Every internally stabilizable multidimensional system admits a doubly coprime factorization

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Abstract

The purpose of this paper is to show how to combine some new results on internal stabilizability [11, 12, 14, 15] with a result on multidimensional linear systems, independently obtained by Byrnes-Spong-Tarn and Kamen-Khargonekar-Tannenbaum in [2, 6], in order to prove a conjecture of Z. Lin [7, 8, 9]. In particular, we shall show that every internal stabilizable multidimensional linear system (in the sense of the structural stability) admits a doubly coprime factorization, and thus, all stabilizing controllers of an internally stabilizable multidimensional linear system can be parametrized by means of the Youla-Kučera parametrization.

Keywords Multidimensional linear systems, Z. Lin's conjecture, internal stabilizability, doubly coprime factorizations, parametrizations of all stabilizing controllers, robust stabilization, lattices.

1 Introduction

Let $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n \,|\, |z_i| \leq 1,\, i = 1, \ldots, n\}$ be the closed unit polydisc in $\mathbb{C}^n$ and let us introduce the ring of *structural stabilizable multidimensional linear systems (or n-D linear systems)* defined by:

$$\mathbb{R}[z_1, \ldots, z_n]_s = \{a/b \,|\, 0 \neq b,\, a \in \mathbb{R}[z_1, \ldots, z_n],\, b(z) = 0 \Rightarrow z \in \mathbb{C}^n \backslash \overline{\mathbb{D}}^n\}.$$

We recall that this ring plays an important role in stabilization problems of multidimensional linear systems. See [7, 8, 9, 18] and the references therein for more details.

In the literature of multidimensional linear systems, the following well-known problems, stated by Z. Lin in [9] (see also [7, 8, 18] for more information), are still open:

- **Problem 1**: Determine whether or not an internally stabilizable n-D linear system defined by a transfer matrix P with entries in $\mathbb{R}(z_1, \ldots, z_n)$ admits a doubly coprime factorization over $\mathbb{R}[z_1, \ldots, z_n]_s$.

  Z. Lin's Conjecture: If P is an internally stabilizable n-D linear system, then P admits a doubly coprime factorization.

- **Problem 2**: Parametrize all stabilizing controllers for a stabilizable n-D linear system.

- **Problem 5**: Prove the existence of doubly coprime factorizations over $\mathbb{R}[z_1, z_2, z_3]_s$ for the class of 3-D linear systems defined in [8].

The purpose of this paper is to show how to use some recent results on internal stabilizability, doubly coprime factorizations, parametrizations of all stabilizing controllers [12, 14, 15] and a result on multidimensional systems, independently obtained by Byrnes-Spong-Tarn and Kamen-Khargonekar-Tannenbaum in [2, 6], in order to solve the previous three open problems. In particular, we shall prove that every internally stabilizable plant (in the sense of the structural stability) defined by a transfer matrix P with entries in $\mathbb{R}(z_1, \ldots, z_n)$ admits a doubly coprime factorization over $\mathbb{R}[z_1, \ldots, z_n]_s$, and thus, all stabilizing controllers of P are parametrized by means of the Youla-Kučera parametrization.

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2 Internal stabilizability & (weakly) doubly coprime factorizations

Let us recall some well-known definitions. See [11, 15, 18, 19] and the references therein for more details.

Definition 1. Let A be a commutative integral domain of stable SISO plants (e.g., $A = RH_\infty$, $\mathbb{R}[z_1, \ldots, z_n]_s$, $H_\infty(\mathbb{C}_+)$, W_+, $\hat{\mathcal{A}}$ [3, 9, 11, 12, 18, 19]) and $K = Q(A) \triangleq \{a/b \,|\, 0 \neq b,\, a \in A\}$ the quotient field of A.

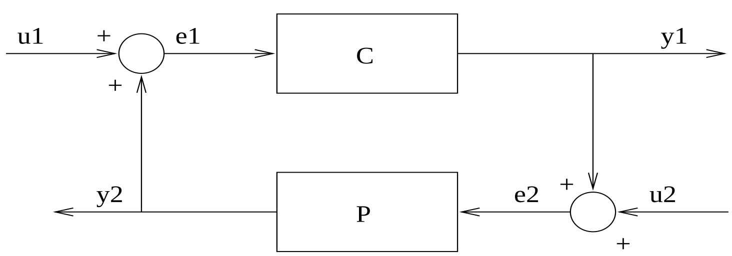
- A transfer matrix $P \in K^{q\times r}$ is said to be *internally stabilizable* if there exists a controller $C \in K^{r\times q}$ such that all the entries of the following transfer matrix

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P,\, C) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

  (see Figure 1) belong to A or, in other words, if we have:

$$\begin{aligned} H(P,\, C) = \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} &= \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \in A^{(q+r)\times(q+r)} \\ &= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \in A^{(q+r)\times(q+r)}. \end{aligned} \tag{1}$$

  Then, C is called a *stabilizing controller* of P.



