A historical journey through the internal stabilization problem

Alban Quadrat

INRIA Sophia Antipolis, APICS project,

2004 route des lucioles, BP 93, 06902 Sophia Antipolis cedex, France.

`Alban.Quadrat@sophia.inria.fr`

`www-sop.inria.fr/cafe/Alban.Quadrat/index.html`

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Introduction

- The purpose of this talk is twofold:
  - Give a tribute to the scientific career of Frank M. Callier.
  - Give a historical but personal journey through the internal stabilization problem.

- Based on Frank's contributions (1972-1980), we study the evolutions of the internal stabilization problem focusing on the results due to Frank, Desoer, Vidyasagar, Zames, Francis...

- Science historians should pay more attention to the historical developments of this central concept in control theory.

- M. Vidyasagar "A brief history of the Graph topology", European Journal of Control, 2 (1996), 80-87.

Frank's main contributions

- Distributed parameter systems theory
- Algebras of transfer functions
- Internal stabilizability, tracking & disturbance rejection problems
- Graphical stability tests, Nyquist theorem
- Poles and zeros
- Robustness
- Spectral factorization, LQ-optimal regulation, Riccati equations
- Operator theory
- Optimization...

- More than 63 papers in MathSciNet and 2 books.

Going back to an epic time...

- Desoer-Wu (1968): Extension of the Nyquist's theorem

$$\mathcal{A}(\sigma) = \{\, f + \sum_{i=0}^{+\infty} a_i\, \delta(t - t_i) \mid f\, e^{-\sigma\, t} \in L_1(\mathbb{R}_+), \\ (a_i\, e^{-\sigma\, t_i})_{i \geq 0} \in l_1(\mathbb{Z}+),\ 0 = t_0 < t_1 < \ldots \}$$

$$\hat{\mathcal{A}}(\sigma) = \{\, \mathcal{L}(f) \mid f \in \mathcal{A}(\sigma)\}.$$

Consider a transfer function of the form:

$$p(s) = \frac{r}{(s - \sigma)} + h(s), \quad h \in \hat{\mathcal{A}}(\sigma).$$

Then, we have $\inf_{\Re(s) \geq \sigma} |1 + p(s)| > 0 \Rightarrow p/(1 + p) \in \hat{\mathcal{A}}(\sigma).$

$p/(1 + p)$: input-output transfer function for a unity feedback.

Extension to $P(s) = R/s + H(s), \quad R \in \mathbb{R}^{n \times n}, \quad H \in \hat{\mathcal{A}}(\sigma)^{n \times n}.$

- Baker-Vakharia (1970): $h \in \mathcal{A}(0),\ a_i \in \mathbb{R},\ b_k \in \mathbb{C},\ \Re(c_k) \geq 0$

$$p(t) = h(t) + \sum_{i=0}^{n} a_i\, t^i + \sum_{k=0}^{m} b_k\, t^{\alpha_k}\, e^{c_k t},$$

$$\Rightarrow \hat{p}(s) = \hat{h}(s) + \sum_{i=0}^{n} a_i\, \frac{i!}{s^{i+1}} + \sum_{k=0}^{m} b_k\, \frac{\alpha_k!}{(s - c_k)^{\alpha_k+1}}.$$

"following a method very similar to that used by Desoer and Wu":

$$\hat{d} = \frac{s^{n+1}}{(s+1)^{n+1}} \prod_{k=0}^{m} \frac{(s-c_k)^{\alpha_k+1}}{(s+1)^{\alpha}} \in \hat{\mathcal{A}}(0), \quad \alpha = \sum_{k=0}^{m} (\alpha_k + 1),$$

$$\Rightarrow n \triangleq \hat{d}\, \hat{p} \in \hat{\mathcal{A}}(0) \quad (\Rightarrow \hat{p} = \hat{n}/\hat{d},\ \hat{n},\ \hat{d} \in \hat{\mathcal{A}}(0)).$$

$$\inf_{\Re(s)\geq 0} |1 + \hat{p}(s)| > 0 \Rightarrow \inf_{\Re(s)\geq 0} |\hat{d}(s) + \hat{n}(s)| > 0$$

$$\Rightarrow (\hat{d} + \hat{n})^{-1} \in \hat{\mathcal{A}}(0) \Rightarrow \hat{p}/(1+\hat{p}) = \hat{n}/(\hat{d}+\hat{n}) \in \hat{\mathcal{A}}(0).$$

Generalization of Desoer-Wu's result: $\hat{p}(s) = \hat{r}(s) + \hat{h}(s),\ \hat{r} \in \mathbb{R}(s)$.

