Towards an effective study of the algebraic parameter estimation problem

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Abstract: The paper aims at developing the first steps toward a symbolic computation approach to the algebraic parameter estimation problem defined by Fliess and Sira-Ramirez and their coauthors. In this paper, within the algebraic analysis approach, we first give a general formulation of the algebraic parameter estimation for signals which are defined by ordinary differential equations with polynomial coefficients such as the standard orthogonal polynomials (Chebyshev, Jacobi, Legendre, Laguerre, Hermite, ... polynomials). Based on a result on holonomic functions, we show that the algebraic parameter estimation problem for a truncated expansion of a function into an orthogonal basis of L^2 defined by orthogonal polynomials can be studied similarly. Then, using symbolic computation methods such as Gröbner basis techniques for (noncommutative) polynomial rings, we first show how to compute ordinary differential operators which annihilate a given polynomial and which contain only certain parameters in their coefficients. Then, we explain how to compute the intersection of the annihilator ideals of two polynomials and characterize the ordinary differential operators which annihilate a first polynomial but not a second one. These results, which are at the core of the algebraic parameter estimation, are implemented in the Non-A package built upon the OreModules software.

Keywords: Parameter estimation problem, algebraic systems theory, symbolic computation, annihilators, orthogonal polynomials, expansion into a basis, ring of differential operators.

1. INTRODUCTION

In this paper, we consider the problem of estimating constant parameters θ of a signal x when the signal

$$y(t) = x(\theta, t) + \gamma(t) + \varpi(t) \tag{1}$$

is observed, and where γ is a perturbation and ϖ a zero-mean noise. Many different approaches have been developed for this problem in the literature of signal processing and control theory. In this paper, we follow the algebraic approach developed by Fliess, Sira-Ramirez, Mboup, and their coauthors. See Fliess et al. (2003); Sira-Ramírez et al. (2014); Mboup et al. (2009); Ushirobira et al. (2012, 2013), and the references therein. In the absence of noise, i.e., $\varpi = 0$, the algebraic approach provides explicit and exact formulas for the constant parameters θ in terms of integrals of the observed signal y. The use of integrals helps to filter the noise ϖ. The explicit expressions of the parameters θ can then be used to do real-time estimation.

In the algebraic approach to parameter estimation, the dynamics of the signal x is defined by a time-invariant ordinary differential (OD) system (Fliess et al. (2003); Sira-Ramírez et al. (2014); Mboup et al. (2009); Ushirobira et al. (2012, 2013)). In Fliess et al. (2010), there is a remark about the possibility to extend this framework to ODEs with polynomial coefficients (see Remark 1 in page 133). Being not aware of this remark but inspired by works developed in the symbolic computation community (Chyzak et al. (2005)), in Ushirobira et al. (2016), we show that the algebraic approach can be extended to signal x which is defined by an OD system with polynomial coefficients. Indeed, within the algebraic approach to parameter estimation, the signal is studied in the frequency domain by means of the operational calculus. The Laplace transform can still be used for OD systems with polynomial coefficients since the time variable t is then transformed into $-\partial_s$, where $\partial_s = \frac{d}{ds}$ is the derivation with respect to the Laplace variable s. Within the *algebraic analysis approach* (Kashiwara et al. (1986); Chyzak et al. (2005)), the Laplace transform is an automorphism of the *Weyl algebra* $A_1(k) = k\langle t, \partial_t \,|\, \partial_t\, t = t\, \partial_t + 1\rangle$ of OD operators with polynomial coefficients in t over a field k of characteristic zero (e.g., $k = \mathbb{Q}, \mathbb{R}$), i.e., the k-linear map $\mathscr{L} : A_1(k) \longrightarrow A_1(k)$ defined by $\mathscr{L}(t) = -\partial_s$ and $\mathscr{L}(\partial_t) = s$ is an automorphism of k-algebras, where the target Weyl algebra is in the complex variable s, i.e., $A_1(k) = k\langle s, \partial_s \,|\, \partial_s\, s = s\, \partial_s + 1\rangle$. The algebraic approach to parameter estimation can then be developed similarly to the linear time-invariant OD systems. In Ushirobira et al. (2016), we initiated the study of signals x which are defined by dynamical systems with polynomial coefficients or signals which can be expanded into orthogonal bases of L^2 defined by *orthogonal polynomials* such as *Hermite*, *Jacobi*, *Laguerre*, ..., *polynomials* (Abramowitz et al. (1964); Chyzak et al. (2005)). Note that the algebraic parameter estimation problem was studied for Taylor expansion series (Mboup et al. (2009)) and initiated for Fourier expansions, i.e., expansions into the orthogonal basis $\{e^{i\,n\,\theta}\}_{n\in\mathbb{Z}}$ of

$L^2(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit torus of $\mathbb{C}$ (Ushirobira et al. (2012, 2013)).

The first goal of this paper is to give a formulation of the algebraic parameter estimation for signals x which satisfy an OD equation with polynomial coefficients. Then, we show that the case of a signal x which is a truncated expansion of a function into a basis defined by a family of orthogonal polynomials can similarly be studied. Finally, using computer algebra techniques (e.g., *Gröbner basis techniques*), we initiate an effective study of the computations of annihilators of polynomials, a problem which plays a key role in the algebraic parameter estimation.

2. ALGEBRAIC PARAMETER ESTIMATION

2.1 Classes of signals x under study

Let a signal x satisfy the following OD equations (ODE)

$$\sum_{i=0}^{n} a_i(t)\, x^{(i)}(t) = 0, \tag{2}$$

where the a_i's are polynomials in t and with coefficients in a commutative ring K of *constants* (namely, $\dot{c} = 0$ for $c \in K$), i.e., $a_i \in K[t]$ for $i = 0, \ldots, n$. Using the following standard results on the Laplace transform $\mathscr{L}$

- $\mathscr{L}\left(f^{(n)}\right)(s) = s^n\, \mathscr{L}(f)(s) - \sum_{i=0}^{n-1} s^{n-i-1}\, f^{(i)}(0)$,
- $\mathscr{L}(t^n\, f)(s) = (-1)^n\, \partial_s^n\, (\mathscr{L}(f)(s))$,

with the notation $\widehat{x} = \mathscr{L}(x)$, we obtain:

$$\sum_{i=0}^{n} a_i(-\partial_s)\left(s^i\, \widehat{x}(s) - \sum_{j=0}^{i-1} s^{i-j-1}\, x^{(j)}(0)\right) = 0.$$

The above identity can be rewritten as:

$$\sum_{i=0}^{n}\left(a_i(-\partial_s)\, s^i\right)\, \widehat{x}(s) - \sum_{i=0}^{n}\sum_{j=0}^{i-1}\left(a_i(-\partial_s)\, s^{i-j-1}\right)\, x^{(j)}(0) = 0.$$

Expending the second term of the above identity, we get

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{j=0}^{i-1}\left(a_i(-\partial_s)\, s^{i-j-1}\right)\, x^{(j)}(0) \\
&= \sum_{j=1}^{n}\left(\sum_{i=j}^{n} a_i(-\partial_s)\, s^{i-j}\right)\, x^{(j-1)}(0) \\
&= \sum_{k=0}^{n-1}\left(\sum_{i=k+1}^{n} a_i(-\partial_s)\, s^{i-k-1}\right)\, x^{(k)}(0),
\end{aligned}$$

and thus, we obtain:

$$\begin{aligned}
&\sum_{i=0}^{n}\left(a_i(-\partial_s)\, s^i\right)\, \widehat{x}(s) \\
&- \sum_{k=0}^{n-1}\left(\sum_{i=k+1}^{n} a_i(-\partial_s)\, s^{i-k-1}\right)\, x^{(k)}(0) = 0.
\end{aligned}$$

If we note

$$R = \sum_{i=0}^{n} a_i(-\partial_s)\, s^i, \quad S_k = -\sum_{i=k+1}^{n} a_i(-\partial_s)\, s^{i-k-1},$$

for $k = 0, \ldots, n-1$, then the above identity becomes:

$$R(s,\, \partial_s)\, \widehat{x}(s) + \sum_{k=0}^{n-1} S_k(s)\, x^{(k)}(0) = 0. \tag{3}$$

Let $D = K[s]\langle\partial_s\rangle$ be the noncommutative polynomial ring of OD operators with coefficients in the commutative ring $A = K[s]$, namely, the ring of noncommutative polynomials of the form $\sum_{i=0}^{n} a_i\, \partial_s^i$, where $a_i \in A$ for $i = 0, \ldots, n$, which satisfy the following relation:

$$\forall\, a \in A, \quad \partial_s\, a = a\, \partial_s + \frac{da}{ds}.$$

Now, using the operator identity $\partial_s\, s = s\, \partial_s + 1$ in D, which corresponds to the standard Leibniz identity, i.e.,

$$(\partial_s\, s)(z(s)) = \frac{d}{ds}\,(s\, z(s)) = s\, \frac{d}{ds}\, z(s) + z(s) = (s\, \partial_s + 1)\, z(s),$$

the terms $a_i(-\partial_s)\, s^k$ can be rewritten as an OD operator of the form $\sum_{l=0}^{m} b_l(s)\, \partial_s^l$, i.e., as an element of D. For more details, see, e.g., Kashiwara et al. (1986); McConnell et al. (2000).

