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ON A GENERAL STRUCTURE OF THE STABILIZING CONTROLLERS BASED ON STABLE RANGE*

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Abstract. In this paper, we prove that some stabilizing controllers of a plant, which admits a left/right-coprime factorization, have a special form where their stable and unstable parts are separated. The dimension of the unstable part depends on the algebraic concept of stable range of the ring A of SISO stable plants. Moreover, we prove that, if the stable range of A is equal to 1, then every plant—defined by a transfer matrix with entries in the quotient field of A and admitting a left/right-coprime factorization—can be stabilized by a stable controller (strong stabilization). In particular, using a result of Treil proving that the stable range of $H_\infty(\mathbb{D})$ is equal to 1, we show that every stabilizable plant—defined by a transfer matrix with entries in the quotient field of $H_\infty(\mathbb{D})$ or $H_\infty(\mathbb{C}_+)$—is strongly stabilizable and, equivalently, every couple of stabilizable plants can be simultaneously stabilized by a controller (simultaneous stabilization). Finally, using the fact that the topological stable range of $H_\infty(\mathbb{D})$ is equal to 2, a result due to Suárez, we show that every unstabilizable SISO plant—defined by a transfer function with entries in the quotient field of $H_\infty(\mathbb{D})$—is as close as we want to a stabilizable plant in the product topology.

Key words. stabilizing controllers, strong stabilization, simultaneous stabilization, stable range, k-stability, topological stable range, unit 1-stable range, n-fold rings, H_∞, K-theory

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1. Introduction.

The fractional representation approach to analysis and synthesis problems was developed in the eighties in order to express in a unique mathematical framework several questions on stabilization problems. In that framework, we can study internal stabilization (existence of an internally stabilizing controller), parametrization of all stabilizing controllers, strong stabilization (possibility of stabilizing a plant by means of a stable controller), simultaneous stabilization (possibility of stabilizing a set of plants by means of a single controller), metrics of robustness (gap or graph topologies), H_∞ or H_2-optimal controllers, etc. See [2, 6, 42] for more details.

Recently, the reformulation of the fractional representation approach to analysis and synthesis problems within an *algebraic analysis approach* has allowed us to obtain new necessary and sufficient conditions for internal stabilizability and for the existence of (weakly) left/right/doubly coprime factorizations in the general setting [25, 26, 24]. Moreover, all the rings of SISO stable plants (used in this framework) over which one of the previous properties is satisfied were completely characterized [25, 26, 24]. In [27, 28], a new parametrization of all stabilizing controllers of a stabilizable plant was developed. It generalizes the Youla–Kučera parametrization [42] for stabilizable plants which do not necessarily admit doubly coprime factorizations. All these results show that a natural mathematical framework for the study of stabilization problems is the so-called *K-theory* [22, 32]. See [29] for more details.

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The purpose of this paper is to show that the concept of *stable range* developed in K-theory also plays an important role in the study of the strong and simultaneous stabilization problems [42]. Using the fractional representation approach to synthesis problems [6, 42], we show that, if the transfer matrix P, with entries in the quotient field of an integral domain A of SISO stable plants (e.g., $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$ or W_+), admits a left-coprime factorization $P = D^{-1} N$, then there exist some stabilizing controllers of P having separated stable and unstable parts. In particular, we show that the dimension of the unstable part is related to the concept of k-*stability* of the matrix $R = (D : \, -N)$ with entries in A [17, 41]. Moreover, using some relations between the k-stability of a matrix with entries in A and the concept of *stable range* $\mathrm{sr}(A)$ *of* A [1, 7, 41], we prove that there exist some stabilizing controllers of P which are such that their unstable parts are defined by $\mathrm{sr}(A) - 1$ unstable rows. Therefore, if the stable range $\mathrm{sr}(A)$ of A is 1, then every transfer matrix which admits a left-coprime factorization is strongly stabilizable; i.e., it is internally stabilized by a stable controller. In particular, using the fact that the stable range of $H_\infty(\mathbb{D})$ is equal to 1 (see [38]), we prove that every stabilizable plant, defined by means of a transfer matrix with entries in the quotient field of $H_\infty(\mathbb{D})$ or $H_\infty(\mathbb{C}_+)$, is strongly stabilizable (strong stabilization). Let us notice that this result answers one of the questions asked in [9]. Moreover, using a result of Vidyasagar [42], we prove that every couple of plants, defined by transfer matrices with entries in $H_\infty(\mathbb{D})$ or $H_\infty(\mathbb{C}_+)$, is simultaneously stabilized by a controller (simultaneous stabilization). Finally, introducing the concept of *topological stable range*, we show that every unstabilizable SISO plant, defined by a transfer function $p = n/d$, with $0 \neq d, \; n \in H_\infty(\mathbb{D})$, is as close as we want to a stabilizable plant in the product topology.

Plan of the paper. In section 2, we give the definition of the stable range of a ring A and present some examples which will be used in the rest of the paper. In section 3, we introduce the concept of k-stability of a matrix with entries in a ring A. We recall the fractional representation approach to analysis and synthesis problems in section 4. In section 5, we give the first main result of this paper concerning the form of certain stabilizing controllers (Theorem 5.1) and examples in order to illustrate this result. Exploiting the relations between k-stability of a matrix with entries in a ring A and the stable range of A, we give the second main result of the paper (Corollary 6.4) and its corollaries (Corollaries 6.5 and 6.6). In the last section, we introduce the definitions of topological stable range, unit 1-stable range, and n-fold ring, and give some applications of these concepts to some stabilization problems.

Notation. A will denote a commutative ring with a unit [33], $A^{q\times p}$ the set of $q \times p$ matrices with entries in A, I_p the identity matrix of $A^{p\times p}$, and

$$\mathrm{GL}_p(A) = \{R \in A^{p\times p} \mid \exists \, S \in A^{p\times p} : \; R\,S = S\,R = I_p\}$$

the group of invertible elements of $A^{p\times p}$. If $R \in A^{q\times p}$, then $R^T \in A^{p\times q}$ is the transposed matrix. If A is an integral domain (i.e., $a\,b = 0, \; a \neq 0 \Rightarrow b = 0$), then we shall denote the *field of fractions* of A by $K = Q(A) = \{n/d \mid d \neq 0, \, n \in A\}$. Finally, p and q will always denote two positive integers satisfying $p \geq q$ ($p - q$ will denote the number of input variables for the transfer matrices) and $\triangleq$ will mean "by definition."

2. Stable range of a commutative ring.

2.1. Definition. Let us give some definitions that will be constantly used in this paper.

Definition 2.1 (see [1, 4, 7, 41]). *We have the following definitions and notation:*

- *A vector* $a = (a_1 : \cdots : a_n) \in A^{1\times n}$ *is said to be* unimodular *if there exists a vector* $b = (b_1 : \ \cdots : \ b_n) \in A^{1\times n}$ *such that* $a\, b^T = \sum_{i=1}^n a_i\, b_i = 1$.
- *We denote the set of all the unimodular vectors of* $A^{1\times n}$ *by* $\mathrm{U}_n(A)$.

Let us notice that $\mathrm{U}_1(A)$ is the set of the units $\mathrm{U}(A) = \{a \in A \mid a^{-1} \in A\}$ of A.

Example 2.1. Let us take $A = H_\infty(\mathbb{C}_+)$, where $H_\infty(\mathbb{C}_+)$ is the algebra of $\mathbb{C}$-valued holomorphic functions on the open right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$ which are bounded w.r.t. the norm $\| f \|_\infty = \sup_{s\in\mathbb{C}_+} |f(s)|$. See [5] for more details. The vector $a = (\frac{s-1}{s+1} : \ \frac{e^{-s}}{s+1}) \in A^{1\times 2}$ is unimodular because we have

$$\left(\frac{s-1}{s+1}\right) \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(\frac{e^{-s}}{s+1}\right) 2\, e = 1, \qquad 1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right),\ 2\, e \in A.$$

Definition 2.2 (see [1, 4, 7, 41]). *A vector* $a = (a_1 : \cdots : a_n) \in \mathrm{U}_n(A)$ *is called* stable *(or* reductible*) if there exists an* $(n-1)$*-tuple* $b = (b_1 : \ \cdots : b_{n-1}) \in A^{1\times(n-1)}$ *such that*

$$(a_1 + a_n\, b_1 : \ \cdots : a_{n-1} + a_n\, b_{n-1}) \in \mathrm{U}_{n-1}(A);$$

i.e., there exists $(c_1 : \cdots : c_{n-1}) \in A^{1\times(n-1)}$ *such that* $\sum_{i=1}^{n-1} (a_i + a_n\, b_i)\, c_i = 1$.

Example 2.2. We have the following examples:

- Let us consider $A = H_\infty(\mathbb{C}_+)$ and $a = (1 - e^{-2s} : \ 1 + e^{-2s}) \in A^{1\times 2}$. We have

$$\frac{1}{2}(1 - e^{-2s}) + \frac{1}{2}(1 + e^{-2s}) = 1 \Rightarrow (1 - e^{-2s}) + (1 + e^{-2s}) = 2 \in \mathrm{U}_1(A), \tag{2.1}$$

  and thus, a is a stable vector of $\mathrm{U}_2(A)$.
- Let $A = RH_\infty = \mathbb{R}(s) \cap H_\infty(\mathbb{C}_+)$ be the $\mathbb{R}$-algebra of proper and stable real rational functions [42]. The vector

$$a = \left(\frac{(s-1)(s-2)}{(s+1)^2} : \ \frac{s}{(s+1)^2}\right) \in A^{1\times 2}$$

  is stable because we have

$$\frac{(s-1)(s-2)}{(s+1)^2} + \frac{6\, s}{(s+1)^2} = \frac{(s+2)}{(s+1)} \in \mathrm{U}_1(A). \tag{2.2}$$

Remark 2.1. If a vector $(a_1 : \ a_2) \in \mathrm{U}_2(A)$ is stable, then, in general, this is not the case for $(a_2 : \ a_1) \in \mathrm{U}_2(A)$. For instance, if $A = \mathbb{R}[s]$, then $(s^2 + 1 : \ s) \in A^{1\times 2}$ is a stable vector because we have $(s^2 + 1) + s\,(-s) = 1 \in \mathrm{U}_1(A)$, whereas the vector $(s : \ s^2 + 1) \in U_2(A)$ is not stable because there does not exist $b \in A$ such that $r \triangleq s + (s^2 + 1)\, b(s) \in A$ is invertible, i.e., is a nonzero real constant (the degree of the polynomial r is at least 1).

