SIAM J. CONTROL OPTIM.
Vol. 42, No. 1, pp. 300–320

THE FRACTIONAL REPRESENTATION APPROACH TO SYNTHESIS PROBLEMS: AN ALGEBRAIC ANALYSIS VIEWPOINT PART II: INTERNAL STABILIZATION*

A. QUADRAT†

Abstract. In this second part of the paper [A. Quadrat, *SIAM J. Control Optim.*, 40 (2003), pp. 266–299], we show how to reformulate the fractional representation approach to synthesis problems within an algebraic analysis framework. In terms of modules, we give necessary and sufficient conditions for internal stabilizability. Moreover, we characterize all the integral domains A of SISO stable plants such that every MIMO plant—defined by means of a transfer matrix whose entries belong to the quotient field $K = Q(A)$ of A—is internally stabilizable. Finally, we show that this algebraic analysis approach allows us to recover on the one hand the approach developed in [M. C. Smith, *IEEE Trans. Automat. Control*, 34 (1989), pp. 1005–1007] and on the other hand the ones developed in [K. Mori and K. Abe, *SIAM J. Control Optim.*, 39 (2001), pp. 1952–1973; S. Shankar and V. R. Sule, *SIAM J. Control Optim.*, 30 (1992), pp. 11–30; V. R. Sule, *SIAM J. Control Optim.*, 32 (1994), pp. 1675–1695; M. Vidyasagar, H. Schneider, and B. A. Francis, *IEEE Trans. Automat. Control*, 27 (1982), pp. 880–894; M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985].

Key words. fractional representation approach to synthesis problems, internal stabilization, Prüfer domains, Youla–Kučera parametrization of the stabilizing controllers, (weakly) left/right/doubly coprime factorizations, coherent Sylvester domains, $H_\infty(\mathbb{C}_+)$, algebraic analysis, module theory, homological algebra

AMS subject classifications. 93C05, 93D25, 93B52, 93B25, 93C20, 93C23, 16D40, 16E60

PII. S0363012902417139

Introduction. Using the algebraic analysis viewpoint of the fractional representation approach to analysis and synthesis problems [5, 28, 29], developed in the first part of the paper [17], we give necessary and sufficient conditions for *internal stabilizability*. Moreover, using these results, we prove that every multi-input multi-output (MIMO) plant—defined by means of a transfer matrix $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}$, where $R = (D :\ -N)$ and $\tilde{R} = (\tilde{N}^T :\ \tilde{D}^T)^T$ are matrices whose entries belong to an integral domain A of single input single output (SISO) stable plants—is internally stabilizable iff A is a *Prüfer domain* [6, 23]. From the fact that the intersection between coherent Sylvester domains (see [17] for more details) and Prüfer domains are just Bézout domains, we also recover the result of Vidyasagar [29]: every MIMO plant admits doubly coprime factorizations iff A is a *Bézout domain.* Hence, if the algebra A is a Prüfer domain but not a Bézout domain, there exist plants which are internally stabilizable but fail to admit doubly coprime factorizations. Therefore, it is not possible to parametrize all their stabilizing controllers by means of the Youla–Kučera parametrization [4, 28]. These results allow us to explain the counterexamples exhibited in [1, 12]. We prove that, over a *projective-free domain* A (e.g., $H_\infty(\mathbb{C}_+)$, RH_∞), every stabilizable system admits doubly coprime factorizations. Finally, we show that the previous results allow us to recover, on the one hand, the results of [25] and, on

*Received by the editors November 15, 1999; accepted for publication (in revised form) November 4, 2002; published electronically April 17, 2003. This work was supported by grant HPMF-CT-1999-00095 during my stay at the University of Leeds.
http://www.siam.org/journals/sicon/42-1/41713.html
†INRIA Sophia Antipolis, CAFE project, 2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis cedex, France (Alban.Quadrat@sophia.inria.fr).

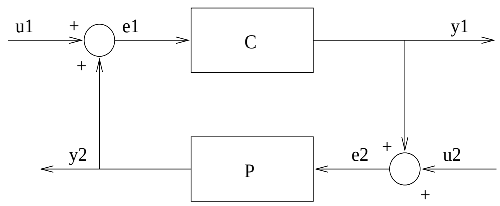

FIG. 1. *Closed-loop.*

the other hand, the ones developed in [12, 13, 24, 26, 27, 28, 29]. We refer to [17] for the development of the algebraic analysis approach used in this second part, as well as for some results that will be continually used in what follows.

Notation. In the course of the text, A denotes a commutative integral domain ($a\,b = 0, a \neq 0 \Rightarrow b = 0$) with a unit, $M_{q\times p}(A)$ (resp., $M_p(A)$), the set of $q \times p$ (resp., $p \times p$) matrices with entries in A and I_p the identity matrix. If $R \in M_{q\times p}(A)$, then R^T is the transposed matrix. *By convention, every vector with entries in A is a row vector.* The positive integers $p, q \in \mathbb{Z}_+$ will always satisfy $p \geq q$. If M and N are two A-modules, then $M \cong N$ means that M and N are isomorphic as A-modules, $\hom_A(M, N)$ is the A-module of the A-morphisms (i.e., A-linear maps) from M to N, and $M^* = \hom_A(M, A)$. Finally, $(a_1, \dots, a_n)$ denotes the ideal $A\,a_1 + \cdots + A\,a_n$ and $\triangleq$ means "by definition."

1. Closed-loop systems. Let A be an algebra of SISO stable systems which forms an integral domain and let $K = Q(A)$ be its field of fractions. Let us consider the closed-loop formed by a plant $P \in M_{q\times(p-q)}(K)$ and a controller $C \in M_{(p-q)\times q}(K)$ as it is shown in Figure 1. The equations of the closed-loop are

$$(1.1)\qquad \begin{cases} e_1 = u_1 + P\,e_2, \\ e_2 = u_2 + C\,e_1, \\ y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{cases}$$

DEFINITION 1.1 (see [5, 28, 29]). *The plant $P \in M_{q\times(p-q)}(K)$ is* internally stabilizable *if there exists a controller $C \in M_{(p-q)\times q}(K)$ such that all the entries of the following transfer matrix*

$$(1.2)\quad H(P, C) = \begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_{p-q} + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}$$

are stable, i.e., $H(P, C) \in M_p(A)$.

Let us write P and C in the form $P = {D_p}^{-1}\,N_p$ and $C = {D_c}^{-1}\,N_c$, where $R_p = (D_p : -N_p) \in M_{q\times p}(A)$ and $R_c = (-N_c : D_c) \in M_{(p-q)\times p}(A)$. Thus, we have

$$(1.3)\qquad (1.1) \Leftrightarrow \begin{cases} D_p\,e_1 - N_p\,e_2 - D_p\,u_1 = 0, \\ -N_c\,e_1 + D_c\,e_2 - D_c\,u_2 = 0, \\ y_1 - e_2 + u_2 = 0, \\ y_2 - e_1 + u_1 = 0. \end{cases}$$

Let us define the matrices

$$R = \begin{pmatrix} D_p & -N_p & -D_p & 0 \\ -N_c & D_c & 0 & -D_c \end{pmatrix} \in M_{p\times 2p}(A)$$

and

$$R_s = \begin{pmatrix} D_p & -N_p & -D_p & 0 & 0 & 0 \\ -N_c & D_c & 0 & -D_c & 0 & 0 \\ 0 & -I_{p-q} & 0 & I_{p-q} & I_{p-q} & 0 \\ -I_q & 0 & I_q & 0 & 0 & I_q \end{pmatrix} \in M_{2p\times 3p}(A),$$

as well as the following A-modules

$$\begin{cases} M_p = A^p/A^q\, R_p, \\ M_c = A^p/A^{p-q}\, R_c, \\ M = A^{2p}/A^p\, R, \\ M_s = A^{3p}/A^{2p}\, R_s. \end{cases}$$

LEMMA 1.2. *We have $M_s = M \cong M_p \oplus M_c$, and thus*

$$M_s/t(M_s) = M/t(M) \cong M_p/t(M_p) \oplus M_c/t(M_c), \tag{1.4}$$

or equivalently

$$A^{3p}/\overline{A^{2p}\, R_s} = A^{2p}/\overline{A^p\, R} \cong A^p/\overline{A^q\, R_p} \oplus A^p/\overline{A^{p-q}\, R_c},$$

where, for instance, $\overline{A^p\, R}$ is the A-closure of $A^p\, R$ in A^{2p} (see [17] *for more details).*

Proof. We have the following equality:

$$\begin{pmatrix} D_p & -N_p & -D_p & 0 \\ -N_c & D_c & 0 & -D_c \end{pmatrix} \begin{pmatrix} 0 & 0 & I_q & 0 \\ 0 & I_{p-q} & 0 & 0 \\ -I_q & 0 & I_q & 0 \\ 0 & I_{p-q} & 0 & -I_{p-q} \end{pmatrix} = \begin{pmatrix} D_p & -N_p & 0 & 0 \\ 0 & 0 & -N_c & D_c \end{pmatrix}.$$

The second matrix in the left-hand side of the previous equality is unimodular, and thus, invertible. Let us denote this matrix by U. Then, from the previous equality, i.e., $R\,U = R_p \oplus R_c$, we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & & A^p & \xrightarrow{.\,R} & A^{2p} & \xrightarrow{\pi} & M \longrightarrow 0 \\ & & \downarrow .I_p & & \downarrow .U & & \\ 0 \longrightarrow & & A^p & \xrightarrow{.\,(R_p \oplus R_c)} & A^{2p} & \xrightarrow{\pi'} & M_p \oplus M_c \longrightarrow 0. \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

From the previous commutative exact diagram, we deduce that there exists an isomorphism $\phi : M \to M_p \oplus M_c$, defined by $\phi(m) = \pi'(z\,U)$, where $z \in A^{2p}$ is such that $\pi(z) = m$, and thus, $M \cong M_p \oplus M_c$. Moreover, using the equations which define the A-module M_s, we can easily check that $M_s = M$. Finally, using the fact that $M_s = M \cong M_p \oplus M_c$, we obtain $t(M_s) = t(M) \cong t(M_p) \oplus t(M_c)$, and thus, $M/t(M) \cong M_p/t(M_p) \oplus M_c/t(M_c)$. $\square$

2. Internal stabilization: A particular case. We refer the reader to [17] for the definition of a weakly left/right/doubly coprime factorization.

