# Controllability and differential flatness of linear analytic ordinary differential systems

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***Abstract*— Based on an extension of Stafford's classical theorem in noncommutative algebra [24] obtained in [4], the purpose of this paper is to show that every controllable linear ordinary differential system with convergent power series coefficients (i.e., germs of real analytic functions) and at least two inputs is differentially flat. This result extends a result obtained in [20], [21] for linear ordinary differential systems with polynomial coefficients. We show how the algorithm developed in [21] for the computation of injective parametrizations and bases of free differential modules with polynomial or rational function coefficients can be used to compute injective parametrizations and flat outputs for these classes of differentially flat systems. This algorithm allows us to remove singularities which naturally appear in the computation of injective parametrizations and bases obtained by means of Jacobson normal form computations.**

## I. INTRODUCTION

Within algebraic analysis ([1], [2], [4], [17]), if $R \in D^{q\times p}$ denotes a $q \times p$ matrix with entries in a ring $D$ and $\mathcal{F}$ a left $D$-module, then the *linear system* or *behaviour*

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$$

can be studied via the quotient/factor left $D$-module

$$M \triangleq D^{1\times p}/(D^{1\times q}\,R),$$

where $D^{1\times q}\,R \triangleq \{\mu\,R \mid \mu \in D^{1\times q}\}$ is the left $D$-submodule of $D^{1\times p}$ formed by the left $D$-linear combinations of the rows of $R$. Indeed, Malgrange remarked ([14]) that we have $\ker_{\mathcal{F}}(R.) \cong \hom_D(M,\mathcal{F})$, where $\hom_D(M,\mathcal{F})$ is the abelian group of left $D$-homomorphisms (i.e., left $D$-linear maps) from $M$ to $\mathcal{F}$ and $\cong$ an isomorphism ([22]). If $\{f_j\}_{j=1,\ldots,p}$ is the *standard basis* of $D^{1\times p}$, namely, $f_j$ is the row vector of length $p$ formed by 1 in its $j^{\text{th}}$ entry and 0 elsewhere, and $\pi : D^{1\times p} \longrightarrow M$ is the *canonical projection* onto $M = D^{1\times p}/(D^{1\times q}\,R)$, namely, the left $D$-homomorphism which maps $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda)$ in $M$, then $\{y_j \triangleq \pi(f_j)\}_{j=1,\ldots,p}$ is a family of generators of $M$ since every $m \in M$ has the form $m = \pi(\lambda)$ for some $\lambda \in D^{1\times p}$, which yields:

$$m = \pi\left(\sum_{j=1}^{p} \lambda_j\, f_j\right) = \sum_{j=1}^{p} \lambda_i\, y_j.$$

This paper is dedicated to Prof. Ulrich Oberst on the occasion of his 70th birthday.

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If $R_{i\bullet}$ denotes the $i^{\text{th}}$ row of $R$, then $\pi(R_{i\bullet}) = 0$ since $R_{i\bullet} \in D^{1\times q}\,R$. Hence, the family of generators $\{y_j\}_{j=1,\ldots,p}$ of $M$ satisfies the left $D$-linear relations

$$\pi(R_{i\bullet}) = \pi\left(\sum_{j=1}^{p} R_{ij}\, f_j\right) = \sum_{j=1}^{p} R_{ij}\, y_j = 0, \; i = 1,\ldots,q.$$

The left $D$-module $M$ is said to be *finitely presented* by $R$ and *defined by generators and relations*. The $\mathbb{Z}$-isomorphism $\chi : \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M,\mathcal{F})$ pointed out by Malgrange is defined by $\chi(\eta)(\pi(\lambda)) = \lambda\,\eta$ for all $\eta \in \ker_{\mathcal{F}}(R.)$ and $\lambda \in D^{1\times p}$, and its inverse $\chi^{-1} : \hom_D(M,\mathcal{F}) \longrightarrow \ker_{\mathcal{F}}(R.)$ is $\chi^{-1}(\phi) = (\phi(y_1) \;\ldots\; \phi(y_p))^T$ for all $\phi \in \hom_D(M,\mathcal{F})$. For more details, see [2], [14], [17].

The previous remark shows that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the underlying finitely presented left $D$-module $M = D^{1\times p}/(D^{1\times q}\,R)$. If the left $D$-module $\mathcal{F}$, called *signal space*, is rich enough, more precisely, if $\mathcal{F}$ is an *injective cogenerator* left $D$-module ([22]), then a complete duality between the system and module properties holds (see, e.g., [2], [19], [17] and the references therein). Roughly speaking, this condition plays a similar role as the classical algebraically closed field condition in algebraic geometry. A dictionary between system and module properties has been developed in the past years (see, e.g., [2], [6], [7], [8], [9], [16], [18], [17], [19], [21]). In particular, it was shown that the concept of controllability for different classes of linear systems is related to the concept of torsion-free modules (see, e.g., [2], [7], [9], [16], [18], [19], [20]) and the concept of differential flatness ([8], [16]) is related to the concept of free modules ([2], [8], [16]).

Based on the development of constructive homological algebra, effective algorithms were obtained in [2], [6], [21] which check whether or not a finitely presented module $M$ over certain classes of noncommutative polynomial rings of functional operators (e.g., the so-called *Ore algebras*) satisfies different module properties (e.g., being torsion, torsion-free, reflexive, projective, stably free, free). These algorithms were implemented in the packages OREMODULES, QUILLENSUSLIN and STAFFORD ([3], [6], [21]). In [20], [21], it was shown that a controllable ordinary differential (OD) linear system with polynomial coefficients and at least two inputs was differentially flat. Using the computation of a

Jacobson normal form of the system matrix $R$, an injective parametrization of the linear system $\ker_{\mathcal{F}}(R.)$ can always be computed (see, e.g., [5]). However, local singularities generally appear. The previous result shows that these local singularities are *removable*, i.e., can always be removed, as soon as the system admits at least two inputs. This result is a consequence of a well-known but difficult result in noncommutative algebra due to Stafford ([24]) which asserts that projective (i.e., stably free) modules of rank at least two over the Weyl algebras $A_n(k)$ and $B_n(k)$ of partial differential (PD) operators in $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$, with coefficients in $k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$ are free whenever $k$ is a field of characteristic zero (e.g., $k = \mathbb{Q}$, $\mathbb{R}$).