Remark 1. Within the framework developed in Fliess et al. (2003), the algebraic estimation problem is stated for a general time-invariant linear control system defined by:

$$\sum_{i=0}^{n} a_i\, x^{(i)}(t) = \sum_{j=0}^{n-1} b_j\, u^{(j)}(t).$$

The presence of an input u does not bring substantial differences from the input free case ($u = 0$). Indeed, an extra term coming from the contribution of u has to be added. To simplify, in this paper, we shall only consider the case $u = 0$ as well as the single-output case. For the general case, we refer the reader to Quadrat (2017).

Let us now consider the signal $z = x + \gamma$, where γ is a perturbation which admits a Laplace transform $\widehat{\gamma} = \mathscr{L}(\gamma)$. Then, combining $\widehat{z} = \widehat{x} + \widehat{\gamma}$ with (3), we obtain:

$$R(s, \partial_s)\, \widehat{z}(s) + \sum_{k=0}^{n-1} S_k(s)\, x^{(k)}(0) - R(s, \partial_s)\, \widehat{\gamma}(s) = 0.$$

To simplify the notation, let $\vartheta_k = x^{(k)}(0)$, $k = 0, \ldots, n-1$, and $S = \sum_{k=0}^{n-1} S_k(s)\, \vartheta_k \in K[s]$. Hence, we obtain:

$$R(s,\, \partial_s)\, \widehat{z}(s) + S(s) - R(s,\, \partial_s)\, \widehat{\gamma}(s) = 0. \tag{4}$$

When γ satisfies an ODE, then the perturbation γ is called *structured* (see Fliess et al. (2003)). If γ satisfies an ODE of order m with polynomial coefficients, i.e., $\sum_{i=0}^{m} a_i'(t)\, \gamma^{(i)}(t) = 0$, where $a_i' \in K[t]$ for $i = 0, \ldots, m$, then, similarly as above, $\widehat{\gamma}$ satisfies the following equation

$$R'(s,\, \partial_s)\, \widehat{\gamma}(s) + S'(s) = 0, \tag{5}$$

where $R' = \sum_{i=0}^{m} a_i'(-\partial_s)\, s^i$ and:

$$S_k' = -\sum_{i=k+1}^{m} a_i'(-\partial_s)\, s^{i-k-1}, \quad S' = \sum_{k=0}^{m-1} S_k'(s)\, \gamma^{(k)}(0).$$

Example 1. If γ is a constant biais, i.e., $\gamma(t) = \gamma\, H(t)$, where H is the *Heaviside distribution* (i.e., $H(t) = 1$ for $t > 0$ and 0 for $t < 0$) and γ is a real constant, then, within algebraic analysis, it is well-known that this distribution is the solution of the ODE with polynomial coefficients $t\, \partial_t\, (\gamma(t)) = 0$. This can be checked again by using the Laplace transform which yields $\partial_s\, (s\, \widehat{\gamma}(s)) = 0$ and its solution $\widehat{\gamma}(s) = \gamma/s$ is the Laplace transform of $\gamma(t)$. Similarly, if $\gamma(t)$ is an impulse, i.e., $\gamma(t) = \gamma\, \delta(t)$, where δ is the *Dirac distribution* and γ is a real constant, then, within algebraic analysis, it is known that this distribution satisfies the equation $t\, \gamma(t) = 0$. This can be checked again since, using the Laplace transform, we get $\partial_s \widehat{\gamma}(s) = 0$, i.e., $\widehat{\gamma}(s) = \gamma$, which is the Laplace transform of $\gamma(t) = \gamma\, \delta(t)$.

To simplify, in this paper, we shall only consider the perturbation γ to be a constant biais (see Example 1). For the general case, see Quadrat (2017). Using the Laplace transform of γ, i.e., $\widehat{\gamma}(s) = \gamma/s$, and (4), we then get:

$$R(s, \partial_s)\, \widehat{z}(s) + S(s) - R(s, \partial_s)\, \frac{\gamma}{s} = 0. \tag{6}$$

We can clean the denominator of (6) to obtain an OD operator with polynomial coefficients in s, i.e., an element of the ring D. We first have:

$$R(s, \partial_s)\, \frac{\gamma}{s} = \gamma \left(\sum_{i=1}^{n} a_i(-\partial_s)\, s^{i-1} + a_0(-\partial_s)\, \frac{1}{s} \right).$$

If the degree of $a_0(-\partial_s)$ in ∂_s is n_0, then using the identity

$$\frac{d^{n_0}}{ds^{n_0}}\, s^{-1} = -n_0!\, s^{-(n_0+1)},$$

we get that $a_0(-\partial_s)\, s^{-1} = d_0(s)/s^{n_0+1}$, where $d_0 \in k[s]$. Therefore, multiplying (6) by s^{n_0+1}, we obtain:

$$s^{n_0+1}\, R(s, \partial_s)\, \widehat{z}(s) + s^{n_0+1}\, S(s) \\ -\gamma \left(s^{n_0+1} \sum_{i=0}^{n} a_i(-\partial_s)\, s^{i-1} \right) = 0.$$

If we note

$$\left\{ \begin{array}{l} P = s^{n_0+1}\, R(s, \partial_s) \in D = K[s]\langle \partial_s \rangle, \\ Q = s^{n_0+1}\, S(s) = s^{n_0+1} \displaystyle\sum_{k=0}^{n-1} S_k(s)\, \vartheta_k \in K[\vartheta_1, \ldots, \vartheta_{n-1}, s], \\ \overline{Q} = -s^{n_0+1} \displaystyle\sum_{i=0}^{n} a_i(-\partial_s)\, s^{i-1}\, \gamma \in K[\gamma, s], \end{array} \right. \tag{7}$$

then we finally obtain that the signal z satisfies:

$$P(s,\, \partial_s)\, \widehat{z}(s) + Q(s) + \overline{Q}(s) = 0. \tag{8}$$

Example 2. If $x = \sum_{k=0}^{m} \frac{a_k}{k!}\, t^k$ is a polynomial of degree m, then we have $x^{(m+1)}(t) = 0$, and thus $n = m+1$, $a_n = 1$, $a_i = 0$ for $i = 0, \ldots, n-1$, $\vartheta_k = x^{(k)}(0)$ for $k = 0, \ldots, n-1$, $n_0 = 0$, and:

$$P = s^{m+2}, \quad Q = -s \sum_{k=0}^{m} s^{m-k}\, \vartheta_k, \quad \overline{Q} = -\gamma\, s^{m+1}.$$

Example 3. If $x = e^{-i\, \frac{2\pi}{T}\, n\, t}$ is an element of the *Fourier orthogonal basis* $\{e^{-i\, \frac{2\pi}{T}\, n\, t}\}_{n \in \mathbb{Z}}$ of $L^2(\mathbb{R})$, then x satisfies the first order ODE $\dot{x} + i\, \nu\, n\, x = 0$, where $\nu = 2\, \pi/T$, i.e., $a_1 = \partial_s$, $a_0 = i\, \nu\, n$ and $n_0 = 0$. Thus, we get (8), where:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0), \\ P = s\, (s + i\, \nu\, n)\,, \\ Q = -s\, \vartheta_0, \\ \overline{Q} = -\gamma\, (s + i\, \nu\, n)\,. \end{array} \right.$$

