Two pendula mounted on a cart (Polderman, Willems)

Definition of Ore algebra and ODE system:

```
> with(OreModules):
> Alg := diff_algebra([Dt,t], polynom={t}, comm={m1, m2, M, L1, L2, g}):
> R := evalm([[m1*L1*Dt^2, m2*L2*Dt^2, -1, (M+m1+m2)*Dt^2],
>      [m1*L1^2*Dt^2-m1*L1*g, 0, 0, m1*L1*Dt^2], [0, m2*L2^2*Dt^2-m2*L2*g, 0, m2*L2*Dt^2]]);
```

$$R := \begin{bmatrix} m1\ L1\ Dt^2 & m2\ L2\ Dt^2 & -1 & (M + m1 + m2)\ Dt^2 \\ m1\ L1^2\ Dt^2 - m1\ L1\ g & 0 & 0 & m1\ L1\ Dt^2 \\ 0 & m2\ L2^2\ Dt^2 - m2\ L2\ g & 0 & m2\ L2\ Dt^2 \end{bmatrix}$$

Computing $\operatorname{ext}^1_D(N, D)$, where $D = Alg$: (CPU: $< 1s$)

```
> Ext1 := exti(adjoint(R, Alg), Alg, 1):  Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longleftrightarrow$ $\operatorname{ext}^1_D(N, D) \cong t(M) = \{0\}$.

$\Longleftrightarrow$ System is controllable $\Longleftrightarrow$ System is parametrizable.

`exti` also constructs a parametrization of the system:

```
> Ext1[3];
```

$$\left[\begin{array}{c} g\,Dt^2 - L2\,Dt^4 \\ g\,Dt^2 - Dt^4\,L1 \\ -Dt^4\,g\,L1\,m2 - L2\,Dt^4\,g\,m1 - L2\,Dt^4\,g\,M + L2\,Dt^6\,L1\,M + g^2\,m2\,Dt^2 - g\,L1\,M\,Dt^4 + g^2\,m1\,Dt^2 + g^2\,M\,Dt^2 \\ -L2\,Dt^2\,g + L2\,Dt^4\,L1 - g\,L1\,Dt^2 + g^2 \end{array}\right]$$

If this parametrization admits a left inverse, we can construct a *flat output*:

```
> L := LI(Ext1[3], Alg);
```

$$L := \left[\begin{array}{cccc} \frac{L1^2}{g^2\,(L1 - L2)} & -\frac{L2^2}{g^2\,(L1 - L2)} & 0 & -\frac{-L1 + L2}{g^2\,(L1 - L2)} \end{array}\right]$$

Now

$$L \cdot (w_1,\, w_2,\, w_3,\, w_4)^T = \frac{L1^2\, w_1 - L2^2\, w_2 + (L1 - L2)\, w_4}{g^2\,(L1 - L2)}$$

is a flat output.

Linear DAE

Definition of Ore algebra and ODE system:

```
> with(OreModules):
> Alg := diff_algebra([D[t],t], polynom={t}):
> R := evalm([[-t*D[t]+1, t^2*D[t], -1, 0], [-D[t], t*D[t]+1, 0, -1]]);
```

$$R := \begin{bmatrix} -t\,\mathrm{D}_t + 1 & t^2\,\mathrm{D}_t & -1 & 0 \\ -\mathrm{D}_t & t\,\mathrm{D}_t + 1 & 0 & -1 \end{bmatrix}$$

```
> Ext1 := exti(adjoint(R, Alg), Alg, 1);
```

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \mathrm{D}_t & -t\,\mathrm{D}_t \\ 1 & -t & -1 & t \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ t^2\,\mathrm{D}_t & -t\,\mathrm{D}_t + 1 \\ t\,\mathrm{D}_t + 1 & -\mathrm{D}_t \end{bmatrix}, \begin{bmatrix} 0 & 1 & -t^2\,\mathrm{D}_t - 2\,t & -t\,\mathrm{D}_t \\ 1 & 0 & t\,\mathrm{D}_t + 2 & \mathrm{D}_t \end{bmatrix} \right]$$