The purpose of this paper is to extend the previous result on the relations between controllability and differential flatness by showing that projective left $D$-modules of rank at least two are free over the ring $D$ of OD operators with coefficients in either the ring $k[\![t]\!]$ of formal power series where $k$ is a field of characteristic zero or the ring $k\{t\}$ of locally convergent power series over the field $k = \mathbb{R}$ or $\mathbb{C}$ (i.e., germs of real analytic or holomorphic functions at 0). We show how this result is a consequence of a result of Coutinho and Holland obtained in [4]. A consequence of this result is that any controllable linear OD system with coefficients in the above rings $k[\![t]\!]$ or $k\{t\}$ with at least two inputs is flat. Finally, we show how the algorithm developed in [21] can be extended to these classes of linear OD systems to compute injective parametrizations of differentially flat systems and their flat outputs (the latter correspond to bases of the underlying free left $D$-module $M$).

## II. MODULE THEORY

We recall a few definitions of noncommutative algebra and module theory ([15], [22]). A ring $D$ is called *prime* if $d_1\, D\, d_2 = 0$, i.e., $d_1\, d\, d_2 = 0$ for all $d \in D$, yields $d_1 = 0$ or $d_2 = 0$. A ring $D$ is called a *domain* if $d_1\, d_2 = 0$ implies $d_1 = 0$ or $d_2 = 0$. Hence, a domain is always a prime ring. A commutative domain is called an *integral domain*. An element $d \in D$ is said to be *regular* if $d_1\, d = 0$ and $d\, d_2 = 0$ yields $d_1 = d_2 = 0$. If $D$ is a domain, then every non-zero element of $D$ is regular. A ring $D$ is *left noetherian* (resp., *right noetherian*) if every left ideal (resp., right ideal) of $D$ is finitely generated as a left (resp., right) $D$-module. A ring is simply called *noetherian* if it is left and right noetherian.

*Definition 1 ([4]):* A ring $D$ is called *very simple* if $D$ is prime and noetherian and, if for any $d_1, \ldots, d_4 \in D$ with $d_4$ a regular element of $D$, then there exist $\lambda$, $\mu \in D$ such that $D\, d_1 + D\, d_2 + D\, d_3 = D\,(d_1 + d_4\, \lambda\, d_3) + D\,(d_2 + d_4\, \mu\, d_3)$.

If $D$ is very simple, then for any $d_1$, $d_2$, $d_3 \in D$, there exist $\lambda$, $\mu \in D$ such that:

$$D\, d_1 + D\, d_2 + D\, d_3 = D\,(d_1 + \lambda\, d_3) + D\,(d_2 + \mu\, d_3). \quad (1)$$

*Theorem 1 ([24]):* If $k$ is a field of characteristic zero, then the *Weyl algebras* $A_n(k)$ and $B_n(k)$, namely the $k$-algebras of PD operators in $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$ with coefficients in $k[x_1, \ldots, x_n]$ and $k(x_1, \ldots, x_n)$ respectively, are very simple.

Theorem 1 is a difficult result of noncommutative algebra due to Stafford which has recently been made constructive in [12], [11] and implemented in the packages *Dmodules* ([12]) of Macaulay2 and STAFFORD ([21]) of Maple.

*Definition 2:* Let $M$ be a finitely generated left module over a noetherian domain $D$.

1) $M$ is a *free* left $D$-module if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $M \cong D^{1\times r}$.
2) $M$ is a *stably free* left $D$-module if there exist $r$, $s \in \mathbb{Z}_{\geq 0}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left $D$-modules.
3) $M$ is a *torsion-free* left $D$-module if the *torsion left $D$-submodule* of $M$, i.e.,

$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} :\; d\, m = 0\},$$

is trivial, i.e. the zero module. The elements of $t(M)$ are called the *torsion elements* of $M$.

The *rank* of a stably free left $D$-module $M$ is defined to be $r - s$. Hence, the rank of a free left $D$-module $M \cong D^{1\times r}$ is equal to $r$. A free module is stably free (take $s = 0$) and a stably free module is torsion-free since it can be embedded into a free module, which is clearly torsion-free.

*Definition 3:* 1) The *general linear group* $\mathrm{GL}_p(D)$ consists of the invertible elements of the ring $D^{p\times p}$:

$$\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\; U\, V = V\, U = I_p\},$$

where $I_p$ denotes the $p \times p$ *identity matrix*.

2) The *elementary group* $\mathrm{E}_p(D)$ of $D$ is the subgroup of $\mathrm{GL}_p(D)$ generated by the matrices of the form

$$I_p + d\, E_{ij}, \quad d \in D, \quad i \neq j,$$

and $E_{ij}$ is the matrix defined by 1 in the $(i, j)$-entry and 0 elsewhere.

*Example 1:* Upper and lower triangular matrices whose diagonal entries are 1 belong to the elementary group ([15]).

Let us recall classical results of module theory (see [2], [6], [21] for proofs).

*Theorem 2:* Let $M$ be a finitely generated left $D$-module.

1) $M$ is free of rank $p - q$ if and only if there exist two matrices $R \in D^{q\times p}$ and $U \in \mathrm{GL}_p(D)$ satisfying $R\, U = (I_q \quad 0)$ and $M \cong D^{1\times p}/(D^{1\times q}\, R)$.
2) $M$ is stably free of rank $p - q$ if and only if there exist two matrices $R \in D^{q\times p}$ and $S \in D^{p\times q}$ satisfying $R\, S = I_q$ and $M \cong D^{1\times p}/(D^{1\times q}\, R)$.
3) If the *$D$-closure* $\overline{D^{1\times q}\, R}$ of $D^{1\times q}\, R$ in $D^{1\times p}$ is

$$\overline{D^{1\times q}\, R} = \{\lambda \in D^{1\times p} \mid \exists\, d \in D \backslash \{0\} :\; d\, \lambda \in D^{1\times q}\, R\},$$

then we have:

$$\begin{cases} t(M) = \left(\overline{D^{1\times q}\, R}\right) / (D^{1\times q}\, R), \\ M/t(M) = D^{1\times p} / \left(\overline{D^{1\times q}\, R}\right). \end{cases}$$

For different classes of (not necessarily commutative) polynomial rings, it was shown in [2] how to explicitly obtain a matrix $R' \in D^{q' \times p}$ satisfying $\overline{D^{1 \times q} R} = D^{1 \times q'} R'$ by computing the first extension left $D$-module $\mathrm{ext}^1_D(N, D)$, where the right $D$-module $N = D^q/(R\, D^p)$ is called the *Auslander transpose* of $M$. See [3] for an implementation of the corresponding algorithm in the package OREMODULES based on noncommutative Gröbner basis techniques.