Example 4. Let x be a signal satisfying a second ODE

$$a_2(t)\, \ddot{x}(t) + a_1(t)\, \dot{x}(t) + a_0(t)\, x(t) = 0, \tag{9}$$

where $a_i \in K[t]$, $i = 0, 1, 2$. If $\deg_t a_0 = n_0$, then (7) gives:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0), \quad \vartheta_1 = \dot{x}(0), \\ P = s^{n_0+1} \left(a_0(-\partial_s)\, 1 + a_1(-\partial_s)\, s + a_2(-\partial_s)\, s^2 \right), \\ Q = -s^{n_0+1} \left(Q_0(s)\, \vartheta_0 + Q_1(s)\, \vartheta_1 \right), \\ \quad = -s^{n_0+1} \left((a_1(-\partial)\, 1 + a_2(-\partial_s)\, s)\, \vartheta_0 + a_2(-\partial_s)\, \vartheta_1 \right), \\ \overline{Q} = -\gamma\, s^{n_0+1} \left(a_0(-\partial_s)\, s^{-1} + a_1(-\partial_s)\, 1 + a_2(-\partial_s)\, s \right). \end{array} \right.$$

Now, let us consider the following case

$$\left\{ \begin{array}{l} a_2(t) = a_{22}\, t^2 + a_{21}\, t + a_{20}, \\ a_1(t) = a_{11}\, t + a_{10}, \\ a_0(t) = a_{00}, \end{array} \right.$$

where a_{22}, a_{21}, a_{20}, a_{11}, a_{10}, $a_{00} \in K$. Then, we have $n_0 = \deg_t a_0 = 0$ and:

$$\left\{ \begin{array}{l} P = s\, (a_{22}\, s^2\, \partial_s^2 + s\, (-a_{21}\, s + 4\, a_{22} - a_{11})\, \partial_s \\ \qquad + a_{20}\, s^2 + (a_{10} - 2\, a_{21})\, s + (a_{00} - a_{11} + 2\, a_{22}), \\ Q = -s\, ((a_{20}\, s + a_{10} - a_{21})\, \vartheta_0 + a_{20}\, \vartheta_1), \\ \overline{Q} = -(a_{20}\, s^2 + (a_{10} - a_{21})\, s + a_{00})\, \gamma. \end{array} \right.$$

(1) If $x = A\, \sin(\omega\, t + \phi)$, then $a_2 = 1$, $a_1 = 0$ and $a_0 = \omega^2$, and thus we obtain (8), where:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0),\ \vartheta_1 = \dot{x}(0), \\ P = s\, (s^2 + \omega^2), \\ Q = -s\, \left(s\, \vartheta_0 + \vartheta_1 \right), \\ \overline{Q} = -(s^2 + \omega^2)\, \gamma. \end{array} \right.$$

(2) If $x = T_n$ is the n^{th} *Chebyshev polynomial of the first kind*, then we have $a_2 = -t^2+1$, $a_1 = -t$ and $a_0 = n^2$, and thus we get (8), where:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0),\ \vartheta_1 = \dot{x}(0), \\ P = -s\, \left(s^2\, \partial_s^2 + 3\, s\, \partial_s - s^2 - n^2 + 1 \right), \\ Q = -s\, \left(s\, \vartheta_0 + \vartheta_1 \right), \\ \overline{Q} = -(s^2 + n^2)\, \gamma. \end{array} \right.$$

If $x = U_n$ is the n^{th} *Chebyshev polynomial of the second kind*, then we have $a_2 = -t^2 + 1$, $a_1 = -3\, t$ and $a_0 = n\, (n+2)$, and thus we get (8), where:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0),\ \vartheta_1 = \dot{x}(0), \\ P = -s\, \left(s^2\, \partial_s^2 + s\, \partial_s - s^2 - (n+1)^2 \right), \\ Q = -s\, \left(s\, \vartheta_0 + \vartheta_1 \right), \\ \overline{Q} = -(s^2 + n\, (n+2))\, \gamma. \end{array} \right.$$

(3) If $x = P_n^{\alpha,\beta}$ is the n^{th} *Jacobi polynomial* which depends on the parameters α and β, then we have $a_2 = -t^2 + 1$, $a_1 = -(\alpha + \beta + 2)\, t + \beta - \alpha$ and $a_0 = n\, (n + \alpha + \beta + 1)$, and thus we get (8), where:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0),\ \vartheta_1 = \dot{x}(0), \\ P = s\, (-s^2\, \partial_s^2 + (\beta + \alpha - 2)\, s\, \partial_s + s^2 + (\beta - \alpha)\, s \\ \qquad + (n+1)\, (n + \alpha + \beta)), \\ Q = -s\, (\vartheta_0\, (s + \beta - \alpha) + \vartheta_1), \\ \overline{Q} = -\left(s^2 + (\beta - \alpha)\, s + n\, (n + \alpha + \beta + 1) \right)\, \gamma. \end{array} \right.$$

The n^{th} *Legendre polynomial* is a particular case of the n^{th} Jacobi polynomial for which $\alpha = \beta = 0$, i.e.:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0),\ \vartheta_1 = \dot{x}(0), \\ P = s\, \left(-s^2\, \partial_s^2 - 2\, s\, \partial_s + s^2 + (n+1)\, n \right), \\ Q = -s\, \left(\vartheta_0\, s + \vartheta_1 \right), \\ \overline{Q} = -\left(s^2 + n\, (n+1) \right)\, \gamma. \end{array} \right.$$

(4) If $x = L_n^{(\alpha)}$ is the n^{th} *Laguerre polynomial* which depends on α, then we have $a_2 = t$, $a_1 = -t + \alpha + 1$ and $a_0 = n$, and thus we get (8), where:

$$\left\{ \begin{array}{l} \vartheta_0 = x(0),\ \vartheta_1 = \dot{x}(0), \\ P = s\, (s\, (1 - s)\, \partial_s + (\alpha - 1)\, s + n + 1), \\ Q = -s\, \alpha\, \vartheta_0, \\ \overline{Q} = -(\alpha\, s + n)\, \gamma. \end{array} \right.$$

(5) If $x = H_n$ is the n^{th} *Hermite polynomial*, then $a_2 = 1$, $a_1 = -2\,t$, $a_0 = 2\,n$, and we get (8), where:

$$\begin{cases} \vartheta_0 = x(0),\ \vartheta_1 = \dot{x}(0), \\ P = s\,(2\,s\,\partial_s + s^2 + 2\,(n+1)) \\ Q = -s\,(s\,\vartheta_0 + \vartheta_1) \\ \overline{Q} = -(s^2 + 2\,n)\,\gamma. \end{cases}$$

Remark 2. The signal x in (1) could also be a *wavelet*. This case will be studied in a future publication. Wavelets satisfy *functional equations* such as dilation equations. Classes of functional equations can be studied by means of the so-called *Ore algebras*. For more details, see Chyzak et al. (2005) and the references therein.

2.2 Estimation of an expansion of x into a basis

Till now, we have consider x to be the solution of an ODE with polynomial coefficients. More generally, x can be the output of a linear OD system with polynomial coefficients. An intermediate case is a signal $x = \sum_{k=0}^{N} \lambda_k\, x_k$ which is a linear combination of signals x_k which satisfy ODEs of the form (2) and the λ_k's are constants. This case corresponds to the case of uncoupled linear OD system since the x_k's are independent from one another. For instance, $\{x_k\}_{k=0,\ldots,N}$ can be the first N terms of the Taylor basis, i.e., $x_k = t^k/k!$. They can also be the first generators of a basis of L^2 such as the Fourier orthogonal basis or an orthogonal basis defined by a family of orthogonal polynomials. In this case, x is an approximation of the expansion of a function into an orthogonal basis. An important remark is that x then satisfies an ODE of the form of (2). This result a consequence of the concept of *holonomic functions* developed in algebraic analysis (Kashiwara et al. (1986)). See the next paragraph for a simple explanation in the particular of ODEs with rational functions coefficients. As a consequence for the algebraic parameter estimation problem, we can always assume that x is defined by (2), i.e., we do not have to make a distinction between the case of a single signal x and of a finite linear combination of signals x_k, when each signal x_k satisfies an ODE with polynomial coefficients.