• Vidyasagar (1972): "As soon as I read this, my immediate reaction was 'What is so difficult about handling that case? All one has to do is to write the unstable part as a ratio of two stable rational functions!'"

$$\Rightarrow p(s) = \frac{n(s)}{d(s)}, \quad 0 \neq d,\ n \in \hat{\mathcal{A}}(\sigma).$$

LC-transmission line: $p(s) = 1/\cosh(s) = 2\, e^{-s}/(1 + e^{-2\, s})$.

Sufficient condition:

$$\inf_{\Re(s) \geq \sigma} |d(s) + n(s)| > 0 \Rightarrow p/(1+p) \in \hat{\mathcal{A}}(\sigma).$$

"But what about a *necessary* condition? It was clear to me that some sort of 'no cancellation condition' was needed – otherwise the sufficient condition could not possibly be necessary".

Generalization to the case of $P = \widetilde{N}\, \widetilde{D}^{-1},\ \widetilde{D},\ \widetilde{N} \in \hat{\mathcal{A}}(\sigma)^{n \times n}$.

- Nasburg-Baker (1972).
- Desoer-Callier (1972):

$T = P\,(I_n + P)^{-1} \in \hat{\mathcal{A}}(\sigma)^{n\times n}$ iff there exist $\widetilde{N}$, $\widetilde{D} \in \hat{\mathcal{A}}(\sigma)^{n\times n}$ s.t.:

$$P = \widetilde{N}\,\widetilde{D}^{-1}, \quad \inf_{\Re(s)\geq\sigma} |\det(\widetilde{N} + \widetilde{D})| > 0. \quad (\star)$$

Proof:

$$T \in \hat{\mathcal{A}}(\sigma)^{n\times n} \Rightarrow P = T\,(I_n - T)^{-1}, \quad T + (I_n - T) = I_n.$$

Stabilizability by a unit feedback $\Rightarrow \exists$ a coprime factorization of P.

$$(\star) \Rightarrow T = \widetilde{N}\,(\widetilde{N} + \widetilde{D})^{-1},\ (\widetilde{N} + \widetilde{D})^{-1} \in \hat{\mathcal{A}}(\sigma)^{n\times n} \Rightarrow T \in \hat{\mathcal{A}}(\sigma)^{n\times n}.$$

Necessary and sufficient conditions for stability in the case of:

$$P = \sum_{i=1}^{m} \frac{R_i}{(s-\sigma)^i} + H, \quad R_i \in \mathbb{C}^{n\times n}, \quad H \in \hat{\mathcal{A}}(\sigma)^{n\times n}.$$

• Callier-Desoer (1973):

$$P(s) = R(s) + H(s), \quad R \in \mathbb{R}(s)_p^{n \times n}, \quad H \in \hat{\mathcal{A}}(\sigma)^{n \times n}.$$

Constant feedback $C \in \mathbb{C}^{n \times n} \Rightarrow P\,(I_n + C\,P)^{-1} \in \hat{\mathcal{A}}(\sigma)^{n \times n}$.

Introduction of coprime factorizations of $R = \widetilde{N}\,\widetilde{D}^{-1}$ over $\mathbb{R}[s]$.

Necessary and sufficient condition for the closed-loop stability.

• Vidyasagar (1975): "... I thought of using the concept of coprimeness, but over the set $\mathcal{A}$. There was only one difficulty: no one knew whether these objects actually *existed*."

$$P(s) = \widetilde{N}(s)\,\widetilde{D}(s)^{-1}, \quad \widetilde{N},\, \widetilde{D} \in \hat{\mathcal{A}}(\sigma)^{n \times n}.$$

Introduction of coprime factorizations of P over $\hat{\mathcal{A}}(\sigma)$.

For $C \in \hat{\mathcal{A}}(\sigma)^{n \times n}$, a NS condition is obtained so that:

$$P(s)\,(I_n + C(s)\,P(s))^{-1} \in \hat{\mathcal{A}}(\sigma)^{n \times n}.$$

Strong stabilizability $\Leftrightarrow \inf_{\Re(s) > 0} |\det(\widetilde{D}(s) + C(s)\,\widetilde{N}(s))| > 0$.

