begin{pmatrix} y \\ u \end{pmatrix} = 0,$$

with $$R = \begin{pmatrix} \underbrace{\begin{matrix} \frac{s-1}{s+1} & 0 \\ 0 & \left(\frac{s-1}{s+1}\right)^2 \end{matrix}}_{D} & \underbrace{\begin{matrix} -\frac{e^{-s}}{s+1} \\ -\frac{e^{-s}}{(s+1)^2} \end{matrix}}_{-N} \end{pmatrix} \in A^{2\times 3}.$$

- We have:

$$P = D^{-1}\, N \in K^2.$$

- **Properties of P can be studied by means of the matrix R with entries in the Banach algebra A**

$\Rightarrow$ by means of **an A-module associated with R.**

Module theory

Finitely presented modules

• Let A be a commutative integral domain, $R \in A^{q \times p}$.

The vectors of A^p and A^q are row vectors.

Let $.R$ be the A-morphism defined by:

$$\begin{array}{ccl} A^q & \xrightarrow{.R} & A^p \\ \mu & \longmapsto & \mu\, R = (\mu_1 \; \ldots \; \mu_q) \left(\begin{array}{ccc} R_{11} & \ldots & R_{1p} \\ \ldots & \ldots & \ldots \\ R_{q1} & \ldots & R_{qp} \end{array} \right) \end{array}$$

• $\operatorname{im} .R = A^q\, R$ is **the module of the A-linear combinations of the rows of R**:

$$\forall\, \lambda \in A^q\, R, \; \exists\, \mu \in A^q : \; \lambda = \mu\, R.$$

• **In algebraic analysis**, we use the A-module:

$$M = \operatorname{coker} .R = A^p / \operatorname{im} .R = A^p / A^q\, R.$$

We can prove that:

M is defined by the A-linear combinations of the equations $R\, z = 0$,

where z_i **corresponds to the class** in $M = A^p / A^q\, R$ of $e_i = (0 \ldots 1 \ldots 0) \in A^p$ (canonical basis of A^p).

Example

- Let $A = H_\infty(\mathbb{C}_+)$ and R be the following matrix:

$$R = \left(\begin{array}{ccc} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{array} \right) \in A^{2\times 3}.$$

- Let us consider the A-morphism $.R$:

$$\begin{array}{ccl} A^2 & \xrightarrow{.R} & A^3 \\ (a_1 \, : \, a_2) & \longrightarrow & \left(a_1 \frac{(s-1)}{(s+1)} \, : \, a_2 \frac{(s-1)^2}{(s+1)^2} \right. \\ & & \qquad \left. -a_1 \frac{e^{-s}}{(s+1)} - a_2 \frac{e^{-s}}{(s+1)^2}\right). \end{array}$$

- $\left\{ \begin{array}{l} y_1 = \pi(e_1), \quad y_2 = \pi(e_2), \\ \\ u = \pi(e_3). \end{array} \right. \qquad \pi : A^3 \to A^3/A^2\, R.$

- $M = A^3/A^2\,R$ **is defined by the equations**:

$$\left\{ \begin{array}{l} \frac{(s-1)}{(s+1)}\, y_1 - \frac{e^{-s}}{(s+1)}\, u = 0, \\ \\ \frac{(s-1)^2}{(s+1)^2}\, y_2 - \frac{e^{-s}}{(s+1)^2}\, u = 0, \end{array} \right.$$

and their A-linear combinations.

Classification of Modules

• Definition: Let M be a finitely generated A-module.

$a)$ M is **free** if $\exists\, r \in \mathbb{Z}_+ :\ M \cong A^r$.

$b)$ M is **stably-free** if $\exists\, r,\, s \in \mathbb{Z}_+$: $M \oplus A^s \cong A^r$.

$c)$ M is **projective** if $\exists\, r \in \mathbb{Z}_+$ and an A-module P:

$$M \oplus P \cong A^r.$$

$d)$ M is **reflexive** if

$$\begin{array}{rrcl} \epsilon : & M & \longrightarrow & \hom_A(\hom_A(M, A), A), \\ & m & \longrightarrow & \epsilon(m),\ \epsilon(m)(f) = f(m), \end{array}$$

is an isomorphism.

$e)$ M is **torsion-free** if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq a \in A : a\, m = 0\} = 0.$$

$m \in t(M)$ is called a **torsion element** of M.

$f)$ M is **torsion** if $M = t(M)$.

Definitions & Results

• Theorem:

free $\Rightarrow$ **stably-free** $\Rightarrow$ **projective**
$\Rightarrow$ **reflexive** $\Rightarrow$ **torsion-free**.

• Definition: 1. A ring is **projective-free** if every finitely generated projective A-module is free.

2. A ring is **Hermite** if $\forall\, n \in \mathbb{Z}_+$,

$$\forall\, a \in U_n(A) = \{a \in A^n \mid \exists\, b \in A^n : a\, b^t = 1\},$$

$$\exists\, V \in \mathrm{GL}_n(A) : \; a = (1 : \; 0 : \; \ldots : \; 0)\, V.$$

3. A is a **Bézout domain** if every finitely generated ideal $I = \sum_{i=1}^{n} A\, a_i$ of A is **principal**, i.e. $I = A\, a$ for a certain $a \in A$.

• Theorem:

1. A is a Hermite ring iff every finitely generated **stably-free** A-module is **free**.

2. A is a Bézout domain iff every finitely generated **torsion-free** A-module is **free**.

3. **Bézout domain** $\Rightarrow$ **projective-free** $\Rightarrow$ **Hermite.**

Doubly weakly coprime factorizations

Unstable poles/zeros cancelations

- Let us consider the system Σ_1 defined by:

$$\begin{cases} \ddot{z}(t) + 4\,\dot{z}(t) + 4\,z(t) = \dot{u}(t) - u(t), \\ \ddot{z}(0) = \dot{z}(0) = \dot{u}(0) = 0. \end{cases}$$

$$u(t) \longrightarrow \boxed{\Sigma_1} \longrightarrow z(t)$$

- By Laplace transform, we obtain $\widehat{z} = \frac{(s-1)}{(s+2)^2}\,\widehat{u}$.

- Let us consider the system Σ_2 defined by:

$$\begin{cases} \dot{y}(t) - \dot{y}(t) = \dot{z}(t) + 2\,z(t), \\ \dot{y}(0) = 0. \end{cases}$$

$$z(t) \longrightarrow \boxed{\Sigma_2} \longrightarrow y(t)$$

- By Laplace transform, we obtain $\widehat{y} = \frac{(s+2)}{(s-1)}\,\widehat{z}$.

- Let us consider the **interconnection** of $\Sigma_1 \& \Sigma_2$:

$$u(t) \longrightarrow \boxed{\Sigma_1} \longrightarrow z(t) \longrightarrow \boxed{\Sigma_2} \longrightarrow y(t)$$

- The transfer function of $\Sigma_1 \& \Sigma_2$ is given by:

$$\widehat{y} = \frac{(s+2)}{(s-1)}\,\frac{(s-1)}{(s+2)^2}\,\widehat{u} = \frac{1}{(s+2)}\,\widehat{u}.$$

• In the transfer function, we have cancelled the common factor $s-1$ which has an **unstable zero** $1 \in \mathbb{C}_+$.

$\Rightarrow$ **Engineering experience**: loss of stability in $\Sigma_1 \& \Sigma_2$.

• **Explanation**: $\Sigma_1 \& \Sigma_2$ is defined by:

$$\begin{cases} \ddot{z}(t) + 4\,\dot{z}(t) + 4\,z(t) = \dot{u}(t) - u(t), \\ \dot{y}(t) - \dot{y}(t) = \dot{z}(t) + 2\,z(t), \\ \ddot{z}(0) = \dot{z}(0) = \dot{u}(0) = \dot{y}(0) = 0. \end{cases}$$

Eliminating z in the equations, we obtain:

$$\ddot{y}(t) + \dot{y}(t) - 2\,y(t) = \dot{u}(t) - u(t)$$

$$\Leftrightarrow \begin{cases} x(t) = \dot{y}(t) + 2\,y(t) - u(t), \\ \dot{x}(t) = x(t), \end{cases}$$

$$\Rightarrow x(t) = (2\,y(0) - u(0))\,e^t.$$

We have:

$$\lim_{t \to +\infty} x(t) = \begin{cases} +\infty & \text{if } 2\,y(0) \neq u(0), \\ 0 & \text{if } 2\,y(0) = u(0). \end{cases}$$

$\Rightarrow$ $\Sigma_1 \& \Sigma_2$ is **generically unstable**: there is an **unobservable variable** x which is not exponentially stable $\Rightarrow$ concept of **a stabilizable Kalman system**.

• **Problem**: Does it exist a framework allowing to predict and to take into account only the unstable pole/zero cancelations in the transfer matrices?

Examples of torsion (torsion-free) (sub)-modules

- $RH_\infty = \left\{\frac{n(s)}{d(s)} \in \mathbb{R}(s) \mid \ \deg n(s) \leq \deg d(s),\ d(s) = 0 \Rightarrow \mathrm{Re}(s) < 0 \right\}$

- Let us consider the following system:

$$\left(\frac{s+1}{s+2}\right) y - \frac{(s+1)}{(s+2)^2}\, u = 0 \iff \left(\frac{s+1}{s+2}\right)\left(y - \frac{1}{s+2}\, u\right) = 0.$$

$$\left(\frac{s+1}{s+2}\right)^{-1} \in RH_\infty \Rightarrow \ \left(\frac{s+1}{s+2}\right) y - \frac{(s+1)}{(s+2)^2}\, u = 0$$

$$\Updownarrow$$

$$y - \frac{1}{(s+2)}\, u = 0.$$

- If we note:

$$R = \left(\frac{s+1}{s+2} : \ -\frac{s+1}{(s+2)^2}\right), \quad M = A^2/A\,R,$$
$$R' = \left(1 : \ -\frac{1}{s+2}\right), \quad N = A^2/A\,R',$$

then, **in terms of RH_∞-modules**, we have $M = N$.

- **In terms of transfer matrices**, we have:

$$h(s) = \frac{\frac{(s+1)}{(s+2)^2}}{\frac{(s+1)}{(s+2)}} = \frac{1}{(s+2)}.$$

• Let us consider the following system:

$$\left(\frac{s-1}{s+2}\right)\, y - \frac{(s-1)}{(s+2)^2}\, u = 0$$
$$\Longleftrightarrow \left(\frac{s-1}{s+2}\right)\left(y - \frac{1}{s+2}\, u\right) = 0.$$

$$\left(\frac{s-1}{s+2}\right)^{-1} \notin RH_\infty \Rightarrow \begin{cases} z = y - \frac{1}{s+2}\, u, \\ \left(\frac{s-1}{s+2}\right)\, z = 0. \end{cases}$$

• **In terms of RH_∞-modules**, we have:

$$R = \left(\frac{s-1}{s+2} : \; -\frac{s-1}{(s+2)^2}\right), \;\; M = A^2/A\, R,$$

$$t(M) = \{z = y - \frac{1}{(s+2)}\, u \mid \left(\frac{s-1}{s+2}\right)\, z = 0\} \neq 0.$$

$$\begin{array}{ccc} h(s) = \dfrac{\frac{(s-1)}{(s+2)^2}}{\frac{(s-1)}{(s+2)}} & = & \dfrac{1}{(s+2)} \\ \Downarrow & & \Downarrow \\ \left(\frac{s-1}{s+2}\right)\, y - \frac{s-1}{(s+2)^2}\, u = 0 & \neq & y - \frac{1}{s+2}\, u = 0 \\ \Updownarrow & & \Updownarrow \\ M & \neq & M/t(M). \end{array}$$

• <u>Conclusion</u>: **The use of RH_∞-modules is a framework which only allows the cancelations by common proper & stable factors in a transfer matrix.**

Weak primeness

- Definition: A matrix $R \in A^{q \times p}$ is **weakly left-prime** if

$$\boxed{K^q\, R \cap A^p = A^p\, R,}$$

where $K = \{\, p = \frac{n}{d} \mid 0 \neq d,\, n \in A\}$, i.e.:

$$\forall\, \mu \in K^q :\ \mu\, R \in A^p \Rightarrow \exists\, \nu \in A^q :\ \mu\, R = \nu\, R.$$

- Dually for **weakly right-prime**:

R is weakly right-prime $\Leftrightarrow$ R^T is weakly left-prime.

- Definition: A matrix $R \in A^{q \times p}$ is **full row rank** if its rows are A-linearly independent.

- If $R \in A^{q \times p}$ is a **full row rank matrix**, then R is **weakly left-prime** iff:

$$\forall\, \mu \in K^q :\ \mu\, R \in A^p \Rightarrow \mu \in A^p.$$

- Let us consider the matrix ($A = H_\infty(\mathbb{C}_+)$):

$$R = \left(\begin{array}{ccc} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{array}\right) \in A^{2 \times 3}.$$

R is **not weakly left-coprime** because we have:

$$\left(\frac{1}{s-1} \ :\ -\frac{s+1}{s-1}\right) R = \left(\frac{1}{s+1} \ :\ -\frac{s-1}{s+1} \ :\ 0\right) \in A^3 \not\Rightarrow \left(\frac{1}{s-1} \ :\ -\frac{s+1}{s-1}\right) \in A^2.$$

A-closure & Torsion-freeness

- Definition: If X is a submodule of A^p, then

$$\overline{X} = \{\lambda \in A^p \mid \exists\, 0 \neq a \in A : a\,\lambda \in X\}$$

is called the **A-closure of X in A^p**.

- Proposition: Let $R \in A^{q \times p}$ and the A-modules $A^q\,R \subseteq A^p$ and $M = A^p/A^q\,R$. Then, we have:

1. $\overline{A^q\,R} = K^q\,R \cap A^p$.

2. $t(M) = (K^q\,R \cap A^p)/A^q\,R = \overline{A^q\,R}/A^q\,R$.

3. $M/t(M) = A^p/(K^q\,R \cap A^p) = A^p/\overline{A^q\,R}$.

- Corollary: We have the following equivalences:

1. $R \in A^{q \times p}$ is **weakly left-prime**,

2. $\overline{A^q\,R} = A^q\,R$,

3. $M = A^p/A^q\,R$ is **torsion-free**.

$\boxed{\textbf{Example}}$

- Let us consider $A = H_\infty(\mathbb{C}_+)$ and:

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{pmatrix} \in A^{2\times 3}.$$

- R is **not weakly left prime** because:

$$\left(\tfrac{1}{s-1} \,:\, -\tfrac{s+1}{s-1}\right) R = \left(\tfrac{1}{s+1} \,:\, -\tfrac{s-1}{s+1} \,:\, 0\right) \in A^3 \not\Rightarrow \left(\tfrac{1}{s-1} \,:\, -\tfrac{s+1}{s-1}\right) \in A^2.$$

- The A-module $M = A^3/A^2\, R$ **is defined by**

$$\begin{cases} \frac{s-1}{s+1}\, y_1 - \frac{e^{-s}}{s+1}\, u = 0, & (1) \\ \left(\frac{s-1}{s+1}\right)^2\, y_2 - \frac{e^{-s}}{(s+1)^2}\, u = 0, & (2) \end{cases}$$

and their A-linear combinations.

- $\frac{1}{(s+1)}\,(1) - (2) \Rightarrow \frac{(s-1)}{(s+1)^2}\, y_1 - \left(\frac{s-1}{s+1}\right)^2\, y_2 = 0 \; (3).$

$$(3) \Leftrightarrow \frac{(s-1)}{(s+1)}\, \underbrace{\left(\frac{1}{(s+1)}\, y_1 - \frac{(s-1)}{(s+1)}\, y_2\right)}_{z} = 0,$$

$$\Rightarrow \begin{cases} z = \frac{1}{(s+1)}\, y_1 - \frac{(s-1)}{(s+1)}\, y_2, \\ \frac{(s-1)}{(s+1)}\, z = 0, \end{cases}$$

is a **torsion element of M**, i.e. $z \in t(M)$.

Transfer matrices

• Theorem: Let $P = D_1^{-1} N_1 = D_2^{-1} N_2$ with:

$$\begin{cases} R_1 = (D_1 : \ -N_1) \in A^{q \times p}, \\ R_2 = (D_2 : \ -N_2) \in A^{q \times p}. \end{cases}$$

Then, we have

$$\overline{A^q \, R_1} = \overline{A^q \, R_2} \subset A^p,$$

i.e. $\overline{A^q \, R_i}$ **only depends on** P**.**

• Similarly for $P = \tilde{N}_1 \, \tilde{D}_1^{-1} = \tilde{N}_2 \, \tilde{D}_2^{-1}$.