Definition 2.3 (see [31, 34, 38, 41]). *We call the* stable range $\mathrm{sr}(A)$ *of* A *the smallest* $n \in \mathbb{N} \cup \{+\infty\}$ *such that every vector of* $\mathrm{U}_{n+1}(A)$ *is stable.*

Let us notice that the stable range $\mathrm{sr}(A)$ is also called the *stable rank* of A.

Remark 2.2. Let us notice that if $\mathrm{sr}(A) = n$, then, for $m \geq n$, every element of $U_m(A)$ is stable [11]. Indeed, if $(a_1 : \cdots : a_{n+2}) \in U_{n+2}(A)$, then there exist $b_1, \ldots, b_{n+2} \in A$ such that $\sum_{i=1}^{n+2} a_i\, b_i = 1$. Hence, the vector

$$(a_1 : \cdots : a_n : \ a_{n+1}\, b_{n+1} + a_{n+2}\, b_{n+2}) \in A^{1\times(n+1)}$$

is unimodular. Using the fact that $\mathrm{sr}(A) = n$, there exist $c_1, \ldots, c_n \in A$ such that the vector

$$(a_1 + c_1\,(a_{n+1}\, b_{n+1} + a_{n+2}\, b_{n+2}) : \cdots : a_n + c_n\,(a_{n+1}\, b_{n+1} + a_{n+2}\, b_{n+2})) \in A^{1\times n}$$

is unimodular; i.e., there exist $d_1, \ldots, d_n \in A$ such that

$$\sum_{i=1}^{n} (a_i + c_i\,(a_{n+1}\,b_{n+1} + a_{n+2}\,b_{n+2}))\,d_i = 1$$

$$\Rightarrow \sum_{i=1}^{n} (a_i + c_i\,a_{n+2}\,b_{n+2})\,d_i + a_{n+1}\left(\sum_{i=1}^{n} b_{n+1}\,c_i\,d_i\right) = 1,$$

which shows that $(a_1+(c_1\,b_{n+2})\,a_{n+2} : \cdots : a_n+(c_n\,b_{n+2})\,a_{n+2} : a_{n+1})$ is unimodular, and thus the vector $(a_1 : \cdots : a_{n+2}) \in U_{n+2}(A)$ is a stable vector. The result directly follows by induction on n.

Example 2.3. We have the following interpretations of $\mathrm{sr}(A) = 2$ and $\mathrm{sr}(A) = 1$:

- A ring A has a stable range $\mathrm{sr}(A) = 2$ iff, $\forall\, n \geq 3$, every element of $\mathrm{U}_n(A)$ is stable and there exists a vector $(a_1 : a_2) \in \mathrm{U}_2(A)$ such that, for every $b \in A$, $a_1 + a_2\,b \notin \mathrm{U}_1(A)$, i.e., $a_1 + a_2\,b$ is not invertible.
- A ring A has a stable range $\mathrm{sr}(A) = 1$ iff, for every $(a_1 : a_2) \in \mathrm{U}_2(A)$, there exists $b \in A$ such that $a_1 + a_2\,b \in \mathrm{U}_1(A)$, i.e., $a_1 + a_2\,b$ is invertible.

2.2. Examples.

THEOREM 2.4 (see [38]). *If $\mathbb{D}$ denotes the open unit disc and $H_\infty(\mathbb{D})$ the ring of $\mathbb{C}$-valued holomorphic functions on $\mathbb{D}$ which are bounded w.r.t. the norm $\| f \|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$, then we have*

$$\mathrm{sr}(H_\infty(\mathbb{D})) = 1.$$

COROLLARY 2.5. *With the notation of Example* 2.1*, we have*

$$\mathrm{sr}(H_\infty(\mathbb{C}_+)) = 1.$$

Proof. Let us consider a unimodular matrix $a = (a_1 : a_2) \in U_2(H_\infty(\mathbb{C}_+))$. Let us denote by $(b_1 : b_2)^T \in H_\infty(\mathbb{C}_+)^{2\times 1}$ a right-inverse of a; i.e., we have

$$a_1(s)\,b_1(s) + a_2(s)\,b_2(s) = 1. \tag{2.3}$$

The fractional linear transformation $s = \psi(z) = (1+z)/(1-z)$ bijectively maps the open unit disc $\mathbb{D}$ on the open right half plane $\mathbb{C}_+$ and $z = \psi^{-1}(s) = (s-1)/(s+1)$. Moreover, from Lemma A.6.15 of [5], we have $f \in H_\infty(\mathbb{C}_+) \Leftrightarrow f \circ \psi \in H_\infty(\mathbb{D})$. Thus, from (2.3), we deduce

$$(a_1 \circ \psi)(z)\,(b_1 \circ \psi)(z) + (a_2 \circ \psi)(z)\,(b_2 \circ \psi)(z) = 1 \circ \psi = 1, \tag{2.4}$$

i.e., $(a_1 \circ \psi : a_2 \circ \psi) \in \mathrm{U}_2(\mathrm{H}_\infty(\mathbb{D}))$. By Theorem 2.4, we know that $\mathrm{sr}(H_\infty(\mathbb{D})) = 1$, and thus there exist $c,\, d \in H_\infty(\mathbb{D})$ such that

$$((a_1 \circ \psi)(z) + (a_2 \circ \psi)(z)\,c(z))\,d(z) = 1 \Leftrightarrow (a_1(s) + a_2(s)\,c(\psi^{-1}(s)))\,d(\psi^{-1}(s)) = 1;$$

i.e., $a = (a_1 : a_2)$ is 1-stable, and thus $\mathrm{sr}(H_\infty(\mathbb{C}_+)) = 1$. □

THEOREM 2.6 (see [1]). *If A is a principal ideal domain, namely, an integral domain such that every ideal of A can be generated by a single element of A, then* $\mathrm{sr}(A) \leq 2$.

COROLLARY 2.7. *Let RH_∞ be the ring of proper and stable real rational functions. Then, we have*

$$\mathrm{sr}(RH_\infty) = 2.$$

Proof. It is well known that RH_∞ is a principal ideal domain [42]. Therefore, by Theorem 2.6, we obtain that $\mathrm{sr}(RH_\infty) \leq 2$. Finally, let $(d:\ n) \in \mathrm{U}_2(RH_\infty)$ with $d \neq 0$ and let us define the transfer function $P = n/d \in \mathbb{R}(s) = Q(RH_\infty)$. Let us notice that $P = n/d$ is a coprime factorization of P because $(d:\ n) \in \mathrm{U}_2(RH_\infty)$. Now, it is also well known that there exists $c \in RH_\infty$ such that $d + c\, n$ is a unit of RH_∞ iff P has the *parity interlacing property* [2, 42], namely, P has an even number of real poles between every pair of real zeros in $\{\mathrm{Re}\, s \geq 0\} \cup \{\infty\}$. Hence, there exist vectors $(d:\ n) \in \mathrm{U}_2(RH_\infty)$ which are not stable in the sense of Definition 2.2 (e.g., $((s-1)/(s+1):\ s/(s+1)^2) \in \mathrm{U}_2(RH_\infty)$ is not stable because the transfer function $P = s/((s+1)\,(s-1))$ does not have the parity interlacing property—see Example 4 of section 3.2 of [42]). Therefore, we have $\mathrm{sr}(RH_\infty) = 2$. $\square$

Let us give more examples of stable ranges of integral domains.

THEOREM 2.8. *We have the following results:*

- [12, 41] $\mathrm{sr}(\mathbb{R}[x_1, \ldots, x_n]) = n + 1$.
- [19] *The ring of entire functions*

$$E(k) = \left\{ f(s) = \sum_{n=0}^{+\infty} a_n\, s^n \;\middle|\; s \in \mathbb{C},\ a_n \in k,\ \lim_{n \to +\infty} |a_n|^{1/n} = 0 \right\}$$

  satisfies $\mathrm{sr}(E(k)) = 1$ *if* $k = \mathbb{C}$ *and* 2 *if* $k = \mathbb{R}$.
- [20] *The disc algebra $A(\mathbb{D})$, i.e., the ring of functions which are holomorphic in the open unit disc $\mathbb{D}$ and continuous on the unit circle $\mathbb{T}$, satisfies* $\mathrm{sr}(A(\mathbb{D})) = 1$.
- [34] *If we denote by W_+ the Wiener algebra defined by*

$$W_+ = \left\{ \sum_{n=0}^{+\infty} a_n\, z^n \;\middle|\; \sum_{n=0}^{+\infty} |\, a_n \,| < +\infty \right\},$$

  then we have $\mathrm{sr}(W_+) = 1$.

Let us recall that the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ is used in the study of multidimensional systems, W_+ represents the sets of l_∞-stable (bounded input bounded output stability) shift-invariant causal digital filters [42], and the disc algebra $A(\mathbb{D})$ is used for interpolation problems and discrete-time control systems [42]. Finally, $E(\mathbb{R})$ is used in the study of a certain class of time-delay systems $\mathcal{E} = E(\mathbb{R}) \cap \mathbb{R}(s)[e^{-s}]$ [21].

3. *k*-stability for matrices. Let us extend the definition of k-stability for matrices with entries in A.