- A transfer matrix $P \in K^{q\times r}$ is said to admit a *left-coprime factorization* if there exist two matrices $R = (D \;:\; -N) \in A^{q\times(q+r)}$ and $S = (X^T \;:\; Y^T) \in A^{(q+r)\times q}$ such that $\det D \neq 0$, $P = D^{-1}\,N$ and $R\,S = D\,X - N\,Y = I_q$.

- A transfer matrix $P \in K^{q\times r}$ is said to admit a *right-coprime factorization* if there exist two matrices $\tilde{R} = (\tilde{N}^T \;:\; \tilde{D}^T)^T \in A^{(q+r)\times r}$ and $\tilde{S} = (-\tilde{Y} \;:\; \tilde{X}) \in A^{r\times(q+r)}$ such that $\det \tilde{D} \neq 0$, $P = \tilde{N}\,\tilde{D}^{-1}$ and $\tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r$.

- A transfer matrix $P \in K^{q\times r}$ is said to admit a *doubly coprime factorization* if P admits left- and right-coprime factorizations.

In [11, 12, 14, 15], new necessary and sufficient conditions for internal stabilizability were given for general rings of SISO stable plants. Let us recall some of these results.

Theorem 1. *[14, 15] A plant, defined by a transfer matrix $P \in K^{q\times r}$, is internally stabilizable iff one of the following equivalent assertions is satisfied:*

1. *There exists $S = (U^T \;:\; V^T)^T \in A^{(q+r)\times q}$ which satisfies $\det U \neq 0$ and:*

   (a) $S\,P = \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{(q+r)\times r}$,

   (b) $(I_q \;:\; -P)\,S = U - P\,V = I_q$.

Then, the controller $C = V\,U^{-1}$ *internally stabilizes the plant* P *and we have:*

$$\begin{cases} U = (I_q - P\,C)^{-1}, \\ V = C\,(I_q - P\,C)^{-1}. \end{cases}$$

2. *There exists* $T = (-X \,:\, Y) \in A^{r\times(q+r)}$ *which satisfies* $\det Y \neq 0$ *and:*
   (a) $P\,T = (-P\,X \,:\, P\,Y) \in A^{q\times(q+r)}$,

   (b) $T\begin{pmatrix} P \\ I_r \end{pmatrix} = -X\,P + Y = I_r.$

   Then, the controller $C = Y^{-1}\,X$ *internally stabilizes the plant* P *and we have:*

$$\begin{cases} Y = (I_r - C\,P)^{-1}, \\ X = (I_r - C\,P)^{-1}\,C. \end{cases}$$

We have the following corollary of Theorem 1 which allows us to check whether or not a controller $C \in K^{r\times q}$ internal stabilizes a plant $P \in K^{q\times r}$.

Corollary 1. *A plant* $P \in K^{q\times r}$ *is internally stabilized by a controller* $C \in K^{r\times q}$ *iff one of the following conditions is satisfied:*

- *The matrix*
$$\Pi_1 = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \tag{2}$$
is an idempotent (or a projector) of $A^{(q+r)\times(q+r)}$, *i.e., we have* $\Pi_1^2 = \Pi_1 \in A^{(q+r)\times(q+r)}$.
- *The matrix*
$$\Pi_2 = \begin{pmatrix} -P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \tag{3}$$
is an idempotent (or a projector) of $A^{(q+r)\times(q+r)}$, *i.e., we have* $\Pi_2^2 = \Pi_2 \in A^{(q+r)\times(q+r)}$.