```
> mult(R, Ext1[3], Alg);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

```
> LI(Ext1[3], Alg);
```

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Wind tunnel model (Manitius)

Definition of Ore algebra and differential time-delay system:

```
> with(OreModules):
> Alg := diff_algebra([D[1],x[1]], [delta,x[2]],
>       polynom={x[1],x[2]}, comm={a, omega, zeta, k}):
> R := evalm([[D[1]+a,-k*a*delta,0,0], [0,D[1],-1,0],
>       [0,omega^2,D[1]+2*zeta*omega,-omega^2]]);
```

$$R := \begin{bmatrix} \mathrm{D}_1 + a & -k\,a\,\delta & 0 & 0 \\ 0 & \mathrm{D}_1 & -1 & 0 \\ 0 & \omega^2 & \mathrm{D}_1 + 2\,\zeta\,\omega & -\omega^2 \end{bmatrix}$$

Computing $\mathrm{ext}^1_D(N, D)$, where $D = Alg$: (CPU: $< 1s$)

```
> Ext1 := exti(adjoint(R, Alg), Alg, 1):  Ext[1];
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longleftrightarrow$ $\mathrm{ext}^1_D(N, D) \cong t(M) = \{0\}$.

$\Longleftrightarrow$ System is controllable $\Longleftrightarrow$ parametrizable.

`exti` also constructs a parametrization of the system:

```
> Ext1[3];
```

$$\begin{bmatrix} -\omega^2\,k\,a\,\delta \\ -\mathrm{D}_1\,\omega^2 - a\,\omega^2 \\ -\omega^2\,\mathrm{D}_1{}^2 - \omega^2\,a\,\mathrm{D}_1 \\ -\mathrm{D}_1{}^3 - 2\,\mathrm{D}_1{}^2\,\zeta\,\omega - a\,\mathrm{D}_1{}^2 - \mathrm{D}_1\,\omega^2 - 2\,a\,\mathrm{D}_1\,\zeta\,\omega - a\,\omega^2 \end{bmatrix}$$

Computing $\mathrm{ext}^2_D(N, D)$, where $D = Alg$: (CPU: $< 1s$)

```
> Ext2 := exti(adjoint(R, Alg), Alg, 2): Ext2[1];
```

$$\begin{bmatrix} \delta \\ \mathrm{D}_1 + a \end{bmatrix}$$

We find: $\mathrm{ext}^2_D(N, D) \neq \{0\}$.

$$\left.\begin{array}{l} \mathrm{ext}^1_D(N, D) = \{0\} \\ \mathrm{ext}^2_D(N, D) \neq \{0\} \end{array}\right\} \Longrightarrow$$

System controllable, but neither reflexive, nor projective, nor free (which are all equivalent in this case).

Hence, we find a polynomial $\pi(\delta)$ such that the system is π-flat:

```
> pi_polynomial(adjoint(R, Alg), Alg, [delta]);
```

$$[\delta]$$

A two reflector antenna (Mounier)

Definition of Ore algebra and differential delay system:

```
> with(OreModules):
> Alg:=diff_algebra([D[1],x[1]], [delta,x[2]], polynom={x[1],x[2]}, comm={K1,K2,Te,Kp,Kc}):
> R := evalm([[D[1],-K1,0$7], [0,D[1]+K2/Te,0$4,-Kp/Te*delta,-Kc/Te*delta,-Kc/Te*delta],
>        [0,0,D[1],-K1,0$5], [0$3,D[1]+K2/Te,0,0,-Kc/Te*delta,-Kp/Te*delta,-Kc/Te*delta],
>        [0$4,D[1],-K1,0$3], [0$5,D[1]+K2/Te,-Kc/Te*delta,-Kc/Te*delta,-Kp/Te*delta]]);
```

$$R := \begin{bmatrix} D_1 & -K1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_1 + \frac{K2}{Te} & 0 & 0 & 0 & 0 & -\frac{Kp\,\delta}{Te} & -\frac{Kc\,\delta}{Te} & -\frac{Kc\,\delta}{Te} \\ 0 & 0 & D_1 & -K1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1 + \frac{K2}{Te} & 0 & 0 & -\frac{Kc\,\delta}{Te} & -\frac{Kp\,\delta}{Te} & -\frac{Kc\,\delta}{Te} \\ 0 & 0 & 0 & 0 & D_1 & -K1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_1 + \frac{K2}{Te} & -\frac{Kc\,\delta}{Te} & -\frac{Kc\,\delta}{Te} & -\frac{Kp\,\delta}{Te} \end{bmatrix}$$

Computing $\text{ext}^1_D(N, D)$, where $D = Alg$: (CPU: $1s$)

```
>  Ext1 := exti(adjoint(R, Alg), Alg, 1):  Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Longleftrightarrow$ $\text{ext}^1_D(N, D) \cong t(M) = \{0\}$.

$\Longleftrightarrow$ System is controllable

$\Longleftrightarrow$ System is parametrizable.

`exti` also constructs a parametrization of the system:

```
>  Ext1[3];
```

$$\begin{bmatrix} 0 & 0 & \%4 \\ 0 & 0 & \%3 \\ 0 & \%4 & 0 \\ 0 & \%3 & 0 \\ \%4 & 0 & 0 \\ \%3 & 0 & 0 \\ \%1 & \%1 & \%2 \\ \%1 & \%2 & \%1 \\ \%2 & \%1 & \%1 \end{bmatrix}$$

$$\begin{aligned} \%1 &:= Te\, \mathrm{D}_1{}^2\, Kc + K2\, \mathrm{D}_1\, Kc \\ \%2 &:= -Kp\, \mathrm{D}_1{}^2\, Te - Kp\, \mathrm{D}_1\, K2 \\ &\quad - Te\, \mathrm{D}_1{}^2\, Kc - K2\, \mathrm{D}_1\, Kc \\ \%3 &:= -Kp^2\, \mathrm{D}_1\, \delta - Kp\, Kc\, \delta\, \mathrm{D}_1 \\ &\quad +2\, \delta\, Kc^2\, \mathrm{D}_1 \\ \%4 &:= 2\, \delta\, Kc^2\, K1 - Kp\, Kc\, \delta\, K1 \\ &\quad -Kp^2\, K1\, \delta \end{aligned}$$

Computing $\operatorname{ext}^2_D(N,D)$, where $D = Alg$: (CPU: $1s$)

```
> Ext2 := exti(adjoint(R, Alg), Alg, 2):  Ext2[1];
```

$$\begin{bmatrix} \delta & 0 & 0 \\ Te\,\mathrm{D}_1{}^2 + K2\,\mathrm{D}_1 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & Te\,\mathrm{D}_1{}^2 + K2\,\mathrm{D}_1 & 0 \\ 0 & 0 & \delta \\ 0 & 0 & Te\,\mathrm{D}_1{}^2 + K2\,\mathrm{D}_1 \end{bmatrix}$$

We find: $\operatorname{ext}^2_D(N,D) \neq \{0\}$.

$$\left.\begin{array}{l} \operatorname{ext}^1_D(N,D) = \{0\} \\ \operatorname{ext}^2_D(N,D) \neq \{0\} \end{array}\right\} \Longrightarrow \begin{array}{l}\text{System controllable, but neither reflexive, nor projective,} \\ \text{nor free (which are all equivalent in this case).}\end{array}$$

Hence, we find a polynomial $\pi(\delta)$ such that the system is π-flat:

```
> pi_polynomial(adjoint(R, Alg), Alg, [delta]);
```

$$[\delta]$$

Flexible Rod (Mounier)