DEFINITION 3.1 (see [11, 17, 41]). *A matrix $R \in A^{q \times p}$ is* unimodular *if there exists a matrix $S \in A^{p \times q}$ such that $R\, S = I_q$, i.e., R has a right-inverse S.*

Remark 3.1. First, let us notice that the previous concept of a unimodular matrix is standard in commutative algebra, whereas, in control theory, a unimodular matrix usually denotes a square matrix $R \in A^{p \times p}$ such that there exists $S \in A^{p \times p}$ satisfying $R\, S = S\, R = I_p$. The reader should be careful not to confuse these two different definitions (only Definition 3.1 will be used in the course of the paper).

Second, if $R \in A^{q \times p}$ is a unimodular matrix, then it is clear that R has *full row rank*, namely its rows are A-linearly independent. Moreover, the A-submodule $A^{1 \times q}\, R$ of $A^{1 \times p}$ generated by the A-linear combinations of the rows of R is isomorphic to $A^{1 \times q}$, and thus we have $1 \leq q \leq p$.

If $R_i \in A^{q \times 1}$ is a column vector, then we shall denote by $\mathrm{col}(R_1, \ldots, R_p)$ the $q \times p$ matrix R whose first column is R_1, whose second one is $R_2, \ldots,$ and whose last column is R_p.

LEMMA 3.2. *$R = \mathrm{col}(R_1 : \cdots : R_p) \in A^{q\times p}$ is unimodular iff the A-module*

$$R\,A^p \triangleq \sum_{i=1}^{p} R_i\,A = \left\{\sum_{i=1}^{p} R_i\,a_i \in A^q \mid a_i \in A\right\}$$

is equal to A^q.

Proof. $\Rightarrow$ Let R be unimodular. Then there exists $S \in A^{p\times q}$ such that $R\,S = I_q$. Therefore, for every $\lambda \in A^q$, the vector $\mu = S\,\lambda \in A^p$ is such that $\lambda = R\,\mu$, and thus $\lambda = \sum_{i=1}^{p} R_i\,\mu_i \in R\,A^p$, where $\mu = (\mu_1 : \cdots : \mu_p)^T$. Hence, we have $R\,A^p = A^q$.

$\Leftarrow$ Let us suppose that $R\,A^p = A^q$. Then, for every $\lambda \in A^q$, there exists $(a_i)_{1\le i\le p}$, with $a_i \in A$, such that $\lambda = \sum_{i=1}^{p} R_i\,a_i$. In particular, for $j = 1,\ldots,q$, let us consider the vector e_j of A^q defined by 1 in the jth component and 0 elsewhere. Then, for $j = 1,\ldots,q$, there exists $S_j \in A^p$ such that $e_j = R\,S_j$, and thus, if we define $S = \mathrm{col}(S_1 : \cdots : S_q) \in A^{p\times q}$, then we have $R\,S = I_q$; i.e., R is unimodular. ∎

Let us introduce the concept of k-stability for unimodular matrices.

DEFINITION 3.3 (see [17, 41]). *A unimodular matrix $R = \mathrm{col}(R_1,\ldots,R_p) \in A^{q\times p}$ is called* k-stable $(1 \le k \le p-q)$ *if there exists a $(p-k)$-tuple $(c_i)_{1\le i\le p-k}$ belonging to the A-module*

$$R_{p-k+1}\,A + \cdots + R_p\,A \triangleq \left\{\sum_{i=1}^{k} R_{p-k+i}\,b_i \mid b_i \in A\right\} \tag{3.1}$$

such that the matrix

$$\mathrm{col}(R_1 + c_1 : R_2 + c_2 : \cdots : R_{p-k} + c_{p-k}) \in A^{q\times(p-k)}$$

is unimodular.

Remark 3.2. Let us notice that a vector $a \in \mathrm{U}_n(A)$ is 1-stable iff a is stable in the sense of Definition 2.2.

LEMMA 3.4. *A unimodular matrix $R \in A^{q\times p}$ is k-stable iff there exists a matrix $T_k \in A^{k\times(p-k)}$ such that the matrix*

$$R_k = \mathrm{col}(R_1 : \cdots : R_{p-k}) + \mathrm{col}(R_{p-k+1} : \cdots : R_p)\,T_k \in A^{q\times(p-k)} \tag{3.2}$$

is unimodular.

Proof. $\Rightarrow$ Let R be a k-stable matrix; then there exists a $(p-k)$-tuple $(c_i)_{1\le i\le p-k}$ of elements of the A-module (3.1) such that $\mathrm{col}(R_1 + c_1 : \cdots : R_{p-k} + c_k) \in A^{q\times(p-k)}$ is a unimodular matrix. By definition of the c_i, there exists $b_{ij} \in A$ such that

$$c_i = \sum_{j=1}^{k} R_{p-k+j}\,b_{i(p-k+j)}.$$

Therefore, we have

$$\mathrm{col}(R_1 + c_1 : \cdots : R_{p-k} + c_k) = \mathrm{col}(R_1 : \cdots : R_{p-k}) + \mathrm{col}(R_{p-k+1} : \cdots : R_p)\,T_k,$$

where $T_k \in A^{k\times(p-k)}$ is defined by

$$T_k = \begin{pmatrix} b_{1(p-k+1)} & b_{2(p-k+1)} & \cdots & b_{(p-k)(p-k+1)} \\ \vdots & \vdots & & \vdots \\ b_{1p} & b_{2p} & \cdots & b_{(p-k)p} \end{pmatrix}.$$

$\Leftarrow$ All the columns c_i of the matrix $\mathrm{col}(R_{p-k+1} : \cdots : R_p)\, T_k$ belong to the A-module (3.1). Thus, R_k has the form $\mathrm{col}(R_1 + c_1 : \cdots : R_{p-k} + c_k)$; i.e., R is k-stable. □

Example 3.1. Let us consider $A = RH_\infty$ and the following matrix:

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \\ \frac{1}{s+1} & -\frac{s}{s+1} & 0 \end{pmatrix} \in A^{2\times 3}.$$

The matrix

$$R_1 = \begin{pmatrix} \frac{s+2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & -\frac{s}{s+1} \end{pmatrix} = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ \frac{1}{s+1} & -\frac{s}{s+1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{s+1} \\ 0 \end{pmatrix} (-3 : \ -1) \tag{3.3}$$

is invertible ($\det R_1 = -1$), and thus R is 1-stable.

PROPOSITION 3.5. *If R is k-stable, then R is $(k-1)$-stable.*

Proof. Using the fact that R is k-stable, then there exist

$$c_1, \ldots, c_{p-k} \in R_{p-k+1}\, A + \cdots + R_p\, A$$

such that $R_k = \mathrm{col}(R_1 + c_1 : \cdots : R_{p-k} + c_{p-k})$ is unimodular. Let us decompose c_i as $c_i = d_i + e_i$, where $d_i \in R_{p-k+1}\, A$ and $e_i \in R_{p-k+2}\, A + \cdots + R_p\, A$, and let us define $R_{k+1} = \mathrm{col}(R_1 + e_1 : \cdots : R_{p-k} + e_{p-k} : R_{p-k+1})$. Then we claim that R_{k+1} is unimodular, and thus R is $(k-1)$-stable. Indeed, we have

$$\sum_{i=1}^{p-k} (R_i + c_i)\, A \subseteq \sum_{i=1}^{p-k} (R_i + e_i)\, A + R_{p-k+1}\, A \subseteq A^q.$$

Then, applying Lemma 3.2 to R_k, we obtain that $\sum_{i=1}^{p-k} (R_i + c_i)\, A = A^q$, and thus $\sum_{i=1}^{p-k} (R_i + e_i)\, A + R_{p-k+1}\, A = A^q$, which proves that R_{k+1} is unimodular by Lemma 3.2. □

4. Internal stabilization. Let A be an integral domain and let its *field of fractions* be

$$K = Q(A) = \{n/d \mid n \in A,\ 0 \neq d \in A\}.$$

In *the fractional representation approach to analysis and synthesis problems* [5, 6, 42], we consider a class of plants which are defined by means of transfer matrices whose entries belong to the quotient field $K = Q(A)$ of an integral domain of stable SISO plants (see [25, 26, 24, 27] for more details).

Example 4.1. We have the following examples of algebras of SISO stable plants:

- For finite-dimensional systems, we usually consider the integral domain of proper and stable real rational functions $A = RH_\infty = \mathbb{R}(s) \cap H_\infty(\mathbb{C}_+)$ and $K = \mathbb{R}(s)$ [42]. Then, A corresponds to the set of proper and stable real rational transfer functions, whereas an element of $K \backslash A$ represents either an unstable or an improper transfer function. For instance,

  $$P = s/((s-1)\,(s-2)) \in \mathbb{R}(s)$$

  belongs to $K = Q(A)$ because we have $P = n/d$, where $n = s/(s+1)^2 \in A$ by $d = ((s-1)\,(s-2))/(s+1)^2 \in A$.

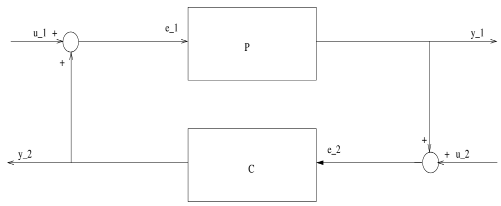

FIG. 4.1. *Closed-loop.*

- For infinite-dimensional systems, we can consider $A = H_\infty(\mathbb{C}_+)$ [36], which gives a class of unstable plants defined by transfer matrices with entries in the quotient field $K = Q(H_\infty(\mathbb{C}_+))$. For instance, the transfer function

$$P = (1 + e^{-2\,s})/(1 - e^{-2\,s})$$

of a wave equation (see, e.g., Exercise 4.24 of [5]) satisfies $P = n/d$, where $n = 1 + e^{-2\,s} \in A$ and $d = 1 - e^{-2\,s} \in A$, and thus we have $P \in K$.