Then, we have $\Pi_1 + \Pi_2 = I_{q+r}$ *and*

$$\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = \Pi_1 \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}, \quad \begin{pmatrix} y_2 \\ e_2 \end{pmatrix} = \Pi_1 \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}, \tag{4}$$

where e_1, e_2, u_1, u_2, y_1 *and* y_2 *are defined in Figure 1.*

Proof. $1 \Rightarrow$ Let us suppose that $C \in K^{r\times q}$ internally stabilizes $P \in K^{q\times r}$. Then, by 1 of Theorem 1, there exists $S = (U^T \,:\, V^T)^T \in A^{(q+r)\times r}$ satisfying 1.a and 1.b of Theorem 1. Let us denote by:

$$\Pi_1 = S\,(I_q \,:\, -P) = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}.$$

By 1.a of Theorem 1, we obtain that $\Pi_1 \in A^{(q+r)\times(q+r)}$ and, by 1.b of Theorem 1, we have:

$$\Pi_1^2 = S\,(I_q \,:\, -P)\,S\,(I_q \,:\, -P) = S\,((I_q \,:\, -P)\,S)\,(I_q \,:\, -P) = S\,(I_q \,:\, -P) = \Pi_1,$$

i.e., Π_1 is an idempotent of $A^{(q+r)\times(q+r)}$.

$1 \Leftarrow$ First of all, let us notice that we have $\Pi_1 = \begin{pmatrix} (I_q - P\,C)^{-1} \\ C\,(I_q - P\,C)^{-1} \end{pmatrix} (I_q \,:\, -P)$. Thus, we have

$$\Pi_1^2 = \begin{pmatrix} (I_q - P\,C)^{-1} \\ C\,(I_q - P\,C)^{-1} \end{pmatrix} ((I_q - P\,C)^{-1} - P\,C\,(I_q - P\,C)^{-1})\,(I_q \,:\, -P) = \Pi_1,$$

i.e., Π_1 is an idempotent of $K^{(q+r)\times(q+r)}$. Now, if Π_1 is an idempotent of $A^{(q+r)\times(q+r)}$, then

$$(I_q - P\,C)^{-1} \in A^{q\times q},\ (I_q - P\,C)^{-1}\,P \in A^{q\times r},\ C\,(I_q - P\,C)^{-1} \in A^{r\times q},\ C\,(I_q - P\,C)^{-1}\,P \in A^{r\times r},$$

which implies that $H(P, C) \in A^{(q+r)\times(q+r)}$, where $H(P, C)$ is defined in Definition 1, and thus, C internally stabilizes P. 2 can be proved similarly.

Finally, using the following well-known identities (see e.g., [19])

$$\begin{cases} (I_q - P\,C)^{-1} = P\,(I_r - C\,P)^{-1}\,C + I_q, \\ (I_q - P\,C)^{-1}\,P = P\,(I_r - C\,P)^{-1}, \\ C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C, \\ C\,(I_q - P\,C)^{-1}\,P = (I_r - C\,P)^{-1} - I_r, \end{cases}$$

we check that we have $\Pi_1 + \Pi_2 = I_{q+r}$ and, using (1) and $y_1 = C\,e_1$ and $y_2 = P\,e_2$, we obtain (4). □

Let us point out that for $A = H_\infty(\mathbb{C}_+)$, the fact that Π_1 and Π_2 are two projectors satisfying $\Pi_1 + \Pi_2 = I_{q+r}$ was already known to be equivalent to internal stabilizability [4]. However, let us insist on the fact that no left- or right-coprime factorizations of the plant P were used in the proof of Corollary 1. To finish, let us notice that the *robustness radius* is defined by:

$$b_{P,C} \triangleq \|\,\Pi_1\,\|_\infty^{-1} = \|\,\Pi_2\,\|_\infty^{-1}. \tag{5}$$

See [4] and the references therein for more information. Finally, let us stress out that the robustness radius also plays an important role in the *loop-shaping procedure* [5].

We shall see later the interest to reinterpret the results of Theorem 1 in a more abstract (intrinsic) way using the concept of *lattices of vector spaces* developed in module theory [1, 16]. We refer to [14, 15] for more details on this natural mathematical framework for the study of stabilization problems. In order to do that, let us firstly give some definitions coming from module theory [1, 16].

Definition 2.
- The *rank* of an A-module M is defined by $\mathrm{rank}_A(M) = \dim_K(K \otimes_A M)$, where $K \otimes_A M$ denotes the K-vector space generated by the A-module M by extending the scalars to the quotient field $K = Q(A)$ of A. If $\mathrm{rank}_A(M)$ is finite, then M is said to be *finitely generated*.
- A finitely generated A-module M is said to be *free* of rank r if M is isomorphic to a A^r, i.e., we have $M \cong A^r$, where $\cong$ denotes an isomorphism between two A-modules. Equivalently, M is a free A-module of rank r if M has a basis with r elements.
- A finitely generated A-module M is said to be *projective* if there exist an A-module N and a positive integer $r \in \mathbb{Z}_+$ such that $M \oplus N \cong A^r$, where $\oplus$ denotes the direct sum of A-modules.

