Abstract: The purpose of this paper is to give new necessary and sufficient conditions for internal stabilizability and existence of left/right/doubly coprime factorizations for linear MIMO systems. In particular, we generalize the Youla-Kučera parametrization of all stabilizing controllers for every internally stabilizable MIMO plant which does not necessarily admit doubly coprime factorizations.

Keywords: Youla-Kučera parametrization of all stabilizing controllers, internal stabilizability, left/right/doubly coprime factorizations, lattices.

1. INTRODUCTION

For finite-dimensional linear systems (i.e. ordinary differential equations), it is well known that a transfer matrix is internally stabilizable if and only if it admits a doubly coprime factorization (Vidyasagar, 1985). However, this result is generally not true for infinite-dimensional linear systems (e.g. differential time-delay systems, partial differential equations) (Quadrat, 2003; Vidyasagar, 1985) or for multidimensional linear systems (Mori, 2002; Sule, 1994).

The Youla-Kučera parametrization of all stabilizing controllers was developed for transfer matrices which admit doubly coprime factorizations (Vidyasagar, 1985). The fact that this parametrization is affine in a matrix of free parameters highly simplifies the research of all optimal stabilizing controllers. Indeed, this parametrization allows us to transform this non-linear optimal problem with constraints into a free affine, and thus, convex optimal problem.

The purpose of this paper is to give new general necessary and sufficient conditions for the existence of left/right/doubly coprime factorizations and internal stabilizability. In order to do that, we shall introduce the concept of lattices on vector spaces (Bourbaki, 1989) into the fractional representation approach to analysis and synthesis problems (Vidyasagar, 1985). Using these results, we shall exhibit the general parametrization of all the stabilizing controllers for a stabilizable plant which does not necessarily admit doubly coprime factorizations. In particular, if the transfer matrix admits a doubly coprime factorization, then the previous parametrization becomes the Youla-Kučera one. These results generalize for MIMO systems the results obtained in (Quadrat, 2003).

2. FRACTIONAL REPRESENTATION APPROACH

Let us recall the fractional representation approach to synthesis problems (Vidyasagar, 1985). Let us consider a commutative integral domain $A$ of (proper) stable SISO plants.

Example 1. For instance, we have the following examples of integral domains of stable systems
\[ RH_\infty = \{ n/d | n, d \in \mathbb{R}[s], \deg d \leq \deg d \] \\
\quad \text{d}(s) = 0 \Rightarrow \Re s^\ast < 0 \}, \\
H_\infty(\mathbb{C}_+) = \{ f \in \mathcal{H}(\mathbb{C}_+) | \] \\
\quad \| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty \}, \\
A = \{ f(t) + \sum_{i=0}^{\infty} a_i (t - t_i) | f \in L_1(\mathbb{R}_+), \] \\
\quad (a_i)_{i\geq 0} \in l_1(\mathbb{Z}_+), 0 = t_0 \leq t_1 \leq \ldots \}, \] \\
where \mathbb{C}_+ = \{ s \in \mathbb{C} | \Re s > 0 \}, \mathcal{H}(\mathbb{C}_+) is the ring of holomorphic functions in \mathbb{C}_+ (Quadrat, 2003; Vidyasagar, 1985).

A transfer function \( p \) belongs to \( RH_\infty \) (resp. \( H_\infty(\mathbb{C}_+) \), \( \tilde{A} = \{ \mathcal{L}(f) | f \in \mathcal{A} \} \), where \( \mathcal{L}(f) \) denotes the Laplace transform) if \( p \) is the transfer function of an exponentially stable (resp. \( L_2(\mathbb{R}_+) \)-stable, \( L_\infty(\mathbb{R}_+) \)-stable) linear time-invariant finite-dimensional (resp. infinite-dimensional) system.