Let us consider a plant defined by the transfer matrix $P \in K^{q\times(p-q)}$, a controller defined by $C \in K^{(p-q)\times q}$, and the closed-loop given by Figure 4.1. We have the following equations:

$$\begin{pmatrix} I_{p-q} & -C \\ -P & I_q \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

DEFINITION 4.1 (see [5, 6, 42]). *A plant defined by the transfer matrix $P \in K^{q\times(p-q)}$ is* internally stabilizable *if there exists a controller $C \in K^{(p-q)\times q}$ such that all the entries of the matrix*

$$(4.1)\quad \begin{pmatrix} I_{p-q} & -C \\ -P & I_q \end{pmatrix}^{-1} = \begin{pmatrix} (I_{p-q} - C\,P)^{-1} & (I_{p-q} - C\,P)^{-1}\,C \\ P\,(I_{p-q} - C\,P)^{-1} & I_q + P\,(I_{p-q} - C\,P)^{-1}\,C \end{pmatrix}$$

$$(4.2)\quad = \begin{pmatrix} I_{p-q} + C\,(I_q - P\,C)^{-1}\,P & C\,(I_q - P\,C)^{-1} \\ (I_q - P\,C)^{-1}\,P & (I_q - P\,C)^{-1} \end{pmatrix}$$

belong to A. Such a controller, $C \in K^{(p-q)\times q}$, is called a stabilizing controller *of P.*

Example 4.2. The controller $C = -(s-1)/(s+1)$ is not a stabilizing controller of the plant $P = s/(s-1)$ because we have

$$\begin{cases} e_1 = \frac{(s+1)}{(2\,s+1)}\,u_1 + \frac{(-s+1)}{(2\,s+1)}\,u_2, \\ e_2 = \frac{s\,(s+1)}{(2\,s+1)(s-1)}\,u_1 + \frac{(s+1)}{(2\,s+1)}\,u_2, \end{cases}$$

and the transfer function between e_2 and u_1 has the unstable pole 1; i.e., it does not belong to RH_∞.

DEFINITION 4.2. *We have the following definitions* [5, 6, 42]:

- *A transfer matrix $P \in K^{q\times(p-q)}$ admits a* left-coprime factorization *if there exist $R = (D :\, -N) \in A^{q\times p}$ and $S = (X^T : Y^T)^T \in A^{p\times q}$ such that*

$$\begin{cases} P = D^{-1}\,N, \\ R\,S = D\,X - N\,Y = I_q. \end{cases}$$

- *A transfer matrix $P \in K^{q\times(p-q)}$ admits a* right-coprime factorization *if there exist $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in A^{p\times(p-q)}$ and $\tilde{S} = (-\tilde{Y} : \tilde{X}) \in A^{(p-q)\times p}$ such that*

$$\begin{cases} P = \tilde{N}\,\tilde{D}^{-1}, \\ \tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} - \tilde{X}\,\tilde{D} = I_{p-q}. \end{cases}$$

- *A transfer matrix $P \in K^{q\times(p-q)}$ admits a* doubly coprime factorization *if P admits both a left and right-coprime factorization.*

PROPOSITION 4.3 (see [42, Theorem 25, p. 105]). *Every transfer matrix $P \in K^{q\times(p-q)}$ which admits a left-coprime factorization $P = D^{-1}\,N$, $D\,X - N\,Y = I_q$, $\det X \neq 0$, is internally stabilized by the controller $C = Y\,X^{-1}$.*

If $P = D_1^{-1}\,N_1 = D_2^{-1}\,N_2$ are two left-coprime factorizations of P and $R_i = (D_i : -N_i)$, for $i = 1, 2$, then there exists a matrix $U \in \mathrm{GL}_q(A)$ such that $R_2 = U\,R_1$. Hence, we deduce that R_q is k-stable iff R_2 is k-stable. A similar result also holds for right-coprime factorizations.

DEFINITION 4.4. *We have the following definitions* [2, 42]:

- *A plant $P \in K^{q\times(p-q)}$ is* strongly stabilizable *if there exists a stable controller $C \in A^{(p-q)\times q}$ which internally stabilizes P.*
- *Two plants $P_1, P_2 \in K^{q\times(p-q)}$ are* simultaneously stabilizable *if there exists a controller $C \in K^{(p-q)\times q}$ which internally stabilizes P_1 and P_2.*

The next proposition is a reformulation of Lemma 7 of section 5.3 of [42] (we thank an anonymous associate editor for pointing out this reference to us).

PROPOSITION 4.5. *A transfer matrix $P \in K^{q\times(p-q)}$ is strongly stabilizable iff P admits a doubly coprime factorization $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ such that the matrices $(D : -N) \in A^{q\times p}$ and $(\tilde{D}^T : \tilde{N}^T) \in A^{(p-q)\times p}$ are, respectively, $(p-q)$ and q-stable.*

In particular, $P \in K(A)$ is strongly stabilizable iff there exists a coprime factorization $P = n/d$ such that the vector $(d : n) \in U_2(A)$ is 1-stable.

Proof. Let us suppose that there exists a stable controller $C \in A^{(p-q)\times q}$ which internally stabilizes P. Then, all the entries of the matrix (4.1) belong to A and, in particular, $P\,(I_{p-q} - C\,P)^{-1} = (I_q - P\,C)^{-1}\,P = V \in A^{q\times(p-q)}$.

Then, from the fact that

$$I_{p-q} + C\,V = I_{p-q} + C\,(I_q - P\,C)^{-1}\,P = (I_{p-q} - C\,P)^{-1},$$

we deduce that $I_{p-q} + C\,V$ is an invertible matrix, and thus we have

$$P\,(I_{p-q} - C\,P)^{-1} = V \Leftrightarrow P = V\,(I_{p-q} + C\,V)^{-1}.$$

Then, P admits the right-coprime factorization $P = V\,(I_{p-q} + C\,V)^{-1}$ because

$$(-C : I_{p-q}) \begin{pmatrix} V \\ I_{p-q} + C\,V \end{pmatrix} = I_{p-q}.$$

The matrix $((I_{p-q} + C\,V)^T : V^T)$ is q-stable because $I_{p-q} + V^T\,C^T - V^T\,C^T = I_{p-q}$.

Moreover, from the fact that

$$I_q + V\,C = I_q + P\,(I_{p-q} - C\,P)^{-1}\,C = (I_q - P\,C)^{-1},$$

we deduce that $I_q + V\,C$ is an invertible matrix, and thus we have

$$(I_q - P\,C)^{-1}\,P = V \Leftrightarrow P = (I_q + V\,C)^{-1}\,V.$$

Then, P admits the left-coprime factorization $P = (I_q + V\,C)^{-1}\,V$, and the matrix $(I_q + V\,C :\ -V)$ satisfies $I_q + V\,C - V\,C = I_q$; i.e., $(I_q + V\,C :\ -V)$ is $(p-q)$-stable.

Conversely, if P admits a left-coprime factorization $P = D^{-1}\,N$ such that the matrix $R = (D :\ -N) \in A^{q\times p}$ is $(p-q)$-stable, then there exists $T_1 \in A^{(p-q)\times q}$ such that $U \triangleq D - N\,T_1 \in \mathrm{GL}_q(A)$. In particular, we have $D\,U^{-1} - N\,(T_1\,U^{-1}) = I_q$, where $U^{-1} \in A^{q\times q}$. Thus, by Proposition 4.3, $C = (T_1\,U^{-1})\,(U^{-1})^{-1} = T_1$ is a stable controller which internally stabilizes P, and thus P is strongly stabilizable. $\square$

5. A general structure of the stabilizing controllers. In the next theorem, we show that there exists a stabilizing controller C of P such that the dimension of its unstable part depends on the k-stability of the matrix $R = (D :\ -N) \in A^{q\times p}$, where $P = D^{-1}\,N$ is a left-coprime factorization of P. Moreover, the unstable part of C is isolated into a single transfer matrix $V\,U^{-1} \in K^{r\times(p-q)}$, where $r = p - q - k$.

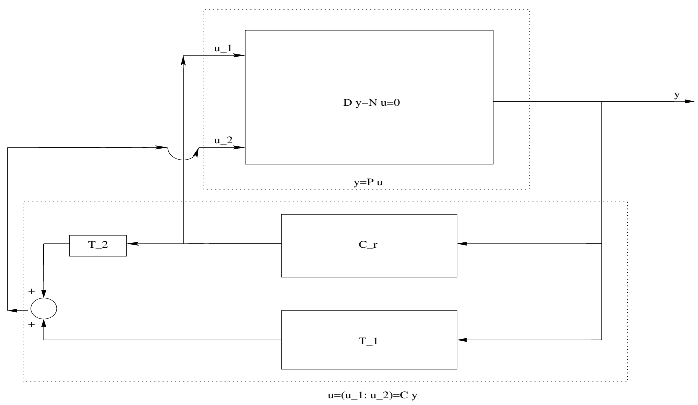

FIG. 5.1. *Closed-loop $y = P\,u$ and $u = C\,y$.*

THEOREM 5.1. *Let A be an integral domain of SISO stable plants, $K = Q(A)$, and let $P \in K^{q\times(p-q)}$ be a transfer matrix admitting a left-coprime factorization $P = D^{-1}\,N$ with $R = (D :\ -N) \in A^{q\times p}$. If R is k-stable and $r \triangleq p - q - k \geq 0$, then there exist two stable matrices*

$$\left\{\begin{array}{l} T_1 \in A^{k\times q}, \\ T_2 \in A^{k\times r} \end{array}\right. \tag{5.1}$$

such that the matrix $R_k = (D - \Lambda\,T_1 :\ -(N_r + \Lambda\,T_2)) \in A^{q\times(p-k)}$ admits a right-inverse with entries in A, with the notation

$$R = (D :\ -N) = (\underset{\substack{\leftrightarrow\\ q}}{D} \ :\underset{\substack{\leftrightarrow\\ r}}{-N_r}\ :\underset{\substack{\leftrightarrow\\ k}}{-\Lambda}) \in A^{q\times p}. \tag{5.2}$$