A projective A-module can be interpreted as a direct summand of a free A-module. Let us point out that it is known in algebra (see e.g., K-theory [17]) that projectors over a ring A and projective A-modules are two equivalent forms of the same mathematical concept (it explains the similar denominations). Moreover, if A is a *commutative Banach algebra* (e.g., $A = H_\infty(\mathbb{C}_+)$, W_+, $\hat{\mathcal{A}}$ [3, 19]), then they are also equivalent to the topological concept of *vector bundle* over the *maximal ideals space* of A (Swan's theorem [17]). See [11] and forthcoming publications for more details. Hence, let us give necessary and sufficient conditions for internal stabilizability in terms of projective A-modules.

Corollary 2. *[14, 15] A plant $P \in K^{q\times r}$ is internally stabilizable iff one of the following equivalent assertions is satisfied:*

1. $(I_q\,:\,-P)\,A^{q+r}$ *is a* projective lattice of K^q, *namely* $(I_q\,:\,-P)\,A^{q+r}$ *is a projective A-submodule of K^q of rank q.*
2. $A^{1\times(q+r)}\begin{pmatrix} P \\ I_r \end{pmatrix}$ *is a* projective lattice of $K^{1\times r}$, *namely* $A^{1\times(q+r)}\begin{pmatrix} P \\ I_r \end{pmatrix}$ *is a projective A-submodule of $K^{1\times r}$ of rank r.*

The next theorem gives some intrinsic necessary and sufficient conditions for a plant to admit a left- or a right-coprime factorization. We refer to [14, 15] for the proof.

Theorem 2. *1. $P \in K^{q\times r}$ admits a left-coprime factorization iff there exists a matrix $D \in A^{q\times q}$ such that* $\det D \neq 0$ *and*

$$(I_q \,:\, -P)\, A^{q+r} = D^{-1}\, A^q,$$

i.e., iff $(I_q \,:\, -P)\, A^{q+r}$ is a free lattice *of K^q, namely $(I_q \,:\, -P)\, A^{q+r}$ is a free A-submodule of K^q of rank q. Then, $P = D^{-1}\, N$, where $N = D\, P \in A^{q\times r}$, is a left-coprime factorization of P.*

2. $P \in K^{q\times r}$ admits a right-coprime factorization iff there exists a matrix $\tilde{D} \in A^{r\times r}$ such that $\det \tilde{D} \neq 0$ *and*

$$A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} = A^{1\times r}\, \tilde{D}^{-1},$$

i.e., iff $A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$ is a free lattice *of $K^{1\times r}$, namely $A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$ is a free A-submodule of $K^{1\times r}$ of rank r. Then, $P = \tilde{N}\, \tilde{D}^{-1}$, where $\tilde{N} = P\, \tilde{D} \in A^{q\times r}$, is a right-coprime factorization of P.*

In commutative algebra, a lattice of $K = Q(A)$ is called a *fractional ideal of A*. We refer to [12] for SISO versions of Theorems 1 and 2 and Corollary 2 using fractional ideals. We also refer to the pioneering work of V. R. Sule [18] for different necessary and sufficient conditions for internal stabilizability and for the existence of coprime factorizations.

Using Definition 2, it is easy to see that a free A-module is also a projective A-module (take simply $N = 0$). Therefore, using Corollary 2 and Theorem 2, we easily deduce that every plant which admits a left- or a right-coprime factorization is internally stabilizable.