Let us define the quotient field of \( A \), namely:
\[ K = Q(A) = \{ n/d | 0 \neq d, n \in A \}. \]

\( K = Q(A) \) corresponds to the class of \( A \)-stable and \( A \)-unstable SISO plants. For instance, we have \( p = e^{-s}/(s - 1) \notin H_\infty(\mathbb{C}_+) \) because \( p \) has an unstable pole in \( \mathbb{C}_+ \). But, \( p \in Q(H_\infty(\mathbb{C}_+)) \) because we have \( p = n/d \), where:
\[ n = e^{-s}/(s + 1), d = (s - 1)/(s + 1) \in H_\infty(\mathbb{C}_+). \]

More generally, we can consider the class of MIMO plants defined by transfer matrices with entries in \( K = Q(A) \). If we have \( P \in K^{q \times (p-q)} \), then we can always write \( P \) as \( P = D^{-1} N = \tilde{N} \tilde{D}^{-1} \), where:
\[ \begin{cases} R = (D : -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in A^{p \times (p-q)}. \end{cases} \]

**Definition 1.** (Vidyasagar, 1985) Let \( A \) be an integral domain of stable SISO plants and \( K = Q(A) \) its quotient field.

- A transfer matrix \( P \in K^{q \times (p-q)} \) is **internally stabilizable** if there exists a stabilizing controller \( C \in K^{(p-q) \times q} \) of \( P \), namely a controller \( C \in K^{q \times (p-q)} \times q \) such that all the entries of the following matrix belong to \( A \):
\[ H(P, C) = \left( \begin{array}{cc} I_q & -P \\ -C & I_{p-q} \end{array} \right)^{-1} \]
\[ = \left( (I_q - PC)^{-1} (I_q - PC)^{-1} P \\ C(I_q - PC)^{-1} (I_q - PC)^{-1} (I_q - PC)^{-1} P \right) \] (1)
\[ = \left( (I_q + (I_{p-q} - C P)^{-1} C (I_{p-q} - C P)^{-1} C P (I_{p-q} - C P)^{-1} C)^{-1} (I_{p-q} - C P)^{-1} C (I_{p-q} - C P)^{-1} P \right) \] (2)

- A transfer matrix \( P \in K^{q \times (p-q)} \) admits a **left-coprime factorization** if there exist
\[ \begin{cases} R = (D : -N) \in A^{q \times p}, \\ S = (X^T : Y^T)^T \in A^{p \times q}, \end{cases} \]
such that \( \det D \neq 0, P = D^{-1} N \) and:
\[ R S = D X - N Y = I_q. \]

-- A transfer matrix \( P \in K^{q \times (p-q)} \) admits a **right-coprime factorization** if there exist
\[ \begin{cases} \tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in A^{p \times (p-q)}, \\ \tilde{S} = (-Y : \tilde{X}) \in A^{(p-q) \times p}, \end{cases} \]
such that \( \det \tilde{D} \neq 0, P = \tilde{N} \tilde{D}^{-1} \) and:
\[ \tilde{S} \tilde{R} = -\tilde{Y} \tilde{N} + \tilde{X} \tilde{D} = I_{p-q}. \]

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**3. COPRIME FACTORIZATION**

**Definition 2.** (Bourbaki, 1989) Let \( V \) be a finite-dimensional \( K = Q(A) \)-vector space. An \( A \)-submodule \( M \) of \( V \) is called a **lattice on \( V \)** if there exist free \( A \)-submodules \( L_1 \) and \( L_2 \) of \( V \) such that:
\[ \begin{cases} L_1 \subseteq M \subseteq L_2, \\ \rk_A(L_1) = \dim_K(V). \end{cases} \]

**Proposition 1.** (Bourbaki, 1989) An \( A \)-submodule \( M \) of \( V \) is a lattice on \( V \) iff the \( K \)-vector space
\[ K M \equiv \{ k m | k \in K, m \in M \} = V \]
and \( M \) is contained in a finitely generated \( A \)-submodule of \( V \).