• Theorem: Let $P = D_1^{-1} N_1 = D_2^{-1} N_2$ with:

$$\begin{cases} R_1 = (D_1 : \ -N_1) \in A^{q \times p}, \\ R_2 = (D_2 : \ -N_2) \in A^{q \times p}. \end{cases}$$

Then, we have

$$A^p \, R_1^T \cong A^p \, R_2^T,$$

i.e. $A^p \, R_i^T$ **only depend** on P up to an isomorphism.

• Similarly for $P = \tilde{N}_1 \, \tilde{D}_1^{-1} = \tilde{N}_2 \, \tilde{D}_2^{-1}$.

• Corollary: **The structural properties of** $P = D^{-1} \, N$, $R = (D : \ -N) \in A^{q \times p}$, **only depend on** $\overline{A^q \, R}$ and **on** $A^p \, R^T$ **up to an isomorphism**.

$\boxed{\textbf{Doubly weakly coprime factorizations}}$

- $\begin{cases} P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times (p-q)}, \\ R = (D : \, -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T : \, \tilde{D}^T)^T \in A^{p \times (p-q)}. \end{cases}$

- Definition: P admits a **weakly left coprime factorization** if there exists a **weakly left prime matrix** $R' = (D' : \, N') \in A^{q \times p}$ such that:

$$P = D'^{-1}\, N'.$$

- Definition: P admits a **weakly right coprime factorization** if there exists a **weakly right prime matrix** $R' = (D'^T : \, N'^T)^T \in A^{p \times p-q}$ such that:

$$P = \tilde{N}'\, \tilde{D}'^{-1}.$$

- Definition: P admits a **doubly weakly coprime factorization** if P has **weakly left and right coprime factorizations**:

$$P = D'^{-1}\, N' = \tilde{N}'\, \tilde{D}'^{-1}.$$

- Theorem: P admits a **weakly left coprime factorization** iff the A-module $\overline{A^q\, R}$ is free of rank q.

- Corollary: P admits a **doubly weakly coprime factorization** iff the A-modules $\overline{A^q\, R}$ and $\overline{A^{p-q}\, \tilde{R}^T}$ are **free** of rank respectively q and $p-q$.

Noetherian Banach algebras

• Definition: **A ring A is noetherian if any ideal I of A is finitely generated**, i.e. there exist $a_1, ..., a_n \in A$:

$$I = \sum_{i=1}^{n} A\, a_i.$$

• Examples: RH_∞, $k[s]$, $k[\chi_1, \ldots, \chi_n]$ $(k = \mathbb{R}, \mathbb{C})$.

• Definition: **An A-module M is noetherian if any A-submodule N of M is finitely generated**, i.e. there exists a finite family $\{n_i\}_{1 \le i \le k}$ of N such that:

$$\forall\, n \in N,\ \exists\, a_i \in A : n = \sum_{i=1}^{k} a_i\, n_i.$$

• Proposition: If A is a noetherian ring, then an A-module M is noetherian iff M is finitely generated.

• Proposition: **The class of the finitely generated A-modules over a noetherian ring A is stable by**:

$$+,\ \oplus,\ /,\ \cap,\ \ker \cdot,\ \mathrm{im} \cdot,\ \otimes_A,\ \hom_A(\cdot, \cdot)...$$

• Theorem (Sinclair-Tullo 74): **Noetherian Banach algebras are finite-dimensional ones**.

$\Rightarrow$ The Banach algebras $H_\infty(\mathbb{C}_+), L_\infty(\mathbb{R}),\ \mathcal{A},\ \hat{\mathcal{A}},$ $l_1(\mathbb{Z}_+),\ L_1(\mathbb{R}_+)$... are not noetherian domains.

• Conclusion: **Algebra & module theory seem to be useless to study infinite-dimensional systems.**

Coherent Rings & Modules

• Definition: **A ring is coherent if for any finitely generated ideal $I = (a_1, ..., a_n)$ of A, the module**

$$S(I) = \{(r_1 : \ldots : r_n) \in A^n \mid \sum_{i=1}^{n} r_i\, a_i = 0\}$$

is finitely generated, i.e.:

$$\exists\, m \in \mathbb{Z}_+,\ \exists\, R \in A^{m \times n} :\ S(I) = A^m\, R.$$

• Examples: Noetherian or Bézout domains, $\mathbb{R}[\chi_i]_{i \in \mathbb{N}}$.

• Definition: **An A-module M is coherent** if:
– M is a finitely generated A-module,
– for any A-morphism $\phi : A^n \to M$, the A-module

$$\ker \phi = \{(r_1 : \ldots : r_n) \in A^n \mid \sum_{i=1}^{n} r_i\, \phi(e_i) = 0\}$$

is finitely generated ($\{e_i\}_{1 \leq i \leq n}$: canonical basis of A^n).

• Proposition: If A is a coherent ring, then an A-module M is coherent iff there exists a finite matrix $R \in A^{q \times p}$ such that $M = A^p / A^q\, R$.

• Proposition: **The class of coherent A-modules over a coherent ring A is stable by:**

$+$, $\oplus$, $\cap$, $/$, $\otimes_A$, $\hom_A(\cdot, \cdot)$, $\mathrm{ann}_A(\cdot)$, $\ker \cdot$, $\mathrm{im} \cdot$...

Examples of Coherent Rings

• <u>Theorem</u> (McVoy-Rubel 76, Rosay 77): **The rings** $H_\infty(\mathbb{C}_+),\ L_\infty(\mathbb{R})$ **are coherent**.

• <u>Theorem</u> (Helmer 40): If k is a subfield of $\mathbb{C}$, then **the ring** $E(k)$ **of entire functions in** $\mathbb{C}$

$$E(k) = \{f(s) = \textstyle\sum_{i=0}^{+\infty} a_i\, s^i \mid \quad a_i \in k, \qquad \lim_{i \to +\infty} \mid a_i \mid^{1/i} = 0\}$$

is a Bézout domain, and thus, **a coherent domain**.

• <u>Theorem</u> (Loiseau 96, Glüsing-Lüerßen 97): $\mathcal{E} = E(\mathbb{R}) \cap \mathbb{R}(s)[e^{-\theta}]$ **is** a Bézout domain, and thus, **a coherent domain.**

• <u>Theorem</u> (Morse 76): RH_∞ **is** a principal ideal domain, and thus, **a coherent domain.**

• Are $\mathcal{A}$, $\hat{\mathcal{A}}$ and $L_1(\mathbb{R}_+) + \mathbb{R}\,\delta$ coherent domains?

• <u>Proposition</u>: Let A be a coherent domain and its quotient field $K = \{\frac{a}{b} \mid 0 \neq b, a \in A\}$. Then, any transfer matrix $T = D^{-1}\, N \in K^{q \times (p-q)}$ defines a coherent A-module $M = A^p / A^q (D :\ -N)$.

• <u>Conclusion</u>: **The classes of infinite dimensional systems over coherent rings are the ones which can be studied by means of module theory.**

Examples of Coherent Rings

• <u>Definition</u>: 1. A topological space X is **completely regular** if X is Hausdorff and $\forall\ U$ closed set and $\forall\, x \in X \backslash U$, $\exists\ f \in C(X)$: $\quad f(x) = 1,\ f(U) = 0$.

2. A completely regular space X is **basically disconnected** if $\forall\, f \in C(X)$:

$$\mathrm{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}} \text{ is open.}$$

• <u>Theorem</u> (Neville 90): X completely regular space:

1. X is basically disconnected $\Leftrightarrow$

2. $C(X)$ is a **semi-hereditary ring**, i.e. every finitely generated ideal of A is projective, $\Leftrightarrow$

3. $C(X)$ is a coherent ring $\Leftrightarrow$

4. $C(X)$ is a **PP ring**, i.e. every principal ideal is projective.

• <u>Theorem</u> (Gillman & Jerison 60): If X is basically disconnected, then X is an F**-space**, i.e. **every finitely generated ideal of $C(X)$ is principal**.

$\Rightarrow A = L_\infty(\mathbb{T})$ or $L_\infty(\mathbb{R})$ are semi-hereditary rings and every finitely generated ideal of A is principal.

(Counter-) Examples of Coherent Rings

• Definition: 1. The **Nevanlinna class** is the algebra N of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ \mid f(r\, e^{i\theta}) \mid d\theta < +\infty,$$

where $\log^+ x = \max(0, \log x)$.

2. A ring A satisfying $H_\infty(\mathbb{D}) \subseteq A \subseteq N$ is of **Nevanlinna-Smirnov type** if:

$$\forall\, f \in A, \exists\; g,\; h \in H_\infty(\mathbb{D}),\; h^{-1} \in A : \;\; f = g/h.$$

• Example of rings of Nevanlinna-Smirnov type: N,

$$N^+ = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic} \mid \exists\; g,\; h \in H_\infty(\mathbb{D}) : \\ f = g/h\},$$

$$N^p = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic} \mid \\ \sup_{0<r<1} \textstyle\int_0^{2\pi} \log^+ \mid f(r\, e^{i\theta}) \mid^p d\theta < +\infty\}.$$

• Theorem (Mortini 89): **Every ring of Nevanlinna type is coherent.**

• Theorem (McVoy & Rubel 76, Mortini & von Renteln 89): **The disc algebra $A(\mathbb{D})$ and the Wiener algebra**

$$W^+ = \{f(z) = \sum_{i=0}^{+\infty} a_n\, z^n \mid \sum_{i=0}^{+\infty} \mid a_n \mid < +\infty\}$$

are not coherent.

$\boxed{\textbf{Algorithm}}$

- *Input*: A **coherent integral domain** A and $R \in A^{q \times p}$.

- *Output*: $R' \in A^{r \times p}$ such that $\overline{A^q\, R} = A^r\, R'$.

1. Start with $R \in A^{q \times p}$.

2. Transpose R to obtain $R^T \in A^{p \times q}$.

3. Find a family of generators of:

$$\ker .R^T = \{\lambda \in A^p \,|\, \lambda\, R^T = 0\}.$$

If $\{\lambda_1, \ldots, \lambda_m\}$ is such a family, then note:

$$R_{-1}^T = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \in A^{m \times p}.$$

4. Tranpose R_{-1}^T to obtain $R_{-1} \in A^{p \times m}$.

5. Find a family of generators of:

$$\ker .R_{-1} = \{\eta \in A^p \,|\, \eta\, R_{-1} = 0\}.$$

If $\{\eta_1, \ldots, \eta_r\}$ is such a family, then note:

$$R' = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix} \in A^{r \times p}.$$

We have $\boxed{\overline{A^q\, R} = A^r\, R'.}$

- This algorithm is obtained from **homological algebra**.

Coherent Sylvester domains

• Definition : A is a **coherent Sylvester domain** if, for every $p \in \mathbb{Z}_+$ and **row column** $R \in A^p$, then

$$\ker\, R. = \{\lambda \in A^p \mid R\,\lambda = 0\} \text{ is } \textbf{free } A\text{-module.}$$

• Example: $k[s, z]$ (k field), every Bézout domain (e.g. $\mathcal{E}$), every principal ideal domain (e.g. RH_∞).

• Theorem: $H_\infty(\mathbb{C}_+)$ is a **coherent Sylvester domain**.

• Definition: A is a **greatest common divisor domain** if every $a, b \in A$ have a greatest common divisor.

• Theorem (Dicks 83): A coherent Sylvester domain is a projective-free greatest common divisor domain.

$\Rightarrow$ $H_\infty(\mathbb{C}_+)$ **is a greatest common divisor domain** (Renteln 77, Smith 89).

• Theorem: We have the equivalences:

1. **Every matrix** P with entries in $K = Q(A)$ **has a doubly weakly coprime factorization**,

2. A is a **coherent Sylvester domain**.

(generalization of a result of M. C. Smith for $H_\infty(\mathbb{C}_+)$).

Example

- Let us consider the transfer matrix ($A = H_\infty(\mathbb{C}_+)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2, \quad K = Q(A).$$

- **Chasing the unstable denominators** of P, we obtain $P = D^{-1}\, N$ with $R = (D :\; -N)$:

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{pmatrix} \in A^{2\times 3}.$$

- $A = H_\infty(\mathbb{C}_+)$ is a **coherent Sylvester domain**.

- There exist a **weakly left matrix** $R' \in A^{2\times 3}$ and **a non-singular matrix** $R'' \in A^{2\times 2}$ such that:

$$R = R''\, R' \Rightarrow (D :\; -N) = (R''\, D' :\; -R''\, N')$$
$$\Rightarrow P = D^{-1}\, N = (R'' D')^{-1}\, R'' N' = D'^{-1}\, N'.$$

- **Algorithm** $\Rightarrow \begin{cases} R'' = \begin{pmatrix} 0 & 1 \\ -\frac{s-1}{s+1} & \frac{1}{s+1} \end{pmatrix} \in A^{2\times 2} \\ R' = \begin{pmatrix} \frac{1}{s+1} & -\frac{s-1}{s+1} & 0 \\ \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \end{pmatrix} \in A^{2\times 3} \end{cases}$

$$\Rightarrow P = \begin{pmatrix} \frac{1}{s+1} & -\frac{s-1}{s+1} \\ \frac{s-1}{s+1} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{e^{-s}}{s+1} \end{pmatrix}$$

is a **weakly left-coprime factorization** of P.

Doubly coprime factorizations

Left & Right-coprime factorizations

- $$\left\{\begin{array}{l} P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1} \in K^{q\times(p-q)},\\ R = (D\,:\,-N) \in A^{q\times p},\\ \tilde{R} = (\tilde{N}^T\,:\;\tilde{D}^T)^T \in A^{p\times(p-q)}.\end{array}\right.$$

- <u>Definition</u>: $P \in K^{q\times(p-q)}$ admits a **left-coprime factorization** if there exist two matrices

$$\left\{\begin{array}{l} R' = (D'\,:\;-N') \in A^{q\times p},\\ S = (X^T\,:\;Y^T)^T \in A^{p\times q},\end{array}\right. \quad \text{such that :}$$

$$\left\{\begin{array}{l} P = {D'}^{-1}\,N',\\ R'\,S = D'\,X - N'\,Y = I_q.\end{array}\right.$$

- <u>Definition</u>: $P \in K^{q\times(p-q)}$ admits a **right-coprime factorization** if there exist two matrices

$$\left\{\begin{array}{l} \tilde{R}' = (\tilde{N}'^T\,:\;\tilde{D}'^T)^T \in A^{p\times(p-q)},\\ \tilde{S} = (-\tilde{Y}\,:\;\tilde{X}) \in A^{(p-q)\times p},\end{array}\right. \quad \text{such that :}$$

$$\left\{\begin{array}{l} P = \tilde{N}'\,\tilde{D}'^{-1},\\ \tilde{S}\,\tilde{R}' = -\tilde{Y}\,\tilde{N}' + \tilde{X}\,\tilde{D}' = I_{p-q}.\end{array}\right.$$

- <u>Theorem</u>: $P \in K^{q\times(p-q)}$ admits a **left-coprime factorization** iff the A-module $A^p/\overline{A^{p-q}\,\tilde{R}^T}$ is **free** of rank q.

- <u>Theorem</u>: $P \in K^{q\times(p-q)}$ admits a **right-coprime factorization** iff the A-module $A^p/\overline{A^q\,R}$ is **free** of rank $p-q$.

$\boxed{\textbf{Doubly coprime factorizations}}$

- $\begin{cases} P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1} \in K^{q\times(p-q)}, \\ R = (D\,:\,-N) \in A^{q\times p}, \\ \tilde{R} = (\tilde{N}^T\,:\,\tilde{D}^T)^T \in A^{p\times(p-q)}. \end{cases}$

- Definition: P has a **doubly coprime factorization** if there exist

$$\begin{cases} R' = (D'\,:\,-N') \in A^{q\times p}, \\ \tilde{R}' = (\tilde{N}'^T\,:\,\tilde{D}'^T)^T \in A^{p\times(p-q)}, \\ S = (X^T\,:\,Y^T)^T \in A^{p\times q}, \\ \tilde{S} = (-\tilde{Y}\,:\,\tilde{X}) \in A^{(p-q)\times p}, \end{cases} \quad \text{such that:}$$

1. $\begin{pmatrix} S & \tilde{R}' \end{pmatrix} \begin{pmatrix} R' \\ \tilde{S} \end{pmatrix} = I_p,$

2. $\begin{pmatrix} R' \\ \tilde{S} \end{pmatrix} \begin{pmatrix} S & \tilde{R}' \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix} = I_p.$

- Theorem: We have the following equivalences:

1. P has a **doubly coprime factorization**.

2. **The A-modules** $A^p/\overline{A^{p-q}\,\tilde{R}^T}$ and $A^p/\overline{A^q\,R}$ **are free** of rank respectively q and $p-q$.