We note that $N = D^q/(R\, D^p)$ is a right $D$-module. It is sometimes convenient to turn the right $D$-module structure of $N$ into a left one by using an *involution* $\theta$ of the ring $D$, namely, an anti-automorphism of $D$, namely, a homomorphism $\theta : D \longrightarrow D$ from the additive group of $D$ to itself which satisfies:

$$\theta^2 = \mathrm{id}_D, \quad \forall\, d_1,\, d_2 \in D, \quad \theta(d_1\, d_2) = \theta(d_2)\, \theta(d_1).$$

*Example 2:* Let $(A, \delta)$ be a *differential ring*, i.e., $A$ is a ring and $\delta : A \longrightarrow A$ satisfies $\delta(a_1 + a_2) = \delta(a_1) + \delta(a_2)$ and $\delta(a_1\, a_2) = a_1\, \delta(a_2) + \delta(a_1)\, a_2$ for all $a_1,\, a_2 \in A$. Moreover, let $D = A\langle \partial \rangle$ be the ring of OD operators with coefficients in a differential ring $(A, \delta)$, namely, $D$ is the noncommutative polynomial ring in $\partial$ with coefficients in $A$: every element $P \in D$ has the form $P = \sum_{i=0}^{r} a_i\, \partial^i$, where the $a_i$'s belong to $A$ and $\partial$ satisfies $\partial\, a = a\, \partial + \delta(a)$ for all $a \in A$. Then, $D$ admits the following involution:

$$\forall\, a \in A, \quad \theta(a) = a, \quad \theta(\partial) = -\partial. \tag{2}$$

If $D$ admits an involution $\theta$ and $R \in D^{q \times p}$, then we define $\theta(R) \triangleq (\theta(R_{ij}))^T \in D^{p \times q}$ and the left $D$-module $\widetilde{N} \triangleq D^{1 \times q}/(D^{1 \times p}\, \theta(R))$, finitely presented by $\theta(R)$, which corresponds to the right $D$-module $N$.

If $\theta$ is the involution of $D = A\langle \partial \rangle$ defined by (2), then the matrix $\theta(R)$ is also known as the *formal adjoint* of the matrix $R$, which we denote by $\widetilde{R}$ in what follows.

*Definition 4 ([4]):* A left $D$-module $M$ is *stably $r$-generated* if for any $p \geq r$ and $m_1, \ldots, m_{p+1} \in M$ such that $M = \sum_{i=1}^{p+1} D\, m_i$, there exist $d_1, \ldots, d_p \in D$ such that:

$$M = \sum_{i=1}^{p} D\, (m_i + d_i\, m_{p+1}).$$

The *stable rank* of $M$ is then the least $r \in \mathbb{N}$ such that $M$ is stably $r$-generated.

*Proposition 1 (Proposition 1.5 of [4], [24]):* Let $D$ be a very simple domain. Then:

1) Every left ideal of $D$ is stably 2-generated.
2) The left $D$-module $D$ has stable rank at most 2.
3) Given any non-zero $d_1,\, d_2 \in D$, there exist $\lambda,\, \mu \in D$ such that:
$$D = D\, d_1\, \lambda + D\, d_2\, \mu.$$
4) A stably free left $D$-module is either free or isomorphic to a left ideal. More precisely, every stably free left $D$-module of reduced rank at least 2 is free.

## III. MAIN RESULTS

Let $k$ be a field of characteristic zero (e.g., $k = \mathbb{Q},\, \mathbb{R},\, \mathbb{C}$). Let $k[\![t]\!]$ be the ring of formal power series with coefficients in $k$, and $k\{t\}$ the ring of convergent power series with coefficients in the field $k = \mathbb{R}$ or $\mathbb{C}$, namely, the subring of $k[\![t]\!]$ formed by elements having a strictly positive radius of convergence. An element $\sum_{i \geq 0} a_i\, t^i$ of $\mathbb{C}\{t\}$ (resp., $\mathbb{R}\{t\}$) can be interpreted as the germ of a holomorphic (resp., real analytic) function at $0$ (see, e.g., [10]).

These two rings are noetherian integral domains and *local*, namely, the ideal $\mathfrak{m} = (t)$ generated by $t$ is the only *maximal ideal* of these rings ([1], [10]). We recall that a maximal ideal of a ring $A$ is a proper ideal of $A$ which is not contained in a larger proper ideal of $A$. An element $\sum_{i \geq 0} a_i\, t^i$ of $k[\![t]\!]$ or $k\{t\}$ is invertible iff $a_0 \neq 0$. Hence, the field of fractions of $k\{t\}$ (resp., $k[\![t]\!]$) is $k\{t\}[t^{-1}]$ (resp., $k[\![t]\!][t^{-1}]$) and can be interpreted as the field of germs of meromorphic functions at $t = 0$. An element of these fields has the form $\sum_{i \geq -n} a_i\, t^i$ for a certain non-negative integer $n$.

These rings are not *artinian*, namely, they admit descending chains of ideals which do not become stationary ([22]). For instance, the descending chain of ideals

$$(t) \supset (t^2) \supset (t^3) \supset \ldots$$

does not become stationary since $(t^i) \supsetneq (t^{i+1})$ for all $i \in \mathbb{N}$.

If $A$ is a local ring, $\mathfrak{m}$ the maximal ideal of $A$, and $k = A/\mathfrak{m}$ the residue field, then a classical result of commutative algebra asserts that $\mathrm{Kdim}(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, where $\mathrm{Kdim}(A)$ denotes the *Krull dimension* of $A$, namely, the supremum of the lengths $d$ of chains

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_d$$

of distinct *prime ideals* of $A$. See, e.g., [22], p. 487. We recall that an ideal $\mathfrak{p}$ of a ring $A$ is prime if $\mathfrak{p} \neq A$, and $a\, b \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Since $\dim_k((t)/(t^2)) = 1$, we get $\mathrm{Kdim}(k[\![t]\!]) \leq 1$ and $\mathrm{Kdim}(k\{t\}) \leq 1$. Moreover, since an integral domain is a field if and only if its Krull dimension is 0 and the integral domains $k[\![t]\!]$ and $k\{t\}$ are not fields, we obtain the following result.