Let us suppose that K is a field and let $D = K(t)\langle\partial\rangle$ be the noncommutative ring of OD operators in ∂_t with coefficients in the field $K(t)$ of rational functions in t with coefficients in K. An element of D is of the form $\sum_{i=0}^{n} a_i(t)\,\partial_t^i$, where $a_i \in K(t)$ for $i = 0, \ldots, n$. A function x is said to be *D-finite* if the left D-module defined by x, namely, $D\,x = \{d\,x \mid d \in D\}$, is a finite-dimensional $K(t)$-vector space (Chyzak et al. (2005)). If x is a D-finite function, then the set $\{\partial_t^i\, x = x^{(i)}\}_{i=0,\ldots,n}$ of cardinality $n+1$ admits at least one relation over $K(t)$, i.e., there exist $a_i \in K(t)$ for $i = 0, \ldots, n$ such that $\sum_{i=0}^{n} a_i(t)\,\partial_t^i\, x(t) = 0$, i.e., x satisfies an ODE with rational function coefficients, and thus an ODE of the form (2). Conversely, if x satisfies an ODE of the form (2), then we get $x^{(n)} = \sum_{i=0}^{n-1} \frac{a_i}{a_n}\, x^{(i)}$, which shows that $D\,x$ is a $K(t)$-vector space of dimension n with the basis $\{x^{(i)}\}_{i=0,\ldots,n-1}$. If x_1 and x_2 are two D-finite functions and λ_1 and λ_2 two constants, then we clearly have

$$I = D\,(\lambda_1\, x_1 + \lambda_2\, x_2) \subseteq D\,x_1 + D\,x_2,$$

which shows that I is a finite-dimensional $K(t)$-vector space since $D\,x_1$ and $D\,x_2$ are both finite-dimensional over $K(t)$, which implies that $\lambda_1\, x_1 + \lambda_2\, x_2$ satisfies an ODE with rational function coefficients, and thus an ODE of the form of (2). For more details, see (Chyzak et al. (2005)). Note that the order n of the ODE for $\lambda_1\, x_1 + \lambda_2\, x_2$, which is equal to the dimension of I, is at most $n_1 + n_2$, where n_i is the order of the ODE defining x_i for $i = 1,\, 2$.

A natural problem is then to compute the ODEs that x satisfies when the ODEs for the x_i's are known. This can be easily done by considering the successive derivatives of $x = \sum_{i=0}^{N} \lambda_i\, x_i$ and to search for $K(t)$-linear combinations among them (Chyzak et al. (2005)).

Example 5. Let $x_j = e^{i\,(\omega_j\, t + \phi_j)}$ for $j = 1, 2, 3$. Clearly, x_j satisfies $\partial_t\, x_j - i\,\omega_j\, x_j = 0$, i.e., x_j is annihilated by $\partial_t - i\,\omega_j$. Let us compute the annihilator of the linear combination $x = \lambda_1\, x_1 + \lambda_2\, x_2 + \lambda_3\, x_3$, where the λ_i's are constants. Differentiating x with respect of t, we get

$$\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ x^{(3)} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i\,\omega_1\,\lambda_1 & i\,\omega_2\,\lambda_2 & i\,\omega_3\,\lambda_3 \\ -\omega_1^2\,\lambda_1 & -\omega_2^2\,\lambda_2 & -\omega_3^2\,\lambda_3 \\ -i\,\omega_1^3\,\lambda_1 & -i\,\omega_2^3\,\lambda_2 & -i\,\omega_3^3\,\lambda_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Let L be the second matrix appearing in the above equation. Linear relations between the rows of L yield OD operators annihilating x. Indeed, if $J = (1 \quad \partial_t \quad \partial_t^2 \ \ldots)^T$ and r is a row vector with entries in K satisfying $r\,L = 0$, then the above equation can be rewritten as $J\,x = L\,x$, and thus we then get $(r\,J)\,x = (r\,L)\,x = 0$, i.e., the OD operator $r\,J$ annihilates x. We can check that there are no relations between the first three rows of L but

$$r = (i\,\omega_1\,\omega_2\,\omega_3 \ \ -(\omega_1\,\omega_2 + \omega_1\,\omega_3 + \omega_2\,\omega_3) \ \ -i\,(\omega_1 + \omega_2 + \omega_3) \ \ 1)$$

is in the left kernel of L, which yields that the OD operator

$$\partial_t^3 - i\,(\omega_1 + \omega_2 + \omega_3)\,\partial_t^2 - (\omega_1\,\omega_2 + \omega_1\,\omega_3 + \omega_2\,\omega_3)\,\partial_t + i\,\omega_1\,\omega_2\,\omega_3$$

annihilates $x = \lambda_1\, x_1 + \lambda_2\, x_2 + \lambda_3\, x_3$.

Example 6. Let x_1 (resp., x_2) be the n^{th} (resp., m^{th}) Hermite polynomial, λ_1 and λ_2 two constants, and the signal $x = \lambda_1\, x_1 + \lambda_2\, x_2$. The functions x_1 and x_2 are respectively defined by the following ODE:

$$\ddot{x}_1(t) - 2\,t\,\dot{x}_1(t) + 2\,m\,x_1(t) = 0,$$
$$\ddot{x}_2(t) - 2\,t\,\dot{x}_2(t) + 2\,n\,x_2(t) = 0.$$

Let us compute the ODE satisfies by x. To do that, we differentiate $2 + 2 = 4$ times x and we get

$$(z \quad \dot{z} \quad \ddot{z} \quad z^{(3)} \quad z^{(4)})^T = L\,(x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2)^T,$$

where the matrix L is defined by:

$$L = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 \\ -2\,m\,\lambda_1 & -2\,n\,\lambda_1 & 2\,t\,\lambda_1 & 2\,t\,\lambda_2 \\ -4\,m\,(2\,t^2 - m)\,\lambda_1 & -4\,n\,(2\,t^2 - n)\,\lambda_2 & 8\,t\,(t^2 - m)\,\lambda_1 & 8\,t\,(t^2 - m)\,\lambda_2 \end{pmatrix}.$$

The left kernel of L is generated by the row vector:

$$r = \begin{pmatrix} 4\,m\,n & -4\,(m+n)\,t & 4\,t^2 + 2\,(m+n) & -4\,t & 1 \end{pmatrix}.$$

Therefore, we get $d(t, \partial_t)\,z(t) = 0$, where d is the OD operator defined by $d = r\,\begin{pmatrix} 1 & \partial_t & \partial_t^2 & \partial_t^3 & \partial_t^4 \end{pmatrix}^T$, i.e.:

$$\partial_t^4 - 4\,t\,\partial_t^3 + (4\,t^2 + 2\,(m+n))\,\partial_t^2 - 4\,(m+n)\,t\,\partial_t + 4\,m\,n.$$

2.3 Statement of the parameter estimation problem

To do study the algebraic parameter estimation problem, we start with (8), i.e., $P(s,\, \partial_s)\, \widehat{z}(s) + Q(s) + \overline{Q}(s) = 0$. Let $\theta = \{\theta_1, \ldots, \theta_r\}$ be a set of independent variables and let us suppose that $K = k[\theta_1, \ldots, \theta_r]$, simply denoted by $K = k[\theta]$, where k is a subfield of K. In other words, we suppose that the polynomials a_i's which define the dynamics of the signal x, i.e., (2), depend on certain constant parameters θ. Similarly, let $\vartheta = \{\vartheta_1, \ldots, \vartheta_s\}$ be another set of independent variables and $\Theta = \theta \cup \vartheta$. We simply denote $k[\theta_1, \ldots, \theta_r, \vartheta_1, \ldots, \vartheta_s]$ by $k[\Theta]$. Now, using (7), we obtain that:

$$P \in k[\theta]\langle \partial \rangle, \quad Q \in k[\theta, \vartheta, s] = k[\Theta, s], \quad \overline{Q} \in k[\theta, \gamma, s].$$

Note that the ϑ_i's (resp., γ) appear linearly in Q (resp., $\overline{Q}$). Since $P = s^{n_0+1} \sum_{i=0}^{n} a_i(-\partial_s)\, s^i$ (see (7)), the coefficients of the polynomials a_i's appear linearly in P. Hence, if the parameters θ_i's appear linearly in the a_j's, then they also appear linearly in P and Q. In what follows, as usually done in the literature, we assume that this condition holds.