3. (Sule 94) **The A-modules** $A^p\,R^T$ and $A^p\,\tilde{R}$ **are free** of rank respectively q and $p-q$.

We cannot comb the hair of a coconut!

• Let $\mathbb{R}_2 = \mathbb{R}[t_0, t_1, t_2]/(t_0^2 + t_1^2 + t_2^2 - 1)$ be **the ring of the polynomials in t_i on the unit sphere S^2**.

• Let x_i be the class of t_i in $\mathbb{R}_2$ and let us consider the "unstable plant"

$$P = \left(-\frac{x_1}{x_0} \ : \ -\frac{x_2}{x_0} \right)$$

and the matrix $R = (x_0 : \ x_1 : \ x_2) \in \mathbb{R}_2^{1\times 3}$.

• The $\mathbb{R}_2$-module $M = \mathbb{R}_2^3/\mathbb{R}_2\, R$ is **stably-free**:

$$R\, R^T = x_0^2 + x_1^2 + x_2^2 = 1.$$

$\Rightarrow P$ has a **normalized left-coprime factorization**
$\Rightarrow P$ **is internally stabilizable** by $C = P^T$.

• **Let us prove that P has no doubly coprime factorizations, i.e. M is not a free $\mathbb{R}_2$-module**.

1. If M were free, then R could be completed into a unimodular matrix

$$U = \begin{pmatrix} x_0 & x_1 & x_2 \\ a & b & c \\ e & f & g \end{pmatrix}, \qquad a, b, c, d, e, f, g \in \mathbb{R}_2.$$

2. Let us consider the **continuous vector field**:

$$\begin{array}{rcl} X : S^2 & \longrightarrow & \mathbb{R}^3 \\ x = (x_0, x_1, x_2) & \longrightarrow & X(x) = (a(x), b(x), c(x))^T. \end{array}$$

3. U is a unimodular matrix, i.e. its determinant

$$e(x_1\, c - x_2\, b) - f\,(x_0\, c - x_2\, a) + g\,(x_0\, b - x_1\, a)$$

is a unit of $\mathbb{R}_2$. Hence, the system

$$\left\{ \begin{array}{l} x_1\, c(x) = x_2\, b(x), \\ x_0\, c(x) = x_2\, a(x), \\ x_0\, b(x) = x_1\, a(x) \end{array} \right.$$

has no solution, i.e. X **is never colinear with** x.

4. Let us consider the continuous vector

$$Y : x \in S^2 \rightarrow \pi_x(X(x)) \in \mathbb{R}^3,$$

where π_x is the orthogonal projection of $X(x)$ onto the tangent plane of S^2 at x.

$\Rightarrow Y$ **is a nowhere vanishing continuous vector field of** S^2.

5. It is not possible because **"the hair of a coconut cannot be combed"** (topological arguments)
$\Rightarrow M$ is not a free $\mathbb{R}_2$-module
$\Rightarrow P$ **has no doubly coprime factorizations as well as no Youla parametrization for its controllers.**

Plants admitting doubly coprime factorizations

• Corollary: $p = \frac{n}{d} \in K = Q(A)$ admits a coprime factorization iff the A-module

$$I = (d,\, n) = A^2\,(d:\ -n)^T$$

is free, i.e. I **is a principal ideal of** A.

• Theorem: (Vidyasagar 85) We have 1 ⇔ 2 ⇔ 3:

1. A **is a Bézout domain,**

2. **Every MIMO plant** – defined by a transfer matrix with entries in $K = Q(A)$ – **admits doubly coprime factorizations,**

3. **Every SISO plant** – defined by a transfer matrix with entries in $K = Q(A)$ – **admits doubly coprime factorizations.**

• Example: **Principal ideal domains** (e.g. $k[s]$, k field, RH_∞), $E(k)$ (k subfield of $\mathbb{C}$),

$$\mathcal{E} = E(\mathbb{R}) \cap \mathbb{R}(s)[e^{-s}] \ldots$$

Internal stabilization

$\boxed{\text{Internal stabilization}}$

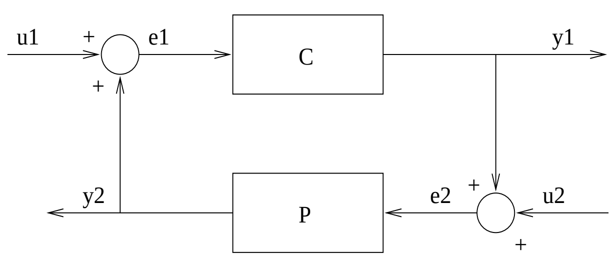
- Let A be an **integral domain of SISO stable plants**.

- $K = \{\frac{n}{d} \mid 0 \neq d,\, n \in A\}$ the field of fractions of A.

- $P \in K^{q \times r}$ **a plant**.

- $C \in K^{r \times q}$ **a controller**.

- The **closed-loop system** is defined by:



$u_1,\, u_2$: external inputs, e_1, e_2: internal inputs, $y_1,\, y_2$: outputs.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I & -P \\ -C & I \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \; \begin{cases} y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{cases}$$

- <u>Definition</u>: C **internally stabilizes** P if the transfer matrix $T = \begin{pmatrix} I & -P \\ -C & I \end{pmatrix}^{-1}$ **exists** and **satisfies**:

$$T = \begin{pmatrix} (I - P\,C)^{-1} & (I - P\,C)^{-1}\,P \\ C(I - P\,C)^{-1} & I + C\,(I - P\,C)^{-1}\,P \end{pmatrix} \in A^{(q+r)\times(q+r)}.$$

- **Internal stability** $\Leftrightarrow \begin{cases} L_2 - L_2 \text{ stability if } A = H_\infty(\mathbb{C}_+), \\ L_\infty - L_\infty \text{ stability if } A = \widehat{\mathcal{A}}. \end{cases}$

Examples

- Example: $A = RH_\infty, \quad K = \mathbb{R}(s)$.

$$\begin{cases} p = \frac{s}{s-1}, \\ \\ c = -\frac{(s-1)}{(s+1)}, \end{cases} \Rightarrow \begin{cases} e_1 = \frac{(s+1)}{(2s+1)}\, u_1 + \frac{s(s+1)}{(2s+1)(s-1)}\, u_2, \\ \\ e_2 = \frac{(-s+1)}{(2s+1)}\, u_1 + \frac{(s+1)}{(2s+1)}\, u_2. \end{cases}$$

$\Rightarrow c$ **does not internally stabilize** p because:

$$\frac{s(s+1)}{(2s+1)(s-1)} \notin RH_\infty \text{ (pole in } 1 \in \mathbb{C}_+).$$

$$u_2 \notin \left(\frac{s-1}{s+1}\right) H_2 \triangleq \left\{\frac{(s-1)}{(s+1)}\, z \mid z \in H_2\right\} \Rightarrow e_1 \notin H_2.$$

$$\text{(e.g. } u_2 = \frac{1}{s+1} \text{ i.e. } \mathcal{L}^{-1}(u_2) = e^{-t}\, Y(t)).$$

The pole/zero cancellation between p and c leads to an unstability.

- Example: $A = RH_\infty, \quad K = \mathbb{R}(s)$.

$$\begin{cases} p = \frac{s}{s-1}, \\ \\ c = 2, \end{cases} \Rightarrow \begin{cases} e_1 = -\frac{(s-1)}{(s+1)}\, u_1 - \frac{s}{(s+1)}\, u_2, \\ \\ e_2 = -2\, \frac{(s-1)}{(s+1)}\, u_1 - \frac{(s-1)}{(s+1)}\, u_2. \end{cases}$$

$\Rightarrow c$ **internally stabilizes** the plant p.

Internal stabilization: results for MIMO plants

- $\begin{cases} P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times (p-q)}, \\ R = (D :\ -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T :\ \tilde{D}^T)^T \in A^{(p-q) \times p}. \end{cases}$

- Theorem: P is **internally stabilizable** iff the A-module $A^p / \overline{A^q\, R}$ (or $A^p / \overline{A^{p-q}\, \tilde{R}^T}$) is **projective**.

- Corollary: $P = D^{-1}\, N$ is **internally stabilizable** iff $\exists\ S = (X^T :\ Y^T)^T \in K^{p \times q}$ such that:

1. $S\, R = \begin{pmatrix} X\, D & -X\, N \\ Y\, D & -Y\, N \end{pmatrix} \in A^{p \times p}$,

2. $R\, S = D\, X - N\, Y = I_q$.

The controller $C = Y\, X^{-1}$ **internally stabilizes** P.

- Corollary: $P \in K^{q \times (p-q)}$ is **internally stabilizable** iff $\exists\ S = (U^T :\ V^T)^T \in A^{p \times q}$ such that:

1. $S\, P = (U\, P :\ V\, P) \in A^{p \times (p-q)}$,

2. $(I_q :\ -P)\, S = U - P\, V = I_q$.

The controller $C = U\, V^{-1}$ **internally stabilizes** P.

Example

- Let us consider the transfer matrix ($A = H_\infty(\mathbb{C}_+)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2, \quad K = Q(A).$$

- **Chasing the unstable denominators** of P, we obtain $P = D^{-1}\, N$ with $R = (D\,:\,-N) \in A^{2\times 3}$:

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{pmatrix}.$$

- The matrix $S = (X^T\,:\,Y^T)^T \in K^{3\times 2}$ defined by

$$S = \begin{pmatrix} b\left(\frac{s-1}{s+1}\right)^2 + \frac{2}{s-1} & 2\,(b-1)\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)}{(s+1)^2} - \frac{1}{s-1} & \frac{2\,b}{s+1} + \frac{s+1}{s-1} \\ -a\,\frac{(s-1)}{(s+1)^2} & -\frac{2\,a}{s+1} \end{pmatrix},$$

$$\text{with} \quad \begin{cases} a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \\ b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A, \end{cases}$$

satisfies:

$$\begin{cases} S\,R \in A^{3\times 3}, \\ R\,S = D\,X - N\,Y = I_2, \end{cases}$$

$\Rightarrow$ P is **internally stabilized** by $C = Y\,X^{-1}$.

The **stabilizing controller** $C = Y\,X^{-1}$ is defined by:

$$
\begin{aligned}
C \; &= \left(-\frac{a\,(s-1)}{(s+1)^2} \; : \; -\frac{2\,a}{(s+1)} \right) \\
&\quad \left(\begin{array}{cc} b\,\left(\frac{s-1}{s+1}\right)^2 + \frac{2}{s-1} & 2\,(b-1)\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)}{(s+1)^2} - \frac{1}{s-1} & \frac{2\,b}{s+1} + \frac{s+1}{s-1} \end{array} \right)^{-1} \\
&= -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3-4\,(5\,s-3)\,e^{-(s-1)})}\,(1 \; : \; 2).
\end{aligned}
$$

$\boxed{\textbf{Example}}$

- Let us reconsider the transfer matrix ($A = H_\infty(\mathbb{C}_+)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2, \quad K = Q(A).$$

- The matrix $S = (U^T \, : \, V^T)^T \in A^{3 \times 2}$ defined by

$$S = \begin{pmatrix} \frac{2}{s+1} + b\left(\frac{s-1}{s+1}\right)^3 & 2\,b\left(\frac{s-1}{s+1}\right)^3 - 2\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\,b\,\frac{(s-1)}{(s+1)^3} \\ -a\,\frac{(s-1)^2}{(s+1)^3} & -2\,a\,\frac{(s-1)^2}{(s+1)^3} \end{pmatrix}$$

with $\left\{ \begin{array}{l} a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \\ b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A, \end{array} \right.$

satisfies

$$\left\{ \begin{array}{l} S\,(I_2 \, : \, -P) \in A^{3 \times 3}, \\ (I_2 \, : \, -P)\,S = U - P\,V = I_2, \end{array} \right.$$

$\Rightarrow P$ is **internally stabilized** by the controller:

$$\begin{aligned} C &= V\,U^{-1} \\ &= -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)})}\,(1 \, : \, 2). \end{aligned}$$

Projective modules

• <u>Definition</u>: If $M = A^p/A^q\,R$ with $R \in A^{q\times p}$, then the i^th **Fitting ideal** $\mathrm{Fitt}_i(M)$ of M the ideal defined by the minors of size $p-i$ of R.

• <u>Theorem</u>: The A-module $M = A^p/A^q\,R$ is **projective** of rank r iff:

$$\begin{cases} \mathrm{Fitt}_r(M) = A, \\ \mathrm{Fitt}_{r-1}(M) = 0. \end{cases}$$

• <u>Theorem</u>: If $R \in A^{q\times p}$ $(q \leq p)$ is a **full row rank matrix**, then $M = A^p/A^q\,R$ is a **projective** A-module **if there exists** $S \in A^{p\times q}$ **such that:**

$$R\,S = I_q \;\; (\Leftrightarrow T(M) = A^q/A^p\,R^T = 0).$$

• <u>Theorem</u>: If A is a **semi-simple Banach algebra**, $X(A)$ its maximal ideal space and $R \in A^{q\times p}$ is a **full row rank** matrix, then $M = A^p/A^q\,R$ is a **projective** A-module iff

$$\inf_{\chi \in X(A)} \sum_{i\in I} \mid \widehat{R}_i(\chi) \mid \geq \delta > 0,$$

where $(\widehat{R}_i)_{i\in I}$ is the minors of size q and $\widehat{\cdot}$ is the **Gelfand transform**.

Example

- Let $A = H_\infty(\mathbb{C}_+)$ and the matrix R defined by:

$$R = \begin{pmatrix} 1 & 0 & -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{pmatrix}.$$

- Let us define the A-module $M = A^4/A^2 R$.

$$\mathrm{Fitt}_2(M) = \left(\frac{s-1}{s+1}, \frac{1}{s+1}, \ldots\right).$$

$\Rightarrow \mathrm{Fitt}_2(M) = A \Rightarrow M$ is a **projective A-module**, and thus, **free** because $H_\infty(\mathbb{C}_+)$ is a **Hermite ring**.

- We have the **generalized Bézout identities**:

$$\begin{pmatrix} 1 & 0 & -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \\ .. & .. & .. & .. \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2\,(\frac{s-1}{s+1}) & : & \frac{e^{-s}}{s+1} & (\frac{s-1}{s+1})^2 \\ 0 & 1 & : & 0 & \frac{1}{s+1} \\ 0 & 0 & : & 1 & 0 \\ 0 & -2 & : & 0 & \frac{s-1}{s+1} \end{pmatrix} = I_4.$$

Algorithm

- *Input*: A coherent domain A, $R = (D : \, -N) \in A^{q \times p}$.
- *Ouput*: Stabilizability or not of $P = D^{-1}\, N \in K^{(p-q)\times q}$.

1. Compute $\overline{A^q\, R}$:

$$\boxed{\begin{array}{c}\text{Find } r \in \mathbb{Z}_+ \text{ and } R' \in A^{r \times p} \text{ such that} \\ \overline{A^q\, R} = A^r\, R'.\end{array}}$$

2. For increasing i, check whether or not:

$$\boxed{1 \in \mathrm{Fitt}_i(A^p/A^r\, R').}$$

If $\exists\, i$ such that $1 \in \mathrm{Fitt}_i(A^p/A^r\, R') \Rightarrow P$ is internally stabilizable, else not.

- **Remark**: In order to be able to check effectively internal stabilizability, we need to:

a. compute the kernel of matrices whose entries belong to A.

b. test whether or not 1 belongs to a finitely generated ideal of A.

- **There exists a general algorithm which computes stabilizing controllers**.

$\boxed{\textbf{Example}}$

- Let us consider the plant ($A = H_\infty(\mathbb{C}_+)$):

$$p = \frac{e^{-s}}{s-1} = \frac{\left(\frac{e^{-s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)} \Rightarrow \left(\frac{s-1}{s+1}\right) y - \left(\frac{e^{-s}}{s+1}\right) u = 0.$$

- Let $R = \left(\frac{s-1}{s+1} \ : \ -\frac{e^{-s}}{s+1}\right) \in A^2$ and the A-module

$$M = A^2/A\,R = M/t(M) \Rightarrow \mathrm{Fitt}_1(M) = \left(\frac{s-1}{s+1}, \frac{e^{-s}}{s+1}\right).$$

- M is **projective** iff $\mathrm{Fitt}_1(M) = A$, i.e. $\exists\, a, b \in A$:

$$\left(\frac{s-1}{s+1}\right) a + \left(-\frac{e^{-s}}{s+1}\right) b = 1. \qquad (1)$$

- By the **Gelfand transform** & **Corona theorem**:

$$(1) \Longleftrightarrow \inf_{\{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 0\}} \left(\left|\frac{s-1}{s+1}\right| + \left|-\frac{e^{-s}}{s+1}\right| \right) > 0.$$

Moreover, we have:

$$\left(\frac{s-1}{s+1}\right) \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + 2\,e \left(\frac{e^{-s}}{s+1}\right) = 1,$$

$\Rightarrow M$ is a **projective** A-module (i.e. a **free** A-module)

$\Rightarrow p$ is **internally stabilizable** and

$$c = -\frac{2\,e}{1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)} = -\frac{2\,e\,(s-1)}{s + 1 - 2\,e^{-(s-1)}}$$

is **a stabilizing controller of** p.