*Lemma 1:* We have $\mathrm{Kdim}(k[\![t]\!]) = 1 = \mathrm{Kdim}(k\{t\})$.

The following lemma is very classical. See, e.g., [1], [13].

*Lemma 2:* If $A = k[\![t]\!]$ or $k\{t\}$ with $k = \mathbb{R}$ or $\mathbb{C}$, then $D = A\langle \partial \rangle$ is a simple noetherian domain.

*Theorem 3 ((ii) of Corollary 6.6.7 of [15]):* Let $A$ be a commutative noetherian ring of finite Krull dimension and $D = A\langle \partial \rangle$ the ring of OD operators with coefficients in a differential ring $A$. If $A$ is not artinian and $D$ is simple, then:

$$\mathrm{Kdim}(D) = \mathrm{Kdim}(A).$$

*Corollary 1:* We have $\mathrm{Kdim}(k[\![t]\!]\langle \partial \rangle) = 1$ and $\mathrm{Kdim}(k\{t\}\langle \partial \rangle) = 1$, where $k = \mathbb{R}$ or $\mathbb{C}$.

We can now state a result of Coutinho and Holland ([4]).

*Theorem 4 (Proposition 1.3 of [4]):* If $D$ is a simple noetherian ring of Krull dimension 1, then $D$ is very simple.

*Corollary 2:* If $A = k[\![t]\!]$ or $k\{t\}$ with $k = \mathbb{R}$ or $\mathbb{C}$, then $D = A\langle \partial \rangle$ is very simple.

*Corollary 3:* If $A = k[\![t]\!]$ or $k\{t\}$ with $k = \mathbb{R}$ or $\mathbb{C}$, then every finitely generated stably free left or right $D = A\langle\partial\rangle$-module $M$ of rank at least 2 is free.

Let us recall a few basic definitions of systems theory.

*Definition 5 ([23]):* Let us consider a time-varying linear OD system of the form:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \tag{3}$$

where $A$ and $B$ are $\mathbb{R}^{n\times n}$- resp. $\mathbb{R}^{n\times m}$-valued functions.

1) (3) is called *analytic* in an interval $\mathcal{I}$ of $\mathbb{R}$ if $A$ and $B$ are analytic on $\mathcal{I}$.
2) (3) is *controllable* on the interval $[a, b]$ of $\mathbb{R}$ if for any $x^\star \in \mathbb{R}$, there exists an essentially bounded function $u$ such that $x^\star = x(b)$, where $x$ satisfies (3) with the initial condition $x(a) = 0$.

*Lemma 3 (2 of Proposition 3.5.16 of [23]):* If the time-varying linear OD system (3) is analytic on $\mathcal{I}$ and $t_0 \in \mathcal{I}$, then (3) is controllable on every nontrivial subinterval of $\mathcal{I}$ if and only if there exists $l \in \mathbb{Z}_{\geq 0}$ such that

$$\mathrm{rank}_{\mathbb{R}}\, C_l(t_0) = n,$$

where $C_l = (B_0, \ldots, B_l)$ and the $B_i$'s are defined by:

$$\forall\, i \in \mathbb{Z}_{\geq 0}, \quad B_{i+1} = A\,B_i - \dot{B}_i, \quad B_0 = B. \tag{4}$$

We note that the controllability of (3) is a local property.

*Proposition 2:* Let $D = k\{t\}\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$. If $A \in k\{t\}^{n\times n}$ and $B \in k\{t\}^{n\times m}$, then the analytic linear OD system $\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$ is controllable in a neighbourhood of $t = 0$ if and only if the finitely presented left $D$-module $M = D^{1\times(n+m)}/(D^{1\times n}\,R)$, where $R = (\partial\, I_n - A \quad -B)$, is stably free.

*Proof:* The left $D$-module $M$ with finite presentation $M = D^{1\times(n+m)}/(D^{1\times n}\,R)$ is stably free if and only if the system matrix $R = (\partial\, I_n - A \quad -B) \in D^{n\times(n+m)}$ admits a right-inverse with entries in $D$, i.e., if and only if, by Proposition 12 of [2], the left $D$-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$, where $\widetilde{R} = (-\partial\, I_n - A^T \quad -B^T)^T \in D^{(n+m)\times n}$, is the zero module. In terms of generators and relations, the left $D$-module $\widetilde{N}$ is defined by:

$$\begin{cases} \dot{\lambda} + A^T\,\lambda = 0, \\ B^T\,\lambda = 0. \end{cases}$$

This system is not formally integrable since the first order equation is not a consequence of the zero order one. Hence, differentiating the second equation, i.e., $B^T\,\dot{\lambda} + \dot{B}^T\,\lambda = 0$, and taking into account the first one, i.e., $\dot{\lambda} = -A^T\,\lambda$, we get the new zero order equation:

$$(B^T\,A^T - \dot{B})\,\lambda = 0.$$

Repeating inductively the same computation with the new zero order equation, we obtain $C_k^T\,\lambda = 0$, where $C_k = (B_0, \ldots, B_k) \in k\{t\}^{n\times m\,k}$ and the $B_i$'s are inductively defined by (4). Since $k\{t\}$ is a noetherian ring, $k\{t\}^{1\times n}$ is a noetherian $k\{t\}$-module and the increasing chain of $k\{t\}$-submodules $L_k = k\{t\}^{1\times m\,k}\,C_k^T$ of $k\{t\}^{1\times n}$ becomes stationary, i.e., there exists $l \in \mathbb{N}$ such that:

$$\forall\, k \geq l, \quad k\{t\}^{1\times m\,k}\,C_k^T = k\{t\}^{1\times m\,l}\,C_l^T.$$

Hence, $M$ is a stably free left $D$-module iff $L_l = k\{t\}^{1\times n}$, i.e., iff the matrix $C_l^T$ admits a left-inverse over $k\{t\}$, i.e., iff the matrix $C_l$ admits a right-inverse over $k\{t\}$. Since all the entries of $C_l$ belong to $k\{t\}$, the last condition is equivalent to $\mathrm{rank}_{\mathbb{R}} C_l(0) = n$, which is then equivalent to the controllability of the analytic linear OD system (3) in a neighbourhood of $t = 0$ by Lemma 3. ∎

We recall that a linear OD system $\ker_{\mathcal{F}}(R.)$ is called *differentially flat* if there exist matrices $Q \in D^{p\times m}$ and $T \in D^{m\times q}$ such that $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$ and $T\,Q = I_m$ ([8]). If so, then, for all $\eta \in \ker_{\mathcal{F}}(R.)$, there exists a unique $\xi = T\,\eta \in \mathcal{F}^m$ such that $\eta = Q\,\xi$, i.e., $\ker_{\mathcal{F}}(R.)$ can be injectively parametrized and the matrix $Q$ is called an *injective parametrization* of $\ker_{\mathcal{F}}(R.)$.