THEOREM 2.1. *Let $P = D_p^{-1} N_p$ and $C = D_c^{-1} N_c$ be two weakly left-coprime factorizations, i.e., $R_p = (D_p : \ -N_p) \in M_{q\times p}(A)$ and $R_c = (-N_c : \ D_c) \in M_{(p-q)\times p}(A)$ are weakly left-prime matrices. Then, $P = D_p^{-1} N_p$ is internally stabilized by the controller $C = D_c^{-1} N_c$ iff*

$$\begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} \in M_p(A), \quad \text{i.e.,} \quad \begin{pmatrix} R_p \\ R_c \end{pmatrix} \in GL_p(A). \tag{2.1}$$

The same result also holds for weakly right-coprime factorizations.

Proof. $\Rightarrow$ By hypothesis, R_p and R_c are two weakly left-prime matrices, and thus, by Corollary 2.5 of [17], the A-modules $M_p = A^p/A^q\, R_p$ and $M_c = A^p/A^{p-q}\, R_c$ are torsion-free. Thus, $t(M) \cong t(M_p \oplus M_c) \cong t(M_p) \oplus t(M_c) = 0$, i.e., M is a torsion-free A-module. Then, by Corollary 2.5 of [17], R is weakly left-prime. Now, the fact that C internally stabilizes P implies (see Definition 1.1)

$$H(P,C) = \begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} = \begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} \begin{pmatrix} D_p & 0 \\ 0 & D_c \end{pmatrix} \in M_p(A).$$

Therefore, we have

$$\begin{aligned} \begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} R &= \begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} \begin{pmatrix} R_p & -D_p & 0 \\ R_c & 0 & -D_c \end{pmatrix} \\ &= \left(I_p \quad -\begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} \begin{pmatrix} D_p & 0 \\ 0 & D_c \end{pmatrix} \right) \in M_{p\times 2p}(A). \end{aligned}$$

Finally, using the fact that R is a weakly left-prime full row rank matrix, we obtain (2.1) (see [17] for more details).

$\Leftarrow$ We have

$$(2.1) \Rightarrow \begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} \begin{pmatrix} D_p & 0 \\ 0 & D_c \end{pmatrix} = \begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} \in M_p(A),$$

i.e., the controller $C = D_c^{-1} N_c$ internally stabilizes the plant $P = D_p^{-1} N_p$. $\square$

COROLLARY 2.2. *Let $P = D_p^{-1} N_p \in M_{q\times(p-q)}(K)$ be a weakly left-coprime factorization of P. Then, P is internally stabilized by a controller $C \in M_{(p-q)\times q}(K)$ which admits a weakly left-coprime factorization $C = D_c^{-1} N_c$ iff P admits a doubly coprime factorization. The same result also holds for a stabilizable plant P admitting a weakly right-coprime factorization.*

Proof. $\Rightarrow$ Let us suppose that the plant $P = D_p^{-1} N_p$ is internally stabilized by a controller $C = D_c^{-1} N_c$ and R_p and R_c are two weakly left-prime matrices. Then, by Theorem 2.1, we have (2.1). Let us note

$$\begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} = \begin{pmatrix} U_1 & V_1 \\ U_2 & V_2 \end{pmatrix} \in M_p(A).$$

Then, we have the following Bézout identities:

$$\begin{pmatrix} D_p & -N_p \\ -N_c & D_c \end{pmatrix} \begin{pmatrix} U_1 & V_1 \\ U_2 & V_2 \end{pmatrix} = I_p, \qquad \begin{pmatrix} U_1 & V_1 \\ U_2 & V_2 \end{pmatrix} \begin{pmatrix} D_p & -N_p \\ -N_c & D_c \end{pmatrix} = I_p. \tag{2.2}$$

In particular, we have

$$\begin{pmatrix} D_p & -N_p \\ 0 & I_{p-q} \end{pmatrix} \begin{pmatrix} U_1 & V_1 \\ U_2 & V_2 \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ U_2 & V_2 \end{pmatrix} \Rightarrow \det D_p \, \det \begin{pmatrix} U_1 & V_1 \\ U_2 & V_2 \end{pmatrix} = \det V_2,$$

and, using the fact that the second matrix is unimodular and $\det D_p \neq 0$, we obtain that $\det V_2 \neq 0$. Finally, from (2.2), we deduce

$$\begin{cases} D_p\, V_1 - N_p\, V_2 = 0, \\ D_p\, U_1 - N_p\, U_2 = I_q, \\ -N_c\, V_1 + D_c\, V_2 = I_{p-q}, \end{cases}$$

which shows that $P = D_p^{-1}\, N_p = V_1\, V_2^{-1}$ is a doubly coprime factorization of P.

$\Leftarrow$ If $P = D_p^{-1}\, N_p = \tilde{N}_p\, \tilde{D}_p^{-1}$ is a doubly coprime factorization of P, then there exist Bézout identities of the form (2.2). Thus, $R_p = (D_p \,:\, -N_p) \in M_{q \times p}(A)$ can be complemented into $(R_p^T \,:\, R_c^T)^T \in GL_p(A)$, with $R_c \in M_{(p-q)\times p}(A)$. The complement $R_c = (-N_c \,:\, D_c)$ to R_p into a unimodular matrix $(R_p^T \,:\, R_c^T)^T$ is not uniquely defined (see Corollary 6.1 on the Youla–Kučera parametrization) and we can choose $D_c \in M_{p-q}(A)$ such that $\det D_c \neq 0$. Finally, R_c admits a right-inverse, i.e., $C = D_c^{-1}\, N_c$ is in particular a weakly left-coprime factorization. Finally, by Theorem 2.1, $C = D_c^{-1}\, N_c$ internally stabilizes P. ☐

The next corollary generalizes a result obtained by Smith for $H_\infty(\mathbb{C}_+)$ [25].

COROLLARY 2.3. *If A is a coherent Sylvester domain (e.g., $A = H_\infty(\mathbb{C}_+)$, RH_∞, Bézout domains), then $P \in M_{q\times(p-q)}(K)$ is internally stabilizable iff P admits a doubly coprime factorization.*

Proof. By Theorem 3.24 of [17], every transfer matrix whose entries belong to $K = Q(A)$ admits a weakly doubly coprime factorization. Then, the result follows directly from Corollary 2.2. ☐

3. Internal stabilization: The general case. In the previous section, we have obtained some results on internal stabilization in the particular case where the transfer matrices admit weakly left- or right-coprime factorizations. In this section, we give some necessary and sufficient conditions for internal stabilizability without any assumption on the transfer matrices.

LEMMA 3.1. *Let $P = D_p^{-1}\, N_p \in M_{q\times(p-q)}(K)$ (resp., $C = D_c^{-1}\, N_c \in M_{(p-q)\times q}(K)$) be a plant (resp., a controller). If C internally stabilizes P, then the A-modules $M_p = A^p/A^q\, R_p$ and $M_c = A^p/A^{p-q}\, R_c$, where $R_p = (D_p \,:\, -N_p) \in M_{q\times p}(A)$, $R_c = (-N_c \,:\, D_c) \in M_{(p-q)\times p}(A)$, satisfy*

$$M_p/t(M_p) \oplus M_c/t(M_c) \cong A^p,$$

i.e., $M_p/t(M_p) = A^p/\overline{A^q\, R_p}$ and $M_c/t(M_c) = A^p/\overline{A^{p-q}\, R_c}$ are projective A-modules.

Proof. By hypothesis, P is internally stabilized by C, and thus, we have

$$H(P, C) = \begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} = \begin{pmatrix} R_p \\ R_c \end{pmatrix}^{-1} \begin{pmatrix} D_p & 0 \\ 0 & D_c \end{pmatrix} = N \in M_p(A).$$

Let us define the following A-modules $M = A^{2p}/A^p\, R$ and $M' = A^{2p}/A^p\, (I_p \,:\, -N)$. By Lemma 2.6 of [17], we have $\overline{A^p\, R} = A^p\, (I_p \,:\, -N)$ because $A^p\, (I_p \,:\, -N)$ is an A-closed submodule of A^{2p}, and thus, we have

$$M/t(M) = A^{2p}/\overline{A^p\, R} = A^{2p}/A^p\, (I_p \,:\, -N) = M'.$$

Moreover, it is easy to see that the A-module M' is free of rank p and thus, that we have $M/t(M) \cong A^p$. Finally, using (1.4), we obtain

$$M/t(M) \cong M_p/t(M_p) \oplus M_c/t(M_c) \cong A^p,$$

which shows that $M_p/t(M_p) = A^p/\overline{A^q\, R_p}$ and $M_c/t(M_c) = A^p/\overline{A^{p-q}\, R_c}$ are projective A-modules. $\square$

THEOREM 3.2. *A plant $P = D_p^{-1}\, N_p \in M_{q\times(p-q)}(K)$ is internally stabilizable iff $M_p/t(M_p) = A^p/\overline{A^q\, R_p}$ is a projective A-module, with $R_p = (D_p :\ -N_p) \in M_{q\times p}(A)$ and $M_p = A^p/A^q\, R_p$.*

Proof. $\Rightarrow$ It was proved in Lemma 3.1.

$\Leftarrow$ Let $M_p/t(M_p)$ be a projective A-module. We have the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & t(M_p) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & A^q & \xrightarrow{.R_p} & A^p & \xrightarrow{\pi} & M_p & & \longrightarrow 0 \\
 & & \downarrow \kappa & & \| & & \downarrow \pi' & & \\
0 \longrightarrow & & \ker\phi & \longrightarrow & A^p & \xrightarrow{\phi} & M_p/t(M_p) & & \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker}\kappa & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{3.1}$$

where $\phi = \pi' \circ \pi$ and $\kappa : A^q \to \ker\phi$ is induced by $\mathrm{id} : A^p \to A^p$ and $\pi' : M_p \to M_p/t(M_p)$. The fact that $M_p/t(M_p)$ is a projective A-module implies that the exact sequence

$$0 \longrightarrow \ker\phi \longrightarrow A^p \xrightarrow{\phi} M_p/t(M_p) \longrightarrow 0 \tag{3.2}$$

splits (see [17]), and thus, $A^p \cong M_p/t(M_p) \oplus \ker\phi$, i.e., $\ker\phi$ is a projective A-module.