The parameter estimation problem aims at "estimating" the parameters Θ (or a given subset of Θ) from the observation of $y = z + \varpi$. In the algebraic parameter estimation problem, "estimating" means, in the noiseless case, i.e., when $\varpi = 0$, exactly and explicitly determine the parameters Θ in terms of y and of its integrals. For more details, see Fliess et al. (2003); Sira-Ramírez et al. (2014); Mboup et al. (2009); Ushirobira et al. (2012, 2013) and the references therein. The use of integrals of y is made to filter the effects of the noise ϖ.

Example 7. If we consider again Example 2, then we can take $\vartheta = \{\vartheta_k\}_{k=0,\ldots,n-1}$ and $\Theta = \vartheta$. Hence, the algebraic estimation problem aims at estimating the ϑ_k's or equivalently the a_k's from the measure of $y = z + \varpi$.

Example 8. If we consider again 1 (resp., 2, resp. 3) of Example 4, then, for instance, we can take $\theta = \{\omega\}$, $\vartheta = \{\vartheta_0, \vartheta_1\}$, and thus $\Theta = \{\omega, \vartheta_0, \vartheta_1\}$ (resp., $\Theta = \{n, \vartheta_0, \vartheta_1\}$, $\Theta = \{\alpha, \beta, \vartheta_0, \vartheta_1\}$). The algebraic estimation problem aims at estimating Θ from $y = z + \varpi$.

To get rid off γ, we have to find *annihilators* of $\overline{Q}$, namely OD operators $\Pi(s, \partial_s)$ satisfying $\Pi(s, \partial_s)\, \overline{Q} = 0$, so that:

$$\Pi(s, \partial_s)\, P(s,\, \partial_s)\, \widehat{z}(s) + \Pi(s, \partial_s)\, Q(s) = 0. \tag{10}$$

In particular, we want to find annihilators Π of $\overline{Q}$ such that the parameters Θ can be obtained explicitly in terms of $\widehat{z}$ and its derivatives. In another words, we want to find annihilators Π_j of $\overline{Q}$, $j = 1, \ldots, r$, such that the OD system

$$\Pi_j(s, \partial_s)\, P(s,\, \partial_s)\, \widehat{z}(s) + \Pi_j(s, \partial_s)\, Q(s) = 0, \; j = 1, \ldots, r, \tag{11}$$

yields $R_1\, \Theta = R_2$, where

$$\begin{cases} R_1 = R_{11}(s, \partial_s)\, \widehat{z}(s) + R_{10}(s), \\ R_2 = R_{21}(s, \partial_s)\, \widehat{z}(s) + R_{20}(s), \end{cases}$$

and R_1 is generically a non-singular matrix so that we get:

$$\Theta = R_1^{-1}\, R_2.$$

We can divise the both R_1 and R_2 by s^N for a large enough integer N so that $R_1' = s^{-N}\, R_1$ and $R_2' = s^{-N}\, R_2$ are polynomials only in s^{-1}. Doing that (i.e., adding integrators in the corresponding expressions) helps to filter the noise ϖ while coming back to the time domain by means of the inverse Laplace transform. Thus, we can consider $\Theta = {R_1'}^{-1}\, R_2'$. Note also that the coefficients of the annihilators Π_j's cannot depend on the parameters to be estimated till they are unknown. For more details, see Fliess et al. (2003); Sira-Ramírez et al. (2014); Mboup et al. (2009); Ushirobira et al. (2012, 2013).

Remark 3. If the parameters θ_i's are not assumed to appear linearly in the a_j's, i.e., in P and Q, then, from (11), algebraic equations have to be solved to get the θ_i's.

To initiate an effective study of the algebraic estimation parameter problem, we propose to investigate the following three problems by means of computer algebra methods:

(1) Let $\overline{Q} \in k[\theta, \gamma, s]$ be the polynomial defined in (7) and let us note $\overline{q} = \overline{Q}/\gamma \in k[\theta, s]$. Compute a set of generators of the following left ideal of $D = k[\theta, s]\langle \partial_s \rangle$
$$\mathrm{ann}_D(.\overline{q}) = \{d \in D \mid d\, \overline{q}(s) = 0\}.$$
(2) If θ' is a subset of θ (e.g., if $\theta' = \emptyset$ or $\theta' = \theta$) and $E = k[\theta', s]\langle \partial_s \rangle$, from Point 1, deduce the left ideal $\mathrm{ann}_D(.\overline{q}) \cap E$ of E.
(3) If $Q \in k[\theta, \vartheta, s]$ is the polynomial defined in (7), then compute $\mathrm{ann}_E(.\overline{q}) \cap \mathrm{ann}_E(.Q)$ and:
$$\mathrm{ann}_E(.\overline{q}) / \left(\mathrm{ann}_E(.\overline{q}) \cap \mathrm{ann}_E(.Q)\right).$$

The first problem aims at computing the annihilators of the polynomial $\overline{Q}$ to get (11). Note that the annihilators of $\overline{Q}$ that we can be used do only have to contain parameters that are already estimated. Thus, the second problem solves this point by controlling the coefficients that can appear in the annihilators. Finally, the last problem solves the problem of recognizing whether or not an annihilator of $\overline{Q}$ also annihilates the polynomial Q which contains the parameters ϑ, i.e., the initial conditions of (2). For instance, if all annihilators of $\overline{Q}$ are also annihilators of Q, then no parameters ϑ can algebraically be estimated.

3. COMPUTATION OF ANNIHILATORS

Let K be a *noetherian ring*, namely, every ideal of K can be generated by a finite set of generators (see, e.g., Rotman (2009)), and $p \in A = K[s]$ a polynomial of degree $q-1$ ($q \geq 1$) in s with coefficients in K. Let $D = A\langle \partial_s \rangle$ be the ring of OD operators in $\partial_s = \frac{d}{ds}$ with coefficients in A. Let us also consider the following column vector:

$$J_q = (1 \quad \partial_s \quad \ldots \quad \partial_s^q)^T \in D^{q \times 1}. \tag{12}$$

Applying J_q to p, we get the following polynomial vector:

$$R = J_q\, p \in A^{q \times 1}.$$

Since A is a noetherian ring, the following A-module

$$\ker_A(.R) = \{\mu \in A^{1 \times q} \mid \mu\, R = 0\}$$

formed by all the A-linear relations among of the rows of R is noetherian, and thus is *finitely generated*, i.e., can be generated by a finite generating set (see Rotman (2009)). Let $\{S_{i\bullet}\}_{i=1,\ldots,r}$ be a set of generators of $\ker_A(.R)$, where $S_{i\bullet} \in A^{1 \times q}$ for $i = 1, \ldots, r$, and $S = (S_{1\bullet}^T \; \ldots \; S_{r\bullet}^T)^T \in A^{r \times q}$ the matrix whose rows are the $S_{i\bullet}$'s. Let us consider the following *A-homomorphisms* (i.e., A-linear maps):

$$\begin{aligned} .R : A^{1 \times q} &\longrightarrow A & .S : A^{1 \times r} &\longrightarrow A^{1 \times q} \\ \mu &\longmapsto \mu\, R, & \nu &\longmapsto \nu\, S. \end{aligned}$$

Then, we obtain $\ker_A(.R) = \mathrm{im}_A(.S)$. If $M = A/(A^{1 \times q}\, R)$ is the A-module *finitely presented* by R, i.e., the factor module of A by the ideal of A generated by the entries of the column vector R, and π the A-homomorphism which maps 1 to its residue class $\pi(1)$, then we have the following *exact sequence* of A-modules (see, e.g., Rotman (2009))

$$A^{1\times r} \xrightarrow{.S} A^{1\times q} \xrightarrow{.R} A \xrightarrow{\pi} M \longrightarrow 0,$$

namely, π is surjective and $\ker_A(.R) = \operatorname{im}_A(.S)$.

Remark 4. If $K = k[x_1, \dots, x_n]$ is a polynomial ring over a *computable field* k (e.g., $k = \mathbb{Q}$, $\mathbb{Q}(y_1, \dots, y_m)$), then *Gröbner basis techniques* can be used to explicitly compute the matrix S (see, e.g., Becker et al. (1993)). For instance, it can be computed by the SyzygyModule command of the OreModules package (Chyzak et al. (2007)).

We have $(S\,J_q)\,p = S\,R = 0$, i.e., $Q = S\,J_q \in D^{r\times 1}$ is a column of OD operators which annihilates p, i.e., $Q\,p = 0$.