Let us define by $S_k = (U^T :\ V^T)^T \in A^{(p-k)\times q}$, $U \in A^{q\times q}$, $V \in A^{r\times q}$ any right-inverse of R_k such that $\det U \neq 0$. Then, the controller $C \in K^{(p-q)\times q}$ defined by

$$C = \begin{pmatrix} V\,U^{-1} \\ T_1 + T_2\,(V\,U^{-1}) \end{pmatrix}, \quad \begin{array}{l} \updownarrow r = p - q - k \\ \updownarrow k \end{array} \tag{5.3}$$

internally stabilizes P (see Figure 5.1). Moreover, if $\det(D - \Lambda\,T_1) \neq 0$, then the controller $C_r = V\,U^{-1} \in K^{r\times q}$ internally stabilizes the plant

$$P_r = (D - \Lambda\,T_1)^{-1}\,(N_r + \Lambda\,T_2) \in K^{q\times r} \tag{5.4}$$

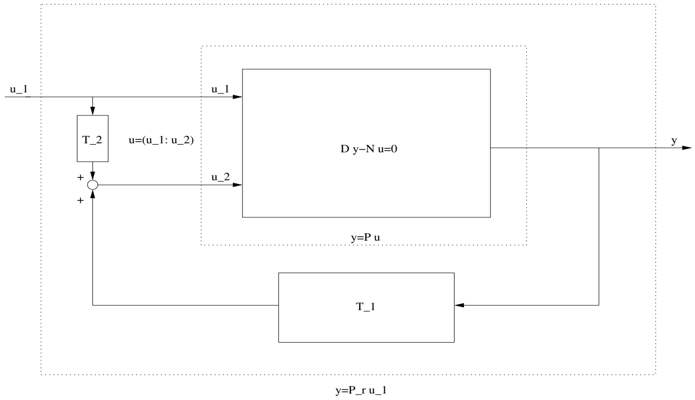

FIG. 5.2. *Plant* $y = P_r\, u_1$.

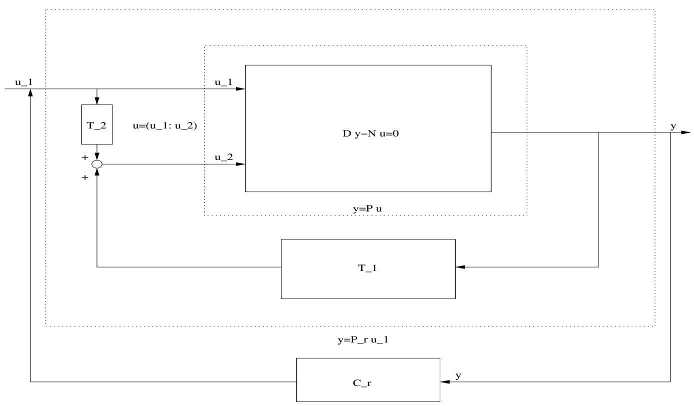

FIG. 5.3. *Closed-loop* $y = P_r\, u_1$ *and* $u_1 = C_r\, y$.

(see Figures 5.2 *and* 5.3*). The unstable part of the controller* (5.3) *corresponds to* $C_r = V\, U^{-1}$*, and its dimension is equal to* $r \times q$*.*

Similar results also hold for a transfer matrix P *admitting a right-coprime factorization* $P = \tilde{N}\, \tilde{D}^{-1}$ $(\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in A^{p\times(p-q)})$.

Proof. P admits a left-coprime factorization $P = D^{-1}\, N$, and thus the matrix $R = (D\,:\,-N) \in A^{q\times p}$ has a right-inverse $S = (X^T\,:\,Y^T)^T \in A^{p\times q}$; i.e., R is unimodular in the sense of Definition 3.1. Also, by hypothesis, R is k-stable, and thus, by Lemma 3.4, there exists $T_k \in A^{k\times(p-k)}$ such that the matrix $R_k \in A^{q\times(p-k)}$ defined by (3.2) is unimodular. Let us denote by $S_k \in A^{(p-k)\times q}$ a right-inverse of R_k; i.e., we have

$$R_k\, S_k = I_q. \tag{5.5}$$

Using expressions (3.2) and (5.5), we obtain that

$$\mathrm{col}(R_1 : \cdots : R_p) \begin{pmatrix} S_k \\ T_k\, S_k \end{pmatrix} = I_q \Leftrightarrow (D:\,-N) \begin{pmatrix} S_k \\ T_k\, S_k \end{pmatrix} = I_q.$$

If we write $S_k = (U_k^T : V_k^T)^T$, with $U_k \in A^{q\times q}$ and $V_k \in A^{r\times q}$, then we have

$$D\,U_k - N \begin{pmatrix} V_k \\ T_k\,S_k \end{pmatrix} = I_q.$$

If $\det U \neq 0$, then by Proposition 4.3, the controller C defined by

$$\begin{aligned} C &= \begin{pmatrix} V_k \\ T_k\,S_k \end{pmatrix} U_k^{-1} = \begin{pmatrix} V_k\,U_k^{-1} \\ T_k \begin{pmatrix} U_k \\ V_k \end{pmatrix} U_k^{-1} \end{pmatrix} \\ &= \begin{pmatrix} V_k\,U_k^{-1} \\ T_k \begin{pmatrix} I_q \\ V_k\,U_k^{-1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} V_k\,U_k^{-1} \\ T_{k1} + T_{k2}\,(V_k\,U_k^{-1}) \end{pmatrix} \end{aligned}$$

internally stabilizes $P = D^{-1}\,N$, where $T_k = (T_{k1} : T_{k2}) \in A^{k\times(q+r)}$ and the dimensions of T_{k1} and T_{k2} are defined by (5.1). With the notation of (5.2), we have

$$\begin{aligned} R_k &= \mathrm{col}(R_1 : \cdots : R_{p-k}) - \Lambda\,(T_{k1} : T_{k2}) \\ &= (\mathrm{col}(R_1 : \cdots : R_q) - \Lambda\,T_{k1} : \mathrm{col}(R_{q+1} : \cdots : R_{p-k}) - \Lambda\,T_{k2}) \\ &= (D - \Lambda\,T_{k1} : -(N_r + \Lambda\,T_{k2})). \end{aligned}$$

Using the fact that $R_k\,S_k = I_q$, by Proposition 4.3, we obtain that $C_r = V_k\,U_k^{-1}$ is a stabilizing controller of the plant $P_r = (D - \Lambda\,T_{k1})^{-1}\,(N_r + \Lambda\,T_{k2})$. $\square$

Example 5.1. Let us consider $A = H_\infty(\mathbb{C}_+)$ and the following transfer matrix:

$$P = \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s-1} \end{pmatrix} \in K^{2\times 2},$$

where $K = Q(A)$. In [25, 26], it is shown that P admits the left-coprime factorization $P = D^{-1}\,N$, where $R = (D : -N) \in A^{2\times 4}$ is defined by

$$R = \begin{pmatrix} 1 & 0 & -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{pmatrix}.$$

The matrix R_1, defined by

$$\begin{aligned} R_1 &= \begin{pmatrix} 1 & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 \end{pmatrix} + \begin{pmatrix} -\frac{s-1}{s+1} \\ -\frac{1}{s+1} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -2 & 0 \end{pmatrix}}_{T_1} \\ &= \begin{pmatrix} 1 & 2\,\frac{(s-1)}{(s+1)} & -\frac{e^{-s}}{s+1} \\ 0 & 1 & 0 \end{pmatrix}, \end{aligned} \tag{5.6}$$

is unimodular because we have

$$\begin{pmatrix} 1 & 2\,\frac{(s-1)}{(s+1)} & -\frac{e^{-s}}{s+1} \\ 0 & 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 - \frac{e^{-s}}{s+1} & -2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \\ -1 & 0 \end{pmatrix}}_{S_1} = I_2. \tag{5.7}$$

Thus, the matrix R is 1-stable, and we can apply Theorem 5.1 to P with $p=4$, $q=2$, $k=1$, and $r=1$. We know that $(S_1^T : (T_1\, S_1)^T)^T$ is a left inverse of R; i.e., we have

$$\left(\begin{array}{cccc} 1 & 0 & -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{array}\right) \left(\begin{array}{cc} 1-\frac{e^{-s}}{s+1} & -2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \\ -1 & 0 \\ 0 & -2 \end{array}\right) = I_3. \tag{5.8}$$

If we define

$$U_1 = \left(\begin{array}{cc} 1-\frac{e^{-s}}{s+1} & -2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right), \quad V_1 = (-1 :\ 0), \quad T_{11} = (0 :\ -2) \in A^{1\times 2}, \quad T_{12} = 0 \in A,$$

then a stabilizing controller C of P has the form

$$C = \left(\begin{array}{c} V_1\, U_1^{-1} \\ T_{11} + T_{12}\,(V_1\, U_1^{-1}) \end{array}\right) = \left(\begin{array}{cc} -\left(1-\frac{e^{-s}}{s+1}\right)^{-1} & -2\,\frac{(s-1)}{(s+1)}\left(1-\frac{e^{-s}}{s+1}\right)^{-1} \\ 0 & -2 \end{array}\right). \tag{5.9}$$

Let us notice that $\inf_{s\in\mathbb{C}_+} |1-\frac{e^{-s}}{s+1}| = 0$ (take the sequence $(s_n = 1/n)_{n\in\mathbb{N}}$), and thus, by the Corona theorem [16], we have $(1-\frac{e^{-s}}{s+1})^{-1} \notin A$. Therefore, the first row of the controller C is unstable, whereas its second row is stable. Now, we may wonder if P is strongly stabilizable. Let us notice that the matrix

$$R_2 = \left(\begin{array}{cc} 1 & 2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right) + \left(\begin{array}{c} -\frac{e^{-s}}{s+1} \\ 0 \end{array}\right) \left(\begin{array}{cc} 0 & 0 \end{array}\right) = \left(\begin{array}{cc} 1 & 2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right)$$

is unimodular because we have

$$\left(\begin{array}{cc} 1 & 2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} 1 & -2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right) = I_2. \tag{5.10}$$