The fact that $\ker\phi$ is a projective A-module is equivalent to the existence of a family $\{a_1, \ldots , a_m\}$ of elements of A satisfying [3, 23]:

1. The ideal $(a_1, \ldots , a_m)$ is equal to A, i.e., $\exists\, x_i \in A : \sum_{i=1}^m x_i\, a_i = 1$.
2. If $S_{a_i} = \{1, a_i, a_i^2, \ldots\}$ is the multiplicative set defined by a_i, then $S_{a_i}^{-1} \ker\phi$ is a free $S_{a_i}^{-1} A$-module (see [17]).

By Proposition 1.10 of [17], we obtain the exact sequence of $S_{a_i}^{-1} A$-modules:

$$0 \longrightarrow S_{a_i}^{-1}(\ker\phi) \longrightarrow (S_{a_i}^{-1} A)^p \xrightarrow{S_{a_i}^{-1}\phi} S_{a_i}^{-1}(M_p/t(M_p)) \longrightarrow 0. \tag{3.3}$$

The fact that $t(M_p)$ is a torsion A-module implies that $K \otimes_A t(M_p) = 0$ (see (1.10) of [17]), and thus, $\operatorname{rank}_A(t(M_p)) = \dim_K(K \otimes_A t(M_p)) = 0$ (see [17] for more details). Applying Proposition 1.10 of [17] to the exact sequence

$$0 \longrightarrow t(M_p) \longrightarrow M_p \longrightarrow M_p/t(M_p) \longrightarrow 0,$$

we obtain $\operatorname{rank}_A(M_p/t(M_p)) = \operatorname{rank}_A(M_p) - \operatorname{rank}_A(t(M_p)) = p - q$. Applying again Proposition 1.10 of [17] to the exact sequence (3.2), we obtain

$$\operatorname{rank}_A(\ker\phi) = p - \operatorname{rank}_A(M_p/t(M_p)) = p - (p - q) = q. \tag{3.4}$$

If we note $S_{a_i}^{-1} A = A_i$, then $S_{a_i}^{-1} \ker\phi$ is a free A_i-module of rank q. Taking a basis of $S_{a_i}^{-1} \ker\phi \cong A_i^q$, there exists a matrix $R_i \in M_{q\times p}(A_i)$ such that (3.3) becomes

$$0 \longrightarrow A_i^q \xrightarrow{.R_i} A_i^p \longrightarrow S_{a_i}^{-1}(M_p/t(M_p)) \longrightarrow 0.$$

By hypothesis, $M_p/t(M_p)$ is a projective A-module, and thus, $S_{a_i}^{-1}(M_p/t(M_p))$ is also a projective A_i-module [3, 23]. Hence, using Proposition 4.2 of [17], the previous exact sequence splits, and thus there exists $S_i \in M_{p\times q}(A_i)$ such that

$$R_i\, S_i = I_q. \tag{3.5}$$

Let us note $R_p = (D_p :\ -N_p) \in M_{q\times p}(A)$ and $R_i = (D_i :\ -N_i) \in M_{q\times p}(A_i)$. First, we prove that $P = D_p^{-1}\, N_p = D_i^{-1}\, N_i$. By localization of (3.1) with respect to S_{a_i}, we obtain the commutative exact diagram ($S_{a_i}^{-1} A$ is a flat A-module [17])

$$\begin{array}{ccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & S_{a_i}^{-1}t(M_p) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_i^q & \xrightarrow{.R_p} & A_i^p & \longrightarrow & S_{a_i}^{-1}M_p & \longrightarrow & 0\\
 & & \downarrow .R_i'' & & \| & & \downarrow & & \\
0 & \longrightarrow & A_i^q & \xrightarrow{.R_i} & A_i^p & \longrightarrow & S_{a_i}^{-1}(M_p/t(M_p)) & \longrightarrow & 0,\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_i^q/A_i^q\, R_i'' & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where $R_i'' \in M_q(A_i)$ corresponds to $S_{a_i}^{-1}\kappa : A_i^q \to S_{a_i}^{-1} \ker\phi \cong A_i^q$. Hence, we have $R_p = R_i''\, R_i$, i.e.,

$$(D_p :\ -N_p) = R_i''\,(D_i :\ -N_i), \tag{3.6}$$

where $R_i'' \in M_q(A_i)$ has full rank and $S_{a_i}^{-1}t(M_p) \cong A_i^q/A_i^q\, R_i''$. Hence, we have

$$P = D_p^{-1}\, N_p = (R_i''\, D_i)^{-1}(R_i''\, N_i) = D_i^{-1}\, N_i.$$

Cleaning the denominators of each R_i and $S_i = (X_i^T :\ Y_i^T)^T$, there exists $\alpha_i \in \mathbb{Z}_+$ such that all the entries of the matrix $a_i^{\alpha_i}\, S_i\, R_i$ are in A. If $\alpha = \max_{1\le i\le m} \alpha_i$, then

$$a_i^{\alpha}\, S_i\, R_i = a_i^{\alpha} \begin{pmatrix} X_i\, D_i & -X_i\, N_i \\ Y_i\, D_i & -Y_i\, N_i \end{pmatrix} \in M_p(A), \qquad i = 1, \dots, m. \tag{3.7}$$

Using the fact that $(a_1, \dots, a_m) = A$, then there exists a family $\{b_1, \dots, b_m\}$ of elements of A such that $\sum_{i=1}^m b_i\, a_i^{\alpha} = 1$. Therefore, we have

$$D_p = R_i''\, D_i \Rightarrow D_p = \sum_{i=1}^m b_i\, a_i^{\alpha}\, D_p = \sum_{i=1}^m b_i\, a_i^{\alpha}\, R_i''\, D_i, \tag{3.8}$$

$$N_p = R_i''\, N_i \Rightarrow N_p = \sum_{i=1}^m b_i\, a_i^{\alpha}\, N_p = \sum_{i=1}^m b_i\, a_i^{\alpha}\, R_i''\, N_i. \tag{3.9}$$

If we define $S = \sum_{i=1}^m b_i\, a_i^\alpha\, S_i\, D_i$, then we have

$$S = \left(\left(\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^T \,:\, \left(\sum_{i=1}^m b_i\, a_i^\alpha\, Y_i\, D_i\right)^T\right)^T.$$

We claim that the controller $C \in M_{q\times(p-q)}(K)$, defined by

$$C = \left(\sum_{i=1}^m b_i\, a_i^\alpha\, Y_i\, D_i\right)\left(\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1},$$

internally stabilizes the plant P; i.e., we have

$$\begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - PC)^{-1} & (I_q - PC)^{-1}P \\ C(I_q - PC)^{-1} & I_{p-q} + C(I_q - PC)^{-1}P \end{pmatrix} \in M_p(A).$$

We easily check that

$$\begin{aligned}
&I_q - PC\\
&\quad = I_q - D_p^{-1}\, N_p\, \left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, Y_i\, D_i\right)\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1}\\
&\quad = D_p^{-1}[D_p\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right) - N_p\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, Y_i\, D_i\right)]\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1}\\
&\quad = D_p^{-1}[\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, (D_p\, X_i - N_p\, Y_i)\, D_i]\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1}\\
&\quad = D_p^{-1}[\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, R_i''\, (D_i\, X_i - N_i\, Y_i)\, D_i]\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1} \quad \text{(by (3.6))}\\
&\quad = D_p^{-1}[\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, R_i''\, D_i]\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1} \quad \text{(by (3.5))}\\
&\quad = D_p^{-1}\, D_p\left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1} \quad \text{(by (3.8))}\\
&\quad = \left(\textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i\right)^{-1}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow (I_q - PC)^{-1} &&= \textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, D_i \in M_q(A),\\
&\Rightarrow C(I_q - PC)^{-1} &&= \textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, Y_i\, D_i \in M_{(p-q)\times q}(A),\\
&\Rightarrow (I_q - PC)^{-1}P &&= \textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, X_i\, N_i \in M_{q\times(p-q)}(A),\\
&\Rightarrow I_{p-q} + C(I - PC)^{-1}P &&= I_{p-q} + \textstyle\sum_{i=1}^m b_i\, a_i^\alpha\, Y_i\, N_i \in M_{p-q}(A). \qquad \Box
\end{aligned}$$

Remark 3.1. Let us note that the proof of Theorem 3.2 seems to be dual to the one given in [13]. The duality between the approach developed in [26], using the A-modules $A^p\, R^T$ and $A^p\, \tilde{R}^T$, and the one developed here, using the A-modules $A^p/\overline{A^q\, R}$ and $A^p/\overline{A^{p-q}\, \tilde{R}^T}$, will be explained in Proposition 3.4 (see also Proposition 2.8 of [17]). We refer the reader to [20] for another proof of Theorem 3.2 and where it is shown that

$$C' = \left(\sum_{i=1}^m b_i a_i^\alpha Y_i R_i^{''-1}\right)\left(\sum_{i=1}^m b_i a_i^\alpha X_i R_i^{''-1}\right)^{-1}$$

is also a stabilizing controller of P.