**Example 2.** If \( P \in K^{q \times (p-q)} \), then the \( A \)-module \( (I_q : -P) A^p \) is a lattice on the \( K \)-vector space \( K^q \). Similarly, the \( A \)-module \( A^1 \times p \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \) is a lattice on the \( K \)-vector space \( K^{1 \times (p-q)} \).

**Definition 3.** (Bourbaki, 1989) Let \( V \) and \( W \) be two finite-dimensional \( K \)-vector spaces and \( M \) (resp. \( N \)) a lattice on \( V \) (resp. \( W \)). Then, we denote by \( N : M \) the \( A \)-submodule of \( \hom_K(V,W) \) formed by the \( K \)-linear maps \( f : V \to W \) which satisfy \( f(M) \subseteq N \).

**Proposition 2.** (Bourbaki, 1989) Let \( V \) and \( W \) be two finite-dimensional \( K \)-vector spaces and \( M \) (resp. \( N \)) a lattice on \( V \) (resp. \( W \)). Then, we have:
\[ \begin{cases} N : M \text{ is a lattice on} \\ \hom_K(V,W) = \{ f : V \to W | f \text{ is a } K \text{- linear map} \}, \end{cases} \]

(1) \( N : M \) is a lattice on \( \hom_K(V,W) \) if \( f : V \to W \) is a \( K \)-linear map,

(2) the canonical map \( N : M \to \hom_A(M,N) \), which maps \( f \in N : M \) into \( f|_{M} \), is bijective.
Example 3. If \( P \in K^{q \times (p-q)} \), then we have:

\[
A : (I_q : -P) A^p = \{ f \in \text{hom}_K(K^q, K) \mid f((I_q : -P) A^p) \subseteq A \}
\]

\[
\{ \lambda \in K^{1 \times q} \mid (I_q : -P) A^p \subseteq A \}
\]

\[
\{ \lambda \in K^{1 \times q} \mid A \subseteq A^{1 \times (p-q)} \}
\]

\[
\{ \lambda \in A^{1 \times q} \mid \lambda P \in A^{1 \times (p-q)} \}
\]

\[
\begin{cases}
\exists \lambda_i \in A^q : f_i = D^{-1} \lambda_i, \ 1 \leq i \leq q, \\
- \ P_j = (I_q : -P) e_{q+j} \in (I_q : -P) A^p = D^{-1} A^q,
\end{cases}
\]

\[
\exists \mu_j \in A^q : p_j = D^{-1} \mu_j, \ 1 \leq j \leq p-q,
\]

\[
D_k^{-1} = D^{-1} f_k \in D^{-1} A^q = (I_q : -P) A^p
\]

\[
\exists \nu_k \in A^p : D_k^{-1} = (I_q : -P) \nu_k, \ 1 \leq k \leq q,
\]

\[
\begin{cases}
\exists D' = (\lambda_1 : \ldots : \lambda_q) \in A^{q \times q} : I_q = D^{-1} D' \\
\Rightarrow D' = D^{-1},
\end{cases}
\]

\[
\begin{cases}
\exists N = (\mu_1 : \ldots : \mu_{p-q}) \in A^{q \times (p-q)} : \\
P = D^{-1} N,
\end{cases}
\]

\[
\begin{cases}
\exists S = (\nu_1 : \ldots : \nu_q) = (X^T : Y^T)^T \in A^{p \times q} : \\
(I_q : -P) S = D^{-1} \Rightarrow D X - N Y = I_q,
\end{cases}
\]

which shows that \( P \) admits a left-coprime factorization \( P = D^{-1} N, D X - N Y = I_q \).

2 can be proved similarly.