Then, the matrix R_1 is 1-stable, and thus R is 2-stable:

$$R_2 = \left(\begin{array}{cc} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{array}\right) + \left(\begin{array}{cc} -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ 0 & -\frac{1}{s+1} \end{array}\right) \left(\begin{array}{cc} 0 & 0 \\ 0 & -2 \end{array}\right) = \left(\begin{array}{cc} 1 & 2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right) \in \mathrm{U}_2(A). \tag{5.11}$$

By Theorem 5.1, we obtain that P is strongly stabilizable ($p=4$, $q=2$, $k=2$, $r=0$). From (5.11), we obtain

$$\left(\begin{array}{cccc} 1 & 0 & -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{array}\right) \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -2 \end{array}\right) \left(\begin{array}{cc} 1 & -2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right) = I_2,$$

which shows that

$$S_2 = U_2 = \left(\begin{array}{cc} 1 & -2\,\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{array}\right), \qquad T_2 = T_{21} = \left(\begin{array}{cc} 0 & 0 \\ 0 & -2 \end{array}\right) \in A^{2\times 2},$$

and thus a stable stabilizing controller C' of P is defined by

$$C' = T_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \in A^{2\times 2}.$$

To finish, let us show how, using parametrization of all stabilizing controllers of the plant $P_1 = (D - \Lambda_1\, T_{11})^{-1}\,(N_1 + \Lambda_1\, T_{12})$, where

$$\Lambda_1 = \begin{pmatrix} \frac{s-1}{s+1} \\ \frac{1}{s+1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} \frac{e^{-s}}{s+1} \\ 0 \end{pmatrix},$$

it was already possible to find C'. First, let us notice that we have

$$R_1 = (D - \Lambda_1\, T_{11} :\ -(N_1 + \Lambda_1\, T_{12})) \in A^{2\times 3}.$$

Now, from (5.7), we know that $S_1 = (U_1^T : V_1^T)^T$ is a right-inverse of R_1. Computing a doubly coprime factorization of P_1, we obtain the following parametrization of all right inverses of R_1 (see [25, 26] for more details):

$$S_1 = \begin{pmatrix} U_1(k_1,\, k_2) \\ V_1(k_1,\, k_2) \end{pmatrix} = \begin{pmatrix} 1 + (k_1 - 1)\, \frac{e^{-s}}{s+1} & -2\, \frac{(s-1)}{(s+1)} + \frac{e^{-s}}{s+1}\, k_2 \\ 0 & 1 \\ k_1 - 1 & k_2 \end{pmatrix} \quad \forall\, k_1,\, k_2 \in A.$$

Therefore, some stabilizing controllers of P are of the form

$$C = \begin{pmatrix} V_1\, U_1^{-1} \\ T_{11} + T_{12}\,(V_1\, U_1^{-1}) \end{pmatrix} = \begin{pmatrix} a\,(k_1 - 1) & a\,(2\,(k_1-1)\,\frac{(s-1)}{(s+1)} + k_2) \\ 0 & -2 \end{pmatrix}, \tag{5.12}$$

where $a = (1 + (k_1 - 1)\, \frac{e^{-s}}{s+1})^{-1}$. Then, taking $k_1 = 1$ and $k_2 = 0$, we recover the stable controller C' of P.

The first difficulty in computing the controllers of the form (5.3) is to be able to determine explicitly the k-stability of a given matrix whose entries belong to a ring A. In section 6, we shall see that it is possible to give a lower bound for it by studying the stable range of the ring A. The second main difficulty is to compute T_k such that R_k, defined by (3.2), satisfies (5.5). In the following corollary of Theorem 5.1, we study the particular case where $T_k = 0$.

COROLLARY 5.2. *Let $P = D^{-1}\, N \in K^{q\times(p-q)}$ be a transfer matrix. If there exists an integer k satisfying $0 \le k \le p - q$ such that $P_r = D^{-1}\, N_r$ admits a left-coprime factorization, $D\, X - N_r\, Y = I_q$, with $\det X \neq 0$ and*

$$R = (D :\ -N) = (\underset{\overset{\leftrightarrow}{q}}{D} \ :\underset{\overset{\leftrightarrow}{r}}{-N_r} \ :\underset{\overset{\leftrightarrow}{k}}{-\Lambda}) \in A^{q\times p},$$

then the controller

$$C = \begin{pmatrix} Y\, X^{-1} \\ 0 \end{pmatrix}, \quad \begin{matrix} \updownarrow r = p - q - k \\ \updownarrow k \end{matrix} \tag{5.13}$$

internally stabilizes $P = D^{-1}\, N$.

Proof. Let us define $T_k = 0$. Then, by hypothesis, the matrix

$$R_k = (D :\ -N_r) - \Lambda\, T_k = (D :\ -N_r)$$

has a left-inverse; i.e., it is unimodular. Therefore, the hypothesis that $P_r = D^{-1} N_r$ admits a left-coprime factorization implies that $R = (D : -N)$ is k-stable. Then, the result directly follows from Theorem 5.1 and $T_k = (T_1 : T_2) = 0$. □

Example 5.2. Let us consider $A = RH_\infty$, $K = Q(A)$, and the transfer matrix

$$P = \begin{pmatrix} \frac{s+1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{s+1}{s-1} \end{pmatrix} \in K^{2\times 2}.$$

P admits a fractional representation $P = D^{-1} N$, where $R = (D : -N) \in A^{2\times 4}$ is defined by

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -1 & 0 \\ \frac{1}{(s+1)^2} & -\frac{(s-1)}{(s+1)} & 0 & 1 \end{pmatrix}.$$

The matrix formed by the first two columns of R is not unimodular, but

$$R_1 = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -1 \\ \frac{1}{(s+1)^2} & -\frac{(s-1)}{(s+1)} & 0 \end{pmatrix}$$

is unimodular because we have

$$\begin{pmatrix} \frac{s-1}{s+1} & 0 & -1 \\ \frac{1}{(s+1)^2} & -\frac{(s-1)}{(s+1)} & 0 \end{pmatrix} \begin{pmatrix} \frac{s-1}{s+1} & 4 \\ \frac{1}{(s+1)^2} & -\frac{(s+3)}{(s+1)} \\ -\frac{4\,s}{(s+1)^2} & 4\,\frac{(s-1)}{(s+1)} \end{pmatrix} = I_3.$$

Thus, we can apply Corollary 5.2 to P with $p = 4$, $q = 2$, $k = 1$, $r = 1$ to obtain a stabilizing controller C of P defined by

$$C = \begin{pmatrix} Y\,X^{-1} \\ 0 \end{pmatrix} = -4 \begin{pmatrix} \frac{1}{s+1} & 1 \\ 0 & 0 \end{pmatrix}.$$

Finally, let us notice that P is strongly stabilizable because C is stable.

6. A general structure of the stabilizing controllers based on the stable range. In the rest of the paper, we shall need the following definition.

DEFINITION 6.1 (see [17, 41]). *Let p and q be two positive integers which satisfy $1 \leq q \leq p$. The ring A is said to* satisfy $\mathrm{sr}_k(q, p, A)$ *if every unimodular matrix $R \in A^{q\times p}$ is k-stable. If no confusion arises, we shall write* $\mathrm{sr}_k(q, p)$ *for* $\mathrm{sr}_k(q, p, A)$.

In particular, if A satisfies $\mathrm{sr}(A) = n < +\infty$, then A satisfies $\mathrm{sr}_1(1, n + 1)$.

THEOREM 6.2 (see [17, 41]). *We have the following equivalences:*

1. $\mathrm{sr}_1(1, n) \Leftrightarrow \mathrm{sr}_1(1, m)\ \forall\, m \geq n$,
2. $\mathrm{sr}_1(1, n) \Leftrightarrow \mathrm{sr}_k(1, n + k - 1)\ \forall\, k \geq 1$,
3. $\mathrm{sr}_k(1, n) \Leftrightarrow \mathrm{sr}_k(m, n + m - 1)\ \forall\, m \geq 1$.

COROLLARY 6.3. *Let A be a ring satisfying $\mathrm{sr}(A) < +\infty$. Then, for every $p, q \in \mathbb{Z}_+$ which satisfies $p - q \geq \mathrm{sr}(A)$, we have*

$$\mathrm{sr}_{p-q-\mathrm{sr}(A)+1}(q, p);$$

namely, for every unimodular matrix $R = \mathrm{col}(R_1 : \cdots : R_p) \in A^{q\times p}$, there exists a matrix $T_{\mathrm{sr}(A)} \in A^{(p-q-\mathrm{sr}(A)+1)\times(q+\mathrm{sr}(A)-1)}$ such that

$$R_{\mathrm{sr}(A)} = \mathrm{col}(R_1 : \cdots : R_{q+\mathrm{sr}(A)-1}) + \mathrm{col}(R_{q+\mathrm{sr}(A)} : \cdots : R_p)\, T_{\mathrm{sr}(A)} \tag{6.1}$$

is a unimodular matrix.

Proof. Using the fact that we have $\mathrm{sr}(A) = n$, A satisfies $\mathrm{sr}_1(1, n+1)$, and thus, by 1 of Theorem 6.2, we have $\mathrm{sr}_1(1, m)\ \forall\, m \geq n+1$. Then, by 2 of Theorem 6.2, A satisfies $\mathrm{sr}_k(1, m+k-1)$ for $k \geq 1$. Finally, by 3 of Theorem 6.2, A satisfies $\mathrm{sr}_k(l, l+m+k-2)\ \forall\, k,\ l \geq 1$ and $m \geq n+1$.