Example

• Let $A = H_\infty(\mathbb{C}_+)$, $R = \left(e^{-s} : \frac{1}{s+1}\right) \in A^2$ and the A-module $M = A^2/A\,R = M/t(M)$.

$$\Rightarrow \ \mathrm{Fitt}_1(M) = \left(\tfrac{1}{s+1}, e^{-s}\right).$$

• M is **projective** iff $\mathrm{Fitt}_1(M) = A$, i.e. iff $\exists\, a, b \in A$:

$$\frac{1}{(s+1)}\, a + e^{-s}\, b = 1.$$

• **By the Corona Theorem,** we have

$$\inf_{\mathrm{Re}\, s>0} \left(|e^{-s}| + \left|\tfrac{1}{s+1}\right|\right) = 0$$

because if $(x_n)_{n\in\mathbb{N}}$ is a sequence of positive reals such that $\lim_{n\to+\infty} x_n = +\infty$, then:

$$\lim_{n\to+\infty} \left(|e^{-x_n}| + \left|\tfrac{1}{x_n+1}\right|\right) = 0,$$

$\Rightarrow 1 \notin \mathrm{Fitt}_1(M)$

$\Rightarrow M$ is **not a projective** A-module.

A generalization of a result of M. C. Smith

• Corollary: If P admits a **doubly weakly coprime factorization**, then P is **internally stabilizable** iff P **admits a doubly coprime factorization**.

If $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is a doubly coprime factorization of P,

$$\left(\begin{array}{cc} D & -N \\ -\tilde{Y} & \tilde{X} \end{array}\right) \left(\begin{array}{cc} X & \tilde{N} \\ Y & \tilde{D} \end{array}\right) = I_p,$$

then, the controller

$$C = Y\,X^{-1} = \tilde{X}^{-1}\,\tilde{Y}$$

internally stabilizes the plant P.

• Corollary: If A is **coherent Sylvester domain** (e.g. $H_\infty(\mathbb{C}_+),\ k[s,\,z]$ (k field), $\mathbb{Z}[s]$, $\mathcal{E},\ RH_\infty$), then:

$$\boxed{\begin{array}{c}\textbf{internal stabilizability}\\ \Leftrightarrow \\ \textbf{existence of doubly coprime factorization.}\end{array}}$$

(generalization of results of Smith, Vidyasagar . . .).

$\boxed{\textbf{Example}}$

- Let us consider the transfer matrix ($A = H_\infty(\mathbb{C}_+)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2, \quad K = Q(A).$$

- P has a **weakly left-coprime factorization**:

$$P = \begin{pmatrix} \frac{1}{s+1} & -\frac{s-1}{s+1} \\ \frac{s-1}{s+1} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{e^{-s}}{s+1} \end{pmatrix}$$

$\Rightarrow$ P is **internally stabilizable** iff

$$R' = \begin{pmatrix} \frac{1}{s+1} & -\frac{s-1}{s+1} & 0 \\ \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \end{pmatrix} \in A^{2\times 3}$$

has a **right-inverse** S, i.e. $T(M') = A^2/A^3\, R'^T = 0$.

$$\left\{ \begin{array}{l} \frac{1}{(s+1)}\,\lambda_1 + \frac{(s-1)}{(s+1)}\,\lambda_2 = \mu_1, \\ -\frac{(s-1)}{(s+1)}\,\lambda_1 = \mu_2, \\ -\frac{e^{-s}}{(s+1)}\,\lambda_2 = \mu_3, \end{array} \right. \qquad \Leftrightarrow$$

$$\left\{ \begin{array}{l} \lambda_1 = 2\,(-b\left(\frac{s-1}{s+1}\right)^2 + 1)\,\mu_1 - (2\,b\,\frac{(s-1)}{(s+1)^2} + 1)\,\mu_2 + 2\,a\,\frac{(s-1)}{(s+1)^2}\,\mu_3, \\ \lambda_2 = b\,\frac{(s-1)}{(s+1)}\,\mu_1 + \frac{b}{(s+1)}\,\mu_2 - \frac{a}{(s+1)}\,\mu_3, \end{array} \right.$$

$$\text{with} \quad \left\{ \begin{array}{l} a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \\ b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A. \end{array} \right.$$

- A **right-inverse** S of R' is defined by:

$$S = \begin{pmatrix} -2\,b\,\left(\frac{s-1}{s+1}\right)^2 + 2 & b\,\frac{(s-1)}{(s+1)} \\ -2\,b\,\frac{(s-1)}{(s+1)^2} - 1 & \frac{b}{(s+1)} \\ 2\,a\,\frac{(s-1)}{(s+1)^2} & -\frac{a}{(s+1)} \end{pmatrix} \in A^{3\times 2}.$$

- Then, a **stabilizing controller** C of

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix}$$

is defined by:

$$C = \left(2\,a\,\frac{(s-1)}{(s+1)^2} : \; -\frac{a}{(s+1)}\right) \begin{pmatrix} -2\,b\,\left(\frac{s-1}{s+1}\right)^2 + 2 & b\,\frac{(s-1)}{(s+1)} \\ -2\,b\,\frac{(s-1)}{(s+1)^2} - 1 & \frac{b}{(s+1)} \end{pmatrix}^{-1}$$

$$= -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3-4\,(5\,s-3)\,e^{-(s-1)})}\;(1 : \; 2).$$

Internal stabilization: results for SISO plants

• Theorem: We have 1 $\Leftrightarrow$ 2 $\Leftrightarrow$ 3:

1. **The plant** $p = \frac{n}{d}$ is **internally stabilizable**,

2. **The ideal** $I = (n, d)$ **is invertible**, i.e. we have

$$I\,(A : I) \triangleq \{\sum_{i=1}^{n} a_i\, b_i, \,|\, a_i \in I,\, b_i \in (A : I)\} = A,$$

where the **fractional ideal** $A : I$ is defined by:

$$A : I = \{\, c \in K = Q(A) \;\mid\; c\, n \in A,\; c\, d \in A\},$$

3. **There exist** $x, y \in K = Q(A)$ **satisfying**:

$$\left\{ \begin{array}{l} d\, x - n\, y = 1, \\ \\ x\, n,\;\, x\, d,\;\, y\, n,\;\, y\, d \in A. \end{array} \right.$$

Then, $c = y/x$ **internally stabilizes** $p = n/d$ and $A : I = (x, y)$ **is the inverse of** $I = (n, d)$.

• Corollary: The plant $p \in K = Q(A)$ is **internally stabilizable** iff the **fractional ideal** $J = (1, p)$ is **invertible**, i.e. $J\,(A : J) = A$, where

$$A : J = \{a \in A \mid a\, p \in A\},$$

i.e. iff $\exists\, a,\, b \in A$ such that:

$$\left\{ \begin{array}{l} a - b\, p = 1, \\ a\, p \in A. \end{array} \right.$$

Then, the controller $c = b/a$ **internally stabilizes** p.

Example

- Let us consider the transfer function ($A = H_\infty(\mathbb{C}_+)$):

$$p = \frac{e^{-s}}{(s-1)} \in K = Q(A), \quad K = Q(A).$$

- Let us define the **fractional ideal** $J = (1, p)$ of A

$$\Rightarrow A : \; J = \{k \in A \mid k\, p \in A\} = \left(\frac{s-1}{s+1}\right)$$

$$\Rightarrow J\,(A : \; J) = \left(\frac{s-1}{s+1}, \; \frac{e^{-s}}{s+1}\right).$$

- We have the following **Bézout identity**

$$\left(\frac{s-1}{s+1}\right) \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(\frac{e^{-s}}{s+1}\right) 2\, e = 1 \tag{2}$$

$\Rightarrow J\,(A : \; J) = A$, i.e. p is **internally stabilizable**.

- Moreover, we have:

$$(2) \Leftrightarrow \left(\frac{s-1}{s+1}\right) \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(2\, e \left(\frac{s-1}{s+1}\right)\right) p = 1,$$

$$\Rightarrow \begin{cases} a = \left(\frac{s-1}{s+1}\right) \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) \in A : J, \\ b = -2\, e \left(\frac{s-1}{s+1}\right) \in A : \; J. \end{cases}$$

Then, a **stabilizing controller** c of p is given by:

$$c \;= \frac{b}{a} = -\frac{2\, e\,(s-1)}{(s-1)+2\,(1-e^{-(s-1)})}.$$

Tangent bundle of S^1

- Let $\mathbb{R}_1 = \mathbb{R}[t_0, t_1]/(t_0^2 + t_1^2 - 1)$ be **the ring of the polynomials in t_0 and t_1 on the unit circle S^1**.

- Let x_i be the class of t_i in $\mathbb{R}_1$, and the plant:

$$p = \frac{b - x_1}{x_0 - a} \in K = Q(\mathbb{R}_1), \qquad a^2 + b^2 = 1.$$

- $R = (x_0 - a \ : \ x_1 - b)$ is **not weak left-prime**:

$$\begin{pmatrix} \frac{x_0+a}{x_1-b} \end{pmatrix} R = (b - x_1 : x_0 + a) \in \mathbb{R}_1^{1\times 2}, \begin{pmatrix} \frac{x_0+a}{x_1-b} \end{pmatrix} \notin R_1$$

$\Rightarrow$ $\mathbb{R}_1$-module $M = \mathbb{R}_1^2/\mathbb{R}_1 \, R$ is not **torsion-free**

$\Rightarrow$ M is **not a free** $\mathbb{R}_1$-module, i.e. $\nexists \, u, v \in \mathbb{R}_1$:

$$u\,(x_0 - a) + v\,(x_1 - b) = 1$$

$\Rightarrow$ p **has no doubly coprime factorizations**.

- Let $I = (x_0 - a, x_1 - b)$, then $A : \ I = \left(1, \ \frac{x_0+a}{b-x_1}\right)$

$$(x_0 - a)\left(-\frac{1}{2a}\right) + (b - x_1)\left(\frac{x_0 + a}{2a(b - x_1)}\right) = 1$$

$\Rightarrow$ $I.\,(A : \ I) = A$, i.e. I is **invertible**.

$\Rightarrow$ $c = \frac{x_0+a}{b-x_1}$ **stabilizes** $p = \frac{b-x_1}{x_0-a}$ and we have:

$$\begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{a-x_0}{2a} & \frac{x_1-b}{2a} \\ -\frac{x_1+b}{2a} & \frac{a-x_0}{2a} \end{pmatrix} \in \mathbb{R}_1^{2\times 2}.$$

Anantharam counter-example (85)

• Let $A = \mathbb{Z}[i\sqrt{5}]$ and the plant $p = \frac{1+i\sqrt{5}}{2}$.

• $R = (1 + i\sqrt{5} \ : \ -2)$ is **not weak left-prime** $\Rightarrow M = A^2/A\,R$ is **not torsion-free** $\Rightarrow$ **not free** $\Rightarrow \nexists\, a, b \in A$ such that $(1 + i\sqrt{5})\,a + 2\,b = 1$ $\Rightarrow p$ **has no doubly coprime factorizations**.

• The plant p is **internally stabilizable** because the ideal $I = (1 + i\sqrt{5}, 2)$ is **invertible**:

$$A : I = \{c \in \mathbb{Q}(i\sqrt{5}) \mid c\,I \subset A)\} = \left(1, \frac{1-i\sqrt{5}}{2}\right)$$

$$(1+i\sqrt{5})\left(\frac{1-i\sqrt{5}}{2}\right) - 2 \times 1 = 1 \in I\,(A : I) = A.$$

$\Rightarrow c = \frac{1-i\sqrt{5}}{2}$ is a **stabilizing controller** of p and:

$$\begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 + i\sqrt{5} \\ 1 - i\sqrt{5} & 2 \end{pmatrix} \in A^{2 \times 2}.$$

• Or, $p = \frac{3}{1-i\sqrt{5}} = \frac{-(1+i\sqrt{5})}{-2}$ with 3-2=1 $\Rightarrow p$ is internally stabilized by $c = -\frac{1-i\sqrt{5}}{-2} = \frac{1-i\sqrt{5}}{2}$.

• **Any MIMO** defined by a transfer matrix with entries in $K = \mathbb{Q}(i\sqrt{5})$ **is internally stabilizable** because A is a Dedekind (i.e. a Prüfer) domain.

Classes of internal stabilizable plants

• Definition: A domain A is a **Prüfer domain** if it satisfies one of the equivalent assertions:

1. **Every finitely generated torsion-free A-module is projective**,

2. **Every finitely generated ideal is projective**,

3. $\forall\, 0 \neq d, n \in A$, $I = (d, n)$ is **invertible**, i.e.:

$$\exists\, x,\, y \in K = Q(A) : \begin{cases} d\,x - n\,y = 1, \\ \\ x\,n,\ x\,d,\ y\,n,\ y\,d \in A. \end{cases}$$

• Examples: – Bézout or Dedekind domains,

– the affine coordinates ring of a non-singular algebraic surface (e.g. $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$),

– the integral closure of $\mathbb{Z}$ into a finite extension of $\mathbb{Q}$ (e.g. $\mathbb{Z}[i\sqrt{5}]$, $\mathbb{Z}[i\sqrt{23}]$),

– the domain of $\mathbb{Z}$-valued polynomials in $\mathbb{Q}[x]$...

• Theorem: 1. A **is a Prüfer domain** $\Leftrightarrow$

2. **Every MIMO plant** – defined by a transfer matrix with entries in $K = Q(A)$ – **is stabilizable** $\Leftrightarrow$

3. **Every SISO plant** – defined by a transfer function with entries in $K = Q(A)$ – **is stabilizable**.

Youla-Kučera parametrization

• <u>Theorem</u>: If $R \in A^{q\times p}$ is a **full row rank matrix**, then the A-module $M = A^p/A^q\, R$ is **free** iff there exist $R_{-1},\ S,\ S_{-1}$ such that:

1. $\begin{pmatrix} S & R_{-1} \end{pmatrix} \begin{pmatrix} R \\ S_{-1} \end{pmatrix} = I_p,$

2. $\begin{pmatrix} R \\ S_{-1} \end{pmatrix} \begin{pmatrix} S & R_{-1} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix}.$

• <u>Proposition</u>: If $R \in A^{q\times p}$ is a **full row rank matrix** and the A-module $M = A^p/A^q\, R$ is **free**, then:

1. $\begin{pmatrix} S'(Q) & R_{-1} \end{pmatrix} \begin{pmatrix} R \\ S'_{-1}(Q) \end{pmatrix} = I_p,$

2. $\begin{pmatrix} R \\ S'_{-1}(Q) \end{pmatrix} \begin{pmatrix} S'(Q) & R_{-1} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix},$

with $\left\{ \begin{array}{l} S'_{-1}(Q) = S_{-1} + Q\, R, \\ \\ S'(Q) = S - R_{-1}\, Q, \end{array} \right. \quad \forall\, Q \in A^{(p-q)\times q}$.

$\Rightarrow$ We call $Q \in A^{(p-q)\times q} \longrightarrow S'_{-1}(Q)$ **the Youla parametrization** of the complements of R in $\mathrm{GL}_p(A)$.

$\boxed{\textbf{Example}}$

• Let us consider $p = \frac{e^{-s}}{(s-1)}$ and the system:

$$\frac{(s-1)}{(s+1)}\, y - \frac{e^{-s}}{(s+1)}\, u = 0.$$

If $A = H_\infty(\mathbb{C}_+)$ and $R = \left(\frac{s-1}{s+1} \ : \ -\frac{e^{-s}}{s+1}\right)$, then the A-module $M = A^2/A\,R$ is **free**.

$\Rightarrow p$ is **internally stabilizable** and **has doubly coprime factorizations**.

• A few computations give the **Bézout identity**:

$$\left(\begin{array}{cc} \frac{s-1}{s+1} & -\frac{e^{-s}}{s+1} \\ 2\,e + \frac{(s-1)}{(s+1)}\,q & 1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right) - \frac{e^{-s}}{(s+1)}\,q \end{array}\right) \left(\begin{array}{cc} 1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right) - \frac{e^{-s}}{(s+1)}\,q & \frac{e^{-s}}{s+1} \\ -2\,e - \frac{(s-1)}{(s+1)}\,q & \frac{s-1}{s+1} \end{array}\right) = I_2, \ q \in A.$$

All the stabilizing controllers are parametrized by:

$$c = -\frac{2\,e + \frac{(s-1)}{(s+1)}\,q}{1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right) - \frac{e^{-s}}{(s+1)}\,q}, \ q \in A.$$

The degree of freedom q can be used to **optimize** the closed-loop performances (H_2 or H_∞ control).