Since the degree of p is $q-1$, we get $\partial_s^q\,p = 0$, which shows that the last entry of R is 0, and thus that the vector $(0\ \dots\ 0\ 1) \in \ker_A(.R) = \operatorname{im}_A(.S)$. Thus, there exists $\omega \in A^{1\times r}$ such that $(0\ \dots\ 0\ 1) = \omega\,S$, and thus we obtain that $\partial_s^q = (0\ \dots\ 0\ 1)\,J_q = \omega\,S\,J_q = \omega\,Q$.

Let us introduce the left ideal of D defined by all the OD operators which annihilate p, i.e., its annihilator over D:

$$\operatorname{ann}_D(.p) = \{d \in D \mid d\,p = 0\}.$$

If we write $Q = (q_1\ \dots\ q_r)^T$, where $q_i \in D$, then we obtain that $q_i \in \operatorname{ann}_D(p.)$, i.e., $\sum_{i=1}^r D\,q_i \subseteq \operatorname{ann}_D(p.)$. Let us prove that we have $\operatorname{ann}_D(.p) = \sum_{i=1}^r D\,q_i$. If $d \in \operatorname{ann}_D(.p)$, then $d = \sum_{i=0}^t a_i\,\partial_s^i$, where $a_i \in A$. We can write $d = c\,\partial_s^q + d'$, where $d' \in D$ is of degree strictly less than q and $c \in D$. Let us write $d' = \sum_{i=0}^{q-1} b_i\,\partial_s^i$ and let $\beta = (b_0\ \dots\ b_{q-1}) \in A^{1\times q}$. Since the degree of the polynomial p is $q-1$, then $\partial_s^q\,p = 0$, and thus we get

$$\sum_{i=0}^{q-1} b_i\,\partial_s^i\,p = c\,\partial_s^q\,p + \sum_{i=0}^{q-1} b_i\,\partial_s^i\,p = d\,p = 0,$$

which shows that $\beta\,R = (\beta\,J_q)\,p = 0$, i.e., $\beta \in \ker_A(.R) = \operatorname{im}_A(.S)$. Hence, there exists $\gamma \in A^{1\times r}$ such that $\beta = \gamma\,S$, which yields $\beta\,J_q = \gamma\,S\,J_q = \gamma\,Q$, and thus we obtain

$$d = c\,\partial_s^q + \gamma\,Q = (c\,\omega + \gamma)\,Q,$$

where $c\,\omega + \gamma \in D^{1\times r}$, which shows that $d \in \sum_{i=0}^r D\,q_i$.

Lemma 1. With the above notations, we have:

$$\operatorname{ann}_D(.p) = \{d \in D \mid d\,p = 0\} = \sum_{i=1}^r D\,q_i.$$

In other words, the annihilator $\operatorname{ann}_D(.p)$ of p over D is generated by the entries of the column vector Q.

Remark 5. If k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$), then a classical result of J. T. Stafford asserts that any left/right ideal of $A_1(k)$ can be generated by two elements (see Stafford (1978)). For an implementation of this result in the Stafford package, see Quadrat (2007). This result extends to a polynomial extension of the Weyl algebra, i.e., to $A_1(k)[y]$, where y is a commuting indeterminate. If we consider $B_1(k)$, then $B_1(k)$ is a *principal left/right ideal domain*, namely, every left/right ideal of $B_1(k)$ can be generated by one element. This generator can simply be computed by means of gcd computations.

Example 9. Let $p = s^2 + \omega^2 = -\overline{Q}/\gamma \in A = \mathbb{Q}(\omega)[s]$, where $\overline{Q}$ is defined in 1 of Example 4. The degree of p being 2, $q = 3$, and we get $\ker_A(.R) = \operatorname{im}_A(.S)$, where:

$$R = J_3\,p = \begin{pmatrix} p & \dfrac{dp}{ds} & \dfrac{d^2p}{ds^2} & 0 \end{pmatrix}^T = (s^2+\omega^2\quad 2\,s\quad 2\quad 0)^T,$$

$$S = \begin{pmatrix} -2 & s & \omega^2 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ Q = S\,J_3 = \begin{pmatrix} \omega^2\,\partial_s^2 + s\,\partial_s - 2 \\ s\,\partial_s^2 - \partial_s \\ \partial_s^3 \end{pmatrix}.$$

If $D = A_1(\mathbb{Q}(\omega))$, then, we obtain:

$$\operatorname{ann}_D(.p) = D\,(\omega^2\,\partial_s^2 + s\,\partial_s - 2) + D\,(s\,\partial_s^2 - \partial_s) + D\,\partial_s^3. \quad (13)$$

The generators of $\operatorname{ann}_D(.p)$ do not form of *Gröbner basis* (Becker et al. (1993)). Computing a Gröbner basis, we get:

$$\operatorname{ann}_D(.p) = D\,(\omega^2\,\partial_s^2 + s\,\partial_s - 2) + D\,((s^2+\omega^2)\,\partial_s - 2\,s).$$

If $E = B_1(\mathbb{Q}(\omega))$, then we can easily check that

$$\omega^2\,\partial_s^2 + s\,\partial_s - 2 = \frac{1}{(s^2+\omega^2)}\,(\omega^2\,\partial_s + 2)\,((s^2+\omega^2)\,\partial_s - 2\,s),$$

which shows that $\operatorname{ann}_E(.p) = E\,((s^2+\omega^2)\,\partial_s - 2\,s)$.

Now, if we consider $A = \mathbb{Q}[\omega, s]$, then we obtain

$$S = \begin{pmatrix} -2 & s & \omega^2 & 0 \\ -2\,s & s^2+\omega^2 & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (14)$$

and thus $\operatorname{ann}_D(.p)$ is defined by the three generators of (13) and $(s^2+\omega^2)\,\partial_s - 2\,s$, and a Gröbner basis then yields:

$$\operatorname{ann}_D(.p) = D\,((s^2+\omega^2)\,\partial_s - 2\,s) + D\,(s\,\partial_s^2 - \partial_s) + D\,\partial_s^3.$$

Finally, we can check that

$$s\,\partial_s^2 - \partial_s = \frac{1}{2}\,\partial_s^2\,(s^2+\omega^2)\,\partial_s - 2\,s) - \frac{1}{2}\,(s^2+\omega^2)\,\partial_s^3,$$

which shows that $\operatorname{ann}_D(.p) = D\,((s^2+\omega^2)\,\partial_s - 2\,s) + D\,\partial_s^3$.

Example 10. Let us consider $K = \mathbb{Q}[\lambda_0, \lambda_1, \lambda_2]$ and:

$$p = \lambda_0\,s^3 + 2\,\lambda_1\,s^2 - 2\,\lambda_2\,s\,(s^2-4) \in A = K[s].$$

See Ushirobira et al. (2016). Since the degree of p is 3, we have $q = 4$ and we get $\ker_A(.R) = \operatorname{im}_A(.S)$, where the matrix R is defined by

$$R = J_4\,p = \begin{pmatrix} p & \dfrac{dp}{ds} & \dfrac{d^2p}{ds^2} & \dfrac{d^3p}{ds^3} & 0 \end{pmatrix}^T,$$

and S is given in Figure 1. Then, $\operatorname{ann}_D(.p)$ is generated by the entries of $Q = S\,J_4 \in D^{7\times 1}$. If $Q = (q_1\ \dots\ q_7)^T$, then, e.g., $q_2 = s^3\,\partial_s^3 - 3\,s^2\,\partial_s^2 + 6\,s\,\partial_s - 6$ and $q_7 = \partial_s^4$.