*Corollary 4:* Every controllable linear OD system with convergent power series coefficients and at least 2 inputs is differentially flat.

*Proof:* Since the system matrix $R = (\partial\, I_n - A \quad -B)$ of (3) has full row rank, the rows of $R$ are left $D$-linearly independent and the rank of the left $D = k\{t\}\langle\partial\rangle$-module $M = D^{1\times(n+m)}/(D^{1\times n}\,R)$ is $m$, i.e., the number of inputs of (3) is $m$. Using Proposition 2, if (3) is controllable, then $M$ is stably free of rank $m$ and thus free by 4 of Proposition 1 since $m \geq 2$. Therefore, there exist matrices $S \in D^{(n+m)\times n}$, $Q \in D^{(n+m)\times m}$ and $T \in D^{m\times(n+m)}$ such that

$$0 \longrightarrow D^{1\times n} \underset{.S}{\overset{.R}{\rightleftarrows}} D^{1\times(n+m)} \underset{.T}{\overset{.Q}{\rightleftarrows}} D^{1\times m} \longrightarrow 0 \tag{5}$$

is a split short exact sequence, namely, such that we have $R\,S = I_n$, $T\,Q = I_m$ and $S\,R + Q\,T = I_{n+m}$. If $\mathcal{F}$ is a left $D$-module, then a classical result of homological algebra asserts that the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ transforms split short exact sequences of left $D$-modules into split short exact sequences of abelian groups (see, e.g., [22]). Hence, applying $\hom_D(\,\cdot\,, \mathcal{F})$ to (5), we get the following split short exact sequence of abelian groups

$$0 \longleftarrow \mathcal{F}^n \underset{S.}{\overset{R.}{\leftrightarrows}} \mathcal{F}^{(n+m)} \underset{T.}{\overset{Q.}{\leftrightarrows}} \mathcal{F}^m \longleftarrow 0,$$

which yields $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$ and $T\,Q = I_m$ and proves that $\ker_{\mathcal{F}}(R.)$ is flat for all left $D$-modules $\mathcal{F}$. ∎

Since the previous systemic properties are characterized in terms of module theory (e.g., stably freeness, freeness), they do not depend on the presentation of the underlying finitely presented left $D$-module $M$, i.e., they do not depend on the representation of the analytic linear OD system under the state-space representation (3). Hence, if an analytic linear OD system is defined in the polynomial form by $\ker_{\mathcal{F}}(R.)$, i.e., $D = k\{t\}\langle\partial\rangle$, $R = (P \quad -Q)$, $P \in D^{q\times q}$ has full row rank and $Q \in D^{q\times r}$, then $\ker_{\mathcal{F}}(R.)$ is controllable (resp., flat) if and only if the left $D$-module $L = D^{1\times(q+r)}/(D^{1\times q}\, R)$ is stably free (resp., free), and the controllable analytic linear OD system is flat as soon as $r \geq 2$.

## IV. ALGORITHM

*Definition 6:* A column vector $(a_1, \ldots, a_m)^T \in D^m$ is said to be *unimodular* if there exist $b_1, \ldots, b_m \in D$ such that $\sum_{i=1}^m b_i\, a_i = 1$. We denote by $\mathrm{U}_m(D)$ the set of all unimodular vectors of $D^m$.

*Proposition 3:* Let $D$ be a ring of stable rank 2 and let $v = (v_1 \ \ldots \ v_m)^T \in \mathrm{U}_m(D)$ where $m \geq 3$. Then, there exists a matrix $E \in \mathrm{E}_m(D)$ such that $E\, v = (1 \ 0 \ \ldots \ 0)^T$.

*Proof:* Let $I = D\, v_1 + D\, v_2 + D\, v_m$ be the left ideal of $D$ generated by $v_1$, $v_2$ and $v_m$. Since the stable rank of $D$ is 2, there exist $\lambda$, $\mu \in D$ such that:

$$I = D\,(v_1 + \lambda\, v_m) + D\,(v_2 + \mu\, v_m).$$

Hence, since $v \in \mathrm{U}_m(D)$, we obtain

$$I + \sum_{i=3}^{m-1} D\, v_i = \sum_{i=1}^{m} D\, v_i = D,$$

and thus there exist $d_1, \ldots, d_{m-1} \in D$ such that:

$$d_1\,(v_1 + \lambda\, v_m) + d_2\,(v_2 + \mu\, v_m) + \sum_{i=3}^{m-1} d_i\, v_i = 1.$$

Hence, $v' \triangleq (v_1 + \lambda\, v_m \quad v_2 + \mu\, v_m \quad v_3 \quad \ldots \quad v_{m-1})^T$ admits a left-inverse over $D$ and pre-multiplying the equation $\sum_{i=1}^{m-1} d_i\, v'_i = 1$ by $v'_1 - 1 - v_m$, we get the identity:

$$\sum_{i=1}^{m-1} ((v'_1 - 1 - v_m)\, d_i)\, v'_i = (v'_1 - 1 - v_m).$$

If $v''_i \triangleq (v'_1 - 1 - v_m)\, d_i$ for all $i = 1, \ldots, m-1$, and

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \lambda \\ 0 & 1 & 0 & \ldots & 0 & \mu \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}, \tag{6}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ v''_1 & v''_2 & v''_3 & \ldots & v''_{m-1} & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -v'_2 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -v'_{m-1} & 0 & 0 & \ldots & 1 & 0 \\ -v'_1 + 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

then we can easily check that $E = E_4\, E_3\, E_2\, E_1 \in \mathrm{E}_m(D)$ and $E\, v = (1 \ 0 \ \ldots \ 0)^T$. ■

From Proposition 3, we obtain the following algorithm.