K-Theory and Class groups

• Corollary If A is a Prüfer domain which is not a Bézout domain, then **there exist stabilizable plants which have no doubly coprime factorizations**.

$\Rightarrow$ **We cannot parametrize their stabilizing controllers by means of the Youla parametrization**.

$-$ A plant $p = n/d$ is stabilizable iff $I = (n, d)$ is a **projective** A-module, i.e. **invertible**.

$-$ A plant $p = n/d$ has doubly coprime factorizations iff $I = (n, d)$ is a **free** A-module, i.e. **principal**.

$\Rightarrow$ $K_0(A)$ and $\tilde{K}_0(A)$ (or the **class group** $C(A)$) study the **difference between projective and free** A**-modules** (or **invertible and principal ideals**).

• Theorem: **If A is a projective-free domain, then a plant is internally stabilizable iff it has doubly coprime factorizations.**

• **Example**: $K_0(H_\infty(\mathbb{C}_+)) = \mathbb{Z}$, $K_0(\mathcal{E}) = \mathbb{Z}$, $K_0(RH_\infty) = \mathbb{Z}$, $K_0(\widehat{\mathcal{A}}) = ?$.

$\boxed{\textbf{What is next?}}$

• **Parametrization of all the stabilizing controllers of a stabilizable plant which does not admit coprime factorizations** ($\Rightarrow$ generalization of the Youla-Kučera parametrizaton) (SCL).

• **Strong stabilization** (existence of a stable controller) and **simultaneous stabilization** (existence of a controller stabilizing a family of plants) (SIAM).

• **Duality between module and operator approaches** (domains, graphs, unbounded operators . . .) (SCL).

• **Nyquist's theorem for ∞-dimensional systems.**

• **Robustness topology**: graph topology and metric, gap metric, ν-gap metric . . .

• H_2 **and** H_∞**-optimal stabilizing controllers**.

• **Development of algorithms and packages** (e.g. for certain classes of delay systems as e.g. $\mathcal{E}$).

Appendix

Homological Algebra

• <u>Definition</u>: A **free** (resp. **projective, flat**) **resolution** of an A-module M is an exact sequence

$$\cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0 \qquad (3)$$

where the F_i are **free** (resp. **projective, flat**).

• <u>Definition</u>: The minimum length $\mathrm{pd}_A(M)$ (resp. $\mathrm{fd}(A)$) of the **projective** (resp. **flat**) **resolutions** of M is called **projective** (resp. **flat**) **dimension**.

• <u>Definition</u>: The **global dimension** of a ring A is:

$$\mathrm{gl.dim}(A) = \min\{\mathrm{pd}_A(M) \mid \forall\ A-\text{module } M\}.$$

The **weak global dimension** of a ring A is:

$$\mathrm{w.gl.dim}(A) = \min\{\mathrm{fd}_A(M) \mid \forall\ A\text{–module } M\}.$$

• <u>Definition</u>: If M is an A-module with (3) for a **projective resolution**, then **the defects of exactness** of

$$\cdots \xleftarrow{d_4^\star} F_3^\star \xleftarrow{d_3^\star} F_2^\star \xleftarrow{d_2^\star} F_1^\star \xleftarrow{d_1^\star} F_0^\star \longleftarrow 0,$$

where $F_i^\star = \mathrm{hom}_A(F_i, A)$, are defined by

1. $\mathrm{ext}_A^0(M, A) = \ker d_0^\star = \mathrm{hom}_A(M, A)$,

2. $\mathrm{ext}_A^i(M, A) = \ker d_{i+1}^\star / \mathrm{im}\, d_i^\star$, $i \geq 1$,

only depend on M **and not on** (3).

Homological Algebra

• Proposition: If A is a coherent domain, then every A-module of the form $M = A^p/A^q\, R$ has a **finite free resolution**, i.e. $F_i \cong A^{r_i}$ with $r_i \in \mathbb{Z}_+$.

• Proposition: If A is a **coherent domain**, then, for every A-module of the form $M = A^p/A^q\, R$:

$$\mathrm{pd}_A(M) = \mathrm{fd}_A(M).$$

• Theorem: If A is a **coherent domain with a weak global dimension** $\mathrm{w.gl.dim}(A) = n$, then, for any A-module of the form $M = A^p/A^q\, R$, we have:

0. $t(M) \cong \mathrm{ext}^1_A(T(M), A)$,

1. M is torsion-free $\Leftrightarrow$ $\mathrm{ext}^1_A(T(M), A) = 0$,

2. M is reflexive $\Leftrightarrow$ $\mathrm{ext}^i_A(T(M), A) = 0, i = 1, 2$,

$\ldots$

n. M is projective $\Leftrightarrow$ $\mathrm{ext}^i_A(T(M), A) = 0, 1 \leq i \leq n$,

where $T(M) = A^q/A^q\, R^T$ is **the transposed** A**-module** of M.

Coherent Sylvester Domains

- Definition: A projective-free coherent domain with $\mathrm{w.gl.dim}(A) \leq 2$ is a **coherent Sylvester domain**.

- Example: $k[\chi_1, \chi_2]$ (k field), every Bézout domain (e.g. $\mathcal{E}$), every principal ideal domain (e.g. RH_∞).

- Theorem (Dicks & Sontag 78): A is a **coherent Sylvester domain** iff for every **row column** $R \in A^p$, the A-module

$$\ker\, R. = \{\lambda \in A^p \mid R\,\lambda = 0\}$$

is **free**.

- Theorem: $H_\infty(\mathbb{C}_+)$ is a **coherent Sylvester domain**.

- Definition: A is a **greatest common divisor domain** if every $a, b \in A$ have a greatest common divisor.

- Theorem (Dicks 83): A coherent Sylvester domain is a greatest common divisor domain.

$\Rightarrow$ (Renteln 77, Smith 89) $H_\infty(\mathbb{C}_+)$ **is a greatest common divisor domain**.

A generalization of the Youla-Kučera parametrization for stabilizable systems

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Introduction

• Theorem: (Morse, Vidyasagar) Every transfer matrix $P \in \mathbb{R}(s)^{q\times r}$ admits a **doubly coprime factorization** over RH_∞, i.e.:

$$P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1},$$

$$\begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I,$$

where D, N, $\tilde{N}$, $\tilde{N}$, X, Y, $\tilde{X}$, $\tilde{Y} \in M(RH_\infty)$.

• Theorem: (Youla, Kučera, Desoer) **All the stabilizing controllers of** $P \in \mathbb{R}(s)^{q\times r}$ **have the form:**

$$\boxed{C(Q) = (\tilde{X} - Q\,N)^{-1}\,(\tilde{Y} - Q\,D) = (Y + \tilde{D}\,Q)\,(X + \tilde{N}\,Q)^{-1}}$$

for every $Q \in RH_\infty^{r\times q}$ such that:

$$\det(\tilde{Y} - Q\,N) \neq 0, \quad \det(X - \tilde{N}\,Q) \neq 0.$$

• Interest: Find the controllers $C \in \mathbb{R}(s)^{r\times q}$ s.t.:

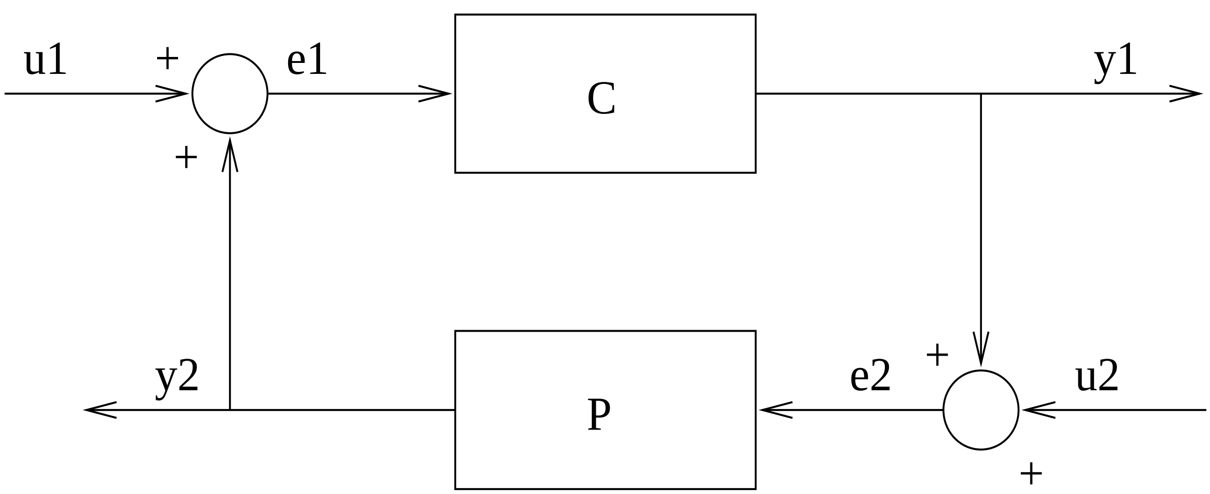
$$\inf_{C \in \mathrm{Stab}(P)} \|\, W_1\,(I - P\,C)^{-1}\,W_2\,\|_\infty,$$

$$\mathrm{Stab}(P) = \{\, C \in \mathbb{R}(s)^{r\times q} \,|\, (I - P\,C)^{-1},\ (I - P\,C)^{-1}\,P,$$
$$C\,(I - P\,C)^{-1},\ C\,(I - P\,C)^{-1}\,P \in M(RH_\infty)\}$$

This **non-linear problem** becomes the **convex** one:

$$\inf_{Q \in RH_\infty^{r\times q}} \|\, W_1\,(X + \tilde{N}\,Q)\,D\,W_2\,\|_\infty\ .$$

Closed-loop



Extension to other classes of systems

“ The foregoing results about **rational functions** are so elegant that one can hardly resist the temptation **to try to generalize them to non-rational functions**.

But to what class of functions?

Much attention has been devoted in the engineering literature to the **identification of a class that is wide enough to encompass all the functions of physical interest and yet enjoys the structural properties that allow analysis of the robust stabilisation problem**”,

N. Young.

(“Some function-theoretic issues in feedback stabilization”, in *Holomorphy Spaces*, MSRI Publications 33, 1998, 337-349.)

Fractional representation approach

- Let A be an **integral domain** and K its quotient field $Q(A) = \{n/d \mid 0 \neq d, \, n \in A\}$.

- <u>Definition</u>: $P \in M(K)$ has a **doubly coprime factorization** over A if there exist

$$\exists \, D, \, N, \, \tilde{D}, \, \tilde{N}, \, X, \, Y, \, \tilde{X}, \, \tilde{Y} \in M(A) \text{ such that:}$$

$$P = D^{-1} \, N = \tilde{N} \, \tilde{D}^{-1},$$

$$\begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I.$$

- <u>Definition</u>: $P \in K^{q \times r}$ is A**-internally stabilizable** if $\exists \, C \in K^{r \times q}$ such that:

$$\begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \in M(A).$$

- $\boxed{\text{existence of a doubly coprime factorization over } A \Rightarrow A\text{-internal stabilizability.}}$

$\Rightarrow P \in M(\mathbb{R}(s))$ **is** RH_∞**-internally stabilizable**.

- <u>Theorem</u>: (Smith) If $A = H_\infty(\mathbb{C}_+)$, then:

$$\boxed{\begin{gathered} H_\infty(\mathbb{C}_+)\text{-internal stabilizability} \\ \Leftrightarrow \\ \text{existence of doubly coprime factorizations} \\ \Rightarrow \exists \text{ \textbf{Youla-Kučera parametrization.}} \end{gathered}}$$

Open questions

• **Does A-internal stabilizability imply the existence of doubly coprime factorizations over**:

$$A = \hat{\mathcal{A}} = \{\mathcal{L}(f)(s) + \textstyle\sum_{i=0}^{+\infty} a_i\, e^{-t_i\, s} \mid f \in L_1(\mathbb{R}_+)$$
$$(a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\ 0 = t_0 \leq t_1 \leq t_2 \leq \ldots\},$$

(ring of BIBO-stable time-invariant systems)

$$A = W_+ = \left\{\textstyle\sum_{i=0}^{\infty} a_i\, z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty\right\},$$

(ring of BIBO-stable causal digital filters)

$$A = M_{\mathbb{D}^n} = \{r/s \mid 0 \neq s,\, r \in \mathbb{R}[x_1, \ldots, x_n],$$
$$s(\underline{x}) = 0 \Rightarrow \underline{x} \notin \mathbb{D}^n\}$$

(ring of nD systems with structural stability) . . . ?

• If it is not the case:

Is it possible to parametrize all stabilizing controllers of a stabilizable plant which does not admit doubly coprime factorizations?

• In this talk, we shall solve the last question.

$$\boxed{\textbf{SISO systems}}$$

Theory of fractional ideals

- Let A and $K = Q(A) = \{n/d \,|\, 0 \neq d,\, n \in A\}$.

- Definition: A **fractional ideal** J **of** A is an A-submodule of K such that $\exists\, 0 \neq d \in A$ satisfying:

$$\boxed{(d)\, J \triangleq \{a\, d \,|\, a \in J\} \subseteq A.}$$

J of A is **integral** if $J \subseteq A$ and **principal** if $\exists\, k \in K$:

$$\boxed{J = (k) \triangleq A\, k = \{a\, k \,|\, a \in A\}.}$$

- Proposition: Let $\mathcal{F}(A)$ be the set of non-zero fractional ideals of A and $I,\, J \in \mathcal{F}(A)$. Then:

$$\begin{cases} I\, J = \{\sum_{\text{finite}} a_i\, b_i \mid a_i \in I,\, b_i \in J\} \in \mathcal{F}(A), \\ I : J = \{k \in K = Q(A) \mid (k)\, J \subseteq I\} \in \mathcal{F}(A). \end{cases}$$

- Example: Let $A = RH_\infty,\; H_\infty(\mathbb{C}_+),\; \widehat{\mathcal{A}},\; W_+ \ldots$ and $p \in K = Q(A)$ be a **transfer function**. Then,

$$\boxed{J = A + A\, p = \{\lambda + \mu\, p \,|\, \lambda,\, \mu \in A\} \in \mathcal{F}(A)}$$

$$(p = n/d,\, d,\, n \in A \Rightarrow (d)\, J = A\, n + A\, d \subseteq A).$$

- Definition: $J \in \mathcal{F}(A)$ is **invertible** if $\exists\, I \in \mathcal{F}(A)$:

$$\boxed{I\, J = A.}$$

$$\Rightarrow I = J^{-1} = A : J = \{k \in K \,|\, (k)\, J \subseteq A\}.$$

$\boxed{\textbf{Stabilization problems}}$

- Theorem: Let $p \in K = Q(A)$ and:

$$J \triangleq (1,\, p) = A + A\,p \in \mathcal{F}(A).$$

1. p has a **weakly coprime factorization** $p = n/d$

$(0 \neq d,\, n \in A, \quad \forall\, k \in K : \; k\,n,\; k\,d \in A \Rightarrow k \in A)$

$$\Leftrightarrow \boxed{A : J \triangleq \{a \in A \,|\, a\,p \in A\} = A\,d.}$$

2. p is **internally stabilizable** $\Leftrightarrow$ $\boxed{J \text{ is invertible}}$, i.e.

$$\boxed{\exists\, a,\, b \in A : \; \begin{cases} a - b\,p = 1, \\ a\,p \in A. \end{cases}}$$

Then, $c = b/a$ internally stabilizes p, $\;J^{-1} = (a,\, b)$.