4. INTERNAL STABILIZABILITY

Theorem 2. A plant, defined by a transfer matrix \( P \in K^{q \times (p-q)} \), is internally stabilizable iff one of the following equivalent assertions is satisfied:

(1) There exists \( S = (U^T : V^T)^T \in A^{p \times q} \) which satisfies \( \text{det} U \neq 0 \) and:

(a) \( S P = (U P, V P) \in A^{p \times (p-q)} \),

(b) \( (I_q : -P) S = U - P V = I_q \).

Then, the controller \( C = V U^{-1} \) internally stabilizes the plant \( P \).

(2) There exists \( T = (-X : Y) \in A^{(p-q) \times p} \) which satisfies \( \text{det} Y \neq 0 \) and:

(a) \( P T = (U P, V P) \in A^{q \times p} \),

(b) \( T \begin{pmatrix} P & I_{p-q} \end{pmatrix} = -X P + Y = I_{p-q} \).

Then, the controller \( C = Y^{-1} X \) internally stabilizes the plant \( P \).

(3) There exist \( S = (U^T : V^T)^T \in K^{p \times q} \) and \( T = (-X : Y) \in K^{(p-q) \times p} \) which satisfy \( \text{det} U \neq 0 \), det \( Y \neq 0 \) and:

\[
\begin{pmatrix}
U & I_{p-q} \\
V & 1
\end{pmatrix}
\begin{pmatrix}
I_q \\
I_{p-q}
\end{pmatrix}

= \begin{pmatrix}
I_q & -P \\
-X & Y
\end{pmatrix} = I_p,
\]

\[
\begin{pmatrix}
U \\
V
\end{pmatrix}
\begin{pmatrix}
P \\
I_{p-q}
\end{pmatrix}

= \begin{pmatrix}
X \\
Y
\end{pmatrix} = I_{p \times p}.
\]

Then, the controller \( C = V U^{-1} = Y^{-1} X \) internally stabilizes the plant \( P \).

Proof. 1 \( \Rightarrow \) Let us suppose that \( P \in K^{q \times (p-q)} \) is internally stabilizable, i.e. there exists a controller \( C \in K^{(p-q) \times q} \) such that we have:

\[
A_1 = (I_q - P C)^{-1} \in A^{q \times q},
\]

\[
A_2 = (I_q - P C)^{-1} P \in A^{q \times (p-q)},
\]

\[
A_3 = C (I_q - P C)^{-1} \in A^{(p-q) \times q},
\]

\[
A_4 = I_{p-q} + C (I_q - P C)^{-1} P \in A^{(p-q) \times (p-q)}.
\]

(8)
Moreover, from (9), we obtain $C = A_3 A_1^{-1}$. If we define $S = (A_3^T : A_3^T)^T \in A^{p \times q}$, then we have:

$$S(I_q : -P) = \begin{pmatrix} A_1 - A_1 P \\ A_3 - A_3 P \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & -A_2 \\ A_3 & I_{p-q} - A_4 \end{pmatrix} \in A^{p \times p},$$

$$(I_q : -P) S = A_1 - P A_3$$

$$= (I_q - PC)^{-1} - PC (I_q - PC)^{-1}$$

$$= I_q.$$

1 \leftarrow Let us suppose that there exists a matrix $S = (U^T : V^T)^T \in A^{p \times q}$ satisfying $\det U \neq 0$, and 1.a and 1.b. If we define $C = V^{-1} U \in K^{(p-q) \times q}$, then, using point 1.b, we obtain:

$$I_q - PC = (U - PV)V^{-1} = U^{-1}$$

$$= (I_q - PC)^{-1} = U \in A^{q \times q}.$$

Hence, using point 1.a and (1), we obtain

$$H(P, C) = \begin{pmatrix} (I_q - PC)^{-1} & (I_q - PC)^{-1} P \\ C(I_q - PC)^{-1} & I_{p-q} + C(I_q - PC)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} U & U P \\ V & V_{p-q} + V \end{pmatrix} \in A^{p \times p},$$

i.e. $C = V^{-1} U$ internally stabilizes the plant $P$.