Example 3.1. Let $A = H_\infty(\mathbb{C}_+)$ and let us consider the following transfer matrix:

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} & a \\ b & \frac{1}{s-1} \end{pmatrix} \in M_2(K), \tag{3.10}$$

where $a,\, b \in A$. The A-system (see [17]) which corresponds to P is defined by

$$\begin{cases} \frac{(s-1)}{(s+1)}\, y_1 - \frac{e^{-s}}{(s+1)}\, u_1 - a\, \frac{(s-1)}{(s+1)}\, u_2 = 0, \\ \frac{(s-1)}{(s+1)}\, y_2 - b\, \frac{(s-1)}{(s+1)}\, u_1 - \frac{1}{(s+1)}\, u_2 = 0, \end{cases}$$

i.e., $R\, z = 0$, where $z = (y_1 :\ y_2 :\ u_1 :\ u_2)^T$ and R is the matrix defined by

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} & -a\, \frac{(s-1)}{(s+1)} \\ 0 & \frac{s-1}{s+1} & -b\, \frac{(s-1)}{(s+1)} & -\frac{1}{s+1} \end{pmatrix} \in M_{2\times 4}(A). \tag{3.11}$$

Let us check whether or not the A-module $M = A^4/A^2\, R$ is projective. We have $\mathrm{Fitt}_0(M) = 0$, $\mathrm{Fitt}_1(M) = 0$, and

$$\mathrm{Fitt}_2(M) = \left(\left(\frac{s-1}{s+1}\right)^2, \frac{(s-1)}{(s+1)^2}, \frac{e^{-s}}{(s+1)^2}, \frac{(s-1)\, e^{-s}}{(s+1)^2} \right).$$

Then, we have

$$\begin{cases} \left(\frac{s-1}{s+1}\right)^2 + 2\, \frac{(s-1)}{(s+1)^2} = \frac{s-1}{s+1} \in \mathrm{Fitt}_2(M), \\ \frac{(s-1)\, e^{-s}}{(s+1)^2} + 2\, \frac{e^{-s}}{(s+1)^2} = \frac{e^{-s}}{s+1} \in \mathrm{Fitt}_2(M). \end{cases}$$

Moreover,

$$\left(\frac{s-1}{s+1}\right) \left(1 + 2\, \left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + 2\, e\, \left(\frac{e^{-s}}{s+1}\right) = 1 \in \mathrm{Fitt}_2(M) \Rightarrow \mathrm{Fitt}_2(M) = A, \tag{3.12}$$

and thus, by Proposition 4.4 of [17], M is a projective A-module of rank 2. Thus, by Theorem 3.2, P is internally stabilizable. Let us find a controller C using the construction given in the proof of Theorem 3.2. First, let us notice that the fact that $M = A^4/A^2\, R$ is a projective A-module implies that $M/t(M) = M = A^4/A^2\, R$. Second, from (3.12), with the notations of the proof of Theorem 3.2, we have

$$\begin{cases} a_1 = \frac{s-1}{s+1} \in \mathrm{Fitt}_2(M), \\ a_2 = \frac{e^{-s}}{s+1} \in \mathrm{Fitt}_2(M), \\ b_1 = 1 + 2\frac{(1-e^{-(s-1)})}{(s-1)} \in A, \\ b_2 = 2\, e \in A. \end{cases}$$

In $A_{\frac{s-1}{s+1}}$, we have the following right-inverse $S_{\frac{s-1}{s+1}}$ of $R_{\frac{s-1}{s+1}} = R$:

$$\begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} & -a\, \frac{(s-1)}{(s+1)} \\ 0 & \frac{s-1}{s+1} & -b\, \frac{(s-1)}{(s+1)} & -\frac{1}{s+1} \end{pmatrix} \begin{pmatrix} \frac{s+1}{s-1} & 0 \\ 0 & \frac{s+1}{s-1} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In $A_{\frac{e^{-s}}{s+1}}$, we have the following right-inverse $S_{\frac{e^{-s}}{s+1}}$ of $R_{\frac{e^{-s}}{s+1}} = R$:

$$\begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} & -a\, \frac{(s-1)}{(s+1)} \\ 0 & \frac{s-1}{s+1} & -b\, \frac{(s-1)}{(s+1)} & -\frac{1}{s+1} \end{pmatrix} \begin{pmatrix} 0 & -2\, a \\ -b\, \frac{(s+1)}{e^{-s}} & 1 \\ -\frac{(s+1)}{e^{-s}} & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, S is defined by

$$S=\left(\frac{(s-1)}{(s+1)}\left(1+2\,\frac{(1-e^{-(s-1)})}{(s-1)}\right)\begin{pmatrix}\frac{s+1}{s-1} & 0\\ 0 & \frac{s+1}{s-1}\\ 0 & 0\\ 0 & 0\end{pmatrix}+\frac{e^{-s}}{(s+1)}\,2\,e\begin{pmatrix}0 & -2\,a\\ -b\,\frac{(s+1)}{e^{-s}} & 1\\ -\frac{(s+1)}{e^{-s}} & 0\\ 0 & -2\end{pmatrix}\right)\frac{(s-1)}{(s+1)}\,I_2$$

$$=\frac{(s-1)}{(s+1)}\begin{pmatrix}1+2\frac{(1-e^{-(s-1)})}{(s-1)} & -4\,a\,\frac{e^{-(s-1)}}{s+1}\\ -2\,e\,b & 1+2\,\frac{(1-e^{-(s-1)})}{(s-1)}+2\,\frac{e^{-(s-1)}}{(s+1)}\\ -2\,e & 0\\ 0 & -4\,\frac{e^{-(s-1)}}{s+1}\end{pmatrix}.$$

Then, a stabilizing controller C of P is defined by

$$C=\begin{pmatrix}-2\,e & 0\\ 0 & -4\,\frac{e^{-(s-1)}}{s+1}\end{pmatrix}\begin{pmatrix}1+2\,\frac{(1-e^{-(s-1)})}{(s-1)} & -4\,a\,\frac{e^{-(s-1)}}{(s+1)}\\ -2\,e\,b & 1+2\,\frac{(1-e^{-(s-1)})}{(s-1)}+2\,\frac{e^{-(s-1)}}{(s+1)}\end{pmatrix}^{-1}.$$

Remark 3.2. Dually to Theorem 3.2, $P=\tilde{N}_p\,\tilde{D}_p^{-1}\in M_{q\times(p-q)}(K)$ is internally stabilized by $C=\tilde{X}_c^{-1}\,\tilde{Y}_c\in M_{(p-q)\times q}(K)$ iff $\tilde{M}_p=A^p/A^{p-q}\,\tilde{R}_p^{\;T}$ is such that $\tilde{M}_p/t(\tilde{M}_p)$ is a projective A-module, with $\tilde{R}_p^{\;T}=(\tilde{N}_p^T\,:\,\tilde{D}_p^T)^T\in M_{p\times(p-q)}(A)$. In order to shorten the paper, we let the readers check this result themselves. (We can use the fact that C internally stabilizes P iff C^T internally stabilizes P^T.)

Corollary 3.3. *If $P=D_p^{-1}\,N_p\in M_{q\times(p-q)}(K)$ is a weakly left-coprime factorization of P, then P is internally stabilizable iff the A-module $M_p=A^p/A^q\,R_p$ is stably free, i.e., iff $P=D_p^1 N_p$ is a left-coprime factorization of P. Moreover, a stabilizing controller C of P has the form*

$$C=Y_c\,X_c^{-1},$$

where $S=(X_c^T\,:\,Y_c^T)^T\in M_{p\times q}(A)$ is a right inverse of R_p, i.e., $D_p\,X_c-N_p\,Y_c=I_q$.

Proof. $\Rightarrow$ If $P=D_p^{-1}\,N_p$ is internally stabilizable, then, by Theorem 3.2, the A-module $A^p/\overline{A^q\,R_p}$ is a projective A-module, where $R_p=(D_p\,:\,-N_p)\in M_{q\times p}(A)$. Using the fact that $P=D_p^{-1}\,N_p$ is a weakly left-coprime factorization of P, then, by Lemma 2.6 and Theorem 2.11 of [17], we have $\overline{A^q\,R_p}=A^q\,R_p$. Thus, the A-module $M_p=A^p/A^q\,R_p$ is projective and, using the fact that M_p is a projective A-module and R_p is a full row rank matrix, the exact sequence $0\longrightarrow A^q\xrightarrow{.R_p}A^p\longrightarrow M_p\longrightarrow 0$ splits [3, 23]. Thus, we have $M_p\oplus A^q\cong A^p$, i.e., M_p is a stably free A-module.

$\Leftarrow$ Let us suppose that M_p is a stably free A-module. In particular, $M_p=M_p/t(M_p)$ is a stably free A-module, and thus, by Theorem 3.2, P is internally stabilizable.

Moreover, we have the exact sequence $0\longrightarrow A^q\xrightarrow{.R_p}A^p\longrightarrow M_p\longrightarrow 0$. Using the fact that M_p is a stably free A-module, then this exact sequence splits, i.e., there exists $S=(X_c^T\,:\,Y_c^T)^T\in M_{p\times q}(A)$ such that $R_p\,S=I_q$. We check that $C=Y_c\,X_c^{-1}$ is a stabilizing controller of $P=D_p^{-1}\,N_p$ by computing (1.2) [29]. (We can also use the construction of the stabilizing controller given in the proof of Theorem 3.2: $\ker\phi=A^q$, $C=(Y_c\,D_p)\,(X_c\,D_p)^{-1}=Y_c\cdot X_c^{-1}$.) $\square$

Example 3.2. Let us reconsider the transfer matrix P defined by (3.10). In Example 3.1, we proved that the $A = H_\infty(\mathbb{C}_+)$-module $M = A^4/A^2\, R$, where R is defined by (3.11), is projective. Let us check whether or not the A-module M is stably free. The A-module $T(M) = A^2/A^4\, R^T$ is defined by the following equations:

$$\text{(3.13)} \qquad \begin{cases} \frac{(s-1)}{(s+1)}\, \lambda_1 = 0, \\ \frac{(s-1)}{(s+1)}\, \lambda_2 = 0, \\ -\frac{e^{-s}}{(s+1)}\, \lambda_1 - b\, \frac{(s-1)}{(s+1)}\, \lambda_2 = 0, \\ -a\, \frac{(s-1)}{(s+1)}\, \lambda_1 - \frac{1}{(s+1)}\, \lambda_2 = 0. \end{cases}$$

If we denote by $\mu = (\mu_1 :\ \mu_2 :\ \mu_3 :\ \mu_4)^T$ the second member of (3.13), we have

$$\begin{cases} \lambda_1 = (1 + 2\, \frac{(1-e^{-(s-1)})}{(s-1)})\, \mu_1 - 2\, e\, b\, \mu_2 - 2\, e\, \mu_3, \\ \lambda_2 = -2\, a\, \mu_1 + \mu_2 - 2\, \mu_4, \end{cases}$$

which proves that, from (3.13), we can deduce $\lambda_1 = \lambda_2 = 0$, i.e., $T(M) = 0$, and thus, by 2 of Proposition 4.2 of [17], M is a stably free A-module. Moreover, a right-inverse S of R, i.e., $R\, S = I_2$, is defined by

$$\text{(3.14)} \qquad S = \begin{pmatrix} 1 + 2\, \frac{(1-e^{-(s-1)})}{(s-1)} & -2\, a \\ -2\, e\, b & 1 \\ -2\, e & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus, a stabilizing controller C of P is defined by

$$C = \begin{pmatrix} -2\, e & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 + 2\, \frac{(1-e^{-(s-1)})}{(s-1)} & -2\, a \\ -2\, e\, b & 1 \end{pmatrix}^{-1}.$$

The next example shows a situation where Corollary 3.3 cannot be used to construct a stabilizing controller for a plant.