- Desoer-Vidyasagar (1975): The results were summed up in:

  Feedback Systems: Input-Output Properties, Academic Press.

- Callier (1975): Frank considered unstable controllers!

Definition: C completely stabilizes P if:

$$P\,(I_n + C\,P)^{-1},\ (I_n + C\,P)^{-1} \in \hat{\mathcal{A}}(\sigma)^{n\times n}.$$

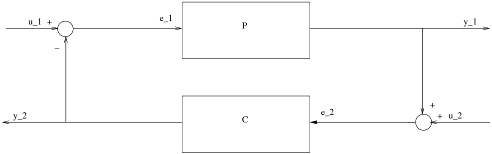
We need to study input-error and input-output stabilities.

Necessary and sufficient conditions for complete stabilizability are obtained in the case of a plant and a controller of the form:

$$\frac{R}{(s-\sigma)} + H(s), \quad R \in \mathbb{R}^{n\times n}, \quad H \in \hat{\mathcal{A}}(\sigma)^{n\times n}.$$

- Desoer-Chan (1975):

Introduction of the central concept of internal stabilizability.



$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} I_n & C \\ -P & I_n \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Definition: $P \in \mathbb{R}(s)^{n \times n}$ is closed-loop stabilized by $C \in \mathbb{R}(s)^{n \times n}$ iff all the entries of the transfer matrix

$$\begin{pmatrix} I_n & C \\ -P & I_n \end{pmatrix}^{-1} = \begin{pmatrix} (I_n + C\,P)^{-1} & -C\,(I_n + C\,P)^{-1} \\ P\,(I_n + C\,P)^{-1} & (I_n + P\,C)^{-1} \end{pmatrix}$$

are exponentially stable.

- Callier-Desoer (1976):

Study of the internal stabilization problem for distributed linear systems admitting pseudo-coprime factorizations over $\hat{\mathcal{A}}(\sigma)$.

Vidyasagar: "... I invented the notion of 'pseudo-coprimeness,' which is now deservedly forgotten."

$$\exists\, x,\, y,\, u \in \hat{\mathcal{A}}(\sigma) : \quad d\, x + n\, y = u, \quad \forall\, \Re(s) \geq 0 : \; u(s) \neq 0.$$

Necessary and sufficient conditions for internal stabilizability.

Necessary and sufficient conditions for internal stabilizability are obtained in the case of a plant and a controller of the form:

$$R(s) + H(s), \quad R(s) \in \mathbb{C}(s)^{n \times n}, \quad H \in \hat{\mathcal{A}}(\sigma)^{n \times n}.$$

Use of a graphical stability test (Callier-Desoer 1972).

- Callier-Desoer (1978-1980):

The class of systems should be an algebra if we want to do parallel and series interconnections of transfer functions (Zames, Morse).

$\{r+h \mid r \in \mathbb{R}(s),\ h \in \hat{\mathcal{A}}(\sigma)\}$ does not have a ring structure.

$\Rightarrow$ Introduction of the Callier-Desoer algebra $\hat{\mathcal{B}}(\sigma)$:

$$\hat{\mathcal{A}}_{-}(\sigma) = \{f \mid f \in \hat{\mathcal{A}}(\beta),\ \beta < \sigma\} = \varinjlim_{\beta<\sigma} \hat{\mathcal{A}}(\beta),$$

$$\hat{\mathcal{A}}_{\infty}(\sigma) = \{f \mid \exists\, \rho > 0 : \inf_{\{\Re(s)\geq\sigma, |s|\geq\rho\}} |f(s)| > 0\},$$

$$\hat{\mathcal{B}}(\sigma) = \hat{\mathcal{A}}_{-}(\sigma)\,(\hat{\mathcal{A}}_{\infty}(\sigma))^{-1} = \left\{\frac{n}{d} \mid n \in \hat{\mathcal{A}}_{-}(\sigma),\ d \in \hat{\mathcal{A}}_{\infty}(\sigma)\right\}.$$

Theorem: $p \in \hat{\mathcal{B}}(\sigma)$ iff there exist $h \in \hat{\mathcal{A}}(\sigma)$ and $r \in \mathbb{R}(s)$, where r is strictly proper and its poles belong to $\{s \in \mathbb{C} \mid \Re(s) \geq \sigma\}$, s.t.:

$$p = r + h.$$

- Desoer-Liu-Murray-Saeks (1980):

This paper develops the fractional representation approach.