2 can be proved similarly.

3 \Rightarrow Let us suppose that $P \in K^{q \times (p-q)}$ is internally stabilized by $C \in K^{(p-q) \times q}$. Following the proofs of 1 \Rightarrow and 2 \Rightarrow, we obtain that $S = (A_2^T : A_2^T)^T$ (resp. $T = (-B_3 : B_4)$, where $B_3 = (I_{p-q} - C P)^{-1} C$ and $B_4 = (I_{p-q} - C P)^{-1}$) satisfies (6) (resp. (7)) and

$$\begin{pmatrix} I_q & -P \\ -B_3 B_4 & A_1 \end{pmatrix} \begin{pmatrix} A_1 & P \\ A_3 & I_{p-q} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ -B_3 A_1 + B_4 A_3 I_{p-q} \end{pmatrix} = I_p,$$

because we have:

$$B_3 A_1 = ((I_{p-q} - C P)^{-1} C)(I_q - PC)^{-1}$$

$$= (I_{p-q} - C P)^{-1} (C(I_q - PC)^{-1})$$

$$= B_4 A_3.$$

Moreover, from (9), we obtain

$$\begin{pmatrix} I_q & -P \\ -B_3 B_4 & A_1 \end{pmatrix} \begin{pmatrix} A_1 & P \\ A_3 & I_{p-q} \end{pmatrix} = \begin{pmatrix} I_q & -P \\ -B_3 B_4 & A_1 \end{pmatrix}$$

which proves (5).

3 \leftarrow Let us suppose that there exist

$$S = (U^T : V^T)^T \in K^{p \times q},$$

$$T = (-X : Y) \in K^{(p-q) \times p},$$

which satisfy (5), (6) and (7). Hence, $S$ (resp. $T$) satisfies 1.a and 1.b (resp. 2.a and 2.b), and thus, by point 1 (resp. point 2), $C_1 = V^{-1} U$ (resp. $C_2 = Y^{-1} X$) is a stabilizing controller of $P$. From (5), we have $X U = YV$, and thus, $C_1 = V^{-1} U = Y^{-1} X = C_2$ is a stabilizing controller of $P$.

Definition 4. (1) $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is a short exact sequence of $A$-modules if the $A$-linear maps $f$ and $g$ satisfy that $f$ is injective, $g$ is surjective and $\ker g = \text{id}_M$.

(2) (Bourbaki, 1989) A short exact sequence is a split exact sequence if one of the following equivalent assertions is satisfied:

- there exists $h : M'' \rightarrow M$, $A$-linear map, which satisfies $g \circ h = \text{id}_{M''}$,
- there exists $k : M \rightarrow M'$, $A$-linear map, which satisfies $k \circ f = \text{id}_{M'}$,
- there exist two $A$-linear maps

$$\begin{cases}
\phi = \begin{pmatrix} g \\ k \end{pmatrix} : M \rightarrow M'' \oplus M', \\
\psi = (h : f) : M'' \oplus M' \rightarrow M
\end{cases}$$

which satisfy:

$$\begin{cases}
\phi \circ \psi = \begin{pmatrix} g \\ k \end{pmatrix}, \\
\psi \circ \phi = (h : f) \begin{pmatrix} g \\ k \end{pmatrix} = \text{id}_M.
\end{cases}$$

In this case, we say that $M$ is isomorphic to $M'' \oplus M'$, denoted by $M \cong M'' \oplus M'$. We shall denote a split exact sequence by:

$$0 \rightarrow M'' \xrightarrow{g} M \xrightarrow{f} M' \rightarrow 0.$$  (10)

(3) (Bourbaki, 1989) An $A$-module $M$ is projective if there exist an $A$-module $P$ and $r \in \mathbb{Z}$ such that we have $M \oplus P \cong A'$, i.e. $M$ is a summand of a finite free $A$-module (namely, a finite product of $A$).