Example 3.3. Let us consider the ring $A = \mathbb{R}[t_0, t_1]/(t_0^2 + t_1^2 - 1)$ of polynomials on the unit circle and x_i the class of t_i in A. We have $A = \mathbb{R}[x_0, x_1]$ with the relation $x_0^2 + x_1^2 = 1$. Let $0 \neq a,\, b \in \mathbb{R}$ be such that $a^2 + b^2 = 1$ and let us consider

$$\text{(3.15)} \qquad p = (b - x_1)/(x_0 - a) \in K = Q(A).$$

It is easy to check that $R = (x_0 - a :\ x_1 - b) \in M_{1\times 2}(A)$ is not weakly left-prime:

$$\left(\frac{x_0+a}{x_1-b}\right)\, (x_0 - a :\ x_1 - b) = (-(x_1 + b) :\ x_0 + a) \in A \quad \not\Rightarrow \quad (x_0 + a)/(x_1 - b) \in A.$$

Therefore, by Corollary 2.5 of [17], the A-module $M = A^2/A\, R$ is not torsion-free. We can show that the torsion submodule $t(M)$ of M is generated by

$$z = (x_1 + b)\, y - (x_0 + a)\, u,$$

which satisfies $(x_1 - b)\, z = 0$. In particular, M is not a free A-module, a fact that implies that there do not exist r and s in A such that $(b - x_1)\, s - (x_0 + a)\, r = 1$, i.e.,

p does not admit a coprime factorization. Moreover, we have $M/t(M) = A^2/A^2\,R'$, where R' is defined by

$$R' = \begin{pmatrix} x_0 - a & x_1 - b \\ x_1 + b & -x_0 - a \end{pmatrix} \in M_2(A) \tag{3.16}$$

and we easily check that

$$\begin{cases} \mathrm{Fitt}_0(M/t(M)) = (-x_0^2 + a^2 - x_1^2 + b^2) = 0, \\ \mathrm{Fitt}_1(M/t(M)) = (x_0 - a, x_0 + a, x_1 - b, x_1 + b). \end{cases}$$

Moreover, we have

$$(x_0 + a)/2\,a - (x_0 - a)/2\,a = 1 \in \mathrm{Fitt}_1(M/t(M)) \Rightarrow \mathrm{Fitt}_1(M/t(M)) = A. \tag{3.17}$$

Thus, by Proposition 4.4 of [17], we obtain that $M/t(M)$ is a projective A-module of rank 1 and, then, by Theorem 3.2, p is internally stabilizable. Hence, we are in a situation where Corollary 3.3 cannot be used to determine a stabilizing controller of p because p does not admit any weakly coprime factorization. ($\overline{A\,R} = A^2\,R'$ and $A^2\,R'$ is not a free A-module.)

We show how to construct a stabilizing controller c for p by following the explicit construction given in the proof of Theorem 3.2. Using the fact $M/t(M) = A^2/A^2\,R'$, we obtain that $\ker\phi$ defined by (3.2) satisfies $\ker\phi = A^2\,R'$, where

$$A^2\,R' = \{\lambda_1\,(x_0 - a :\; x_1 - b) + \lambda_2\,(x_1 + b :\; -(x_0 + a)) \mid \lambda_1, \lambda_2 \in A\}.$$

Let $\alpha = (x_0 - a :\; x_1 - b)$ and $\beta = (x_1 + b :\; -(x_0 + a))$. We have the relations

$$\begin{cases} (x_0 + a)\,\alpha + (x_1 - b)\,\beta = 0, \\ (x_1 + b)\,\alpha - (x_0 - a)\,\beta = 0. \end{cases}$$

$A_{x_0+a} \otimes_A \ker\phi$ is a free A_{x_0+a}-module generated by β because we have

$$\alpha = -[(x_1 - b)/(x_0 + a)]\,\beta.$$

Thus, we have $A_{x_0+a} \otimes_A (M/t(M)) = A^2_{x_0+a}/A_{x_0+a}\,(x_1 + b :\; -(x_0 + a))$ and we have

$$\begin{cases} -\frac{(x_1-b)}{(x_0+a)}\,(x_1 + b :\; -(x_0 + a)) = (x_0 - a :\; x_1 - b) \Rightarrow R''_{x_0+a} = -\frac{(x_1-b)}{(x_0+a)}, \\ (x_1 + b :\; -(x_0 + a)) \begin{pmatrix} 0 \\ \frac{-1}{x_0+a} \end{pmatrix} = 1. \end{cases}$$

$A_{x_0-a} \otimes_A \ker\phi$ is a free A_{x_0-a}-module generated by α because we have

$$\beta = [(x_1 + b)/(x_0 - a)]\,\alpha.$$

Thus, we have $A_{x_0-a} \otimes_A (M/t(M)) = A^2_{x_0-a}/A_{x_0-a}\,(x_0 - a :\; x_1 - b)$, and

$$\begin{cases} (x_0 - a :\; x_1 - b) = (x_0 - a :\; x_1 - b) \Rightarrow R''_{x_0-a} = 1, \\ (x_0 - a :\; x_1 - b) \begin{pmatrix} \frac{1}{x_0-a} \\ 0 \end{pmatrix} = 1. \end{cases}$$

Hence, from (3.17), we obtain

$$S = \tfrac{(x_0+a)}{2\,a} \begin{pmatrix} 0 \\ \dfrac{-1}{x_0 + a} \end{pmatrix} (x_1 + b) - \tfrac{(x_0-a)}{2\,a} \begin{pmatrix} \dfrac{1}{x_0 - a} \\ 0 \end{pmatrix} (x_0 - a) = -\tfrac{1}{2\,a} \begin{pmatrix} x_0 - a \\ x_1 + b \end{pmatrix},$$

and thus, the controller defined by

$$c = \left(-\tfrac{(x_1+b)}{2\,a}\right) / \left(-\tfrac{(x_0-a)}{2\,a}\right) = \tfrac{(x_1+b)}{(x_0-a)}$$

internally stabilizes p. We can easily check that we have

$$\begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = -\frac{1}{2\,a} \begin{pmatrix} x_0 - a & -x_1 + b \\ x_1 + b & x_0 - a \end{pmatrix} \in M_2(A).$$

Remark 3.3. Let us notice that Corollary 2.3 also follows from Corollary 3.3: If A satisfies the conditions of Corollary 2.3, then, by Corollary 3.22 of [17], there exists a weakly left-prime matrix $R'_p = (D'_p : \ -N'_p) \in M_{q\times p}(A)$ such that $P = {D'_p}^{-1}\, N'_p$. By Corollary 3.3, P is internally stabilizable iff P admits a left-coprime factorization, i.e. the A-module $M'_p = A^p/A^q\, R'_p$ is a stably free A-module (see Proposition 4.7 of [17]). Using the fact that A is a projective-free ring, and thus, a Hermite ring, then M'_p is a free A-module and, by Proposition 4.9 of [17], P is internally stabilizable iff P admits a doubly coprime factorization.

PROPOSITION 3.4. *Let $R \in M_{q\times p}(A)$ and $M = A^p/A^q\, R$ be an A-module. Then, $M/t(M) = A^p/\overline{A^q\, R}$ is a projective A-module iff $A^p\, R^T$ is a projective A-module.*

Proof. $\Rightarrow$ Let $M/t(M)$ be a projective A-module. We have the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & 0 & & \\
 & & & & \downarrow & & & & \downarrow & & \\
 & & & & \ker\kappa & & 0 & & t(M) & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longrightarrow \ker .R \longrightarrow & & & A^q & \xrightarrow{.R} & A^p & \xrightarrow{\pi} & M & \longrightarrow 0 \\
(3.18) & & & & \downarrow \kappa & & \| & & \downarrow \pi' & & \\
 & 0 \longrightarrow & & & \ker\phi & \longrightarrow & A^p & \xrightarrow{\phi} & M/t(M) & \longrightarrow 0, \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{coker}\kappa & & 0 & & 0 & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

where $\phi = \pi' \circ \pi$ and $\kappa : A^q \to \ker\phi$ is induced by $\mathrm{id} : A^p \to A^p$ and $\pi' : M \to M/t(M)$. Thus, by the snake lemma [3, 23], we obtain $\ker\kappa \cong \ker .R$ and $\operatorname{coker}\kappa \cong t(M)$. $M/t(M)$ is a projective A-module, and thus, the last horizontal exact sequence splits and $A^p \cong \ker\phi \oplus M/t(M)$. Then, $\ker\phi$ is a finitely generated projective A-module. Therefore, its dual $(\ker\phi)^\star \triangleq \hom_A(\ker\phi, A)$ is also a projective A-module [3, 23]. Dualizing the previous diagram and using the fact that $t(M)^\star \triangleq \hom_A(t(M), A) = 0$, we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
0 \longleftarrow & A^p\, R^T & \xleftarrow{.R^T} & A^p & \longleftarrow & M^\star & \longleftarrow 0 \\
 & & & \| & & \uparrow & \\
0 \longleftarrow & (\ker\phi)^\star & \longleftarrow & A^p & \longleftarrow & (M/t(M))^\star & \longleftarrow 0. \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

Hence, we deduce that $A^p\, R^T \cong (\ker \phi)^\star$, and thus, $A^p\, R^T$ is a projective A-module.