3. $c \in K$ **internally stabilizes** p

$$\Leftrightarrow \boxed{(1,\, p)\,(1,\, c) = (1 - p\,c).}$$

4. p admits a **coprime factorization** $p = n/d$

$(0 \neq d,\, n \in A, \quad \exists\, x,\, y \in A : \; d\,x - n\,y = 1)$

$$\Leftrightarrow \boxed{J = (1/d)}.$$

5. p is **strongly stabilizable**

$$\Leftrightarrow \boxed{\exists\, c \in A : \; J = (1 - p\,c).}$$

Example

- Let us consider $A = H_\infty(\mathbb{C}_+)$ and:

$$p = \frac{e^{-s}}{(s-1)} = \frac{\frac{e^{-s}}{(s+1)}}{\frac{(s-1)}{(s+1)}} \in K = Q(A).$$

- Let us define the **fractional ideal** $J = (1,\, p)$ of A

$$\Rightarrow A:\ J = \{d \in A \mid d\, p \in A\} = \left(\frac{s-1}{s+1}\right),$$

because A is a **GCDD** and $\gcd\left(\frac{e^{-s}}{s+1},\, \frac{s-1}{s+1}\right) = 1.$

- p is **internally stabilizable** iff $\exists\ a,\, b \in A : J$ s.t.:

$$a - b\, p = 1 \Leftrightarrow \exists\, x,\ y \in A:\ \left\{ \begin{array}{l} a = \frac{(s-1)}{(s+1)}\, x, \\ b = \frac{(s-1)}{(s+1)}\, y, \\ a - b\, p = 1. \end{array} \right.$$

$$\begin{array}{l} b = \frac{(a-1)}{p} = \left(\frac{s-1}{s+1}\right) \left(\frac{(s-1)\, x - (s+1)}{e^{-s}}\right) \\ \Leftrightarrow y = \frac{(s-1)\, x - (s+1)}{e^{-s}} \\ \Leftrightarrow x = \frac{(s+1) + e^{-s}\, y}{s-1} \end{array}$$

$$\Rightarrow ((s+1) + e^{-s}\, y(s))(1) = 0 \Rightarrow y(1) = -2\, e.$$

- Taking $y(s) = y(1) = -2\, e \in A$, then:

$$x = \frac{(s+1) - 2\, e^{-(s-1)}}{s-1} = 1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right) \in A.$$

• Therefore, we have:

$$\begin{cases} a = \left(\frac{s-1}{s+1}\right)\left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) \in A : J, \\ b = -2\,e\left(\frac{s-1}{s+1}\right) \in A :\; J, \\ a - b\,p = 1, \qquad\qquad (\star) \end{cases}$$

$\Rightarrow$ a **stabilizing controller** c of p is defined by:

$$c \;= \frac{b}{a} = -\frac{2\,e\,(s-1)}{(s-1)+2\,(1-e^{-(s-1)})} = -\frac{2\,e\,(s-1)}{s+1-2\,e^{-(s-1)}}.$$

• $J = (1,\, p)$ is **invertible**, $J^{-1} = A : J = \left(\frac{s-1}{s+1}\right)$

$\Rightarrow J = (J^{-1})^{-1} = \left(\frac{s+1}{s-1}\right)$ is **principal**

$\Rightarrow p = \dfrac{\frac{e^{-s}}{(s+1)}}{\frac{(s-1)}{(s+1)}}$ is a **coprime factorization**:

$$(\star) \quad \Leftrightarrow \left(\frac{s-1}{s+1}\right)\left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(2\,e\left(\frac{s-1}{s+1}\right)\right)p = 1,$$

$$\Leftrightarrow \left(\frac{s-1}{s+1}\right)\left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) - \left(\frac{e^{-s}}{s+1}\right)(-2\,e) = 1,$$

$$\Rightarrow \begin{cases} x = 1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right) \in A, \\ y = -2\,e \in A, \\ d\,x - n\,y = 1. \end{cases}$$

Robust stabilization

- $c \in K = Q(A)$ **internally stabilizes** $p \in K$ iff:

$$\boxed{(1,\, p)\,(1,\, c) = (1 - p\,c).}$$

- Let $\delta \in A$. c internally stabilizes p and $p + \delta$

$$\text{iff} \quad \begin{cases} (1,\, p)\,(1,\, c) = (1 - p\,c), \\ \\ (1,\, p + \delta)\,(1,\, c) = (1 - (p + \delta)\,c). \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,\, p)\,(1,\, c) = (1 - p\,c), \\ \\ (1,\, p)\,(1,\, c) = (1 - (p + \delta)\,c). \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,\, p)\,(1,\, c) = (1 - p\,c), \\ \\ \left(\frac{1-(p+\delta)\,c}{1-p\,c}\right) = \left(1 - \frac{\delta\,c}{1-p\,c}\right) = A, \end{cases}$$

$\Leftrightarrow$ c stabilizes p and $(1 - \delta\,c)/(1 - p\,c) \in \mathrm{U}(A)$.

- If A is a **Banach algebra**, then (small gain thm):

$$\boxed{\| \, 1 - a \, \|_A < 1 \Rightarrow a \in \mathrm{U}(A).}$$

$\Rightarrow$ **a sufficient condition of robust stabilization**:

$$\boxed{\| \, \delta \, \|_A < (\| \, c/(1 - p\,c) \, \|_A)^{-1}}$$

Example

• Let A be the ring of **BIBO-stable causal filters**:

$$A = W_+ = \{f(z) = \sum_{i=0}^{+\infty} a_i\, z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty\}.$$

• Let us consider the **transfer function** $p = e^{-\left(\frac{1+z}{1-z}\right)}$:

$$\begin{cases} n = (1-z)^3\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \\ d = (1-z)^3 \in A, \end{cases} \Rightarrow p = n/d \in Q(A).$$

• Let us consider the **fractional ideal** $J = (1,\, p)$ of A.

$$A : J = \{k \in K \mid k,\ k\, p \in A\} = \{d \in A \mid d\, p \in A\}.$$

• The ideal $A : J$ **is not finitely generated**

(R. Mortini & M. Von Renteln, **Ideals in Wiener algebra**, J. Austral. Math. Soc., 46 (1989), 220-228),

i.e. $\nexists$ **finite family** $\{d_1, \ldots, d_r\}$, $d_i \in A$, such that:

$$\forall\, d \in A : J, \quad \exists\, a_i \in A : \quad d = \sum_{i=1}^{r} a_i\, d_i.$$

$\Rightarrow p$ **has not weakly coprime factorizations**

$\Rightarrow p$ **does not admit coprime factorizations**
$\&$ p **is not internally stabilizable**

Parametrizations

• <u>Theorem</u>: Let $p \in K = Q(A)$ be a **stabilizable plant** and $J = (1, p)$. Then, **all the stabilizing controllers** of p have the form

$$\boxed{c(q_1, q_2) = \frac{b + r_1 q_1 + r_2 q_2}{a + r_1 p q_1 + r_2 p q_2}, \quad (\star)}$$

where $c = b/a$ is a stabilizing controller of p, i.e.

$$a - b p = 1, \quad a p \in A,$$

$J^{-2} = (r_1, r_2)$ (e.g. $r_1 = a^2, r_2 = b^2$)
and $\forall q_1, q_2 \in A : a + r_1 p q_1 + r_2 p q_2 \neq 0$.

1. $(\star)$ has **only one free parameter**

$\Leftrightarrow p^2$ **admits a coprime factorization** $p^2 = s/r$

$$\boxed{(\star) \Leftrightarrow c(q) = \frac{b + r q}{a + r p q}, \quad q \in A : a + r p q \neq 0.}$$

2. If p admits a **coprime factorization** $p = n/d$,

$$0 \neq d, n \in A, \quad d x - n y = 1, \quad \text{then:}$$

$$(\star) \Leftrightarrow c(q) = \frac{y + d q}{x + n q}, \quad \forall q \in A : x + n q \neq 0.$$

$$(a = d x, \quad b = d y, \quad r = d^2)$$

Youla-Kučera parametrization

Example

- Let us consider the ring $A = \mathbb{R}[x^2, x^3]$ of **discrete time delay systems without the unit delay**.

- A has been used for high-speed circuits, computer memory devices... (K. Mori).

- Let us consider $p = (1 - x^3)/(1 - x^2) \in Q(A)$.

- Let us consider the **fractional ideal** $J = (1, p)$.

- Using the relation in A

$$(1 - x^3)(1 + x^3) = (1 - x^2)(1 + x^2 + x^4),$$

we have:

$$p = \frac{(1 - x^3)}{(1 - x^2)} = \frac{(1 + x^2 + x^4)}{(1 + x^3)}.$$

$\Rightarrow A : J = (1 - x^2, 1 + x^3)$ **is not principal** because $x + 1 \notin A$.

$$\boxed{\Rightarrow p \text{ **does not admit a weakly coprime factorization.**}}$$

$\Rightarrow p$ **does not admit a coprime factorization**

$\Rightarrow$ **we cannot parametrize all the stabilizing controllers of p by means of the Youla-Kučera parametrization.**

Example

- $J\,(A:J) = (1-x^2,\ 1+x^3,\ 1-x^3,\ 1+x^2+x^4)$

$\Rightarrow (1+x^3)/2 + (1-x^3)/2 = 1 \in J\,(A:J)$

$\Rightarrow p$ is **internally stabilizable** and $J^{-1} = A:J$.

- $$(1+x^3)/2 + (1-x^3)/2 = 1 \in J\,(A:J)$$

$$\Leftrightarrow$$

$$(1+x^3)/2 + ((1-x^2)/2)\,p = 1$$

$$\Rightarrow \begin{cases} a = (1+x^3)/2 \in J^{-1}, \\ b = -(1-x^2)/2 \in J^{-1}, \end{cases}$$

$\boxed{\Rightarrow c = b/a = -(1-x^2)/(1+x^3)}$
internally stabilizes p.

- $J^{-2} = ((1-x^2)^2,\ (1+x^3)^2)$ **is not principal ideal of** A ($x+1 \notin A$).

- **All the stabilizing controllers** of p have the form

$$\boxed{c(q_1,q_2) = \frac{-(1-x^2)+(1-x^2)^2\,q_1+(1+x^3)^2\,q_2}{(1+x^3)+(1-x^2)\,(1-x^3)\,q_1+(1+x^3)\,(1+x^2+x^4)\,q_2}}$$

for all $q_1,\ q_2 \in A$ such that the denominator exists.

Example

- Let us consider $A = \mathbb{Z}[i\sqrt{5}]$, $K = \mathbb{Q}(i\sqrt{5})$ and:

$$p = (1 + i\sqrt{5})/2 \in K$$

"On stabilization and existence of coprime factorizations", V. Anantharam, , IEEE TAC 30 (1985), 1030-1031.

- Let us define the **fractional ideal** $J = (1,\, p)$.

- Using the relation in A

$$2 \times 3 = (1 + i\,\sqrt{5})\,(1 - i\,\sqrt{5}) = 6,$$

$$\Rightarrow p = (1 + i\sqrt{5})/2 = 3/(1 - i\,\sqrt{5}).$$

$\Rightarrow A : J = (2,\, 1 - i\,\sqrt{5})$ is **not a principal ideal**.

$\Rightarrow p$ **does not admit a (weakly) coprime factorization.**

$\Rightarrow \nexists$ **Youla-Kučera parametrization.**

- $J\,(A : J) = (2,\, 1 + i\,\sqrt{5},\, 1 - i\,\sqrt{5},\, 3) = A$:

$$-2 + 3 = -2 - (-1 + i\,\sqrt{5})\,p = 1$$

$\Rightarrow c = (1 - i\,\sqrt{5})/2$ **internally stabilizes** p.

- $J^{-2} = (A : J)^2 = (2,\, 1 - i\,\sqrt{5})^2 = (2)$

$$\Rightarrow c(q) = \frac{1 - i\,\sqrt{5} - 2\,q}{2 - (1 + i\,\sqrt{5})\,q}, \quad \forall\, q \in A.$$

Picard group

• Definition: Let $\mathcal{P}(A)$ be the **group of non-zero principal fractional ideals of A**:

$$\boxed{\mathcal{P}(A) = \{(k) \triangleq A\,k \mid 0 \neq k \in K\}.}$$

Let $\mathcal{I}(A)$ be the **group of non-zero invertible fractional ideals of A**:

$$\boxed{\mathcal{I}(A) = \{J \in \mathcal{F}(A) \mid \exists\, I \in \mathcal{F}(A) : \ I\,J = A\}.}$$

The **Picard group** of A is the defined by:

$$\boxed{\mathcal{C}(A) = \mathcal{I}(A)/\mathcal{P}(A)}$$

• Proposition: If $\mathcal{C}(A) \cong \mathbb{Z}/2\,\mathbb{Z}$, then every stabilizable plant $p \in Q(A)$ has a **parametrization of all its stabilizing controllers with only one free parameter.**

If $\mathcal{C}(A) \cong 1$, then every stabilizable plant $p \in Q(A)$ has a **Youla-Kučera parametrization** (e.g. $H_\infty(\mathbb{C}_+)$, RH_∞, Bézout domains).

Sensitivity minimization

- Let A be a **Banach algebra** ($H_\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}$, W_+,...)

- Let $p \in K = Q(A)$ be a **stabilizable plant**, then

$$\inf_{c \in \mathrm{Stab}(p)} \| \frac{w}{1 - p\,c} \|_A$$
$$=$$
$$\inf_{q_1,\, q_2 \in A} \| \, w\,(a + a^2\,p\,q_1 + b^2\,p\,q_2) \, \|_A \quad (\star)$$

(convex problem)

where $a,\, b \in A$ satisfy $\quad a - b\,p = 1, \quad a\,p \in A,$
and $c_\star = b/a$ is **a stabilizing controller of p.**

- 1. If $p = n/d$ is a **coprime factorization** of p

$$d\,x - n\,y = 1, \quad x,\, y \in A,$$

$$\Rightarrow a = d\,x, \quad b = d\,y,$$

$$\boxed{\begin{array}{c} \Rightarrow a + a^2\,p\,q_1 + b^2\,p\,q_2 = d\,(x + q\,n), \\ q = x^2\,q_1 + y^2\,q_2. \end{array}}$$

2. $\forall \in A,\ \exists\, q_1,\, q_2 \in A : q = x^2\,q_1 + y^2\,q_2,$

with $q_1 = d^2\,(1 - 2\,n\,y)\,q,\ q_2 = n^2\,(1 + 2\,d\,x)\,q,$

$$\left[(d^2\,(1 - 2\,n\,y))\,x^2 + (n^2\,(1 + 2\,d\,x))\,y^2 = 1 \right].$$

$$\boxed{(\star) \Leftrightarrow \inf_{q \in A} \| \, w\,d\,(x + n\,q) \, \|_A \,.}$$

MIMO systems

Lattices

- Let V be a **finite-dimensional** K**-vector space.**

- Definition: An A-submodule M of V is a **lattice of** V if $\exists\, L_1,\, L_2$ two **free** A**-submodules of** V s.t.:

$$\begin{cases} L_1 \subseteq M \subseteq L_2, \\ \mathrm{rk}_A(L_1) = \dim_K(V). \end{cases}$$

- Example: The **lattices of** $V = K$ are just the non-zero **fractional ideals of** A**.**

- Proposition: An A-submodule M of V is **a lattice of** V iff

$$\begin{cases} K\, M \triangleq \{k\, m \mid k \in K,\ m \in M\} = V, \\ M \subseteq P, \end{cases}$$

where P is a finitely generated A-submodule of V.

- Example: Let $P \in K^{q \times r}$, then the A-module

$$(I_q \, : \, -P)\, A^{q+r}$$

is a **lattice of** the K-vector space K^q.

- Example: Let $P \in K^{q \times r}$, then the A-module

$$A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$$

is a **lattice of** the K-vector space $K^{1 \times r}$.

Lattices

- Let V and W be finite-dimensional K-vector spaces.

- Let M (resp. N) be a lattice of V (resp. W).

- Definition: $N : M$ **is the** A**-submodule of**

$$\hom_K(V, W) = \{f : V \to W \mid f \text{ is a } K\text{−linear map}\}$$

formed by the K**-linear maps** $f : V \to W$ **which satisfy** $f(M) \subseteq N$.

- Proposition: 1. $N : M$ is **a lattice of** $\hom_K(V, W)$.

2. The **map**

$$\begin{array}{rcl} N : M & \to & \hom_A(M, N) = \{f : M \to N \mid \\ & & \qquad\qquad f \text{ is a } A - \text{linear map}\}, \\ f & \mapsto & f_{|M}, \end{array}$$

is bijective.