(Zames) The set of transfer functions has the structure of an algebra (parallel $+$, serie $\circ$, constant feedback . by $\mathbb{R}$).

(Vidyasagar) Let A be an algebra of stable transfer functions which has an integral domain structure.

Then, the class of systems considered is the field of fractions:

$$K = Q(A) = \{p = n/d \mid 0 \neq d,\, n \in A\}.$$

p is A-stable iff $p \in A$: stability = membership problem $\Rightarrow$ algebra.

$p \in K$, admits a coprime factorization if $\exists\, 0 \neq d,\, n,\, x,\, y \in A$:

$$p = n/d, \quad d\,x + n\,y = 1.$$

$$\Rightarrow c(q) = \frac{y + q\,d}{x - q\,n}, \quad q \in A: \; x - q\,n \neq 0.$$

Youla-Kučera parametrization of all stabilizing controllers of p.

- Callier-Desoer (1980):

Every $p \in \hat{\mathcal{B}}(\sigma)$ admits a coprime factorization, i.e.:

$$\exists \; d \in \hat{\mathcal{A}}_{\infty}(\sigma), \; n, \, x, \, y \in \hat{\mathcal{A}}_{-}(\sigma) : \quad p = n/d, \quad d\,x + n\,y = 1.$$

The problems of internal stabilization, tracking & disturbance rejection were successfully developed for MIMO plants over $\hat{\mathcal{B}}(\sigma)$.

This approach does not necessarily deal with square MIMO plants!

For a nice account of these results, see the book:

Curtain & Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer, 1995.

• Vidyasagar-Schneider-Francis (1982): Improvements of the fractional representation approach + graph topology.

$A = RH_\infty$ (Morse/Vidyasagar), W_+, $\hat{\mathcal{A}}(\sigma)$, $H^\infty(\mathbb{C}_+)$, $A(\mathbb{D})\dots$

Basic assumption: existence of doubly coprime factorizations.

How conservative is this assumption?

• Vidyasagar (1985): The fractional representation was explained in the nowadays classical book:

Vidyasagar, Control System Synthesis: A Factorization Approach, MIT Press.

• Inouye (1988-1990) - Smith (1989):

Over the rings $H^\infty(\mathbb{D})$ and $H^\infty(\mathbb{C}_+)$, internal stabilizability is equivalent to the existence of (doubly) coprime factorizations.

Meanwhile. . .

- Zames (1981):

"Normed algebras provide the natural setting for the study of system interconnections such as feedback".

Introduction of $H^\infty(\mathbb{C}_+)$ in control theory!

- If p is stable, then every stabilizing controller of p has the form

$$c(q) = q/(1 - p\,q),$$

where q is any stable plant satisfying $1 - p\,q \neq 0$.

$$\Rightarrow \inf_{c \in \mathrm{Stab}(p)} \| \, w\,(1 + p\,c)^{-1} \, \|_\infty = \inf_{q \text{ stable}} \| \, w\,(1 - p\,q) \, \|_\infty.$$

"the optimal sensitivity depends upon how close $p\,q$ can be made to 1 under the w weighting, i.e., on how invertible p is."

$\Rightarrow$ approximate inverses, singularity measures. . .

• Zames-Francis (1983):

$$p = r + h, \quad r \in \mathbb{R}(s)_p, \quad h \in H^\infty(\mathbb{C}_+).$$

Key idea: Introduction of the fractional transformation:

$$q = c\,(1 + p\,c)^{-1}.$$

We then obtain the controller c under the form $c = q\,(1 - p\,q)^{-1}$.

$$\begin{pmatrix} 1 & c \\ -p & 1 \end{pmatrix}^{-1} = \frac{1}{1 + p\,c} \begin{pmatrix} 1 & -c \\ p & 1 \end{pmatrix} = \begin{pmatrix} 1 - q\,p & -q \\ p\,(1 - q\,p) & 1 - q\,p \end{pmatrix}.$$

Theorem: c internally stabilizes p iff there exists q stable such that "$q\,p$ and $p\,(1 - q\,p)$ both stable".