$\Rightarrow$ Let $A^p\, R^T$ be a projective A-module. Then, the exact sequence

$$0 \longleftarrow A^p\, R^T \stackrel{.R^T}{\longleftarrow} A^p \longleftarrow M^\star \longleftarrow 0$$

splits, and thus, we have $A^p \cong A^p\, R^T \oplus M^\star$, which implies that $M^\star \triangleq \hom_A(M, A)$ is a finitely generated projective A-module, and thus, $M^{\star\star}$ is also a projective A-module [3, 23]. Moreover, using the fact that $M^\star$ is a finitely generated A-module, then $M^\star$ has a finite free resolution [3], and thus, $T(M) = A^q/A^p\, R^T$ has a finite free resolution:

$$0 \longleftarrow T(M) \longleftarrow A^q \stackrel{.R^T}{\longleftarrow} A^p \stackrel{.R^T_{-1}}{\longleftarrow} A^n \stackrel{.R^T_{-2}}{\longleftarrow} A^m \stackrel{.R^T_{-3}}{\longleftarrow} \dots .$$

Dualizing this exact sequence, we obtain the following complex:

$$0 \longrightarrow A^q \stackrel{.R}{\longrightarrow} A^p \stackrel{.R_{-1}}{\longrightarrow} A^n \stackrel{.R_{-2}}{\longrightarrow} A^m \stackrel{.R_{-3}}{\longrightarrow} \dots .$$

Therefore, we have the following exact sequence (see [3] for more details):

$$0 \longrightarrow \operatorname{ext}^1_A(T(M), A) \longrightarrow M \longrightarrow \ker .R_{-2} \longrightarrow \operatorname{ext}^2_A(T(M), A) \longrightarrow 0.$$

Moreover, we have the exact sequence $0 \longleftarrow M^\star \longleftarrow A^n \stackrel{.R^T_{-2}}{\longleftarrow} A^m$, which gives by duality the exact sequence $0 \longrightarrow M^{\star\star} \longrightarrow A^n \stackrel{.R_{-2}}{\longrightarrow} A^m$, from which we deduce that $\ker .R_{-2} = M^{\star\star}$. Hence, we obtain the following exact sequence [14]:

$$0 \longrightarrow \operatorname{ext}^1_A(T(M), A) \longrightarrow M \stackrel{\epsilon}{\longrightarrow} M^{\star\star} \longrightarrow \operatorname{ext}^2_A(T(M), A) \longrightarrow 0.$$

We have $\operatorname{ext}^2_A(T(M), A) \cong \operatorname{ext}^1_A(A^p\, R^T, A) = 0$ because $A^p\, R^T$ is a projective A-module [3, 23]. Using the fact that M is a finitely presented A-module, we have the following commutative exact diagram (see [14] for more explanations):

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & 0 & & 0 & & 0 & t(M) & & \\
 & \downarrow & & \downarrow & & \downarrow & \downarrow & & \\
0 \longrightarrow & \hom_A(T(M), A) & \longrightarrow & A^q & \stackrel{.R}{\longrightarrow} & A^p & \longrightarrow & M & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \hom_A(T(M), K) & \longrightarrow & K^q & \stackrel{.R}{\longrightarrow} & K^p & \longrightarrow & K \otimes_A M & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \hom_A(T(M), K/A) & \longrightarrow & (K/A)^q & \longrightarrow & (K/A)^p & \longrightarrow & (K/A) \otimes_A M & \longrightarrow 0, \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \operatorname{ext}^1_A(T(M), A) & & 0 & & 0 & & 0 & \\
 & \downarrow & & & & & & & \\
 & \operatorname{ext}^1_A(T(M), K) = 0 & & & & & & &
\end{array}$$

where $\operatorname{ext}^1_A(T(M), K) = 0$ because K is an *injective A-module* [3]. Thus, a chase in the diagram shows that $\operatorname{ext}^1_A(T(M), A) \cong t(M)$. Finally, using the fact that $\operatorname{ext}^1_A(T(M), A) \cong t(M)$, we have $M/t(M) \cong M^{\star\star}$. The result follows from the fact that $M^{\star\star}$ is projective, and thus, so is $M/t(M) = A^p/\overline{A^q\, R}$. ∎

Using Theorem 3.2 and Proposition 3.4, we obtain the following corollary.

COROLLARY 3.5 (see [26]). *The system $P = D^{-1}\, N \in M_{q\times(p-q)}(K)$ is internally stabilizable iff the A-module $A^p\, R^T$ is projective, where $R = (D : \ -N) \in M_{q\times p}(A)$.*

From Corollary 3.5, we deduce the next result. We refer to [20] for more details and a direct proof of this result.

COROLLARY 3.6. *The system $P = D^{-1} N \in M_{q\times(p-q)}(K)$ is internally stabilizable iff there exists $S = (X^T : Y^T)^T \in M_{p\times q}(K)$ such that*

1. $S\,R = \begin{pmatrix} X\,D & -X\,N \\ Y\,D & -Y\,N \end{pmatrix} \in M_p(A)$,
2. $R\,S = D\,X - N\,Y = I_q$,

where $R = (D :\ -N) \in M_{q\times p}(A)$. Then, $C = Y\,X^{-1}$ internally stabilizes P.

PROPOSITION 3.7. *Let $P = (P_1 + P_2) \in M_{q\times(p-q)}(K)$ be a transfer matrix where $P_1 \in M_{q\times(p-q)}(A)$ is the stable part of P and $P_2 \in M_{q\times(p-q)}(K)$ the instable one. Then, we have the following results:*

(1) *P is internally stabilizable iff P_2 is internally stabilizable.*

(2) *If $P_2 = D_2^{-1}\,N_2$ admits a left-coprime factorization and $S_2 = (X_2^T :\ Y_2^T)^T$ is a right-inverse of $R_2 = (D_2 :\ -N_2) \in M_{q\times p}(A)$, then a stabilizing controller of P is given by $C = C_2\,(I_q + P_1\,C_2)^{-1}$, where $C_2 = Y_2\,X_2^{-1}$ is a stabilizing controller of P_2. A similar result exists if P_2 admits a right-coprime factorization.*

Proof. (1) Let us suppose that $P_2 = D_2^{-1}\,N_2$ is a fractional representation of P_2 where $R_2 = (D_2 :\ -N_2) \in M_{q\times p}(A)$. Then, P has the following fractional representation: $P = D_2^{-1}(D_2\,P_1 + N_2)$ with $R = (D_2 :\ -(D_2\,P_1 + N_2)) \in M_{q\times p}(A)$. Let $M = A^p/A^q\,R$ and $M_2 = A^p/A^q\,R_2$; then we have to prove that the A-module $M/t(M)$ is projective iff $M_2/t(M_2)$ is projective or, equivalently by Proposition 3.4, that the A-module $A^q\,R^T$ is projective iff $A^q\,R_2^T$ is also projective. But, we have trivially $A^q\,R^T = A^q\,R_2^T$.

(2) The A-module $T(M_2) = A^q/A^p\,R_2^T$ is defined by the following equations:

$$\begin{cases} D_2^T\,\lambda = 0, \\ -(N_2^T + P_1^T\,D_2^T)\,\lambda = 0. \end{cases}$$

Putting a second member $\mu = (\mu_1^T :\ \mu_2^T)^T$ in the previous equations and using the fact that S_2 is a right-inverse of R_2, we obtain $\lambda = (X_2^T + Y_2^T\,P_1^T)\,\mu_1 + Y_2^T\,\mu_2$, i.e., $S = ((X_2 + P_1\,Y_2)^T :\ Y_2^T)^T$ is a right-inverse of R. Therefore, by Corollary 3.3,

$$\begin{aligned} C &= Y_2\,(X_2 + P_1\,Y_2)^{-1} = Y_2\,((I + P_1\,Y_2\,X_2^{-1})\,X_2)^{-1} \\ &= Y_2\,X_2^{-1}\,(I + P_1\,(Y_2\,X_2^{-1}))^{-1} = C_2\,(I + P_1\,C_2)^{-1} \end{aligned}$$

is a stabilizing controller of P. □

PROPOSITION 3.8. *A system of the form $P \in M_{1\times(p-1)}(K)$ is internally stabilizable iff one of the following assertions is satisfied:*

- *The ideal $I = (a_1, \ldots, a_p)$ is* invertible [22, 23], *namely we have*

$$I\,(A : I) \triangleq \left\{ \sum_{i=1}^{n} a_i\,b_i \,|\, a_i \in I,\, b_i \in (A : I) \right\} = A, \tag{3.19}$$

  where $(A : I) = \{k \in K = Q(A) \ \ |\ \ (k)\,I \subseteq A\}$ is a fractional ideal of A *and $P = d^{-1}\,N$, $0 \neq d \in A$, $N \in M_{1\times(p-1)}(A)$, $a_1 = d$, and $a_i = N_i$ for $2 \leq i \leq p$.*
- *For $i = 1, \ldots, p$, there exist $x_i \in K = Q(A)$ such that*

$$\begin{cases} \sum_{i=1}^{p} a_i\,x_i = 1, \\ a_i\,x_j \in A, \qquad i, j = 1, \ldots, p. \end{cases} \tag{3.20}$$

Then, the inverse $I^{-1} \triangleq A : I$ *of* I *is defined by* $I^{-1} = (x_1, \ldots, x_p)$ *and*

$$C = -\left(x_2/x_1 : \ldots : x_p/x_1\right)^T \in M_{(p-1)\times 1}(K) \tag{3.21}$$

internally stabilizes P.

Proof. By Theorem 3.2, a plant defined by $P = d^{-1} N \in M_{1\times(p-1)}(K)$ is internally stabilizable iff the A-module $M = A^p/A\,R$ is such that the A-module $M/t(M)$ is projective, where $R = (d : \; -N) = (a_1 : \ldots : a_p) \in M_{1\times p}(A)$. $A^p\,R^T$ is the ideal $I = (a_1, \ldots, a_p)$ of A. Thus, by Proposition 3.4, $M/t(M)$ is a projective A-module iff the ideal $I = (a_1, \ldots, a_p)$ is also a projective A-module. Using the fact that $I \neq 0$, then I is a projective A-module iff I is an invertible ideal, i.e., $I\,(A : I) = A$ [2, 22, 23]. Finally, (3.20) is just (3.19) written in terms of equations (see [23]). □

4. Internal stabilization of SISO plants. The following corollary of Proposition 3.8 gives a characterization of internal stabilization for SISO plants.