4. TOWERS OF ANNIHILATORS

Let θ' be a subset of θ and $K' = k[\theta']$ the subring of $K = k[\theta]$. From the computation of the annihilators of $p \in K[s]$ over the ring $A = K[s] = k[\theta, s]$ (see Section 3), in this section, we show how to obtain the annihilators of p with coefficients in the subring $B = K'[s] = k[\theta', s]$ of A. In other words, if $E = B\langle\partial_s\rangle$ (resp., $D = A\langle\partial_s\rangle$), is the ring of OD operators in ∂_s with coefficients in B (resp., A), then, from the knowledge $\operatorname{ann}_D(.p)$, we explain how to compute the left E-module $\operatorname{ann}_E(.p) = \{e \in E \mid e\,p = 0\}$. By Lemma 1, the annihilators of p over D are completely determined by the left D-module $D^{1\times r}\,Q = \sum_{i=1}^r D\,q_i$. In other words, the entries of Q generate $\operatorname{ann}_D(.p)$. Since $B \subseteq A$, we get $E \subseteq D$, and thus we have:

$$\operatorname{ann}_E(.p) = \operatorname{ann}_D(.p) \cap E.$$

Computing $\operatorname{ann}_D(.p) \cap E$ can be obtained by means of a noncommutative Gröbner basis computation with a

$$S = \begin{pmatrix} -9\,\lambda_0 + 18\,\lambda_2 & 3\,\lambda_0\,s - 6\,\lambda_2\,s - 2\,\lambda_1 & 2\,\lambda_1\,s + 4\,\lambda_2 & 4\,\lambda_2\,s & 0 \\ -6 & 6\,s & -3\,s^2 & s^3 & 0 \\ 6\,\lambda_1 & -4\,\lambda_1\,s & s\,(\lambda_1\,s - 4\,\lambda_2) & 4\,\lambda_2\,s^2 & 0 \\ 0 & -6\,\lambda_0 + 12\,\lambda_2 & 3\,s\,(\lambda_0 - 2\,\lambda_2) & 2\,\lambda_1\,s + 8\,\lambda_2 & 0 \\ 0 & 2\,\lambda_1 & -2\,\lambda_1\,s - 4\,\lambda_2 & s\,(\lambda_1\,s + 4\,\lambda_2) & 0 \\ 0 & 0 & -3\,\lambda_0 + 6\,\lambda_2 & 3\,\lambda_0\,s - 6\,\lambda_2\,s + 2\,\lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in A^{7\times 5}.$$

Fig. 1. Matrix S for $p = \lambda_0\,s^3 + 2\,\lambda_1\,s^2 - 2\,\lambda_2\,s\,(s^2 - 4)$

monomial order which eliminates the elements of $\theta \setminus \theta'$. For more details, see, e.g., Chyzak et al. (2005).

Example 11. We consider again Example 9. Let $K' = \mathbb{Q}$ be the subring of $K = \mathbb{Q}[\omega]$ (i.e., $\theta' = \emptyset$, $\theta = \{\omega\}$), $A = K[s]$, $B = K'[s] = \mathbb{Q}[s]$, $D = A\langle \partial_s \rangle$ and $E = B\langle \partial_s \rangle$. We obtain:

$$\mathrm{ann}_E(.p) = \mathrm{ann}_D(.p) \cap E = E\,(s\,\partial_s^2 - \partial_s) + E\,\partial_s^3.$$

Example 12. We consider again Example 10. Let $K' = \mathbb{Q}$ be the subring of $K = \mathbb{Q}[\lambda_0,\, \lambda_1,\, \lambda_2]$ (i.e., $\theta = \{\lambda_0,\, \lambda_1,\, \lambda_2\}$, $\theta' = \emptyset$), $A = K[s]$, $B = K'[s]$, $D = A\langle \partial_s \rangle$ and $E = B\langle \partial_s \rangle$. We get that $\mathrm{ann}_E(.p) = E\,e_1 + E\,e_2$, where $e_1 = \partial_s^4$ and $e_2 = s^3\,\partial_s^3 - 3\,s^2\,\partial_s^2 + 6\,s\,\partial_s - 6$. Now, if $\theta' = \{\lambda_0, \lambda_2\}$, $K' = \mathbb{Q}[\lambda_0, \lambda_2]$, $B = K'[s]$ and $E = B\langle \partial_s \rangle$, then we obtain that $\mathrm{ann}_E(.p) = E\,e_1 + E\,e_2 + E\,e_3$, where:

$$e_3 = -8\,\lambda_2\,s\,\partial_s^3 + (\lambda_0 - 2\,\lambda_2)\,(3\,s^2\,\partial_s^2 - 12\,s\,\partial_s + 18).$$

5. INTERSECTION OF ANNIHILATORS

With the notations of Section 4, let $p_1, p_2 \in B$ and $\mathrm{ann}_E(.p_i)$ for $i = 1, 2$. Let us show how to effectively compute the left ideal $I = \mathrm{ann}_E(.p_1) \cap \mathrm{ann}_E(.p_2)$ of E. As explained in Section 3 with the ring B instead of A, let $\deg_s p_i = q_i - 1$, $J_i = (1 \;\; \ldots \;\; \partial_s^{q_i})^T$, $R_i = J_i\,p_i$ and $S_i \in B^{r_i \times q_i}$ be such that $\ker_B(.R_i) = \mathrm{im}_B(.S_i)$. If $Q_i = S_i\,J_i = (q_{1,i} \;\ldots\; q_{r_i,i})^T$, then, by Lemma 1, we have:

$$\mathrm{ann}_E(.p_i) = \sum_{j=1}^{r_i} E\,q_{j,i}, \quad i = 1,\, 2.$$

An element $p \in I$ is an element of E which satisfies $p = \sum_{j=1}^{r_i} e_{j,i}\,q_{j,i}$ for $i = 1, 2$ and for certain $q_{j,i}$'s in E. If we note $Q = (Q_1^T \quad Q_2^T)^T \in E^{(r_1+r_2)\times 1}$, then we have:

$$(e_{1,1} \;\ldots\; e_{1,r_1} \;\; -e_{1,2} \;\ldots\; -e_{r_2,2})\,Q = 0.$$

Hence, if $T \in E^{s\times(r_1+r_2)}$ is such that $\ker_E(.Q) = \mathrm{im}_E(.T)$, and $T = (T_1 \quad -T_2)$, where $T_1 \in E^{s\times r_1}$ and $T_2 \in E^{s\times r_2}$, then we have $T_1\,Q_1 = T_2\,Q_2$, and thus we obtain:

$$\mathrm{ann}_E(.p_1) \cap \mathrm{ann}_E(.p_2) = E^{1\times s}\,(T_1\,Q_1) = E^{1\times s}\,(T_2\,Q_2).$$

The computation of the matrix T can be obtained by the SyzygyModule command of OreModules.

Finally, let us explicit characterize the left E-module:

$$P = \mathrm{ann}_E(.p_1)/\left(\mathrm{ann}_E(.p_1) \cap \mathrm{ann}_E(.p_2)\right).$$

Using *Noether's second isomorphism theorem* (see, e.g., Rotman (2009)), we have:

$$\begin{aligned} P \cong P' &= \left(\mathrm{ann}_E(.p_1) + \mathrm{ann}_E(.p_2)\right)/\mathrm{ann}_E(.p_2) \\ &= \left(\sum_{j=1}^{r_1} E\,q_{j,1} + \sum_{j=1}^{r_2} E\,q_{j,2}\right)/\left(\sum_{j=1}^{r_2} E\,q_{j,2}\right) \\ &= \left(E^{1\times(r_1+r_2)}\,Q\right)/\left(E^{1\times r_2}\,Q_2\right). \end{aligned}$$

We note that $e \in \mathrm{ann}_E(.p_1)$ is of the form of $e = \eta\,Q_1$, where $\eta \in E^{1\times r_1}$. If $\tau : \mathrm{ann}_E(.p_1) \longrightarrow P$ (resp., $\kappa : E^{1\times(r_1+r_2)}\,Q \longrightarrow P'$) is the left E-homomorphism defining the canonical projection onto P (resp., P'), then the above isomorphism $\psi : P \longrightarrow P'$ is defined by:

$$\psi\,(\tau(\eta\,Q_1)) = \kappa(\eta\,Q_1).$$

Now, using Lemma 3.1 on pages 349-350 of Cluzeau et al. (2008), we obtain the following isomorphism:

$$\begin{aligned} P' &\cong E^{1\times(r_1+r_2)}/\left(E^{1\times(r_1+s)}\begin{pmatrix} 0 & I_{r_2} \\ T_1 & -T_2 \end{pmatrix}\right) \\ &\cong P'' = E^{1\times r_1}/\left(E^{1\times s}\,T_1\right). \end{aligned}$$

More precisely, if $\sigma : E^{1\times r_1} \longrightarrow P''$ is the isomorphism defining the canonical projection onto P'', then we have:

$$\begin{array}{ll} \phi : P' \longrightarrow P'' & \phi^{-1} : P'' \longrightarrow P' \\ \kappa((\eta \quad \zeta)\,Q) \longmapsto \sigma(\eta), & \sigma(\eta) \longmapsto \kappa(\eta\,Q_1). \end{array}$$

We obtain the isomorphism $\phi \circ \psi : P \longrightarrow P''$ defined by:

$$\forall\,\eta \in E^{1\times r_1}, \quad (\phi \circ \psi)(\tau(\eta\,Q_1)) = \sigma(\eta).$$

Using the isomorphism $\phi \circ \psi$, testing if $e = \eta\,Q_1 \in \mathrm{ann}_E(.p_1)$ belongs to $\mathrm{ann}_E(.p_2)$, i.e., if $\tau(e) = 0$, is equivalent to testing if $\sigma(\eta) = 0$, i.e., if $\eta = \lambda\,T_1$ for a certain $\lambda \in E^{1\times r_1}$. This condition can be checked by means of the Factorize command of the OreModulespackage. We also note that $P \cong P'' = 0$ iff the matrix T_1 admits a left inverse $U_1 \in E^{r_1\times s}$, i.e., $U_1\,T_1 = I_{r_1}$. This condition can be checked by means of the LeftInverse command of the OreModules package. If $P = 0$, then we have $\mathrm{ann}_E(.p_1) \subseteq \mathrm{ann}_E(.p_2)$, i.e., we cannot find an annihilator of p_1 which is not an annihilator of p_2.