Interpolation condition: At any pole a of p of order m in $\Re(s) \geq 0$, $(1 - p\,q)$ must have m or more zeros ($\Rightarrow$ q has m zeros at a).

- Francis-Zames (1984): $RH_\infty = \mathbb{R}(s) \cap H^\infty(\mathbb{C}_+)$.

$p \in \mathbb{R}(s)$ admits a coprime factorization over RH_∞ (Vidyasagar):

$$\exists\, 0 \neq d,\, n,\, x,\, y \in RH_\infty, \quad p = n/d, \quad d\,x + n\,y = 1.$$

- They parametrize all the $q \in RH_\infty$ satisfying:

$$q\,p \in RH_\infty, \quad q\,(1 - p\,q) \in RH_\infty.$$

They obtain the so-called Q-parametrization of the form:

$$c(q) = q/(1 - p\,q), \quad q = d\,y + q_1\,d^2, \quad q_1 \in RH_\infty.$$

Then, the sensitivity transfer function becomes:

$$s = 1/(1 + p\,c(q)) = 1 - p\,(d\,y + q_1\,d^2) = d\,(x - q_1\,n).$$

- We note that we have:

$$c(q) = (d\,y + q_1\,d^2)/(d\,x - q_1\,d\,n) = (y + q_1\,d)/(x - q_1\,n).$$

Q-parametrization v.s. Youla-Kučera parametrization

"The Q-parametrization is convenient here because it displays the relationship between sensitivity and invertibility."

"... the interpretation of Q as an approximate inverse satisfying the interpolation constraints (3.1) is constructive... and offers a guide to finding (possibility suboptimal) compensators."

"Observe that the characterization of Q via interpolation constraints, as in (3.1), avoids any notion of coprime factorization."

Vidyasagar 75 "... the stability results based on coprime factorizations, though they are quite elegant, do not lead readily applicable testing procedures".

The Q-parametrization was still used in:

Desoer-Chen (1981), Desoer-Lin (1983), Desoer-Gustafson (1984), Bhaya-Desoer (1986), Boyd-Barrat-Norman (1990)...

A fractional ideal approach (SCL 03)

- A is an integral domain of SISO stable plants and $K = Q(A)$.
- Let $p \in K$ be a plant and let us introduce the fractional ideal:

$$J = (1,\ p) \triangleq A + A\, p.$$

- J is defined by all the stable linear combinations of 1 and p.
- Why do we need 1? Algebraic answer: the structural properties of a plant p only depend on the system:

$$y - p\, u = 0 \ \Leftrightarrow \ (1 \quad -p) \left(\begin{array}{c} y \\ u \end{array} \right) = 0.$$

Analysis answer: the structural properties of a plant p depend on the graph of the unbounded operator:

$$u \longmapsto y = p\, u.$$

Theory of fractional ideals

"Dedekind's invention of ideals in the 1870s was a major turning point in the development of algebra", Stillvell.

- **Definition:** An A-submodule J of $K = Q(A)$ is a fractional ideal of A if $\exists\, 0 \neq d \in A$ such that $(d)\, J = \{d\, j \mid j \in J\} \subseteq A$.

- A fractional ideal $J \subseteq A$ is called an ideal of A.

- A fractional ideal J is principal if $\exists\, k \in K$ s.t. $J = A\, k = (k)$.

- $I\, J = \{\sum_{i=1}^{n} a_i\, b_i \mid a_i \in I,\ b_i \in J\}$, $A : J = \{k \in K \mid (k)\, J \subseteq A\}$.

- A fractional ideal J is invertible if $\exists\, I \in \mathcal{F}(A)$ such that $I\, J = A$.

If J is invertible then its inverse J^{-1} is unique and defined by $A : J$.

Main results (SCL 03)

- Let A be a ring of stable transfer functions and $K = Q(A)$.
- Let $p \in K$ be a transfer function.
- Let $J = (1,\, p)$ be a fractional ideal, $A : J = \{d \in A \mid d\ p \in A\}$.
- **Theorem**: 1. p is stable iff $J = A$ iff $A : J = A$.

2. p admits a weakly coprime factorization iff:

$$\exists\, 0 \neq d \in A : \quad A : J = (d).$$

Then, $p = n/d$, $(n = d\, p \in A)$, is a weakly coprime factorization.