Definition of Ore algebra and differential time-delay system:

```
> with(OreModules):
> Alg := diff_algebra([D[1],x[1]], [delta,x[2]], polynom={x[1],x[2]}):
> R := evalm([[D[1], -D[1]*delta, -1], [2*D[1]*delta, -D[1]-D[1]*delta^2, 0]]);
```

$$R := \begin{bmatrix} D_1 & -D_1\,\delta & -1 \\ 2\,D_1\,\delta & -D_1 - D_1\,\delta^2 & 0 \end{bmatrix}$$

Computing $\mathrm{ext}^1_D(N, D)$, where $D = Alg$: (CPU: $< 1s$)

```
> exti(adjoint(R, Alg), Alg, 1);
```

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D_1 \end{bmatrix}, \begin{bmatrix} -D_1 & D_1\,\delta & 1 \\ D_1\,\delta & -D_1 & \delta \\ -2\,\delta & 1+\delta^2 & 0 \end{bmatrix}, \begin{bmatrix} 1+\delta^2 \\ 2\,\delta \\ D_1 - D_1\,\delta^2 \end{bmatrix}, \begin{bmatrix} 1+\delta^2 & -2\,\delta & -D_1 + D_1\,\delta^2 \end{bmatrix} \right]$$

We find: $t(M) \cong \mathrm{ext}^1_D(N, D) \neq \{0\}$.

E.g. $m := (-2\,\delta)\,y_1 + (1+\delta^2)\,y_2 \in M$ is torsion element: $D_1\,m = 0$.

An electric transmission line (Mounier)

Definition of Ore algebra and differential delay system:

```
> with(OreModules):
> Alg := diff_algebra([Dt,t], [delta,s], polynom={t,s},
>       comm={a[0],a[1],a[2],a[3],a[4],a[5],b[0]}):
> R := evalm([[Dt+a[0],-(a[4]*Dt+a[0])*delta,-a[0],0,-b[0]*Dt],
>       [-delta*(a[5]*Dt+a[1]),Dt+a[1],0,a[1],0],
>       [a[2],-a[2]*a[4]*delta,Dt,0,-a[2]*b[0]],
>       [a[3]*a[5]*delta,-a[3],0,Dt,0]]);
```

$$R := \begin{bmatrix} Dt + a_0 & -(a_4\, Dt + a_0)\,\delta & -a_0 & 0 & -b_0\, Dt \\ -\delta\,(a_5\, Dt + a_1) & Dt + a_1 & 0 & a_1 & 0 \\ a_2 & -a_2\, a_4\, \delta & Dt & 0 & -a_2\, b_0 \\ a_3\, a_5\, \delta & -a_3 & 0 & Dt & 0 \end{bmatrix}$$

Computing $\text{ext}^1_D(N, D)$, where $D = Alg$: (CPU: $\approx 10s$)

```
> Ext1 := exti(adjoint(R, Alg), Alg, 1):  Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Longleftrightarrow$ $\text{ext}^1_D(N, D) \cong t(M) = \{0\}$.

$\Longleftrightarrow$ System is controllable $\Longleftrightarrow$ parametrizable.

`exti` also constructs a parametrization of the system:

```
> Ext1[3];
```

$$\left[\begin{array}{l} Dt^4 b_0 + a_1 b_0 Dt^3 + a_0 a_2 b_0 Dt^2 + a_1 b_0 Dt^2 a_3 + a_0 a_1 a_2 b_0 Dt + a_1 b_0 a_3 a_2 a_0 \\ a_0 a_1 a_2 Dt\delta b_0 + a_0 a_1 a_2 a_5 b_0 a_3 \delta + a_0 a_2 a_5 Dt^2 \delta b_0 + a_5 Dt^4 \delta b_0 + a_1 a_5 b_0 \delta Dt^2 a_3 + a_1 Dt^3 \delta b_0 \\ a_1 b_0 a_3 a_2 a_0 - a_0 a_1 a_2 a_5 b_0 \delta^2 a_3 + a_0 a_1 a_2 b_0 Dt - a_0 a_1 a_2 Dt\delta^2 b_0 - a_0 a_2 a_5 Dt^2 \delta^2 b_0 + a_0 a_2 b_0 Dt^2 \\ a_0 a_1 a_2 b_0 a_3 \delta - a_0 a_1 a_2 a_5 b_0 a_3 \delta + a_1 b_0 a_3 \delta Dt^2 - a_1 a_5 b_0 \delta Dt^2 a_3 \\ a_1 a_0 Dt^2 + Dt^4 + a_1 Dt^3 + a_0 Dt^3 - a_1 Dt^3 \delta^2 a_4 + a_1 a_3 Dt^2 + a_0 a_2 Dt^2 - a_0 a_1 a_2 \delta^2 Dt a_4 \\ \qquad -a_0 a_1 \delta^2 Dt a_3 a_5 - a_0 a_2 a_5 \delta^2 Dt^2 a_4 - a_0 Dt^3 \delta^2 a_5 + a_0 a_1 a_2 a_3 + a_0 a_1 a_2 Dt \\ \qquad +a_0 a_1 a_3 Dt - a_0 a_1 \delta^2 Dt^2 - a_0 a_1 a_2 a_5 \delta^2 a_3 a_4 - a_5 \delta^2 Dt^4 a_4 - a_1 a_5 \delta^2 Dt^2 a_3 a_4 \end{array}\right]$$

Computing $\operatorname{ext}^2_D(N, D)$, where $D = Alg$: (CPU: $5s$)

```
> Ext2 := exti(adjoint(R, Alg), Alg, 2):  Ext2[1];
```

$$\left[\begin{array}{l} -Dt a_0 a_1 a_2 \delta + \delta a_0 a_5 a_2 a_1 Dt - \delta a_1{}^2 a_5 a_3 Dt + \delta Dt a_1{}^2 a_3 + \delta^3 a_1{}^2 a_2 a_0 + a_0{}^2 a_5{}^2 a_2{}^2 \delta^3 \\ \qquad -2\, a_0 a_5{}^2 a_2 a_1 \delta^3 a_3 + a_1{}^2 a_5{}^2 a_3{}^2 \delta^3 + 2\, a_1 a_5 a_3 a_0 a_2 \delta - \delta a_1{}^2 a_5 a_3{}^2 - a_1{}^2 \delta a_2 a_0 - a_0{}^2 a_5 a_2{}^2 \delta \\ \delta^2 Dt a_1 - Dt^2 - a_5 \delta^2 a_0 a_2 + a_1 a_5 \delta^2 a_3 - a_1 Dt - a_1 a_3 \\ \delta a_2 a_0 + Dt^2 \delta \\ a_1 Dt^3 + a_1{}^2 Dt^2 + a_0 a_2 a_5 Dt^2 - Dt^2 a_1 a_5 a_3 + \delta^2 a_1{}^2 a_2 a_0 + a_0{}^2 a_5{}^2 a_2{}^2 \delta^2 - 2\, a_0 a_5{}^2 a_2 a_1 \delta^2 a_3 \\ \qquad +a_1{}^2 a_5{}^2 a_3{}^2 \delta^2 + Dt a_1{}^2 a_3 + a_0 a_5 a_2 a_1 Dt - a_1{}^2 a_5 a_3 Dt + a_0 a_2 a_5 a_1 a_3 - a_1{}^2 a_5 a_3{}^2 \end{array}\right]$$

Einstein equations (lin. Ricci equations in vacuum)

Underdetermined PDE system – is it *parametrizable*?

$\Longleftrightarrow$ J. Wheeler: Exist potentials for Einstein equations?

Answer (Pommaret 1995): *No.* Let us prove this by using *OreModules*:

Definition of Ore algebra and PDE system:

```
> with(OreModules):
> Alg := diff_algebra([D[1],x1], [D[2],x2], [D[3],x3], [D[4],x4], polynom={x1,x2,x3,x4}):
> R := evalm([ ... ]);
```

$$R := \begin{bmatrix} \%1 & D_1^2 & D_1^2 & -D_1^2 & -2\,D_1\,D_2 & 0 & 0 & -2\,D_1\,D_3 & 0 & 2\,D_1\,D_4 \\ D_2^2 & \%2 & D_2^2 & -D_2^2 & -2\,D_1\,D_2 & -2\,D_2\,D_3 & 0 & 0 & 2\,D_2\,D_4 & 0 \\ D_3^2 & D_3^2 & \%3 & -D_3^2 & 0 & -2\,D_2\,D_3 & 2\,D_3\,D_4 & -2\,D_1\,D_3 & 0 & 0 \\ D_4^2 & D_4^2 & D_4^2 & \%4 & 0 & 0 & -2\,D_3\,D_4 & 0 & -2\,D_2\,D_4 & -2\,D_1\,D_4 \\ 0 & 0 & D_1\,D_2 & -D_1\,D_2 & D_3^2-D_4^2 & -D_1\,D_3 & 0 & -D_2\,D_3 & D_1\,D_4 & D_2\,D_4 \\ D_2\,D_3 & 0 & 0 & -D_2\,D_3 & -D_1\,D_3 & D_1^2-D_4^2 & D_2\,D_4 & -D_1\,D_2 & D_3\,D_4 & 0 \\ D_3\,D_4 & D_3\,D_4 & 0 & 0 & 0 & -D_2\,D_4 & D_1^2+D_2^2 & -D_1\,D_4 & -D_2\,D_3 & -D_1\,D_3 \\ 0 & D_1\,D_3 & 0 & -D_1\,D_3 & -D_2\,D_3 & -D_1\,D_2 & D_1\,D_4 & D_2^2-D_4^2 & 0 & D_3\,D_4 \\ D_2\,D_4 & 0 & D_2\,D_4 & 0 & -D_1\,D_4 & -D_3\,D_4 & -D_2\,D_3 & 0 & D_1^2+D_3^2 & -D_1\,D_2 \\ 0 & D_1\,D_4 & D_1\,D_4 & 0 & -D_2\,D_4 & 0 & -D_1\,D_3 & -D_3\,D_4 & -D_1\,D_2 & D_2^2+D_3^2 \end{bmatrix}$$

$\%1 := D_2^2 + D_3^2 - D_4^2, \quad \%2 := D_1^2 + D_3^2 - D_4^2, \quad \%3 := D_1^2 + D_2^2 - D_4^2, \quad \%4 := D_1^2 + D_2^2 + D_3^2$

Computing $\operatorname{ext}^1_D(N, D)$, where $D = Alg$: (CPU: $22s$)

```
> Ext1 := exti(adjoint(R, Alg), Alg, 1): Ext1[1];
```

$$\underbrace{\begin{bmatrix} \%1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \%1 \end{bmatrix}}_{20} \qquad \%1 := \mathrm{D}_4{}^2 - \mathrm{D}_3{}^2 - \mathrm{D}_2{}^2 - \mathrm{D}_1{}^2$$

We find: $t(M) \cong \operatorname{ext}^1_D(N, D) \neq \{0\}$.

More precisely, $t(M)$ is generated by 20 torsion elements, and these satisfy the *Dalambertian equation* $(\Delta - c^2 \frac{\partial^2}{\partial t^2})\, y = 0$.

$\Rightarrow$ The linearized Ricci equations cannot be parametrized.

Of course, $M/t(M)$ can be parametrized:

```
> Ext1[3];
```

$$\begin{bmatrix} -2\,\mathrm{D}_1 & 0 & 0 & 0 \\ 0 & 0 & -2\,\mathrm{D}_2 & 0 \\ 0 & 0 & 0 & -2\,\mathrm{D}_3 \\ 0 & -2\,\mathrm{D}_4 & 0 & 0 \\ -\mathrm{D}_2 & 0 & -\mathrm{D}_1 & 0 \\ 0 & 0 & -\mathrm{D}_3 & -\mathrm{D}_2 \\ 0 & -\mathrm{D}_3 & 0 & -\mathrm{D}_4 \\ -\mathrm{D}_3 & 0 & 0 & -\mathrm{D}_1 \\ 0 & -\mathrm{D}_2 & -\mathrm{D}_4 & 0 \\ -\mathrm{D}_4 & -\mathrm{D}_1 & 0 & 0 \end{bmatrix}$$

`exti` also gives a generating set for $t(M)$:

```
> Ext1[2];
```

$$
\left[\begin{array}{cccccccccc}
D_2^2 & D_1^2 & 0 & 0 & -2\,D_1\,D_2 & 0 & 0 & 0 & 0 & 0 \\
-D_2\,D_4 & 0 & 0 & 0 & D_1\,D_4 & 0 & 0 & 0 & -D_1^2 & D_1\,D_2 \\
0 & D_1\,D_4 & 0 & 0 & -D_2\,D_4 & 0 & 0 & 0 & -D_1\,D_2 & D_2^2 \\
0 & 0 & 0 & 0 & 0 & -D_1\,D_4 & 0 & D_2\,D_4 & D_1\,D_3 & -D_2\,D_3 \\
-D_2\,D_3 & 0 & 0 & 0 & D_1\,D_3 & -D_1^2 & 0 & D_1\,D_2 & 0 & 0 \\
0 & D_1\,D_3 & 0 & 0 & -D_2\,D_3 & -D_1\,D_2 & 0 & D_2^2 & 0 & 0 \\
0 & 0 & 0 & D_1\,D_2 & D_4^2 & 0 & 0 & 0 & -D_1\,D_4 & -D_2\,D_4 \\
0 & D_4^2 & 0 & D_2^2 & 0 & 0 & 0 & 0 & -2\,D_2\,D_4 & 0 \\
D_4^2 & 0 & 0 & D_1^2 & 0 & 0 & 0 & 0 & 0 & -2\,D_1\,D_4 \\
0 & 0 & 0 & -D_1\,D_3 & 0 & 0 & D_1\,D_4 & -D_4^2 & 0 & D_3\,D_4 \\
0 & 0 & 0 & -D_2\,D_3 & 0 & -D_4^2 & D_2\,D_4 & 0 & D_3\,D_4 & 0 \\
0 & 0 & 0 & 0 & D_3\,D_4 & -D_1\,D_4 & D_1\,D_2 & 0 & 0 & -D_2\,D_3 \\
0 & D_3\,D_4 & 0 & 0 & 0 & -D_2\,D_4 & D_2^2 & 0 & -D_2\,D_3 & 0 \\
D_3\,D_4 & 0 & 0 & 0 & 0 & 0 & D_1^2 & -D_1\,D_4 & 0 & -D_1\,D_3 \\
0 & 0 & D_1\,D_4 & 0 & 0 & 0 & -D_1\,D_3 & -D_3\,D_4 & 0 & D_3^2 \\
0 & 0 & D_2\,D_4 & 0 & 0 & -D_3\,D_4 & -D_2\,D_3 & 0 & D_3^2 & 0 \\
0 & 0 & D_1\,D_2 & 0 & D_3^2 & -D_1\,D_3 & 0 & -D_2\,D_3 & 0 & 0 \\
0 & 0 & D_4^2 & D_3^2 & 0 & 0 & -2\,D_3\,D_4 & 0 & 0 & 0 \\
0 & D_3^2 & D_2^2 & 0 & 0 & -2\,D_2\,D_3 & 0 & 0 & 0 & 0 \\
D_3^2 & 0 & D_1^2 & 0 & 0 & 0 & 0 & -2\,D_1\,D_3 & 0 & 0
\end{array}\right]
$$

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