Corollary 1. A plant $P \in K^{q \times (p-q)}$ is internally stabilizable if one of the following equivalent assertions is satisfied:

1. The $A$-module $(I_q : -P) A^p$ is projective.

2. The $A$-module $A^{1 \times p} \left( \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} \right)$ is projective.

Proof. 1. Let us define the following $A$-linear map:

$$g : A^p \rightarrow (I_q : -P) A^p,$$

$$\lambda \mapsto (I_q : -P) \lambda.$$

The $A$-linear map $g$ is surjective and:

$$\ker g = \{ \lambda = (\lambda_1^T : \lambda_2^T)^T \in A^p \mid \lambda_1 = P \lambda_2 \}$$

$$= \{(P \lambda_2)^T : \lambda_2^T \in A^p \mid \lambda_2 \in A^{p-q} : P \lambda_2 \in A^q\}$$

$$= \left( \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} \right) \{ \lambda_2 \in A^{p-q} \mid P \lambda_2 \in A^q \}$$

$$= \left( \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} \right) A \left( \begin{pmatrix} A^{1 \times p} \left( \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} \right) \right).$$
Therefore, we have the following exact sequence
\[ 0 \rightarrow (I_q : -P) A^p \xrightarrow{\phi} A \left( A^{1 \times p} \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \right) \rightarrow 0, \]
where \( f \) is defined by \( f(\lambda) = (P^T : I_{p-q}^T)^T \lambda \).

Now, from 1 of Theorem 2, we know that \( P \) is internally stabilizable iff there exists a matrix
\[ S = \left( V^T : V^T \right) \in A^{p \times q} \]
such that
\[ \left\{ \begin{array}{l}
S(I_q : -P) \in A^{p \times p}, \\
(I_q : -P)S = I_q,
\end{array} \right. \]
i.e. iff there exists an \( A \)-linear map
\[ h : (I_q : -P) A^p \rightarrow A^p, \quad \mu \mapsto S \mu, \]
satisfying \( gh = \text{id}_{(I_q : -P) A^p} \), i.e. iff (11) is a split exact sequence (see 2 of Definition 4). However, if the exact sequence (11) splits, then we have
\[ A^p \cong (I_q : -P) A^p \oplus \left( A : \left( A^{1 \times p} \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \right) \right), \]
which shows that \( (I_q : -P) A^p \) is a projective \( A \)-module. Conversely, if \( (I_q : -P) A^p \) is a projective \( A \)-module, then (11) is a split exact sequence because (11) ends by a projective \( A \)-module (Bourbaki, 1989), and thus, \( P \) is internally stabilizable iff \( (I_q : -P) A^p \) is projective.

2 can be proved similarly.

**Corollary 2.**

(1) If \( P \in K^{q \times (p-q)} \) admits a left-coprime factorization \( P = D^{-1} N \), where
\[ D X - N Y = I_q, \quad \det X \neq 0, \]
and \( (X^T : Y^T) \in A^{p \times q} \), then the matrix
\[ S = \left( (X D)^T : (Y D)^T \right) \in A^{p \times q} \]
satisfies 1.a and 1.b of Theorem 2 and \( C = Y X^{-1} \) is a stabilizing controller of \( P \).

(2) If \( P \in K^{q \times (p-q)} \) admits a right-coprime factorization \( P = \tilde{N} \tilde{D}^{-1} \), where
\[ -\tilde{Y} \tilde{N} + \tilde{X} \tilde{D} = I_{p-q}, \quad \det \tilde{X} \neq 0, \]
and \( (-\tilde{Y} : \tilde{X}) \in A^{(p-q) \times p} \), then the matrix
\[ T = (-\tilde{D} \tilde{Y} : \tilde{D} \tilde{X}) \in A^{(p-q) \times p} \]
satisfies 2.a and 2.b of Theorem 2 and \( C = \tilde{X}^{-1} \tilde{Y} \) is a stabilizing controller of \( P \).