Now, let $p,\ q \in \mathbb{Z}_+$ such that $p-q \geq \mathrm{sr}(A)$. Let us define $k = p-q-\mathrm{sr}(A)+1 \geq 1$. We have $p = q + (\mathrm{sr}(A)+1) + (p-q-\mathrm{sr}(A)+1) - 2$ and, if we define

$$\begin{cases} l = q \geq 1, \\ n = \mathrm{sr}(A), \\ m = \mathrm{sr}(A) + 1, \\ p = l + m + k - 2, \end{cases}$$

then A satisfies $\mathrm{sr}_k(l, l+m+k-2)$, i.e., $\mathrm{sr}_{p-q-\mathrm{sr}(A)+1}(q, p)$. Finally, from Lemma 3.4, there exists $T_{\mathrm{sr}(A)} \in A^{(p-q-\mathrm{sr}(A)+1)\times(p+\mathrm{sr}(A)-1)}$ such that the matrix $R_{\mathrm{sr}(A)}$ defined by (6.1) is unimodular. ∎

Now, we are in position to state the second main result of this paper.

COROLLARY 6.4. *Let $P \in K^{q\times(p-q)}$ be a transfer matrix which admits a left-coprime factorization $P = D^{-1}N$, $R = (D :\ -N) \in A^{q\times p}$ and satisfies $p-q \geq \mathrm{sr}(A)$. Then, there exist two stable matrices*

$$\begin{cases} T_1 \in A^{(p-q-\mathrm{sr}(A)+1)\times q}, \\ T_2 \in A^{(p-q-\mathrm{sr}(A)+1)\times(\mathrm{sr}(A)-1)} \end{cases} \tag{6.2}$$

such that the matrix $R_{p-q-\mathrm{sr}(A)+1} = (D - \Lambda\, T_1 :\ -(N_{\mathrm{sr}(A)-1} + \Lambda\, T_2)) \in A^{q\times(q+\mathrm{sr}(A)-1)}$ admits a right-inverse, with the notation

$$R = (D :\ -N) = (\underbrace{D}_{q}\ :\underbrace{-N_{\mathrm{sr}(A)-1}}_{\mathrm{sr}(A)-1}\ :\underbrace{-\Lambda)}_{p-q-\mathrm{sr}(A)+1} \in A^{q\times p}. \tag{6.3}$$

Let us denote by $S_{p-q-\mathrm{sr}(A)+1} = (U^T : V^T)^T \in A^{(q+\mathrm{sr}(A)-1)\times q}$ any right-inverse of $R_{p-q-\mathrm{sr}(A)+1}$ such that $\det U \neq 0$. *Then, the controller C defined by*

$$C = \begin{pmatrix} V\,U^{-1} \\ T_1 + T_2\,(V\,U^{-1}) \end{pmatrix}, \quad \begin{matrix} \updownarrow \mathrm{sr}(A) - 1 \\ \updownarrow p - q - \mathrm{sr}(A) + 1 \end{matrix} \tag{6.4}$$

internally stabilizes the plant $P = D^{-1}N$. Moreover, if $\det(D - \Lambda\, T_1) \neq 0$, *then the controller $C_{\mathrm{sr}(A)-1} = V\,U^{-1}$ internally stabilizes the plant*

$$P_{\mathrm{sr}(A)-1} = (D - \Lambda\, T_1)^{-1}\,(N_{\mathrm{sr}(A)-1} + \Lambda\, T_2).$$

Finally, the unstable part of the controller (6.4) *is $C_{\mathrm{sr}(A)-1} = V\,U^{-1}$ and its dimension is equal to* $(\mathrm{sr}(A) - 1) \times q$.

Proof. By Corollary 6.3, every matrix of $A^{q\times p}$ is $k = (p-q-\mathrm{sr}(A)+1)$-stable. Then, the result directly follows from Theorem 5.1. ∎

COROLLARY 6.5. *Let us consider $A = RH_\infty$ and $K = Q(A) = \mathbb{R}(s)$. Then, every transfer matrix $P \in \mathbb{R}(s)^{q\times(p-q)}$ admits a stabilizing controller of the form*

$$C = \begin{pmatrix} V\,U^{-1} \\ T_1 + T_2\,(V\,U^{-1}) \end{pmatrix}, \quad \begin{matrix} \updownarrow 1 \\ \updownarrow p - q - 1 \end{matrix},$$

where

$$\begin{cases} T_1 \in A^{(p-q-1)\times q}, \\ T_2 \in A^{(p-q-1)\times 1}, \end{cases}$$

$P = D^{-1} N$ *is a left-coprime factorization of* P, $S_{p-q-1} = (U^T : V^T)^T \in A^{(q+1)\times q}$ *is any right-inverse of* $R_{p-q-1} = (D - \Lambda T_1 : -(N_1 + \Lambda T_2)) \in A^{q\times(q+1)}$ *such that* $\det U \neq 0$, *and*

$$R = (D : -N) = (\underbrace{D}_{q} : \underbrace{-N_1}_{1} : \underbrace{-\Lambda}_{p-q-1}) \in A^{q\times p}.$$

Proof. Every MIMO transfer matrix P with entries in $K = \mathbb{R}(s)$ admits a doubly coprime factorization $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ over A,

$$\begin{pmatrix} D & -N \\ -\tilde{Y} + Q D & \tilde{X} - Q N \end{pmatrix} \begin{pmatrix} X - \tilde{N} Q & \tilde{N} \\ Y - \tilde{D} Q & \tilde{D} \end{pmatrix} = I,$$

where Q is an arbitrary matrix. See [42] for more details. Then, applying Lemma 17 on page 112 of [42], we obtain that there exists $Q^\star$ such that the matrix $\det(X - \tilde{N} Q^\star) \neq 0$. Using the facts that $\mathrm{sr}(RH_\infty) = 2$ (see Corollary 2.7) and

$$(U^T : V^T)^T = ((X - \tilde{N} Q^\star)^T : (Y - \tilde{D} Q^\star)^T)^T,$$

the result follows from Corollary 6.4. ∎

We have the following straightforward consequence of Corollary 6.4.

Corollary 6.6. *If* $\mathrm{sr}(A) = 1$, *then every transfer matrix which admits a left-coprime factorization is strongly stabilizable (i.e., it is internally stabilized by a stable controller). In particular, this result holds for* $A = W_+$ *or* $A(\mathbb{D})$.

Moreover, every internally stabilizable plant, defined by a transfer matrix P *with entries in the quotient field of* $A = H_\infty(\mathbb{D})$ *or* $H_\infty(\mathbb{C}_+)$, *is strongly stabilizable.*

Proof. The first part of the corollary directly follows from Corollary 6.4 and the fact that $\mathrm{sr}(A) = 1$. Moreover, by Theorem 2.8, we know that $\mathrm{sr}(W_+) = 1$ and $\mathrm{sr}(A(\mathbb{D})) = 1$. Finally, if $A = H_\infty(\mathbb{C}_+)$ or $H_\infty(\mathbb{D})$, then it is well known that P is internally stabilizable iff P admits a doubly coprime factorization [25, 26, 36]. The last result directly follows from this fact, Corollary 6.4, Theorem 2.4, and Corollary 2.5. ∎

Let us notice that the second part of Corollary 6.6 extends Treil's result [38] to MIMO systems. The question of the possibility of having the matrix analogous to Treil's result was asked in [9]. However, the issue consisting in computing effectively the stable stabilizing controllers of a stabilizable plant, defined by a transfer matrix with entries in $K = Q(H_\infty(\mathbb{D}))$ or $K = Q(H_\infty(\mathbb{C}_+))$, is still open.

Corollary 6.7. *If* $\mathrm{sr}(A) = 1$, *then every pair of plants, defined by two transfer matrices* P_0 *and* P_1 *with entries in* $K = Q(A)$, *having the same dimensions, and admitting doubly coprime factorizations, is simultaneously stabilized by a controller (simultaneous stabilization). In particular, this result holds for* $A = W_+$ *or* $A(\mathbb{D})$.

Moreover, if $A = H_\infty(\mathbb{D})$ *or* $H_\infty(\mathbb{C}_+)$ *and* P_0, P_1 *are two stabilizable plants with entries in* $K = Q(A)$, *then* P_0 *and* P_1 *are simultaneously stabilized by a controller.*

Proof. Following the proof of Theorem 14 of section 8.3 of [42], there exists a stabilizing controller of P_0 and P_1 iff there exists a matrix T with entries in A such that $U + V T$ is a square unimodular matrix, where

$$\begin{cases} U = D_1 X_0 - N_1 Y_0, \\ V = -D_1 \tilde{N}_0 + N_1 \tilde{D}_0, \end{cases}$$

and $P_i = D_i^{-1}\, N_i = \tilde{N}_i\, \tilde{D}_i^{-1}$ is a doubly coprime factorization of P_i, $i = 0,\, 1$; i.e.,

$$\left(\begin{array}{cc} D_i & -N_i \\ -\tilde{Y}_i & \tilde{X}_i \end{array}\right)\left(\begin{array}{cc} X_i & \tilde{N}_i \\ Y_i & \tilde{D}_i \end{array}\right) = I, \quad \left(\begin{array}{cc} X_i & \tilde{N}_i \\ Y_i & \tilde{D}_i \end{array}\right)\left(\begin{array}{cc} D_i & -N_i \\ -\tilde{Y}_1 & \tilde{X}_i \end{array}\right) = I.$$

The matrix $(U\,:\,V)$ is unimodular because we have $U\,X - V\,Y = I$, where

$$\left\{\begin{array}{l} X = D_0\, X_1 - N_0\, Y_1, \\ Y = \tilde{Y}_0\, X_1 + \tilde{X}_0\, Y_1. \end{array}\right.$$

Using the fact that $\mathrm{sr}(A) = 1$, by Corollary 6.3, we obtain that there exists T with entries in A such that $U + V\,T$ is a square unimodular matrix, and thus every couple of plants is simultaneously stabilized by a controller. Finally, by Theorem 2.8, we know that $\mathrm{sr}(W_+) = 1$ and $\mathrm{sr}(A(\mathbb{D})) = 1$.