- Example: Let $P \in K^{q \times r}$ and $M = (I_q : -P)\, A^{q+r}$. Then, we have:

$$\begin{aligned} A : M &= \{f : K^q \to K \mid f(M) \subseteq A\} \\ &= \{\lambda \in K^{1 \times q} \mid \lambda\, (I_q : -P)\, A^{q+r} \subseteq A\} \\ &= \{\lambda \in K^{1 \times q} \mid \lambda \in A^{1 \times q},\ \lambda\, P \in A^{1 \times r}\} \\ &= \{\lambda \in A^{1 \times q} \mid \lambda\, P \in A^{1 \times r}\}. \end{aligned}$$

Weakly coprime factorizations

• Definition: $P \in K^{q\times r}$ admits a **weakly left-coprime factorization** if $\exists\ R = (D\ :\ -N) \in A^{q\times(q+r)}$ s.t.:

$$P = D^{-1}\, N,$$
$$\forall\ \lambda \in K^{1\times q}:\ \lambda\, R \in A^{1\times(q+r)} \Rightarrow \lambda \in A^{1\times q}.$$

• Definition: $P \in K^{q\times r}$ admits a **weakly right-coprime factorization** if $\exists\ \tilde{R} = (\tilde{N}^T\ :\ \tilde{D}^T)^T \in A^{(q+r)\times r}$ such that:

$$P = \tilde{N}\, \tilde{D}^{-1},$$
$$\forall\ \lambda \in K^{r}:\ \tilde{R}\, \lambda \in A^{p} \Rightarrow \lambda \in A^{r}.$$

• Proposition: $P \in K^{q\times r}$ **admits a weakly left-coprime factorization** iff $\exists\ D \in A^{q\times q}$ such that

$$\begin{aligned} A : ((I_q\ :\ -P)\, A^{q+r}) &= \{\lambda \in A^{1\times q} \mid \lambda\, P \in A^{1\times r}\} \\ &= A^{1\times q}\, D, \end{aligned}$$

i.e. is a **free lattice of $K^{1\times q}$.**

• Proposition: $P \in K^{q\times r}$ **admits a weakly right-coprime factorization** iff $\exists\ \tilde{D} \in A^{r\times r}$ such that

$$\begin{aligned} A : \left(A^{1\times(q+r)} \begin{pmatrix} P \\ I_{p-q} \end{pmatrix}\right) &= \{\lambda \in A^{r} \mid P\, \lambda \in A^{q}\} \\ &= \tilde{D}\, A^{r}, \end{aligned}$$

i.e. is **free lattice of K^{r}.**

Coprime factorizations

- Let A be an **integral domain** and $K = Q(A)$.

- Proposition: $P \in K^{q \times r}$ admits the **left-coprime factorization**

$$P = D^{-1}\,N, \quad D\,X - N\,Y = I_q,$$

iff $\exists\; D \in A^{q \times q}$ such that

$$\boxed{\begin{aligned}(I_q\,:\,-P)\,A^{q+r} &\triangleq \{\lambda_1 - P\,\lambda_2 \mid \lambda_1 \in A^q,\, \lambda_2 \in A^r\}\\ &= D^{-1}\,A^q,\end{aligned}}$$

i.e. iff $(I_q\,:\,-P)\,A^p$ is a **free lattice of K^q.**

- Proposition: If $P \in K^{q \times r}$ admits a **right-coprime factorization**

$$P = \tilde{N}\,\tilde{D}^{-1}, \quad -\tilde{Y}\,X + \tilde{X}\,\tilde{D} = I_r,$$

iff $\exists\; \tilde{D} \in A^{r \times r}$ such that

$$\boxed{\begin{aligned}A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} &\triangleq \{\lambda_1\,P + \lambda_2 \mid \\ &\qquad\qquad (\lambda_1\,:\,\lambda_2) \in A^{1 \times (q+r)}\}\\ &= A^{1 \times (q+r)}\,\tilde{D}^{-1},\end{aligned}}$$

i.e. iff $A^{1 \times (q+r)}\,(P^T\;:\;I_r)^T$ is a **free lattice of $K^{1 \times r}$.**

$\boxed{\textbf{Stabilizability}}$

• <u>Theorem</u>: $P \in K^{q\times r}$ is **internally stabilizable** iff one of the following conditions is satisfied:

1. $(I_q \,:\, -P)\, A^{q+r}$ is a **projective lattice of** K^q, namely $\exists$ A-module M such that:

$$\boxed{(I_q \,:\, -P)\, A^{q+r} \oplus M \cong A^{q+r}.}$$

2. $A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$ is a **projective lattice of** $K^{1\times r}$, namely $\exists$ A-module N such that:

$$\boxed{A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} \oplus N \cong A^{1\times(q+r)}.}$$

• Let $R = (I_q \,:\, -P), \quad Q = \begin{pmatrix} P \\ I_r \end{pmatrix}, \quad p = q + r,$ then we have the following **split exact sequences**:

$$0 \longleftarrow (I_q \,:\, -P)\, A^p \underset{\xrightarrow{S.}}{\xleftarrow{R.}} A^p \underset{\xrightarrow{T.}}{\xleftarrow{Q.}} A : \left(A^{1\times p} \begin{pmatrix} P \\ I_r \end{pmatrix}\right) \longleftarrow 0,$$

$$0 \longrightarrow A : ((I_q \,:\, -P)\, A^p) \underset{\xleftarrow{.S}}{\xrightarrow{.R}} A^{1\times p} \underset{\xleftarrow{.T}}{\xrightarrow{.Q}} A^{1\times p} \begin{pmatrix} P \\ I_r \end{pmatrix} \longrightarrow 0.$$

$\Rightarrow \Pi_1 = S\,R,\ \Pi_2 = Q\,T$ are **projectors** of $A^{p\times p}$.

Stabilizability

• Theorem: $P \in K^{q \times r}$ is **internally stabilizable** iff one of the following conditions is satisfied:

$\mathbf{C_1}$. $\exists\, S = (U^T : \, V^T)^T \in A^{(q+r) \times q}$ such that:

$$\boxed{\begin{array}{ll} S\,P & = \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{(q+r)\times r}, \\ (I_q : \, -P)\,S & = U - P\,V = I_q. \end{array}}$$

Then, $C = V\,U^{-1}$ is **a stabilizing controller of P.**

$\mathbf{C_2}$. $\exists\, T = (-X : \, Y) \in A^{r \times (q+r)}$ such that:

$$\boxed{\begin{array}{ll} P\,T & = (P\,X : \, P\,Y) \in A^{q \times (q+r)}, \\ T \begin{pmatrix} P \\ I_r \end{pmatrix} & = -X\,P + Y = I_r. \end{array}}$$

Then, $C' = Y^{-1}\,X$ is **a stabilizing controller of P.**

• Proposition: If P is **internally stabilizable**, then $\exists\, S \in A^{(q+r)\times q}$, $T \in A^{r \times (q+r)}$ satisfying $\mathbf{C_1}$, $\mathbf{C_2}$,

$$T\,S = -X\,U + Y\,V = 0,$$

i.e. $\exists$ **a stabilizing controller of P of the form:**

$$\boxed{C = V\,U^{-1} = Y^{-1}\,X.}$$

Example

- Let us consider the transfer matrix ($A = H_\infty(\mathbb{C}_+)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2, \quad K = Q(A).$$

- The matrix $S = (U^T \,:\, V^T)^T \in A^{3\times 2}$ defined by

$$S = \begin{pmatrix} \frac{2}{s+1} + b\left(\frac{s-1}{s+1}\right)^3 & 2\,b\left(\frac{s-1}{s+1}\right)^3 - 2\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\,b\,\frac{(s-1)}{(s+1)^3} \\ -a\,\frac{(s-1)^2}{(s+1)^3} & -2\,a\,\frac{(s-1)^2}{(s+1)^3} \end{pmatrix}$$

with $\begin{cases} a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \\ b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A, \end{cases}$

satisfies

$$\begin{cases} S P \in A^{3\times 1}, \\ (I_2 \,:\, -P)\,S = U - P\,V = I_2, \end{cases}$$

$\Rightarrow$ P is **internally stabilized** by the controller:

$$\begin{aligned} C &= V\,U^{-1} \\ &= -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)})}\,(1 \,:\, 2). \end{aligned}$$

Stabilizability

• Corollary: $P \in K^{q \times r}$ is **internally stabilized by the controller** $C \in K^{r \times q}$ iff one of the following conditions is satisfied:

1. The matrix

$$\Pi_1 = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}$$

is a **projector of** $A^{(q+r)\times(q+r)}$, i.e.:

$$\boxed{\Pi_1^2 = \Pi_1 \in A^{(q+r)\times(q+r)}.}$$

2. The matrix

$$\Pi_2 = \begin{pmatrix} -P\,(I_{p-q} - C\,P)^{-1}\,C & P\,(I_{p-q} - C\,P)^{-1} \\ -(I_{p-q} - C\,P)^{-1}\,C & (I_{p-q} - C\,P)^{-1} \end{pmatrix}$$

is a **projector of** $A^{(q+r)\times(q+r)}$, i.e.:

$$\boxed{\Pi_2^2 = \Pi_2 \in A^{(q+r)\times(q+r)}.}$$

Then, we have $\boxed{\Pi_1 + \Pi_2 = I_{q+r}.}$

• Remark: This result was known for $A = H_\infty(\mathbb{C}_+)$. The **robustness radius** is defined by (loop-shaping):

$$\boxed{b_{P,C} \triangleq \| \Pi_1 \|_\infty^{-1} = \| \Pi_2 \|_\infty^{-1}.}$$

Stabilizability

- Fact 1: P admits a **doubly coprime factorization**

$$\Leftrightarrow (I_q \,:\, -P)\, A^{q+r} \quad \& \quad A^{1\times(q+r)} \left(\begin{array}{c} P \\ I_r \end{array}\right)$$

are **free A-modules.**

- Fact 2: P is **internally stabilizable**

$$\Leftrightarrow (I_q \,:\, -P)\, A^{q+r} \quad \& \quad A^{1\times(q+r)} \left(\begin{array}{c} P \\ I_r \end{array}\right)$$

are **projective A-modules.**

- Fact 3: **A free A-module is projective.**

- Corollary:

If $P \in K^{q\times r}$ admits a **left-coprime factorization**

$$P = D^{-1}\, N, \quad D\, X - N\, Y = I_q,$$

then $S = ((X\, D)^T \,:\, (Y\, D)^T)^T$ satisfies $\mathbf{C_1}$

$$\boxed{\Rightarrow C = (Y\, D)\, (X\, D)^{-1} = Y\, X^{-1} \in \mathsf{Stab}(P).}$$

If $P \in K^{q\times r}$ admits a **right-coprime factorization**

$$P = \tilde{N}\, \tilde{D}^{-1}, \quad -\tilde{Y}\, X + \tilde{X}\, \tilde{D} = I_r,$$

then $T = (-\tilde{D}\, \tilde{Y} \,:\, \tilde{D}\, \tilde{X})$ satisfies $\mathbf{C_2}$

$$\boxed{\Rightarrow C = (\tilde{D}\, \tilde{X})^{-1}\, (\tilde{D}\, \tilde{Y}) = \tilde{X}^{-1}\, \tilde{Y} \in \mathsf{Stab}(P).}$$

Parametrization

- Theorem: Let $P \in K^{q \times r}$ be a **stabilizable plant**. **All the stabilizing controllers of P have the form**

$$\boxed{\begin{array}{ll} C(Q) & = (V + Q)\,(U + P\,Q)^{-1} \\ & = (Y + Q\,P)^{-1}\,(X + Q), \end{array}}$$

where $C = V\,U^{-1} = Y^{-1}\,X$ is **a particular stabilizing controller of P**, i.e.

$$\left\{\begin{array}{l} U - P\,V = I_q, \\ Y - X\,P = I_r, \\ \left(\begin{array}{c} U\,P \\ V\,P \end{array}\right) \in A^{(q+r)\times r}, \\ (-P\,X \,:\, P\,Y) \in A^{q\times(q+r)}, \end{array}\right.$$

and Q is **every matrix which belongs to**

$$\boxed{\begin{array}{ll} \Omega = \{L \in A^{r\times q} \mid & L\,P \in A^{r\times r},\, P\,L \in A^{q\times q}, \\ & P\,L\,P \in A^{q\times r}\} \end{array}}$$

such that $\det(U + P\,Q) \neq 0$ and $\det(Y + Q\,P) \neq 0$.

(Ω is a **projective A-module of rank $q \times r$**).

$\boxed{\textbf{Study of the } A\textbf{-module } \Omega}$

• Open question: **Find a family of generators of the projective A-module of rank $q \times r$**

$$\Omega = \{L \in A^{r \times q} \mid \ \ L\,P \in A^{r \times r},\ P\,L \in A^{q \times q},\\ P\,L\,P \in A^{q \times r}\},$$

i.e. a finite family $\{L_i\}_{1 \leq i \leq s}$ such that:

$$\boxed{\forall\, L \in \Omega,\ \exists\, L = \sum_{i=1}^{s} \lambda_i\, L_i, \quad \lambda_i \in A.}$$

• Proposition: If $P \in Q(A)^{q \times r}$ admits a **left-coprime factorization** $P = D^{-1}\,N$,

$$D\,X - N\,Y = I_q,$$

where $D,\ N,\ X,\ Y \in M(A)$, then:

$$\boxed{\Omega = \{L \in A^{r \times q} \mid \ \ P\,L \in A^{q \times q}\}\,D.}$$

• Proposition: If $P \in Q(A)^{q \times r}$ admits a **right-coprime factorization** $P = \tilde{N}\,\tilde{D}^{-1}$,

$$-\tilde{Y}\,X + \tilde{X}\,\tilde{D} = I_r,$$

where $\tilde{D},\ \tilde{N},\ \tilde{X},\ \tilde{Y} \in M(A)$, then:

$$\boxed{\Omega = \tilde{D}\,\{L \in A^{r \times q} \mid \ \ L\,P \in A^{r \times r}\}.}$$

Youla-Kučera parametrization

• Corollary: Let $P \in Q(A)^{q \times r}$ be a plant which admits a **doubly coprime factorization**:

$$\begin{cases} P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}, \\ \left(\begin{array}{cc} D & -N \\ -\tilde{Y} & \tilde{X} \end{array} \right) \left(\begin{array}{cc} X & \tilde{N} \\ Y & \tilde{D} \end{array} \right) = I_{q+r}. \end{cases}$$

Then, the A-module

$$\begin{array}{ll} \Omega = \{L \in A^{r \times q} \mid & L\, P \in A^{r \times r},\; P\, L \in A^{q \times q}, \\ & P\, L\, P \in A^{q \times r}\} \end{array}$$

is the **free A-module** defined by:

$$\boxed{\begin{array}{ll} \Omega & = \tilde{D}\, A^{r \times q}\, D \\ & = \{L \in A^{r \times q} \,|\, L = \tilde{D}\, R\, D,\; \forall\, R \in A^{r \times q}\}. \end{array}}$$

$\Rightarrow$ **All stabilizing controllers of P have the form**

$$\boxed{C(Q) = (Y + \tilde{D}\, Q)\, (X + \tilde{N}\, Q)^{-1} = (\tilde{X} + Q\, N)^{-1}\, (\tilde{Y} + Q\, D),}$$

where $Q \in A^{r \times q}$ is **every matrix such that:**

$$\det(X + \tilde{N}\, Q) \neq 0, \quad \det(\tilde{X} + Q\, N) \neq 0.$$

Sensitivity minimization

- Let A be a **Banach algebra** ($H_\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}$, W_+,...)

- Let $P \in Q(A)^{q\times r}$ be a **stabilizable plant**, then

$$\inf_{C\in \mathrm{Stab}(P)} \| \, W_1 \, (I_q - P\,C)^{-1} \, W_2 \, \|_A$$
$$=$$
$$\inf_{Q\in\Omega} \| \, W_1 \, (U + P\,Q) \, W_2 \, \|_A \quad (\star)$$

(**convex problem**)

where $(U^T : \, V^T)^T \in A^{(q+r)\times q}$ satisfy

$$U - P\,V = I_q, \quad \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{p\times r},$$

and $C_\star = V\,U^{-1}$ is **a stabilizing controller of** P**.**

- If P admits a **doubly coprime factorization**

$$\begin{cases} P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \\ \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I_{q+r}. \end{cases}$$

$$\Rightarrow \begin{cases} Q \in \Omega & = \tilde{D}\,A^{r\times q}\,D, \\ U + P\,Q & = X\,D + \tilde{N}\,\tilde{D}^{-1}\,(\tilde{D}\,R\,D), \\ & = (X + \tilde{N}\,R)\,D, \end{cases}$$

$$\boxed{(\star) \Leftrightarrow \inf_{R\in A^{r\times q}} \| \, W_1 \, (X + \tilde{N}\,R) \, D \, W_2 \, \|_A \, .}$$

Conclusion

I. Summary:

• We **generalized the Youla-Kučera parametrization** for **MIMO stabilizable plants**.

• This parametrization **does not assume the existence of doubly coprime factorizations.**

II. General comments:

When does a stabilizable plant admit a doubly coprime factorization?

• We proved that this problem is equivalent to:

When is a projective A-module free?

• This is a **difficult problem** studied for years in:

– **algebra**: algebraic K-theory (Serre's conjecture (55) $A = k[x_1, \ldots, x_n]$, solved by Quillen-Suslin (76)),

– **number theory**: number fields,

– **algebraic analysis**: function fields,

– **topology**: triviality of vector bundles,

– **operator theory**: topological K-theory ($C^\star$-algebra).

this problem could be difficult for $\widehat{\mathcal{A}}$, W_+, $M_{\mathbb{D}^n}$...