Corollary 3. *1. If $P \in K^{q\times r}$ admits the left-coprime factorization*

$$P = D^{-1}\, N, \qquad D\, X - N\, Y = I_q, \qquad \det X \neq 0,$$

with $(X^T \,:\, Y^T)^T \in A^{(q+r)\times q}$, then $S = ((X\, D)^T \,:\, (Y\, D)^T)^T \in A^{(q+r)\times q}$ satisfies 1.*a and* 1.*b of Theorem 1, and thus, $C = Y\, X^{-1}$ is a stabilizing controller of P.*

2. If $P \in K^{q\times r}$ admits the right-coprime factorization

$$P = \tilde{N}\, \tilde{D}^{-1}, \qquad -\tilde{Y}\, \tilde{N} + \tilde{X}\, \tilde{D} = I_r, \qquad \det \tilde{X} \neq 0,$$

with $(-\tilde{Y} \,:\, \tilde{X}) \in A^{r\times(q+r)}$, then $T = (-\tilde{D}\, \tilde{Y} \,:\, \tilde{D}\, \tilde{X}) \in A^{r\times(q+r)}$ satisfies 2.*a and* 2.*b of Theorem 1, and thus, $C = \tilde{X}^{-1}\, \tilde{Y}$ is a stabilizing controller of P.*

Using Theorem 2 and Corollary 3, we also deduce easily the following result.

Corollary 4. *[11] If A is a* projective-free ring *− namely a ring such that every finitely generated projective A-module is free −, then every internally stabilizable system admits a doubly coprime factorization. In particular, this result holds for $A = RH_\infty$ and $H_\infty(\mathbb{C}_+)$.*

Let us notice that we do not know whether or not the integral domain

$$\mathcal{A} = \{h(t) = f(t) + \sum_{i=0}^{\infty} a_i\, \delta(t - t_i) \mid f \in L_1(\mathbb{R}_+), (a_i)_{i\geq 0} \in l_1(\mathbb{Z}_+),\; 0 = t_0 \leq t_1 \leq t_2 \leq \ldots\},$$

of *BIBO infinite-dimensional time-invariant linear systems* [3] is projective-free, and thus, whether or not every internally stabilizable plant admits a doubly coprime factorization [11, 12, 15].

Surprisingly, independently to multidimensional systems, the following important theorem was proved in [2, 6] in the study of neutral time-delay systems within a "systems over rings" approach.

Theorem 3. *(Corollary 2.2.4 of [2] and Theorem A.3 of [6]) The ring of structural stabilizable multidimensional linear systems* $\mathbb{R}[z_1,\ldots,z_n]_s = \{a/b \,|\, 0 \neq b,\, a \in \mathbb{R}[z_1,\ldots,z_n],\, b(z)=0 \Rightarrow z \in \mathbb{C}^n \backslash \overline{\mathbb{D}}^n\}$, *where* $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n \,|\, |z_i| \leq 1,\, i=1,\ldots,n\}$ *is the closed unit polydisc in* $\mathbb{C}^n$*, is projective-free.*

Using Corollary 2 and Theorem 3, we can solve Problems 1 and 5 defined in the introduction.

Corollary 5. *Let* $\mathbb{R}[z_1,\ldots,z_n]_s = \{a/b \,|\, 0 \neq b,\, a \in \mathbb{R}[z_1,\ldots,z_n],\, b(z)=0 \Rightarrow z \in \mathbb{C}^n \backslash \overline{\mathbb{D}}^n\}$ *and* $K = \mathbb{R}(z_1,\ldots,z_n)$*. Then, every internally stabilizable multidimensional linear system (n-D linear system) defined by a transfer matrix* P *with entries in* K *admits a doubly coprime factorization.*

Let us notice the fact that the ring of structural stabilizable multidimensional SISO linear systems be projective-free is a non-trivial result (the main part of the proof in [6] comes from a private communication with the famous Fields medalist P. Deligne). This certainly explains why the equivalence between internal stabilizability and the existence of coprime factorizations for multidimensional linear systems was still open till now.

A ring A is said to be a *coherent Sylvester domain* if A is a projective-free *coherent ring* with a *weak global dimension* w.gl.dim(A) less or equal to 2. We refer to [16] for the undefined terms and to [11] for the role this concept plays in stabilization problems. Using the fact that $\mathbb{R}[z_1,z_2]_s$ is a projective-free noetherian ring [16]and w.gl.dim$(\mathbb{R}[z_1,z_2]_s) \leq$ w.gl.dim$(\mathbb{R}[z_1,z_2]) = 2$ because we have

$$\mathbb{R}[z_1,z_2]_s = S^{-1}\,\mathbb{R}[z_1,z_2] \triangleq \{a/b \,|\, a \in \mathbb{R}[z_1,z_2],\ b \in S\},$$

where $S = \{b \in \mathbb{R}[z_1,z_2] \,|\, b(z)=0 \Rightarrow z \in \mathbb{C}^2 \backslash \overline{\mathbb{D}}^2\}$, we obtain that $\mathbb{R}[z_1,z_2]_s$ is a coherent Sylvester domain. Therefore, the following corollary follows from [11].