*Algorithm 1:*
- **Input:** The ring $D = A\langle\partial\rangle$ of OD operators with coefficients in the ring $A = k[t]$ or $k[\![t]\!]$ where $k$ is a field of characteristic 0, or $k\{t\}$ where $k = \mathbb{R}$ or $\mathbb{C}$, a column vector $v \in D^m$ which admits a left-inverse over $D$ with $m \geq 3$.
- **Output:** $E \in \mathrm{E}_m(D)$ such that $E\, v = (1 \ 0 \ \ldots \ 0)^T$.

1) Compute $\lambda$, $\mu \in D$ such that $D\, v_1 + D\, v_2 + D\, v_m = D\,(v_1 + \lambda\, v_m) + D\,(v_2 + \mu\, v_m)$.
2) Define $v'_1 = v_1 + \lambda\, v_m$, $v'_2 = v_2 + \mu\, v_m$ and $v'_i = v_i$ for $i = 3, \ldots, m-1$.
3) Compute $d_1, \ldots, d_{m-1} \in D$ satisfying the Bézout identity $\sum_{i=1}^{m-1} d_i\, v'_i = 1$.
4) Define $v''_i = (v'_1 - 1 - v_m)\, d_i$ for all $i = 1, \ldots, m-1$ and the matrices (6).
5) Return $E = E_4\, E_3\, E_2\, E_1$.

*Example 3:* Let $D = \mathbb{R}\{t\}\langle\partial\rangle$ and $v = (0 \quad \sin(t) \quad \partial)^T$. The vector $v$ admits a left-inverse over $D$ since the check of formal integrability of the OD linear system

$$\begin{cases} \Phi_1 = 0, \\ \Phi_2 = \sin(t)\, y, \\ \Phi_3 = \partial\, y \end{cases}$$

yields $\sin(t)\, \Phi_2 + \cos(t)\,(\partial\, \Phi_2 - \sin(t)\, \Phi_3) = y$, i.e., $v$ admits the left-inverse $w = (0 \quad \cos(t)\,\partial + \sin(t) \quad -\cos(t)\,\sin(t))$ and $D\,0 + D\,\sin(t) + D\,\partial = D$. Hence, taking $\lambda = 1$ and $\mu = 0$, we get $J = D\,(0 + \partial) + D\,\sin(t)$ and thus $v'_1 = \partial$, $v'_2 = \sin(t)$, $d_1 = -\cos(t)\,\sin(t)$, $d_2 = \cos(t)\,\partial + \sin(t)$, $v''_1 = \cos(t)\,\sin(t)$, $v''_2 = -\cos(t)\,\partial - \sin(t)$ and we can define the following four matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos(t)\,\sin(t) & -\cos(t)\,\partial - \sin(t) & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ -\sin(t) & 1 & 0 \\ -\partial + 1 & 0 & 1 \end{pmatrix}.$$

Then, the matrix $E = E_4\, E_3\, E_2\, E_1 \in \mathrm{E}_3(D)$ defined by

$$E = \left( \begin{matrix} 1 - \cos(t)\, \sin(t) \\ \sin(t)\, (\cos(t)\, \sin(t) - 1) \\ (\cos(t)\, \sin(t) - 1)\, \partial + 2\, \cos(t)^2 \end{matrix} \right.$$

$$\left. \begin{matrix} \cos(t)\, \partial + \sin(t) & -\cos(t)\, \sin(t) \\ -\cos(t)\, (\sin(t)\, \partial - \cos(t)) & \sin(t)^2\, \cos(t) \\ -\cos(t)\, (\partial^2 + 1) & \cos(t)\, (\sin(t)\, \partial + 2\, \cos(t)) \end{matrix} \right)$$

satisfies $E\, v = (1 \quad 0 \quad 0)^T$. Finally, we check that $E$ is unimodular since $E^{-1}$ is defined by:

$$\begin{pmatrix} 0 & -\cos(t)\, \partial - \sin(t) & \cos(t)\, \sin(t) \\ \sin(t) & 1 & 0 \\ \partial & \cos(t)\, \partial + \sin(t) & 1 - \cos(t)\, \sin(t) \end{pmatrix}.$$

The following result is a direct consequence of 4 of Proposition 1 (see also [15], [24]) but we give here a constructive proof which follows the one given in [21] for the first Weyl algebra $A_1(k)$. The idea of the proof is simple and builds essentially on Gaussian elimination.

*Theorem 5:* Let $D = A\langle \partial \rangle$ be the ring of OD operators with coefficients in $A = k[t]$ or $k[\![t]\!]$ where $k$ is a field of characteristic 0, or $k\{t\}$ where $k = \mathbb{R}$ or $\mathbb{C}$, and $R \in D^{q \times p}$ a matrix admitting a right-inverse $S \in D^{p \times q}$. If $p - q \geq 2$, then the stably free left $D$-module $M = D^{1 \times p}/(D^{1 \times q}\, R)$ of rank $p - q$ is free.

*Proof:* The matrix $R$ has full row rank, namely, $\ker_D(.R) \triangleq \{\lambda \in D^{1 \times q} \mid \lambda\, R = 0\} = 0$, since $R\, S = I_q$, and the left $D$-module $M$ admits the finite free resolution

$$0 \longrightarrow D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0.$$

Applying the involution $\theta$ of $D$ defined by (2) to $R\, S = I_q$, we get $\theta(S)\, \theta(R) = I_q$, i.e., the matrix $\theta(R) \in D^{p \times q}$ admits the left-inverse $\theta(S) \in D^{q \times p}$ and we have the following split short exact sequence:

$$0 \longleftarrow D^{1 \times q} \xleftarrow{.\theta(R)} D^{1 \times p} \longleftarrow \ker_D(.\theta(R)) \longleftarrow 0.$$

Then, the first column vector of $\theta(R)$ is unimodular and, by Proposition 3, we obtain a matrix $G_1 \in \mathrm{E}_p(D)$ satisfying

$$G_1\, \theta(R) = \begin{pmatrix} 1 & \star \\ 0 & \\ \vdots & \theta(R_2) \\ 0 & \end{pmatrix}, \quad \theta(R_2) \in D^{(p-1) \times (q-1)}.$$