5. A GENERALIZATION OF THE YOULA-KUČERA PARAMETRIZATION

**Lemma 1.** Let us consider the split exact sequence (10). Then, we have:

(1) All the \( A \)-linear maps \( \overline{\kappa} : M'' \rightarrow M \)
satisfying \( g \circ \overline{\kappa} = \text{id}_{M''} \) are of the form \( \overline{\kappa} = h + f \circ l \), where \( l \) is any element of \( \text{hom}_A(M'', M') \), namely any \( A \)-linear map from \( M'' \) to \( M' \).

(2) All the \( A \)-linear maps \( \kappa : M \rightarrow M' \) satisfying \( \kappa \circ f = \text{id}_{M'} \) are of the form \( \kappa = k + l \circ g \), where \( l \) is any element of \( \text{hom}_A(M'', M') \), namely any \( A \)-linear map from \( M'' \) to \( M' \).

(3) For every \( l \in \text{hom}_A(M'', M') \), we have:
\[ \left\{ \begin{array}{l}
\left( g \quad (k - l \circ g) \right) (h + f \circ l : f) = \text{id}_{M''} \oplus \text{id}_{M'}, \\
(h + f \circ l : f) \left( g \quad (k - l \circ g) \right) = \text{id}_{M''},
\end{array} \right. \]

**Theorem 3.** If \( P \in K^{q \times (p-q)} \) is internally stabilizable and \( S = (U^T : V^T)^T \in A^{p \times q} \) (resp. \( T = (-X : Y) \in A^{(p-q) \times p} \)) is a matrix satisfying \( \det U \neq 0 \) (resp. \( \det Y \neq 0 \)) and 1.a and 1.b (resp. 2.a and 2.b) of Theorem 2. Then, all stabilizing controllers of \( P \) are given by
\[ C(Q) = (V + Q)(U + P Q)^{-1} \]
where \( Q \) is every matrix which belongs to
\[ \Omega \triangleq \left\{ \left. A \colon \left( A^{1 \times p} \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \right) \right\} : (I_q : -P) A^p \right\} \]
\[ \Omega = \{ L \in A^{(p-q) \times q} \mid L P \in A^{(p-q) \times (p-q)}, \]
\[ P L \in A^{q \times q}, \quad P L P \in A^{q \times q} \}, \]
and satisfies
\[ \begin{cases}
\det(U + P Q) \neq 0, \\
\det(Y + Q P) \neq 0.
\end{cases} \]

**Proof.** Using the fact that \( P \) is internally stabilizable, then, by 3 of Theorem 2, there exist
\[ S = (U^T : V^T)^T \in A^{p \times q}, \\
T = (-X : Y) \in A^{(p-q) \times p}, \]
which satisfy (5), (6) and (7). Then, the \( A \)-linear map \( h : (I_q : -P) A^p \rightarrow A^p \), defined by (12), satisfies \( \overline{\kappa} \circ h = \text{id}_{(I_q : -P) A^p} \), and thus, (11) becomes the split exact sequence defined by (13), where \( k : A^p \rightarrow A : \left( A^{1 \times p} \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \right) \) is defined by
\[ k(\lambda) = T \lambda, \quad \forall \lambda \in A^p. \]
By Lemma 1, we obtain
\[ \left\{ \begin{array}{l}
\left( g \quad (k - l \circ g) \right) (h + f \circ l : f) = \text{id}_{M''} \oplus \text{id}_{A^{1 \times p}}(P^T : I_{p-q}^T)^T, \\
(h + f \circ l : f) \left( g \quad (k - l \circ g) \right) = \text{id}_{A^p},
\end{array} \right. \]
where \( l \) belongs to (see 2 of Proposition 2):
\[ \text{hom}_A(I_q : -P) A^p, A : \left( A^{1 \times p} \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \right) \]
\[ \cong \left( A : \left( A^{1 \times p} \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \right) \right) : ((I_q : -P) A^p). \]
Therefore, every right inverse of \( g \) has the form \( h + f \circ l \), whereas every left-inverse of \( f \) has the form \( k - l \circ g \), where \( l \in \Omega \). Hence, we have
\[ 0 \leftarrow (I_q : -P) A^p \xleftarrow{q^p} A^p \xrightarrow{h} A : \left( A^{1 \times p} \left( \begin{array}{c} P \\ I_{p-q} \end{array} \right) \right) \leftarrow 0, \]