3. p is internally stabilizable iff J is invertible, i.e., iff:

$$\exists\, a,\, b \in A, \quad a + b\, p = 1, \quad a\, p \in A.$$

If $a \neq 0$, then $c = b/a$ is a stabilizing controller of p and:

$$J^{-1} = (a,\ b), \quad a = 1/(1 + p\, c), \quad b = c/(1 + p\, c).$$

Main results (SCL 03)

4. $c \in K$ internally stabilizes $p \in K$ if we have:

$$(1,\ p)\,(1,\ c) = (1 + p\,c).$$

5. $c \in K$ externally stabilizes $p \in K$, i.e., $p\,c/(1 - p\,c) \in A$, iff:

$$(1,\ p\,c) = (1 + p\,c).$$

6. p is strongly stabilizable iff there exists $c \in A$ such that:

$$(1,\ p) = (1 + p\,c).$$

7. p admits a coprime factorization iff J is principal.

Then, there exists $0 \neq d \in A$ such that $(1,\ p) = (1/d)$ and $p = n/d$ is a coprime factorization of p ($n = d\,p \in A$).

Proof 1

- Let $p \in K$ and $J = (1,\ p)$. If J is invertible, then we have:

$$1 \in J\,(A : J) = (1,\ p)\,(\{d \in A \mid d\,p \in A\}) = \{\alpha + \beta\,p \mid \alpha,\ \beta \in A : J\}$$

$$\Leftrightarrow \exists\ a,\ b \in A : \begin{cases} a + b\,p = 1, \\ a\,p \in A, \quad b\,p \in A. \end{cases}$$

If $a \neq 0$, then $c = b/a \in K$ satisfies:

$$H(p,\ c) = \begin{pmatrix} \frac{1}{1+p\,c} & \frac{p}{1+p\,c} \\ \frac{c}{1+p\,c} & \frac{1}{1+p\,c} \end{pmatrix} = \begin{pmatrix} a & a\,p \\ b & a \end{pmatrix} \in A^{2\times 2},$$

$\Rightarrow c = b/a$ internally stabilizes p $\quad (a = 0 \Rightarrow c = 1 + b$ IS $p)$.

- If p is internally stabilizable, then there exists $c \in K$ s.t.:

$$a = 1/(1+p\,c) \in A, \quad a\,p = p/(1+p\,c) \in A, \quad b = c/(1+p\,c) \in A.$$

Let $I = (a,\ b)$. Then, $a + b\,p = 1 \in I\,J \Rightarrow I\,J = A \Rightarrow I = J^{-1}$.

Proof 2

- $J = (1, \; p)$ is principal iff there exists $0 \neq k \in K$ s.t. $J = (k)$, i.e., iff there exist $0 \neq d, \, n, \, x, \, y \in A$ s.t.:

$$\left\{ \begin{array}{l} 1 = d\,k, \\ p = n\,k, \\ k = x + y\,p \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} k = 1/d, \\ p = n/d, \\ 1/d = x + y\,(n/d), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} p = n/d, \\ d\,x + n\,y = 1. \end{array} \right.$$

p admits a coprime factorization $p = n/d$ iff $J = (1/d)$.

- A principal fractional ideal $J = (k)$ is invertible: $J^{-1} = (1/k)$.

- $(d\,x) + (d\,y)\,p = 1$, i.e., $a = d\,x, \; b = d\,y \in J^{-1} = (d)$,

$$\Rightarrow c = b/a = y/x \in \mathrm{Stab}(p).$$

Solving Zames-Francis' conditions

- Let $p \in Q(A)$ be an internally stabilizable plant

$$\Leftrightarrow \exists\, a,\, b \in A, \quad a + b\, p = 1, \quad a\, p \in A, \quad (\star)$$

$$\Leftrightarrow \exists\, b \in A, \quad b\, p, \quad p\,(1 + b\, p) \in A. \quad \text{(Zames-Francis)}$$

- If $a \neq 0$, then $c = b/a = b/(1 + b\, p)$ internally stabilizes p.