We can now check that the first column vector of $\theta(R_2)$ is unimodular and, applying again Proposition 3 to this column vector, we get a matrix $F_2 \in \mathrm{E}_{p-1}(D)$ satisfying

$$F_2\, \theta(R_2) = \begin{pmatrix} 1 & \star \\ 0 & \\ \vdots & \theta(R_3) \\ 0 & \end{pmatrix}, \quad \theta(R_3) \in D^{(p-2) \times (q-2)}.$$

Hence, if $G_2 = \mathrm{diag}(1, F_2) \in \mathrm{E}_p(D)$, then we obtain:

$$(G_2\, G_1)\, \theta(R) = \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ \vdots & \vdots & \theta(R_3) \\ 0 & 0 & \end{pmatrix}.$$

We iterate this procedure until we obtain $\theta(R_q) \in D^{p-q+1}$. Since $p - q + 1 \geq 3$, this works for every $\theta(R_i)$, including $\theta(R_q)$. We finally obtain a matrix $G \in \mathrm{E}_p(D)$ satisfying:

$$G\, \theta(R) = \begin{pmatrix} 1 & \star & \star & \cdots & \star \\ 0 & 1 & \star & \cdots & \star \\ \vdots & \vdots & \vdots & 1 & \star \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{7}$$

If we introduce $J \in D^{q \times q}$ by $G\, \theta(R) = (J^T \quad 0)^T$, then, using Example 1, we get $J \in \mathrm{E}_q(D)$. Pre-multiplying (7) by the matrix $F = \mathrm{diag}(J^{-1}, I_{p-q}) \in \mathrm{E}_p(D)$, we get:

$$F\, G\, \theta(R) = \begin{pmatrix} J^{-1} & 0 \\ 0 & I_{p-q} \end{pmatrix} \begin{pmatrix} J \\ 0 \end{pmatrix} = \begin{pmatrix} I_q \\ 0 \end{pmatrix}.$$

Defining $U = \theta(F\, G) \in \mathrm{GL}_p(D)$, we have $R\, U = (I_q \quad 0)$, which, according to 1 of Theorem 2, proves that $M$ is a free left $D$-module of rank $p - q$ and the residue classes of the last $p - q$ columns of the matrix $U^{-1} = (R^T \quad T^T)^T \in D^{p \times p}$ form a basis of $M$. ∎

*Algorithm 2:*

- **Input:** The ring $D = A\langle \partial \rangle$ of OD operators with coefficients in $A = k[t]$ or $k[\![t]\!]$, where $k$ is a field of characteristic 0, or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, the involution $\theta$ of $D$ defined by (2), a matrix $R \in D^{q \times p}$, with $p - q \geq 2$, which admits a right-inverse $S \in D^{p \times q}$.
- **Output:** Two matrices $Q \in D^{p \times (p-q)}$ and $T \in D^{(p-q) \times p}$ satisfying $T\, Q = I_{p-q}$ and $\{\pi(T_{i\bullet})\}_{i=1,\ldots,p-q}$ forms a basis of the free left $D$-module $M = D^{1 \times p}/(D^{1 \times q}\, R)$, where $T_{i\bullet}$ is the $i^{\text{th}}$ row of $T$ and $\pi : D^{1 \times p} \longrightarrow M$ the canonical projection onto $M$.

1) Compute $\theta(R) \in D^{p \times q}$ and set $i = 1$, $V = \theta(R)$ and $U = I_p$.
2) Denote by $V_i \in D^{p-i+1}$ the column vector formed by the last $p - i + 1$ elements of the $i^{\text{th}}$ column of $V$.

3) By applying Algorithm 1 to the column vector $V_i$, compute a matrix $F_i \in \mathrm{E}_{p-i+1}(D)$ such that:

$$F_i\, V_i = (1 \;\; 0 \;\; \dots \;\; 0)^T.$$

4) Define the matrix $G_i = \begin{pmatrix} I_{i-1} & 0 \\ 0 & F_i \end{pmatrix} \in \mathrm{E}_p(D)$, where $G_1 = F_1$.
5) If $i < q$, then return to 2 with $V$ replaced by $G_i\, V$, $U$ replaced by $G_i\, U$, and $i$ replaced by $i+1$.
6) Define $G = G_q\, U$ and the matrix $P$ formed by the last $p-q$ rows of $G$.
7) Define $Q = \theta(P) \in D^{p \times (p-q)}$ and compute a left-inverse $T \in D^{(p-q) \times p}$ of $Q$.

*Example 4:* Let us consider the following linear system:

$$\left\{ \begin{array}{l} \dot{x}_2(t) - u_2(t) = 0, \\ \dot{x}_1(t) - \sin(t)\, u_1(t) = 0. \end{array} \right. \tag{8}$$

If we rewrite (8) as $\dot{x}(t) = B(t)\, u(t)$, where $x = (x_1 \quad x_2)^T$ and $u = (u_1 \quad u_2)^T$ and $B = \mathrm{diag}(\sin(t), 1)$, then the first two terms of the controllability distribution yield

$$\mathrm{rank}_{\mathbb{R}}(B(t) \quad \dot{B}(t))(0) = \mathrm{rank}_{\mathbb{R}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 2,$$

which shows that (8) is controllable in a neighbourhood of $t = 0$. Moreover, we can easily check that (8) admits the following injective parametrization:

$$\left\{ \begin{array}{l} u_1(t) = \dfrac{\dot{x}_1(t)}{\sin(t)}, \\ u_2(t) = \dot{x}_2(t). \end{array} \right.$$

This injective parametrization is singular at $t = 0$ since $\sin(t)^{-1} = t^{-1} + t/6 + O(t^2)$ and thus $\{x_1, x_2\}$ is a basis of the free $E = \mathbb{R}\{t\}[t^{-1}]\langle \partial \rangle$-module $L = E^{1 \times 4}/(E^{1 \times 2}\, R)$ of rank 2, where $R$ is the system matrix of (8) defined by:

$$R = \begin{pmatrix} 0 & \partial & 0 & -1 \\ \partial & 0 & -\sin(t) & 0 \end{pmatrix}.$$