Conclusion

- **Complete results for SISO systems:**

"On a generalization of the Youla-Kučera parametrization. Part I: The fractional ideal approach to SISO systems",
A.Q.
Systems & Control Letters, vol. 50 (2003), n. 2, 135-148.

- **Complete results for MIMO systems:**

"On a generalization of the Youla-Kučera parametrization. Part II: The lattice approach to MIMO systems",
A.Q.
Proceedings of TDS03, IFAC Workshop, 08-10/09/03, INRIA Rocquencourt (France),
submitted to Mathematics of Control, Signal and Systems.

- **The dual approach** to these results generalizes the **operator-theoretic approach to stabilizability** (unbounded operators, domains, graphs. . .)

"An algebraic interpretation to the operator-theoretic approach to stabilizability. Part I: SISO systems",
A. Q.
submitted to Acta Applicandæ Mathematicæ.

Study of the strong & simultaneous stabilization problems based on stable range

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$\boxed{\textbf{Stable range of a commutative ring}}$

• **Definition**: $a = (a_1 : \ldots : a_n) \in A^n$ is **unimodular** if there exists $b = (b_1 : \ \ldots : \ b_n)^T \in A^n$:

$$< a, b >= \sum_{i=1}^{n} a_i \, b_i = 1.$$

We denote $\mathsf{U}_n(A) = \{\text{unimodular vectors of } A^n\}$.

• **Definition**: $(a_1 : \ldots : a_n) \in \mathsf{U}_n(A)$ is **stable** if there exists $(b_1 : \ \ldots : b_{n-1}) \in A^{(n-1)}$ such that:

$$(a_1 + b_1 \, a_n : \ \ldots : a_{n-1} + b_{n-1} \, a_n) \in \mathsf{U}_{n-1}(A).$$

• **Definition**: We call **stable range** of A the smallest number $\mathsf{sr}(A) \in \{1, 2, \ldots, +\infty\}$ such that **every vector** $a \in \mathsf{U}_{\mathsf{sr}(A)+1}(A)$ is **stable**.

• $\mathsf{sr}(A) = 2 \Leftrightarrow \forall \, (a_1 : \ a_2 : \ a_3) \in \mathsf{U}_3(A), \exists \, b_1, b_2 \in A :$

$$(a_1 + b_1 \, a_3 : \ a_2 + b_2 \, a_3) \in \mathsf{U}_2(A).$$

$$\exists \, (a_1 : \ a_2) \in \mathsf{U}_2(A) : a_1 + b \, a_2 \notin \mathsf{U}_1(A), \forall \, b \in A.$$

• $\mathsf{sr}(A) = 1 \Leftrightarrow \forall \, (a_1 : \ a_2) \in \mathsf{U}_2(A), \ \exists \, b \in A :$

$$a_1 + b \, a_2 \in \mathsf{U}_1(A) \Leftrightarrow (a_1 + b \, a_2)^{-1} \in A.$$

Examples

• **Example**: Let us consider $A = RH_\infty$ and the vector

$$a = \left(\frac{(s-1)^2}{(s+1)^2} : \; \frac{s}{(s+1)^2} \right) \in A^{2\times 1}.$$

The vector a is **unimodular** because we have:

$$\left(\frac{(s-1)^2}{(s+1)^2} : \; \frac{s}{(s+1)^2} \right) \left(\begin{array}{c} \frac{s^2+3\,s+1}{(s+1)^2} \\ \frac{3\,s^2+1\,s+3}{(s+1)^2} \end{array} \right) = 1$$

Moreover, a is a **stable vector** because we have:

$$\frac{(s-1)^2}{(s+1)^2} + 4\,\frac{s}{(s+1)^2} = 1 \in \mathsf{U}_1(A).$$

• **Example**: Let us consider $A = H_\infty(\mathbb{C}_+)$ and

$$a = \left(1 - e^{-2\,s} : \; 1 + e^{-2\,s}\right) \in A^{2\times 1}.$$

The vector a is **unimodular** because we have:

$$\left(1 - e^{-2\,s} : \; 1 + e^{-2\,s}\right) \left(\begin{array}{c} \frac{3+2\,e^{-2\,s}}{2} \\ \frac{-1+2\,e^{-2\,s}}{2} \end{array} \right) = 1$$

Moreover, a is a **stable vector** because we have:

$$(1 - e^{-2\,s}) + (1 + e^{-2\,s}) = 2 \in \mathsf{U}_1(A).$$

Examples of stable ranges

- **Theorem**: (Bass 64, Vasershtein 71, Jensen 85).

  - If A is a Bézout domain, then sr$(A) \leq\ 2$.

  - sr$(\mathbb{R}[\chi_1, \ldots, \chi_n]) = n + 1$,

  - sr$(E(k)) = \left\{ \begin{array}{l} 1,\ k = \mathbb{C}, \\ 2,\ k = \mathbb{R}. \end{array} \right.$

  - sr$(C^\infty(\mathbb{R}^n)) = n + 1$.

  - sr$(W_+) = 1$, where W_+ is the Wiener algebra:

    $W_+ = \{\sum_{n=0}^{+\infty} a_n\, z^n \mid \sum_{n=0}^{+\infty} \mid a_n \mid < +\infty\}$.

- **Proposition**: (Youla, Vidyasagar) sr$(RH_\infty) = 2$.

- **Theorem**: (Treil 92) sr$(H_\infty(\mathbb{C}_+)) = 1$.

k-stability of a matrix

• **Definition**: A matrix $R \in A^{q \times p}$ is **unimodular** if there exists a matrix $S \in A^{p \times q}$ such that:

$$R\,S = I_q.$$

($\Rightarrow\ 0 < q \leq p$ and the rows of R are A-linearly independent).

• **Definition**: A unimodular matrix $R \in A^{q \times p}$ is $k-$**stable** if there exists $T_k \in A^{k \times (p-k)}$ such that

$$R_k \triangleq \mathrm{col}(R_1 : \ldots : R_{p-k}) + \mathrm{col}(R_{p-k+1} : \ldots : R_p)\,T_k$$

is a **unimodular matrix.**

• **Example**: Let us consider $A = RH_\infty$ and the matrix:

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \\ \frac{1}{s+1} & -\frac{s}{s+1} & 0 \end{pmatrix} \in A^{2 \times 3}.$$

The following matrix

$$R_1 \triangleq \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ \frac{1}{s+1} & -\frac{s}{s+1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{s+1} \\ 0 \end{pmatrix} \underbrace{\begin{pmatrix} -3 & -1 \end{pmatrix}}_{T_1}$$

$$= \begin{pmatrix} \frac{s+2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & -\frac{s}{s+1} \end{pmatrix}$$

is **invertible** ($\det R_1 = -1$) $\Rightarrow R_1$ is **unimodular** $\Rightarrow R$ is **1-stable**.

Strong & Simultaneous stabilizations

- **Definition**:

  - $P \in K^{q\times(p-q)}$ is **strongly stabilizable** if there exists a **stable controller** $C \in A^{(p-q)\times q}$ which internally stabilizes P.

  - $P \in K^{q\times(p-q)}$ is **bistably stabilizable** if there exists a controller $C \in \mathrm{U}_1(A)^{(p-q)\times q}$ which internally stabilizes P.

  - $P_1,\, P_2 \in K^{q\times(p-q)}$ are **simultaneously stabilizable** if there exists a controller $C \in K^{(p-q)\times q}$ which internally stabilizes P_1 and P_2.

- **Theorem**: A plant $P \in K^{q\times(p-q)}$ is **strongly stabilizable** iff P admits a **doubly coprime factorization** $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ such that the **matrices**

$$\begin{cases} (D :\ -N) \in A^{q\times p}, \\ (\tilde{D}^T :\ \tilde{N}^T) \in A^{(p-q)\times p} \end{cases}$$

are respectively $(p-q)$ and q-**stable**.

General structure of the stabilizing controllers

• **Theorem**: Let P be a transfer matrix which admits a **left-coprime factorization** $P = D^{-1}\,N$ such that $R = (D \,:\, -N) \in A^{q\times(p-q)}$ is k**-stable** where $r = p - q - k \geq 0$. Then, there exist

$$\begin{cases} T_1 \in A^{k\times q}, \\ T_2 \in A^{k\times r}, \end{cases}$$

such that the **controller** C, defined by

$$C = \begin{pmatrix} U\,V^{-1} \\ T_1 + T_2\,(U\,V^{-1}) \end{pmatrix} \quad \begin{array}{l} \updownarrow r = p - q - k \\ \updownarrow k \end{array} \quad ,$$

internally stabilizes $P = D^{-1}\,N$.

Moreover, $C_r = V\,U^{-1} \in K^{r\times q}$ is a controller which **internally stabilizes** the plant

$$P_r = (D - \Lambda\,T_1)^{-1}\,(N_r + \Lambda\,T_2) \in K^{q\times r},$$

where:

$$\begin{cases} N_r \triangleq -\mathrm{col}(R_{q+1} : \ldots : R_{p-k}) \in A^{q\times r}, \\ \Lambda \triangleq= -\mathrm{col}(R_{p-k+1} : \ldots : R_p) \in A^{q\times k}, \\ N = (N_r :\, \Lambda) \in A^{q\times(p-q)}, \\ R = (D :\, -N) = -\mathrm{col}(R_1 : \ldots : R_p) \in A^{q\times p}. \end{cases}$$

$\boxed{\textbf{Example}}$

- Let us consider $A = RH_\infty$ and the transfer matrix:

$$P = \begin{pmatrix} \frac{1}{s-1} \\ \frac{1}{s\,(s-1)} \end{pmatrix} \in \mathbb{R}(s)^{2\times 1}.$$

- P admits the **left coprime factorization**:

$$P = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ \frac{1}{s+1} & -\frac{s}{s+1} \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{s+1} \\ 0 \end{pmatrix}.$$

- The matrix $R = (D \,:\, -N)$ defined by

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \\ \frac{1}{s+1} & -\frac{s}{s+1} & 0 \end{pmatrix} \in A^{2\times 3}$$

is **1-stable** because

$$R_1 \triangleq \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ \frac{1}{s+1} & -\frac{s}{s+1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{s+1} \\ 0 \end{pmatrix} \underbrace{\begin{pmatrix} -3 & -1 \end{pmatrix}}_{T_1}$$

$$= \begin{pmatrix} \frac{s+2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & -\frac{s}{s+1} \end{pmatrix}$$

is an **invertible** matrix, i.e. unimodular
$\Rightarrow r = 3-2-1 = 0 \Rightarrow P$ is **strongly stabilizable**.

We have the following **stable stabilizing controller**:

$$C = -(3 \,:\, 1) \in A^{1\times 2}.$$

Definition of $\mathrm{sr}_k(q, p, A)$

• **Definition**: Let $p,\, q \in \mathbb{Z}_+$ such that $1 \leq q \leq p$. We say that A **satisfies** $\mathrm{sr}_k(q, p, A)$ if **every matrix of the form** $R \in A^{q \times p}$ is k**-stable**.

• **Theorem**: (Vasershtein 71, Hong 95) We have:

- $\mathrm{sr}_1(1, n, A) \Leftrightarrow \mathrm{sr}_1(1, m, A),\ \forall\, m \geq n,$
- $\mathrm{sr}_1(1, n, A) \Leftrightarrow \mathrm{sr}_k(1, n+k-1, A),\ \forall\, k \geq 1,$
- $\mathrm{sr}_k(1, n, A) \Leftrightarrow \mathrm{sr}_k(m, n+m-1, A),\ \forall\, m \geq 1.$

• Corollary: Let A be a ring such that $\mathrm{sr}(A) < +\infty$. $\forall\, p,\, q \in \mathbb{Z}_+$ satisfying $p - q \geq \mathrm{sr}(A)$, A **satisfies**:

$$\mathrm{sr}_{p-q-\mathrm{sr}(A)+1}(q, p, A).$$

In particular, **for every unimodular matrix** of the form $R = \mathrm{col}(R_1 : \ldots : R_p) \in A^{q \times p}$, **there exists**

$$T_{\mathrm{sr}(A)} \in A^{(p-q-\mathrm{sr}(A)+1) \times (q+\mathrm{sr}(A)-1)}$$

such that

$$\begin{aligned} R_{\mathrm{sr}(A)} \ \triangleq\ & \mathrm{col}(R_1 : \ldots : R_{q+\mathrm{sr}(A)-1}) \\ & + \mathrm{col}(R_{q+\mathrm{sr}(A)} : \ldots : R_p)\, T_{\mathrm{sr}(A)} \end{aligned}$$

is a **unimodular matrix**.

$\boxed{\textbf{Main results}}$

- $\begin{cases} P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times (p-q)}, \\ R = (D \,:\, -N) \in A^{q \times p}. \end{cases}$

- **<u>Theorem</u>**: Let $P = D^{-1}\, N$ be a transfer matrix which admits a left-coprime factorization, then **there exists a stabilizing controller of the form**:

$$C = \begin{array}{cc} \left(\begin{array}{c} U\, V^{-1} \\ T_1 + T_2\, (U\, V^{-1}) \end{array} \right) & \begin{array}{l} \updownarrow \mathrm{sr}(A) - 1 \\ \updownarrow p - q - \mathrm{sr}(A) + 1 \end{array} \end{array}$$

$$\begin{cases} T_1 \in A^{(p-q-\mathrm{sr}(A)+1) \times q}, \\ T_2 \in A^{(p-q-\mathrm{sr}(A)+1) \times \mathrm{sr}(A)-1} \end{cases} \textbf{(i.e. stable)}.$$

- **<u>Corollary</u>**: If $\mathrm{sr}(A) = 1$, then:

  - Every plant P which admits a left-coprime factorization is internally stabilized by a **stable controller** (**strong stabilization**).

  - If P_1 and P_2 admit doubly coprime factorizations, then there exists a controller C which simultaneously stabilizes P_1 and P_2 (**simultaneous stabilization**).

$$\boxed{H_\infty(\mathbb{C}_+)}$$

• **Corollary**: $A = H_\infty(\mathbb{C}_+)$. **Every stabilizable plant**, defined by a transfer matrix P with entries in $K = Q(A)$, **is stabilized by a stable controller**.

• **Corollary**: $A = H_\infty(\mathbb{C}_+)$. **Every couple of stabilizable plants**, defined by the transfer matrices P_1 et P_2 with entries in $K = Q(A)$, **is stabilized by a same controller**.

• **Wanted**: An algorithm which computes the previous controllers.

• **Corollary**: **Every plant** $P \in \mathbb{R}(s)^{q \times (p-q)}$ is stabilized by a controller of the form

$$C = \begin{array}{c} \left(\begin{array}{c} U\,V^{-1} \\ T_1 + T_2\,(U\,V^{-1}) \end{array} \right) \end{array} \quad \begin{array}{l} \updownarrow 1 \\ \updownarrow p-q-1 \end{array}$$

with $A = RH_\infty$ and:

$$\begin{cases} T_1 \in A^{(p-q-1) \times q}, \\ T_2 \in A^{(p-q-1) \times 1}. \end{cases}$$

Topological stable range

- **Definition**: Let A be a **Banach algebra**. We call **topological stable rank** $\mathrm{tsr}(A)$ of A the smallest $n \in \mathbb{N} \cup \{+\infty\}$ such that $\mathrm{U}_n(A)$ is dense in A^n for the product topology.

- **Proposition**: Let A be a **Banach algebra** such that $\mathrm{tsr}(A) = 2$, then **every system** – defined by a transfer function $p = n/d$ ($0 \neq d,\, n \in A$) – **is as close as we want to a plant admitting a coprime factorization**, i.e.:

$$\forall\, \epsilon > 0,\ \exists\, (d_\epsilon : n_\epsilon) \in \mathrm{U}_2(A) : \begin{cases} \| d - d_\epsilon \|_A \leq \epsilon, \\ \| n - n_\epsilon \|_A \leq \epsilon. \end{cases}$$

- **Theorem**: (Suárez 96) $\mathrm{tsr}(H_\infty(\mathbb{C}_+)) = 2$.

- **Corollary**: Every SISO system – defined by a transfer function $p = n/d$ ($0 \neq d,\, n \in H_\infty(\mathbb{C}_+)$) – is such that $\forall\, \epsilon > 0,\ \exists\, (d_\epsilon : n_\epsilon) \in \mathrm{U}_2(H_\infty(\mathbb{C}_+))$:

$$\begin{cases} \| d - d_\epsilon \|_\infty \leq \epsilon, \\ \| n - n_\epsilon \|_\infty \leq \epsilon. \end{cases}$$

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