COROLLARY 4.1. *A SISO plant, defined by* $p = n/d$ $(0 \neq d,\, n \in A)$, *is internally stabilizable iff one of the following equivalent assertions is satisfied:*

- *The ideal* $I = (n,\, d)$ *is invertible, i.e., we have*

$$I\,(A : I) = A, \tag{4.1}$$

  where $A : I = \{k \in K = Q(A) \mid k\,n,\; k\,d \in A\}$ *is a fractional ideal of* A.
- *There exist* $x,\, y \in K = Q(A)$ *such that*

$$\begin{cases} d\,x - n\,y = 1, \\ d\,x,\; n\,x,\; d\,y,\; n\,y \in A. \end{cases} \tag{4.2}$$

Then, $I^{-1} = A : I = (x, y)$ *and* $c = y/x$ *internally stabilizes* $p = n/d$.

Remark 4.1. We can also check Corollary 4.1 by computing

$$\begin{pmatrix} 1 & -n/d \\ -y/x & 1 \end{pmatrix}^{-1} = \frac{1}{(d\,x - n\,y)} \begin{pmatrix} d\,x & n\,x \\ d\,y & d\,x \end{pmatrix} \in M_2(A),$$

because $d\,x - n\,y = 1$ and $d\,x,\; n\,x,\; d\,y \in A$. We refer to [16, 19] for more characterizations of stabilization problems of SISO plants in terms of fractional ideals.

Example 4.1. Let us consider the ring $A = \mathbb{R}[t_0, t_1]/(t_0^2 + t_1^2 - 1)$ of polynomials on the unit circle S^1. Let x_i be the class of t_i in $\mathbb{R}_1$ and let us reconsider

$$p = (b - x_1)/(x_0 - a) \in K = Q(A), \quad \text{where } a^2 + b^2 = 1, \;\; 0 \neq a, b \in \mathbb{R}.$$

Let us define the ideal $I = (b - x_1,\; x_0 - a)$ of A; then, using the fact that

$$(x_0 - a)\,(x_0 + a) = (b - x_1)\,(b + x_1),$$

we have $A : I = (1, (x_0 + a)/(b - x_1))$ and

$$\left(\tfrac{-1}{2a}\right)(x_0 - a) - \left(-\tfrac{x_0 + a}{2a(b - x_1)}\right)(b - x_1) = 1 \in I\,(A : I) \Rightarrow I\,(A : I) = A.$$

Thus, $c = (x_0{+}a)/(b{-}x_1) = (x_1{+}b)/(x_0{-}a)$ internally stabilizes $p = (b{-}x_1)/(x_0{-}a)$.

Example 4.2. Let us consider $p = (1 + i\,\sqrt{5})/2$ [1]. Let us define the ideal $I = (2, 1 + i\sqrt{5})$ of $A = \mathbb{Z}[i\sqrt{5}]$. Using the fact that $6 = 2 \times 3 = (1 - i\,\sqrt{5})(1 + i\,\sqrt{5})$, we obtain that $A : I = \left(1, (1 - i\,\sqrt{5})/2\right)$. Moreover, we have

$$(-1)\,2 - \left(-\tfrac{1 - i\,\sqrt{5}}{2}\right)(1 + i\,\sqrt{5}) = 1 \in I\,(A : I) \Rightarrow I\,(A : I) = A,$$

and thus, $c = (1 - i\sqrt{5})/2$ is a stabilizing controller of the plant $p = (1 + i\sqrt{5})/2$.

LEMMA 4.2 (see [15]). *Let $I = (n, d)$ be an ideal of A such that $d \neq 0$; then we have*

$$I\,(A : I) = (d :\, n) + (n :\, d),$$

where $(a : b) \triangleq \{c \in A \mid c\,b \in (a)\}$ *for all* $a,\, b \in A$ [24].

Proof. Let us prove that $(d :\, n) + (n :\, d) \subseteq I\,(A : I)$. Let us choose an element $a \in (d : n) = \{b \in A \mid \exists\, k \in A : b\,n = k\,d\}$; then we have

$$\begin{cases} (a/d)\,n = k \in A, \\ (a/d)\,d = a \in A, \end{cases} \Rightarrow (a/d) \in (A : I),\ d \in I \Rightarrow a = d\,(a/d) \in I\,(A : I).$$

Similarly, we prove that $b/n \in (A : I)$, and using the fact that $n \in I$, we obtain that $b = n\,(b/n) \in I\,(A : I)$. Finally, any element $c \in (d :\, n) + (n :\, d)$ can be written as $c = a + b$ with $a \in (d :\, n)$ and $b \in (n :\, d)$, and thus, $c = d\,(a/d) + n\,(b/n) \in I\,(A : I)$, which proves the first inclusion. Second, let us prove that $I\,(A : I) \subseteq (d :\, n) + (n :\, d)$. Any element $c \in I\,(A : I)$ can be written as

$$c = \left(\sum_{i=1}^{l} a_i\, x_i\right) n + \left(\sum_{j=1}^{m} b_j\, x_j\right) d,$$

where $a_i,\, b_j \in A$ and $x_i \in K$ is such that $x_i\, n \in A$ and $x_i\, d \in A$. We have $d\,(\sum_{i=1}^{l} a_i\, x_i\, n) = (\sum_{i=1}^{l} a_i\, x_i\, d)\, n \in (n)$ because $\sum_{i=1}^{l} a_i\, x_i\, d \in A$. In a similar way, we have $n\,(\sum_{j=1}^{m} b_j\, x_j\, d) = (\sum_{j=1}^{m} b_j\, x_j\, n)\, d \in (d)$, and thus, $c \in (d :\, n) + (n :\, d)$, which concludes the proof. $\square$

Using Lemma 4.2, we have the following corollary of Proposition 4.1.

COROLLARY 4.3 (see [24]). *A SISO plant, defined by $p = n/d$ ($0 \neq d,\ n \in A$), is internally stabilizable iff* $(d :\, n) + (n :\, d) = A$.

5. Characterization of the classes of internal stabilizable plants. The following proposition characterizes the integral domains A of SISO stable plants over which every plant is internally stabilizable. We refer to section 3.2 of [17] for the definition of a *Prüfer domain*.

PROPOSITION 5.1 (see [6, 23]). *An integral domain A is a Prüfer domain iff every finitely generated torsion-free A-module M is projective.*

THEOREM 5.2 (see [15]). *We have the equivalences:*

1. *every MIMO plant is internally stabilizable,*
2. *every SISO plant is internally stabilizable,*
3. *A is a Prüfer domain.*

Proof. $1 \Rightarrow 2$ follows from the fact that MIMO plants contain SISO plants.

$2 \Rightarrow 3$ Let us suppose that every SISO system, defined by $p = n/d$, is internally stabilizable. Then, $R = (d :\ -n) \in M_{1\times 2}(A)$ has full row rank. By Theorem 3.2, the A-module $M = A^2/A\,R$ is such that $M/t(M)$ is a projective A-module. But, $A^2\,R^T = (n, d)$ is the ideal of A defined by n and $0 \neq d$. By Proposition 3.4, $M/t(M)$ is a projective A-module iff $I = (n, d)$ is a projective A-module. Hence, every ideal I, generated by two elements n and $0 \neq d$ of A, is a projective A-module, a result which is equivalent to the fact that A is a Prüfer domain (see Lemma 3 of [9]).

$3 \Rightarrow 1$ Let us note $K = Q(A)$ and $P = D^{-1}\,N \in M_{q\times(p-q)}(K)$. Let us define the A-module $M = A^p/A^q\,R$, where $R = (D :\ -N) \in M_{q\times p}(A)$. By hypothesis, A is a

Prüfer domain, and thus, by Proposition 5.1, the torsion-free A-module $M/t(M)$ is projective. Finally, by Theorem 3.2, P is internally stabilizable. $\square$

Example 5.1. We have the following examples of Prüfer domains.

- The domain of entire functions $E(k)$ is a Bézout domain ($k = \mathbb{R}, \mathbb{C}$) [8], and thus, a Prüfer domain [17]. So is $\mathcal{E} = \mathbb{R}(s)[e^{-s}] \cap E(\mathbb{R})$ [11] and RH_∞ [29].
- The integral closure of $\mathbb{Z}$ into a finite extension of $\mathbb{Q}$ is a Dedekind domain, and thus, a Prüfer domain (see section 3.2 of [17] for more details). For instance, the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i\sqrt{5})$ is the Dedekind domain $\mathbb{Z}[i\sqrt{5}]$.
- If A is a one-dimensional Noetherian domain, K is its field of fractions, and L is a finite algebraic extension field of K, then the integral closure of A in L is a Dedekind domain, and thus, a Prüfer domain. In particular, a nonsingular algebraic surface defines a Dedekind affine domain. For instance, the ring $\mathbb{R}[t_0, t_1]/(t_0^2 + t_1^2 - 1)$ of polynomials on the unit circle is a Dedekind domain.
- If X is an affine irreducible nonsingular real algebraic variety of dimension $m+1$ and Y is any subset of X, then the ring $\mathcal{H}_Y(X)$ of rational functions on X, which are locally bounded on Y (i.e., for all $y \in Y$, there exist a neighborhood $\mathcal{V}(y)$ and a positive real number $M(y)$ such that $|n(x)/d(x)| \leq M(y)$ for all $x \in \mathcal{V}(y) \setminus (d^{-1}(0)y)$, is a Prüfer domain and every finitely generated ideal of $\mathcal{H}_Y(X)$ is generated by $m+1$ elements [10]. More generally, the ring of *meromorphic bounded Nash functions* on a *Nash submanifold* of $\mathbb{R}^m$ is a Prüfer domain [10].
- The integral domain $A = \{P \in \mathbb{Q}[x] \mid P(\mathbb{Z}) \subseteq \mathbb{Z}\}$ of $\mathbb{Z}$-valued polynomials in $\mathbb{Q}[x]$ is a Prüfer domain [6].