Corollary 6. *The ring* $\mathbb{R}[z_1,z_2]_s = \{a/b \,|\, 0 \neq b,\, a \in \mathbb{R}[z_1,\ldots,z_2],\, b(z)=0 \Rightarrow z \in \mathbb{C}^2 \backslash \overline{\mathbb{D}}^2\}$ *is a coherent Sylvester domain, and thus, a* greatest common divisor domain *[11]. In particular, every transfer matrix* $P \in \mathbb{R}(z_1,z_2)^{q\times r}$ *admits a* weakly doubly coprime factorization, *namely* P *has the form of* $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$*, where the greatest common divisor of the* $q\times q$*-minors (resp.* $r\times r$*-minors) of* $R = (D\,:\,-N) \in \mathbb{R}[z_1,z_2]_s^{q\times(q+r)}$ *(resp.* $\tilde{R} = (\tilde{N}^T\,:\,\tilde{D}^T)^T \in \mathbb{R}[z_1,z_2]_s^{(q+r)\times r}$*) is 1.*

We refer to [11] for an algorithm which computes weakly doubly coprime factorizations of a transfer matrix if *syzygy modules* [16] can be effectively computed over A.

3 Parametrization of all stabilizing controllers

The Youla-Kučera parametrization of all stabilizing controllers was developed for plants which admitted doubly coprime factorizations (see [19] and the references therein). However, using Corollary 3 and Theorem 2, we obtain that an internally stabilizable system does not generally admit a doubly coprime factorization (indeed, a projective A-module is generally not free). Hence, it is natural to ask whether or not it is possible to parametrize all stabilizing controllers of an internally stabilizable plant which does not necessarily admit a doubly coprime factorization.

Let us recall the main result of [14, 15] which gives a generalization of the Youla-Kučera parametrization of all stabilizing controllers for internally stabilizable plant which does not necessarily admit a doubly coprime factorization (see also [18] for an implicit parametrization of all stabilizing controllers when A is a *unique factorization domain* [1, 16]).

Theorem 4. *Let* A *be an integral domain of SISO stable plants,* $K = Q(A)$ *the quotient field of* A*,* $P \in K^{q\times r}$ *an internally stabilizable system and* $S = (U^T\,:\,V^T)^T \in A^{(q+r)\times q}$ *(resp.* $T = (-X\,:\,Y) \in A^{r\times(q+r)}$*) a matrix satisfying* $\det U \neq 0$ *(resp.* $\det Y \neq 0$*) and* 1.a *and* 1.b *(resp.* 2.a *and* 2.b*) of Theorem 1. Then, all stabilizing controllers of* P *are defined by*

$$C(Q) = (V+Q)\,(U+P\,Q)^{-1} = (Y+Q\,P)^{-1}\,(X+Q), \tag{6}$$

where Q *is any matrix which satisfies* $\det(U+P\,Q) \neq 0$*,* $\det(Y+Q\,P) \neq 0$ *and:*

$$Q \in \Omega \triangleq \{L \in A^{r\times q} \,|\, L\,P \in A^{r\times r},\ P\,L \in A^{q\times q},\ P\,L\,P \in A^{q\times r}\}. \tag{7}$$

Corollary 7. *A more direct way to express Theorem 4 is to say that, if $C_* \in K^{r\times q}$ is a particular internally stabilizing controller of P, then all stabilizing controllers of P are defined by*

$$\begin{aligned} C(Q) &= (C_*\,(I_q - P\,C_*)^{-1} + Q)\,((I_q - P\,C_*)^{-1} + P\,Q)^{-1}, \\ &= ((I_r - C_*\,P)^{-1} + Q\,P)^{-1}\,((I_r - C_*\,P)^{-1}\,C_* + Q), \end{aligned}$$

where Q is any matrix of Ω which satisfies $\det((I_q - P\,C_*)^{-1} + P\,Q) \neq 0$, $\det((I_r - C_*\,P)^{-1} + Q\,P) \neq 0$.

Using (1), (6) and (7), we easily obtain the following result.