Example 13. Let us consider again Example 9. If $p = \overline{Q}/\gamma$, $D = \mathbb{Q}[\omega, \vartheta_0, \vartheta_1]\langle \partial_s \rangle$ and $E = A_1(\mathbb{Q})$, then we have:

$$\begin{aligned} \mathrm{ann}_D(.p) &= D\,((s^2 + \omega^2)\,\partial_s - 2\,s) + D\,(s\,\partial_s^2 - \partial_s) + D\,\partial_s^3, \\ \mathrm{ann}_E(.p) &= E\,(s\,\partial_s^2 - \partial_s) + E\,\partial_s^3, \\ \mathrm{ann}_D(.Q) &= D\,(s^2\,\partial_s^2 - 2\,s\,\partial_s + 2) \\ &\quad + D\,(\vartheta_1\,s\,\partial_s^2 + 2\,\vartheta_0\,s\,\partial_s - 4\,\vartheta_0) \\ &\quad + D\,((2\,\vartheta_0\,s + \vartheta_1)\,\partial_s^2 - 2\,\vartheta_0\,\partial_s) + D\,\partial_s^3, \\ \mathrm{ann}_E(.Q) &= E\,(s^2\,\partial_s^2 - 2\,s\,\partial_s + 2) + E\,\partial_s^3. \end{aligned}$$

Using the OreModules package, we obtain:

$$\begin{aligned} &\mathrm{ann}_E(.p) \cap \mathrm{ann}_E(.Q) = E\,\partial_s^3, \\ &\mathrm{ann}_E(.p)/\left(\mathrm{ann}_E(.p) \cap \mathrm{ann}_E(.Q)\right) \\ &\cong P'' = E^{1\times 2}/\left(D^{1\times 2}\begin{pmatrix} \partial_s & 0 \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

Let us show how to use the above results to solve the parameter estimation problem for $x = A\,\sin(\omega\,t + \phi)$

defined in 1 of Example 4 (Ushirobira et al. (2012)). Applying $d_1 = \partial_s^3 \in \mathrm{ann}_E(.p) \cap \mathrm{ann}_E(.Q)$ to (8) to get rid off of both $\overline{Q}$ and Q, we obtain $d_1(\partial_s)\, P(s)\, \widehat{z}(s) = 0$, i.e., $c_1\, \theta_1 = c_2$, where $\theta_1 = \omega^2$ and:

$$\begin{cases} c_1 = s\, \widehat{z}^{(3)}(s) + 3\, \widehat{z}^{(2)}(s), \\ c_2 = -\left(s^3\, \widehat{z}^{(3)}(s) + 9\, s^2\, \widehat{z}^{(2)}(s) + 18\, s\, \widehat{z}^{(1)}(s) + 6\, \widehat{z}(s)\right). \end{cases}$$

Let $Q_1 = (s\, \partial_s^2 - \partial_s \quad \partial_s^3)^T$ be the column vector formed by the two generators of $\mathrm{ann}_E(.p)$. Since the residue class of $\eta = (1 \quad 0)$ in P'' is not zero, i.e., $\eta \notin D^{1\times 2}\, T_1$, we get $d_2 = \eta\, Q_1 = s\, \partial_s^2 - \partial_s \in \mathrm{ann}_E(.p) \setminus \mathrm{ann}_E(.Q)$. Hence, applying d_2 to (8), we get $c_3\, \theta_1 + \vartheta_1 = c_4$, where:

$$\begin{cases} c_3 = s^2\, \widehat{z}^{(2)}(s) + s\, \widehat{z}^{(1)}(s) - \widehat{z}(s), \\ c_4 = -\left(s^4\, \widehat{z}^{(2)}(s) + 5\, s^3\, \widehat{z}^{(1)}(s) + 3\, s^2\, \widehat{z}(s)\right). \end{cases}$$

We can easily solve these two linear equations to get:

$$\begin{cases} \theta_1 = \dfrac{c_2}{c_1} = \dfrac{s^{-4}\, c_2}{s^{-4}\, c_1}, \\ \vartheta_1 = \dfrac{c_1\, c_4 - c_2\, c_3}{c_1} = \dfrac{s^{-5}\, (c_1\, c_4 - c_2\, c_3)}{s^{-5}\, c_1}. \end{cases}$$

We are now left with the identification of the last parameter ϑ_0 appearing in Q as the coefficient of s^2. Using the results of Section 4, we obtain $\mathrm{ann}_D(.p) \cap \mathrm{ann}_D(.s^2) = D\, (s\, \partial_s^2 - \partial_s) + D\, \partial_s^3$. Hence, considering the first generator $d_3 = (s^2 + \omega^2)\, \partial_s - 2\, s$ of $\mathrm{ann}_D(.p)$ and apply it to (8), we get $2\, \theta_1\, \vartheta_0 = s\, \vartheta_1 + c_5\, \theta_1^2 + c_6\, \theta_1 + c_7$, with the notations:

$$\begin{cases} c_5 = s\, \widehat{z}^{(2)}(s) + 2\, \widehat{z}^{(1)}(s), \\ c_6 = s^4\, \widehat{z}^{(2)}(s) + 7\, s^2\, \widehat{z}^{(1)}(s) + 5\, s\, \widehat{z}(s), \\ c_7 = s^4\, \widehat{z}^{(1)}(s) + s^3\, \widehat{z}(s). \end{cases}$$

Therefore, we obtain:

$$\vartheta_0 = \frac{s\, \vartheta_1 + c_5\, \theta_1^2 + c_6\, \theta_1 + c_7}{2\, \theta_1}.$$

Rewriting the above fraction as a quotient of polynomials in s^{-1} and coming to the time domain by means of the inverse Laplace transform, we obtain explicit formulas for the parameters θ_1, ϑ_0 and ϑ_1 (see Ushirobira et al. (2012)).

The algorithms for solving the three problems are implemented in the Non-A package dedicated to the algebraic estimation problem built upon the OreModules package.

It is well-known that the ring $D = A\langle \partial_s \rangle$ of OD operators is a *left Ore domain*, namely, for all $d_1, d_2 \in D \setminus \{0\}$, there exist $e_1, e_2 \in D \setminus \{0\}$ such that $e_1\, d_1 = e_2\, d_2$ (see, e.g., McConnell et al. (2000)). With the notations of Section 2.1, if $d_1 = R$ and $d_2 = R'$, then there exist $T, T' \in D \setminus \{0\}$ such that $T\, R = T'\, R'$. A set of generators of the OD operators T and T' satisfying the above equality can be computed by means of the SyzygyModule command of the OreModules package. Using (4) and (5), we get

$$T(s, \partial_s)\, R(s, \partial_s)\, \widehat{z}(s) + T(s, \partial_s)\, S(s) + T'(s, \partial_s)\, S'(s) = 0, \tag{15}$$

i.e., $\widehat{\gamma}$ has been eliminated from (4) and (5). The algebraic estimation problem can then be directly studied by means of (15). The main advantage of this approach is that it does not require the integration in closed-form solutions of the dynamics of the perturbation γ, which allows us to consider general type of structured perturbation γ. For more details, See Quadrat (2017).

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