Let P_1 and P_2 be two stabilizable transfer matrices with entries in $A = H_\infty(\mathbb{D})$ or $H_\infty(\mathbb{C}_+)$. Then, from [25, 26, 36], we know that P_1 and P_2 admit doubly coprime factorizations. The results directly follow from Theorem 2.4, Corollary 2.5, and the previous point. □

7. Some more results based on stable range.

7.1. Topological stable range. Let us recall the definition of a Banach algebra.

Definition 7.1 (see [13]). *A k-algebra A ($k = \mathbb{R}, \mathbb{C}$) is a* Banach algebra *if A is a Banach k-vector space w.r.t. the norm $\|\cdot\|_A$ and satisfies*

1. $\| 1 \|_A = 1$,
2. $\| a\, b \|_A \leq \| a \|_A \| b \|_A$ *(continuity of the product in each factor).*

Example 7.1. The Hardy space $H_\infty(\mathbb{C}_+)$ of the holomorphic functions in $\mathbb{C}_+$ bounded w.r.t. the norm $\| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)|$ is a Banach algebra [5]. Moreover, the disc algebra $A(\mathbb{D})$ (resp., the Wiener algebra W_+), defined in Theorem 2.8, with the norm $\| f \|_{A(\mathbb{D})} = \sup_{s \in \overline{\mathbb{D}}} |f(s)|$ (resp., $\| f \|_{W_+} = \sum_{n=0}^{+\infty} |a_n|$), are two Banach algebras [13, 42].

Definition 7.2. *If A is a Banach algebra, then the* topological stable range $\mathrm{tsr}(A)$ *of A is the smallest $n \in \mathbb{N} \cup \{+\infty\}$ such that $\mathrm{U}_n(A)$ is dense in A^n for the product topology.*

As for the stable range, the topological stable range $\mathrm{tsr}(A)$ is sometimes called the *topological stable rank* of A.

Theorem 7.3. *We have the following results:*

- [37] $\mathrm{tsr}(H_\infty(\mathbb{D})) = 2$,
- [31] $\mathrm{tsr}(A(\mathbb{D})) = 2$.

Proposition 7.4. *If A is a Banach algebra such that $\mathrm{tsr}(A) = 2$, then every SISO plant, defined by the transfer function $P = n/d$ ($0 \neq d, n \in A$), satisfies*

$$\forall\, \epsilon > 0,\ \exists\, (d_\epsilon\,:\, n_\epsilon) \in \mathrm{U}_2(A) : \quad \left\{\begin{array}{l} \| n - n_\epsilon \|_A \leq \epsilon, \\ \| d - d_\epsilon \|_A \leq \epsilon. \end{array}\right.$$

If $d_\epsilon \neq 0$, then, in the product topology, P is as close as we want to a transfer function $P_\epsilon = n_\epsilon / d_\epsilon$ which admits a coprime factorization. In particular, this result holds for $A = H_\infty(\mathbb{D})$ or $A(\mathbb{D})$.

Proof. Let us consider the vector $(d\,:\,-n) \in A^{1\times 2}$. Using the fact that $\mathrm{tsr}(A) = 2$, we obtain

$$\forall\, \epsilon > 0,\ \ \exists\, (d_\epsilon\,:\, -n_\epsilon) \in \mathrm{U}_2(A) : \quad \left\{\begin{array}{l} \| d - d_\epsilon \|_A \leq \epsilon, \\ \| n - n_\epsilon \|_A \leq \epsilon. \end{array}\right.$$

Finally, using the fact that $(d_\epsilon : \, -n_\epsilon) \in \mathrm{U}_2(A)$, there exist x_ϵ, $y_\epsilon \in A$ such that we have $d_\epsilon \, x_\epsilon - n_\epsilon \, y_\epsilon = 1$, and thus $p_\epsilon = n_\epsilon / d_\epsilon$ admits a coprime factorization. □

In particular, if P is not internally stabilizable, then there exists a stabilizable plant P_ϵ as close as we want to P in the product topology.

7.2. Unit 1-stable range and *n*-fold. Let us introduce a few definitions.

Definition 7.5. *We have the following definitions* [4, 14, 40]*:*

- [14] *A ring A satisfies* unit 1-stable range *if, for every $a = (a_1 : \, a_2) \in \mathrm{U}_2(A)$, there exists an element $u \in U(A)$ such that $a_1 + a_2 \, u \in U(A)$.*
- [39] *A ring A is said to be n*-fold *if, for every n-tuple $a^i = (a_1^i : \, a_2^i) \in \mathrm{U}_2(A)$, $1 \leq i \leq n$, there exists $b \in A$ such that $a_1^i + a_2^i \, b \in \mathrm{U}(A)$ for $1 \leq i \leq n$.*

Example 7.2. Using a result of Handelman [15], one can easily prove that $\mathrm{sr}(L_\infty(\mathbb{T})) = 1$, where $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ is the unit circle, because $L_\infty(\mathbb{T})$ is a *commutative von Neumann algebra* [23], and thus $L_\infty(\mathbb{T})$ has unit 1-stable range (for a $C^\star$-algebra A with a unit [23], it is well known that $\mathrm{sr}(A) = 1$ is equivalent to A has unit 1-stable range [14]). See [18] for the study of stabilization problems over $A = L_\infty(\mathbb{T})$. For the sake of simplicity, in this paper we have studied only the case of integral domains A of SISO stable plants. However, all the results can be easily extended to any ring A with zero divisors.

Proposition 7.6. *We have the following results:*

1. *If A satisfies unit* 1*-stable range, then any SISO plant—defined by the transfer function $P = n/d$ $(d \neq 0, n \in A)$—admitting a coprime factorization is* bistably stabilizable*; namely it is stabilized by a bistable controller (i.e., a stable and inverstable controller)* [2].
2. *If A is an n-fold ring, then every n-tuple of SISO plants—defined by the transfer function $P_i = n_i/d_i$ $(d_i \neq 0, n_i \in A)$ with $1 \leq i \leq n$—having coprime factorizations is stabilized by a stable controller.*

Proof. 1. Let $P = n/d$ be a plant which has a coprime factorization. We may assume that we have $d \, x + n \, y = 1$ with $x, y \in A$. Thus, we have $(d : \, -n) \in \mathrm{U}_2(A)$. Using the fact that A satisfies unit 1-stable range, there exists $u \in U(A)$ such that $d - n \, u \in \mathrm{U}(A)$, and thus a stabilizing controller is given by $C = u \in \mathrm{U}(A)$; i.e., P is bistably stabilizable.

2. Let $i = 1, \ldots, n$, and let $P_i = n_i/d_i$ be a transfer function admitting a coprime factorization. We may assume that we have $d_i \, x_i + n_i \, y_i = 1$ for certain $x_i, y_i \in A$. Thus, we have $(d_i : \, -n_i) \in \mathrm{U}_2(A)$. Using the fact that A is n-fold, there exists $y \in A$ such that we have $d_i - n_i \, y \in \mathrm{U}(A)$ for $i = 1, \ldots, n$. Thus, the stable controller defined by $C = y$ simultaneously stabilizes the family of plants $\{P_i\}_{1 \leq i \leq n}$. □

Conclusion. In this paper, we have shown that the concept of stable range was an interesting one in the study of the strong and simultaneous stabilization problems. In particular, we proved that a plant, defined by means of a transfer matrix which admits a left-coprime factorization $P = D^{-1} \, N$, is internally stabilized by a controller, where its unstable and stable parts are separated and the dimension of the unstable part depends only on the k-stability of the matrix $R = (D : \, -N) \in A^{q \times p}$. Then, using the fact that the stable range of A gives a lower bound of the k-stability of every matrix with entries in A, we proved that, if the stable range of A is 1, then every plant, defined by a transfer matrix admitting a left-coprime factorization, is strongly stabilizable. In particular, using the fact that the stable range of $H_\infty(\mathbb{D})$ is 1 (see [38]), we proved that every stabilizable plant, defined by a transfer matrix with entries in the quotient field of $H_\infty(\mathbb{C}_+)$ or $H_\infty(\mathbb{D})$, is strongly stabilizable. Moreover, we were able to prove that there always exists a stabilizing controller which stabilizes simultaneously two stabilizable plants defined by a transfer matrix with entries in the quotient field of $H_\infty(\mathbb{C}_+)$ or $H_\infty(\mathbb{D})$. Finally, using the fact that the topological stable range of $H_\infty(\mathbb{D})$ is equal to 2 (see [37]), we proved that every unstabilizable SISO plant, defined by a transfer function with entries in $Q(H_\infty(\mathbb{D}))$, is as close as we want to a stabilizable plant in the product topology.

In this paper, we proved the existence of some particular stabilizing controllers. However, the algorithmical aspects of their constructions were not developed. In forthcoming publications, we shall try to develop this difficult problem.

The concept of a stable range of A was developed by Bass [1] in order to "stabilize" the computation of the group $K_1(A)$ which is the quotient of the group $GL(A)$ of invertible matrices with entries in A by its normal subgroup $EL(A)$ of elementary matrices with entries in A. The connections between the strong stabilization problem and the computation of this group $K_1(A)$ need to be clarified. Moreover, in [35], the obstruction of the simultaneous stabilization of two n-D plants is explicitly expressed in terms of the vanishing of a certain cohomology class. Using the concept of the *Chern character*, it would be interesting to study the links between the results developed in [35] and *topological K-theory*. More generally, it seems that some mathematical tools developed in *algebraic/topological/Hermetian K-theory* are useful for some stabilization problems. Hence, we believe that the study of stabilization problems within a K-theoretical approach should give new interesting results [29].

Finally, a necessary condition for strong stabilizability is the existence of a doubly coprime factorization for the plant (see Proposition 4.5). However, internal stabilizability is generally not equivalent to the existence of doubly coprime factorizations (see [24, 25, 26, 27, 28] and the references therein). Hence, if we do not assume the existence of doubly coprime factorizations for the plants, then the existence of a controller which simultaneously stabilizes two plants P_1 and P_2 is generally not equivalent to the existence of a stable controller for a certain plant P built from P_1 and P_2. For more details, see [30].

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