Corollary 8. *Let $P \in K^{q\times r}$ be an internally stabilizable and $C_* \in K^{r\times q}$ an internally stabilizable controller of P. Then, the transfer matrices of the stable closed-loop system are defined by*

$$\begin{aligned} H(P, C(Q)) &= \begin{pmatrix} (I_q - P\,C_*)^{-1} + P\,Q & (I_q - P\,C_*)^{-1}\,P + P\,Q\,P \\ C_*\,(I_q - P\,C_*)^{-1} + Q & I_r + C_*\,(I_q - P\,C_*)^{-1}\,P + Q\,P \end{pmatrix} \\ &= \begin{pmatrix} I_q + P\,(I_r - C_*\,P)^{-1}\,C_* + P\,Q & P\,(I_r - C_*\,P)^{-1} + P\,Q\,P \\ (I_r - C_*\,P)^{-1}\,C_* + Q & (I_r - C_*\,P)^{-1} + Q\,P \end{pmatrix}, \end{aligned}$$

where Q is any matrix of Ω which satisfies $\det((I_q - P\,C_*)^{-1} + P\,Q) \neq 0$, $\det((I_r - C_*\,P)^{-1} + Q\,P) \neq 0$.

Hence, the transfer matrices $H(P, C(Q))$ are affine in the arbitrary parameter $Q \in \Omega$, and thus, there are convex in Q, namely we have:

$$\forall\, \lambda \in A,\ \forall\, Q_1,\, Q_2 \in \Omega, \quad H(P, C(\lambda\,Q_1 + (1-\lambda)\,Q_2)) = \lambda\,H(P, C(Q_1)) + (1-\lambda)\,H(P, C(Q_2)). \quad (8)$$

Firstly, let us point out (8) holds for any element λ in A and not only necessarily for $\lambda \in \mathbb{R}$.

Secondly, the fact that the general parametrization of all stabilizing controllers (6) is convex in the arbitrary parameter Q allows us to rewrite any convex optimization problem as a convex problem in $Q \in \Omega$. For instance, if A is a Banach algebra with the norm $\|\cdot\|_A$ (e.g., $A = H_\infty(\mathbb{C}_+)$, W_+, $\hat{\mathcal{A}}$), then the sensitivity minimization problem becomes

$$\begin{aligned} \inf_{C\in \mathrm{Stab}(P)} \|\, W_1\,(I_q - P\,C)^{-1}\,W_2\,\|_A &= \inf_{Q\in\Omega} \|\, W_1\,(U + P\,Q)\,W_2\,\|_A \\ &= \inf_{Q\in\Omega} \|\, W_1\,((I_q - P\,C_*)^{-1} + P\,Q)\,W_2\,\|_A, \end{aligned}$$

where $C_* = V\,U^{-1}$ is a particular stabilizing controller of P (see 1 of Theorem 1), $\mathrm{Stab}(P)$ is the set of all stabilizing controllers of P and $W_1, W_2 \in A^{q\times q}$ are two weighted matrices [15]. Similarly, by extension with the case $A = H_\infty(\mathbb{C}_+)$, we can define the optimal robustness radius by

$$b_{\mathrm{opt}}(P) \triangleq \sup_{C\in\mathrm{Stab}(P)} b_{P,C},$$

where robustness radius $b_{P,C}$ is defined by (5). Then, we have

$$\begin{aligned} b_{\mathrm{opt}}(P)^{-1} &= \inf_{C\in\mathrm{Stab}(P)} \|\, \Pi_1(P, C)\,\|_A = \inf_{Q\in\Omega} \|\, \Pi_1(P, C(Q))\,\|_A \\ &= \inf_{Q\in\Omega} \left\| \begin{pmatrix} (I_q - P\,C_\star)^{-1} + P\,Q & -(I_q - P\,C_\star)^{-1}\,P + P\,Q\,P \\ C_\star\,(I_q - P\,C_\star)^{-1} + Q & -C_\star\,(I_q - P\,C_\star)^{-1}\,P + Q\,P \end{pmatrix} \right\|_A, \end{aligned}$$

where the idempotent Π_1 of $A^{(q+r)\times(q+r)}$ is defined by (2). The extension of the concepts of robustness radius and loop-shaping procedure for Banach algebras A will be studied in forthcoming publications.

Finally, in the spirit of the work of U. Oberst [10] on the behavioural approach to multidimensional linear systems, if X is an A-module (e.g., $X = H_2(\mathbb{C}_+)$, $A = H_\infty(\mathbb{C}_+)$ or $X = L_p(\mathbb{R}_+)$ and $A = \mathcal{A}$ [3]), we show in [13] how to use the functor $\hom_A(\cdot, X)$ in order to obtain a duality between the algebraic approach developed in [11, 12, 14, 15] and the operator-theoretic approach developed in [3, 4, 19] (using unbounded operators, graphs, domains. . .).