- $J = (1,\ p)$ is invertible and $J^{-1} = (a,\ b) \Rightarrow J^2 = (1,\ p,\ p^2)$

$$\Rightarrow J^{-2} = (J^2)^{-1} = \{\alpha \in A \,|\, \alpha\, p,\ \alpha\, p^2 \in A\}.$$

$$\Rightarrow J^{-2} = (J^{-1})^2 = (a,\ b)^2 = (a,\ a\, b,\ b^2).$$

- Using $(\star)$, we get $a\, b = (b)\, a^2 + (a\, p)\, b^2 \in (a^2,\ b^2)$

$$\Rightarrow J^{-2} = (a^2, a\, b,\ b^2) = (a^2,\ b^2).$$

Solving Zames-Francis' conditions

- Let us find all the possible a' and b' satisfying:

$$\exists\, a',\, b' \in A, \quad a' + b'\, p = 1, \quad a'\, p \in A, \quad (1)$$

- Using the fact that $a,\, b,\, a\, p \in A$ and $a + b\, p = 1$, we get:

$$(b' - b)\, p = a - a' \in A, \quad (b' - b)\, p^2 = (a - a')\, p \in A,$$

$$\Rightarrow b' - b \in \{\alpha \in A \,|\, \alpha\, p,\ \alpha\, p^2 \in A\} = (a^2,\, b^2),$$

$$\Rightarrow \exists\, q_1,\ q_2 \in A : \quad \begin{cases} b' = b + q_1\, a^2 + q_2\, b^2, \\ a' = a - (q_1\, a^2 + q_2\, b^2)\, p, \end{cases} \quad (2)$$

$$\Rightarrow c' = \frac{b'}{a'} = \frac{b + q_1\, a^2 + q_2\, b^2}{a - (q_1\, a^2 + q_2\, b^2)\, p} \in \mathrm{Stab}(p).$$

- We can check that, for all q_1 and $q_2 \in A$, (2) satisfies (1).

Zames-Francis Q-parametrization

- Let $p = n/d$ be a coprime factorization of p over A:

$$d\,x + n\,y = 1.$$

$$\Rightarrow J = (1/d) \Rightarrow J^{-2} = (d^2),$$

$$\Rightarrow a = d\,x, \;\; b = d\,y \in J^{-1} = (d): \;\; a + b\,p = 1, \quad a\,p \in A.$$

$$\Rightarrow c(q) = \frac{b + q\,d^2}{a - q\,d^2\,p} = \frac{d\,y + q\,d^2}{d\,x - d\,n\,q} = \frac{y + q\,d}{x - n\,q}.$$

- **Conclusion:** We have just found again Zames-Francis and Youla-Kučera parametrizations of all stabilizing controllers of p.

General Q-parametrization (SCL 03)

- **Theorem**: Let c be a stabilizing controller of $p \in Q(A)$,

$$a = 1/(1 + p\,c), \quad b = c/(1 + p\,c), \quad J = (1,\, p).$$

Then, all stabilizing controllers of p are

$$c(q_1,\, q_2) = \frac{b + a^2\, q_1 + b^2\, q_2}{a - a^2\, p\, q_1 - b^2\, p\, q_2}, \quad (\star)$$

where q_1 and q_2 any element of A: $a - a^2\, p\, q_1 - b^2\, p\, q_2 \neq 0$.

1. $(\star)$ depends on only one free parameter

$$\Leftrightarrow p^2 \text{ admits a coprime factorization } p^2 = s/r.$$

$$(\star) \Leftrightarrow c(q) = \frac{b + r\,q}{a - r\,p\,q}, \quad \forall\, q \in A: \; a - r\,p\,q \neq 0.$$

2. If p admits a coprime factorization $p = n/d$, $d\,x + n\,y = 1$:

$$(\star) \Leftrightarrow c(q) = \frac{y + d\,q}{x - n\,q}, \quad \forall\, q \in A: \; x - n\,q \neq 0.$$

Conclusion

• We have shown how some of Frank's contributions have played an important role in the sucessful development of stabilization problems for distributed parameter systems.

• We have also drawn a path starting from classical engineering techniques (Nyquist theorem) to abstract algebraic concepts.

• Having read again Frank's papers for preparing this talk, I realized how much we could still learn from them.

• They will be a source of inspiration for the next generations.

• Nothing would have been possible for me if I had not come across Frank's beautiful algebras, without his precious advices and help.

Chapeau bas l'Artiste et bon vent!

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