(13)

Let us denote \( D^{-1} = (D_i^1 : \ldots : D_i^{q-1}) \), where \( D_i^1 \in A^q \). If \( T \in \Omega \), then \( T D_i^{-1} \in \tilde{D} A^{p-q} \), i.e. there exists \( \lambda_i \in A^{p-q} \) such that \( T D_i^{-1} = \tilde{D} \lambda_i \), \( 1 \leq i \leq q \). Now, if we denote \( \Lambda = (\lambda_1 : \ldots : \lambda_q) \in A^{(p-q) \times q}, \)
then we have \( T D^{-1} = \tilde{D} \Lambda \Rightarrow T \in \Omega \).

Conversely, if \( T = \tilde{D} \Lambda D \), with \( \Lambda \in A^{(p-q) \times q}, \)
then \( T D^{-1} = \tilde{D} \Lambda D \), and thus, we have:

\[ T D^{-1} A^q = \tilde{D} \Lambda D A^q \subseteq \tilde{D} A^{p-q} \Rightarrow T \in \Omega, \]
\[ \Rightarrow \Omega = \{ \tilde{D} \Lambda D | \Lambda \in A^{(p-q) \times q} \} = \tilde{D} A^{(p-q) \times q} D. \]

By Corollary 2, \( S = (X D)^T : (Y D)^T \) satisfies 1.a and 1.b of Theorem 2, and thus, by 1 of Theorem 2, \( C = (Y D) (X D)^{-1} = Y X^{-1} \) is a stabilizing controller of \( P \). Moreover, by Corollary 2, \( T = (\tilde{D} \tilde{Y} : D \tilde{X}) \) satisfies 2.a and 2.b of Theorem 2, and thus, by 2 of Theorem 2,
\[ C' = (\tilde{D} X)^{-1} (\tilde{D} \tilde{Y}) = \tilde{X}^{-1} \tilde{Y} \]
is a stabilizing controller of \( P \). Using (17), we obtain that \(-Y X + \tilde{X} Y = 0 \), and thus, we have \( C' = C \). By Theorem 3, we obtain that all the stabilizing controllers of \( P \) are defined by
\[ C(\Lambda) = (Y D + \tilde{D} \Lambda D) (X D + P \tilde{D} \Lambda D)^{-1} \]
\[ = (Y + \tilde{D} \Lambda) \tilde{D} D^{-1} (X + \tilde{N} \Lambda)^{-1} \]
\[ = (Y + \tilde{D} \Lambda) \tilde{D} (X + \tilde{N} \Lambda)^{-1}, \]
\[ C(\Lambda) = (\tilde{D} \tilde{X} + \tilde{D} \Lambda D P)^{-1} (\tilde{D} \tilde{Y} + \tilde{D} \Lambda D) \]
\[ = (\tilde{X} + \Lambda N)^{-1} \tilde{D}^{-1} \tilde{D} (\tilde{Y} + \Lambda D) \]
\[ = (\tilde{X} + \Lambda N)^{-1} (\tilde{Y} + \Lambda D), \]
where \( \Lambda \in A^{(p-q) \times q} \) is every matrix which satisfies \( \text{det}(X + \tilde{N} \Lambda) \neq 0 \) and \( \text{det}(-X + \Lambda N) \neq 0 \).

REFERENCES