Let us recall that the existence of a left-coprime factorization (resp. right-coprime factorization) does not necessarily imply the existence of a right-coprime factorization (resp. left-coprime factorization). See [11, 19] for more details. The following corollary, which was proved in [15], specifies the set Ω of the arbitrary parameters of (6) when the plant admits a left- or a right-coprime factorization.

Corollary 9. *1. If $P \in K^{q\times r}$ admits a left-coprime factorization $P = D^{-1} N$, then we have:*

$$\Omega = \{T \in A^{r\times q} \,|\, P\,T \in A^{q\times q}\}\, D.$$

2. If $P \in K^{q\times r}$ admits a right-coprime factorization $P = \tilde{N}\,\tilde{D}^{-1}$, then we have:

$$\Omega = \tilde{D}\,\{S \in A^{r\times q} \,|\, S\,P \in A^{r\times r}\}.$$

3. If $P \in K^{q\times r}$ admits a doubly coprime factorization $P = D^{-1} N = \tilde{N}\,\tilde{D}^{-1}$, then we have:

$$\Omega = \tilde{D}\,A^{r\times q}\,D.$$

In fact, [15] shows that Corollary 9 also holds if we use the weaker concepts of weakly left- and weakly right-coprime factorizations (see [11] for more details on these concepts) instead of using left- and right-coprime factorizations. Using Corollaries 3 and 9, we obtain the following result.

Corollary 10. *[14, 15] If $P \in K^{q\times r}$ admits the doubly coprime factorization*

$$P = D^{-1} N = \tilde{N}\,\tilde{D}^{-1}, \; \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} = I_{q+r},$$

then all stabilizing controllers of P are of the form

$$C(\Lambda) = (Y + \tilde{D}\,\Lambda)\,(X + \tilde{N}\,\Lambda)^{-1} = (\tilde{X} + \Lambda\,N)^{-1}\,(\tilde{Y} + \Lambda\,D), \tag{9}$$

where $\Lambda \in A^{r\times q}$ is every matrix such that $\det(X + \tilde{N}\,\Lambda) \neq 0$ *and* $\det(\tilde{X} + \Lambda\,N) \neq 0$.

We find again the *Youla-Kučera parametrization of all stabilizing controllers of P* [19].

Finally, using Corollaries 5 and 10, we obtain the following result which gives an answer to Problem 2 of Z. Lin [9].

Corollary 11. *Let $\mathbb{R}[z_1, \ldots, z_n]_s = \{a/b \,|\, 0 \neq b,\, a \in \mathbb{R}[z_1, \ldots, z_n],\, b(z) = 0 \Rightarrow z \in \mathbb{C}^n \backslash \overline{\mathbb{D}}^n\}$ be the ring of structural stabilizable multidimensional linear systems, where $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n \,|\, |z_i| \leq 1,\, i = 1, \ldots, n\}$ is the closed unit polydisc in $\mathbb{C}^n$, and $K = \mathbb{R}(z_1, \ldots, z_n)$. Then, all stabilizing controllers of an internally stabilizable multidimensional system, defined by a transfer matrix $P \in K^{q\times r}$, are parametrized by means of the Youla-Kučera parametrization (9).*

4 Conclusion

In this paper, we have shown how to use some recent results on stabilization problems [11, 14, 15] and a result of Byrnes-Spong-Tarn/Kamen-Khargonekar-Tannenbaum developed for neutral differential time-delay systems [2, 6] in order to prove a conjecture of Z. Lin on multidimensional systems [9].

Finally, let us notice that the proves given in [2, 6] of the fact that $\mathbb{R}[z_1, \ldots, z_n]_s$ is a projective-free ring do not seem to be effective ones. Hence, the fundamental issue consisting in developing an effective proof of this result is still open (see Problems 3 and 4 of [9]). However, let us notice that if the syzygy modules [16] of a finitely generated $\mathbb{R}[z_1, \ldots, z_n]_s$-module can effectively be computed and an effective version of the Nullstellensatz exists in $\mathbb{R}[z_1, \ldots, z_n]_s$ (i.e., an effective algorithm testing whether or not 1 belongs to an ideal), then we can apply the algorithms developed in [11] in order to compute doubly coprime factorizations. The effective computational issues will be studied in forthcoming publications.

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