This last result can be checked again by computing a Jacobson normal form of the matrix $R$ over the principal left ideal domain $E = \mathbb{R}\{t\}[t^{-1}]\langle \partial \rangle$, namely,

$$\begin{aligned} \begin{pmatrix} -1 & 0 \\ 0 & -\sin(t)^{-1} \end{pmatrix} R \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sin(t)^{-1}\, \partial \\ 1 & 0 & \partial & 0 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{9}$$

and considering the injective parametrization defined by the last two columns of third matrix in (9). Let us now study whether or not (8) admits a non-singular injective parametrization at $t = 0$. To do that, let us consider the left $D = \mathbb{R}\{t\}\langle \partial \rangle$-module $M = D^{1 \times 4}/(D^{1 \times 2}\, R)$. Since $R$ has full row rank, we get $\mathrm{rank}_D(M) = 4 - 2 = 2$. Moreover, $R$ admits the following right-inverse:

$$S = \begin{pmatrix} 0 & \cos(t)\, \sin(t) \\ 0 & 0 \\ 0 & \cos(t)\, \partial - 2\, \sin(t) \\ -1 & 0 \end{pmatrix} \in D^{4 \times 2}.$$

Therefore, $M$ is stably free of rank 2 and thus free by Corollary 3. Let us compute a basis of $M$ by means of Algorithm 2. Applying Algorithm 1 to the first column of

$$\theta(R) = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \\ 0 & -\sin(t) \\ -1 & 0 \end{pmatrix} \in D^{4 \times 2},$$

we can take $\lambda = 1$ and $\mu = 0$ since

$$D\, 0 + D\, (-\partial) + D\, (-1) = D\, (0 - 1) + D\, (-\partial),$$

i.e., $v_1' = -1$, $v_2' = -\partial$ and $v_3' = 0$, and thus $d_1 = -1$, $d_2 = 0$, $d_3 = 0$, $v_1'' = 1$, $v_2'' = 0$ and $v_3'' = 0$, and we obtain

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we have:

$$F_1 = E_4\, E_3\, E_2\, E_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\partial \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{E}_4(D),$$

$$F_1\, \theta(R) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\sin(t) \\ 0 & -\partial \end{pmatrix}.$$

We now apply again Algorithm 1 to $(0 \quad -\sin(t) \quad -\partial)^T$. Up to a sign, this was already done in Example 3. For $E$ defined there, $F_2 = -E$ satisfies $F_2\, (0 \quad -\sin(t) \quad -\partial)^T = (1 \quad 0 \quad 0)^T$. Then, $G_2 = \mathrm{diag}(1, F_2)\, F_1 \in \mathrm{E}_4(D)$ is such that $G_2\, \theta(R) = (I_2^T \quad 0^T)^T$ and thus $R\, V = (I_2 \quad 0)$, where $V = \theta(G_2) \in \mathrm{E}_4(D)$ is defined in (10). The matrix $Q$ formed by the last two columns of $V$ defines an injective parametrization of (8), i.e., $\ker_{\mathcal{F}}(R.) = P\, \mathcal{F}^2$ for all left $D$-modules $\mathcal{F}$, and $T\, Q = I_2$, where the matrix $T \in D^{2 \times 4}$ is defined via $V^{-1} = (R^T \quad T^T)^T$ in (11). Finally, the residue

$$V = \begin{pmatrix} 0 & \cos(t)\,\sin(t) & -\cos(t)\,\sin(t)^2 & \cos(t)\,\sin(t)\,\partial - 1 \\ 0 & -1 + \cos(t)\,\sin(t) & -\sin(t)\,(\cos(t)\,\sin(t) - 1) & (\cos(t)\,\sin(t) - 1)\,\partial - 1 \\ 0 & \cos(t)\,\partial - 2\,\sin(t) & -\cos(t)\,\sin(t)\,\partial - 3\,\cos(t)^2 + 1 & (\cos(t)\,\partial - 2\,\sin(t))\,\partial \\ -1 & (\cos(t)\,\sin(t) - 1)\,\partial + 2\,\cos(t)^2 - 1 & (\sin(t) - \cos(t) + \cos(t)^3)\,\partial - 3\,\cos(t)^2\,\sin(t) + \sin(t) + \cos(t) & (\cos(t)\,\sin(t) - 1)\,\partial^2 - 2\,\sin(t)^2\,\partial \end{pmatrix}. \tag{10}$$

$$V^{-1} = \begin{pmatrix} 0 & \partial & 0 & -1 \\ \partial & 0 & -\sin(t) & 0 \\ \cos(t)\,\partial - 2\,\sin(t) & -\cos(t)\,\partial + 2\,\sin(t) & -1 & 0 \\ -1 + \cos(t)\,\sin(t) & -\cos(t)\,\sin(t) & 0 & 0 \end{pmatrix} \in D^{4\times 4}. \tag{11}$$

classes of the two rows $T_{1\bullet}$ and $T_{2\bullet}$ of $T$ in $M$, namely

$$\begin{cases} z_1 = (\cos(t)\,\partial - 2\,\sin(t))\,x_1 + \\ \qquad (-\cos(t)\,\partial + 2\,\sin(t))\,x_2 - u_1, \\ z_2 = (-1 + \cos(t)\,\sin(t))\,x_1 - \cos(t)\,\sin(t)\,x_2, \end{cases}$$

define a basis $\{z_1, z_2\}$ of the free left $D$-module $M$ of rank 2 and we have $(x_1 \quad x_2 \quad u_1 \quad u_2)^T = P\,(z_1 \quad z_2)^T$.

## V. Future Works

Contrary to the case of $A = k[t]$, where $k$ is a field of characteristic zero ([12], [11]), when $A = k[\![t]\!]$ and $k$ is a field of characteristic zero or $A = \mathbb{R}\{t\}$ or $\mathbb{C}\{t\}$, we do not know in general how to compute $\lambda$ and $\mu \in A\langle\partial\rangle$ satisfying (1). The proof that $D$ is very simple developed in [4] has to be made constructive for these particular differential rings $A$. Moreover, the computation of solutions of Bézout identities $\sum_{i=1}^{m-1} d_i\, v_i' = 1$, where $v_i' \in D$, $i = 1, \ldots, m-1$, are given, also needs to be constructively studied. If these important issues can be solved, then Algorithm 2 will be implemented in an extension of the package Stafford ([21]).

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