6. Youla–Kučera parametrization of the stabilizing controllers. The matrices S and S_{-1} defined in Proposition 4.9 of [17] are defined up to an arbitrary matrix which corresponds to the free parameter in the *Youla–Kučera parametrization* [4, 29].

COROLLARY 6.1. *With the same hypothesis as in Proposition* 4.9 *of* [17]*, we have the following splitting exact sequence:*

$$0 \longrightarrow A^q \quad \underset{\underset{.S(Q)}{\longleftarrow}}{\overset{.R}{\longrightarrow}} \quad A^p \quad \underset{\underset{.S_{-1}(Q)}{\longleftarrow}}{\overset{.R_{-1}}{\longrightarrow}} \quad A^{p-q} \longrightarrow 0, \tag{6.1}$$

with

$$\begin{cases} S_{-1}(Q) = S_{-1} + Q\, R, \\ S(Q) = S - R_{-1}\, Q, \end{cases} \tag{6.2}$$

where $R_{-1}, S,$ and S_{-1} are defined in Proposition 4.9 *of* [17] *and $Q \in M_{(p-q)\times q}(A)$. This is equivalent to the following two Bézout identities:*

(1) $\begin{pmatrix} S(Q) & R_{-1} \end{pmatrix} \begin{pmatrix} R \\ S_{-1}(Q) \end{pmatrix} = I_p,$

(2) $\begin{pmatrix} R \\ S_{-1}(Q) \end{pmatrix} \begin{pmatrix} S(Q) & R_{-1} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix} = I_p.$

Proof. We have the following relations which prove the identities (1) and (2):

- $S(Q)\, R + R_{-1}\, S_{-1}(Q) = S\, R + R_{-1}\, S_{-1} = I_p,$
- $R\, S(Q) = R\, S = I_q,$
- $S_{-1}(Q)\, R_{-1} = S_{-1}\, R_{-1} = I_{p-q},$
-
$$\begin{aligned} S_{-1}(Q)\, S(Q) &= S_{-1}\, S - S_{-1}\, R_{-1}\, Q + Q\, R\, S - Q\, R\, R_{-1}\, Q = Q - Q \\ &= 0. \qquad \square \end{aligned}$$

COROLLARY 6.2. *Let $P \in M_{q\times(p-q)}(K)$ be a transfer matrix which admits a doubly coprime factorization. Then, all the stabilizing controllers of P are parametrized by means of the Youla–Kučera parametrization*

$$C(Q) = Y(Q)\,X(Q)^{-1} = \tilde{X}(Q)^{-1}\,\tilde{Y}(Q),$$

where $Q \in M_{(p-q)\times q}(A)$ is a free parameter such that $\det X(Q) \neq 0$, $\det \tilde{X}(Q) \neq 0$, *and $S_1(Q) = (-\tilde{Y}(Q) : \tilde{X}(Q))$ and $S(Q) = (X(Q)^T : Y(Q)^T)^T$ are defined by* (6.2).

Example 6.1. In Example 4.3 of [17], we proved that the $A = H_\infty(\mathbb{C}_+)$-module $M = A^2/A\,R$, with $R = (\frac{s-1}{s+1} : \frac{e^{-s}}{s+1}) \in M_{1\times 2}(A)$, is projective and thus free because A is a coherent Sylvester domain (see Corollary 3.31 of [17]). Few computations lead to the following Bézout identity ($q \in A$):

$$\begin{pmatrix} \frac{s-1}{s+1} & -\frac{e^{-s}}{s+1} \\ 2\,e + \frac{(s-1)}{(s+1)}\,q & 1 + 2\,\frac{(1-e^{-(s-1)})}{(s-1)} - \frac{e^{-s}}{(s+1)}\,q \end{pmatrix} \begin{pmatrix} 1 + 2\,\frac{(1-e^{-(s-1)})}{(s-1)} - \frac{e^{-s}}{(s+1)}\,q & \frac{e^{-s}}{s+1} \\ -2\,e - \frac{(s-1)}{(s+1)}\,q & \frac{s-1}{s+1} \end{pmatrix} = I_2.$$

Thus, all the stabilizing controllers of $p = e^{-s}/(s-1)$ are parametrized by

$$c(q) = \frac{-(2\,e + \frac{(s-1)}{(s+1)}\,q)}{1 + 2\,\frac{(1-e^{-(s-1)})}{(s-1)} - \frac{e^{-s}}{(s+1)}\,q}, \quad q \in A.$$

THEOREM 6.3. *If A is a projective-free domain, then every internally stabilizable plant, defined by a transfer matrix P with entries in $K = Q(A)$, admits doubly coprime factorizations and all the stabilizing controllers of a stabilizable plant can be parametrized by means of the Youla–Kučera parametrization.*

Proof. Using Theorem 3.2 and the exact sequence (3.2), we obtain that $M_p/t(M_p)$ and $\ker\phi$ are two projective A-modules. Using the fact that A is a projective-free ring, we obtain that $M_p/t(M_p)$ and $\ker\phi$ are two free A-modules. From (3.4), we obtain that $\ker\phi \cong A^q$, and thus, we have the following exact sequence:

$$0 \longrightarrow A^q \xrightarrow{.R'} A^p \longrightarrow M_p/t(M_p) \longrightarrow 0,$$

with $R' \in M_{q\times p}(A)$. Using (3.1), we obtain that there exists a full rank matrix $R'' \in M_q(A)$ such that $R = R''\,R'$, i.e., $(D : -N) = R''\,(D' : -N')$, and thus

$$P = D^{-1}\,N = (R''D')^{-1}\,(R''N') = D'^{-1}\,N'.$$

Therefore, by Proposition 4.9 of [17] and Corollary 6.1, the plant P admits doubly coprime factorizations and all the stabilizing controllers of P are parametrized by the Youla–Kučera parametrization. □

COROLLARY 6.4 (see [25]). *If $A = H_\infty(\mathbb{C}_+)$, then a plant is internally stabilizable iff it admits a doubly coprime factorization.*

Example 6.2. The ring $A = \mathbb{R}[t_0, t_1](t_0^2 + t_1^2 - 1)$ (resp., $A = \mathbb{Z}[i\sqrt{5}]$) is a Dedekind domain which is not a principal ideal domain: The ideal $I = (x_0 - a,\, -x_1 + b)$ (resp., $I = (2,\, 1 + i\,\sqrt{5})$) is not a principal ideal [22]. By Corollary 4.13 of [17], it is not possible to parametrize all the stabilizing controllers of $p = (b - x_1)/(x_0 - a)$ (resp., $p = (1 + i\,\sqrt{5})/2$) by means of the Youla–Kučera parametrization.

It is possible to obtain a parametrization of all the stabilizing controllers which generalizes the Youla–Kučera parametrization for a stabilizable plant which does not admit doubly coprime factorizations. We refer the reader to [19, 20] for more details.

PROPOSITION 6.5. *The intersection between the sets of coherent Sylvester domains and Prüfer domains is exactly the set of Bézout domains.*

Proof. $\Rightarrow$ If A is a Prüfer domain, then every ideal $I = (d,\, n)$, generated by two elements $0 \neq d$ and n of A, is invertible [9]. Using the fact that A is also a coherent Sylvester domain, and thus, a greatest common divisor domain (see Corollary 3.20 of [17]), then $I^{-1} = (1,\, 1/[d,\, n])$, where $[d,\, n]$ denotes the greatest common divisor of d and n, and thus, we have

$$I\, I^{-1} = (d/[d,\, n],\, n/[d,\, n]) = A \Rightarrow \exists\, x,\, y \in A :\; d\, x + n\, y = [d,\, n],$$

which proves that I is a principal ideal of A, and thus, A is a Bézout domain.

$\Leftarrow$ By definition, a Bézout domain is a Prüfer and a coherent Sylvester domain. $\square$

Conclusion. We hope we have convinced the reader that the algebraic analysis framework developed in this paper allows us to generalize some results on internal stabilization and to obtain new ones. Due to a lack of space, it was not possible to develop here the strong and the simultaneous stabilization problems [29]. We refer the reader to [16, 18] for a description of a canonical form, based on the concept of *stable range*, that certain stabilizing controllers possess. This canonical form allows us to show that, over a ring A of SISO stable plants of stable range 1 (e.g., $A = H_\infty(\mathbb{C}_+)$), every plant which admits a doubly coprime factorization is strongly stabilizable (i.e., stabilized by means of a stable controller). We also refer the reader to [19, 20] for other results on synthesis problems using fractional ideal and lattice approaches. In particular, a new parametrization of the stabilizing controllers for plants which do not admit doubly coprime factorizations is obtained. Moreover, in this paper the concept of *class group* $C(A)$ and the group $K_0(A)$ of nontrivial isomorphism classes of projective A-modules [22] are introduced. The computations of these groups allow us to check whether or not every internally stabilizable plant admits a doubly coprime factorization (e.g., $\mathcal{C}(\mathbb{R}[t_0, t_1]/(t_0^2 + t_1^2 - 1)) \cong \mathbb{Z}/2\,\mathbb{Z} \neq 0$ and $\mathcal{C}(\mathbb{Z}[i\sqrt{5}]) \cong \mathbb{Z}/2\,\mathbb{Z} \neq 0$ [22] showing that there exist internal stabilizable plants which do not admit a doubly coprime factorization). Finally, in [21], from the algebraic analysis point of view, we show how to recover the operator-theoretic approach developed in [7] (graphs, domains, unbounded operators, etc.) and to obtain new results.

Acknowledgments. I would like to thank H. Lombardi (University of Franche-Comté, France) for discussions on Prüfer domains and for making his paper, "Platitude, localisation et anneaux de Prüfer: une approche constructive," Publications Mathématiques de Besançon, Théorie des nombres, 2002, available to me. I am also grateful to J. J. Loiseau (IRCCyN, France) and C. Bonnet (INRIA Rocquencourt, France) for their continuous interest in this work and to those without whom this paper would never have been written, specially to Céline. I would like to thank the University of Leeds for its hospitality and, specially, J. R. Partington and H. G. Dales.

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