Symbolic computation for operators with matrix coefficients

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Introduction

Our goal is to provide an algebraic framework to compute with operators having matrix coefficients of **generic size**. For example, consider a system of homogeneous LODE

$$Y' - AY = 0 \quad \text{where} \quad A \in C^\infty(\mathbb{R})^{n\times n}.$$

Let $L = \partial - A$ where ∂ denotes the derivation operator and A denotes the multiplication operator $F \mapsto AF$ induced by the matrix A. If Φ is a fundamental matrix of $Ly = 0$, then by the Leibniz rule, independently of the size n, the following identity holds

$$L \circ \Phi = \Phi \circ \partial + \partial\Phi - A\Phi = \Phi \circ \partial.$$

In what follows, let (R, ∂) be a differential ring with **ring of constants**

$$K = \{c \in R \mid \partial c = 0\}.$$

Considering the coefficient ring R and its ring of constants K, we have two cases.
Scalars: R and K are commutative and hence R and $R\langle\partial\rangle$ are K-**algebras**.
Matrices: R and K are noncommutative and $rc \neq cr$, so R and $R\langle\partial\rangle$ are **only** K-**rings**.

Tensor ring

We use the **tensor ring for modelling additive operators** and interpret $\otimes$ as composition of operators. The tensor product of K-bimodules $M \otimes_K N$ over an arbitrary ring K satisfies

$$(m+m')\otimes n = m\otimes n + m'\otimes n, \quad m\otimes(n+n') = m\otimes n + m\otimes n', \quad mk\otimes n = m\otimes kn.$$

Note that in general $km \otimes n \neq m \otimes kn$. The n-fold tensor product of a K-bimodule M is denoted by $M^{\otimes n} = M \otimes_K \cdots \otimes_K M$ such that $M^{\otimes 0} = K\varepsilon$, where ε denotes the empty tensor. Then the K-**tensor ring** over the K-bimodule M is defined as a K-bimodule by

$$K\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n}.$$

Every $t \in K\langle M\rangle$ can be written in the form

$$t = k\varepsilon + \sum_{i=1}^{p} m_{i,1} \otimes \cdots \otimes m_{i,n_i}.$$

The K-tensor ring over the free K-bimodule on a set X is generated as an additive group by

$$\{k_1x_1 \otimes k_2x_2 \otimes \cdots \otimes k_nx_nk_{n+1} \mid n \in \mathbb{N}_0, x_i \in X, k_j \in K\}$$

and elements of the tensor ring do not have a unique representation. In contrast, if K is commutative and M is a free left K-module over some set X, elements of the K-tensor algebra $K\langle M\rangle$ have a unique representation as K-linear combinations of products $x_1 \otimes x_2 \otimes \cdots \otimes x_n$.

Tensor reduction systems

Identities of operators which cannot be covered by the tensor ring are modelled by **tensor reduction rules** $r_W = (W, h)$ where $h\colon M_W \to K\langle M\rangle$ is a K-bimodule homomorphism.

Example: Differential operators

Let (R, ∂) be a differential ring with ring of constants K. We consider the K-bimodules

$$M = R \oplus K\partial, \qquad M_{\mathsf{K}} = K, \qquad M_{\mathsf{R}} = R, \qquad M_{\mathsf{D}} = K\partial$$

and reduction rules acting on M_{K}, $M_{\mathsf{R}} \otimes M_{\mathsf{R}}$, $M_{\mathsf{D}} \otimes M_{\mathsf{R}}$ defined by

$$(\mathsf{K}, 1 \mapsto \varepsilon), \quad (\mathsf{RR}, f \otimes g \mapsto fg), \quad (\mathsf{DR}, \partial \otimes f \mapsto f \otimes \partial + \partial f).$$

These rules induce the **two-sided reduction ideal**

$$J = (1 - \varepsilon, f \otimes g - fg, \partial \otimes f - f \otimes \partial - \partial f \mid f, g \in R),$$

which is used to define the K-ring of differential operators as $R\langle\partial\rangle = K\langle M\rangle/J$.

More formally, the framework for tensor reduction systems is the following. Let $(M_z)_{z\in Z}$ be a family of a K-subbimodules of M and let $X \subseteq Z$ be sets where $M = \bigoplus_{x\in X} M_x = \sum_{z\in Z} M_z$. To a word $W = w_1 \dots w_n \in \langle Z\rangle$ we associate the K-bimodule $M_W := M_{w_1} \otimes \cdots \otimes M_{w_n}$. So

$$K\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n} = \bigoplus_{W\in\langle X\rangle} M_W = \sum_{W\in\langle Z\rangle} M_W.$$

Theorem (confluence criterion, [2, 4]):

Given a tensor reduction system, every tensor $t \in K\langle M\rangle$ has a unique normal form $t\!\downarrow$ iff all **ambiguities are resolvable**. In that case, the tensor ring factored by the reduction ideal is isomorphic to the ring of irreducible tensors with multiplication $s \cdot t := (s \otimes t)\!\downarrow$.

Integro-differential operators with linear substitutions

Recall the fundamental theorem of calculus and the definition of linear substitution operators:

$$\frac{d}{dx}\int_a^x f(s)ds = f(x), \quad f(a) = f(x) - \int_a^x f'(s)ds, \quad \sigma_{a,b}(f(x)) = f(ax - b).$$

Based on these identities we have the following general definition.

Definition:

An **integro-differential ring with linear substitutions** $(R, \partial, \int, S)$ is a differential ring (R, ∂) with the ring of constants K where $\int$ is a K-bimodule homomorphism of R such that $\partial \int f = f$, the evaluation $\mathrm{E}f := f - \int \partial f$ is multiplicative, i.e. $\mathrm{E}fg = (\mathrm{E}f)\mathrm{E}g$, and S is a group of multiplicative K-bimodule homomorphisms $\sigma_{a,b}\colon R \to R$ on R fixing the constants K and satisfying $\partial\sigma_{a,b}f = a\sigma_{a,b}\partial f$.

Let $(R, \partial, \int, \mathrm{E}, S)$ be an integro-differential ring with linear substitutions and

$$M = R \oplus K\partial \oplus K\textstyle\int \oplus K\mathrm{E} \oplus KS.$$

Let M_{K}, M_{R}, M_{D} as before, $M_{\mathsf{I}} = K\int$, $M_{\mathsf{E}} = K\mathrm{E}$, $M_{\mathsf{G}} = KS$, and $M_{\mathsf{N}} = K\sigma_{1,0}$. Then we call $R\langle\partial, \int, \mathrm{E}, S\rangle := K\langle M\rangle/J$ the **ring of integro-differential operators with linear substitutions**, where J is the two-sided reduction ideal induced by the reduction system:

K	$1 \mapsto \varepsilon$	IRE	$\int \otimes f \otimes \mathrm{E} \mapsto \int f \otimes \mathrm{E}$
RR	$f \otimes g \mapsto fg$	IRD	$\int \otimes f \otimes \partial \mapsto f - \int \otimes \partial f - (\mathrm{E}f)\mathrm{E}$
DR	$\partial \otimes f \mapsto f \otimes \partial + \partial f$	IRI	$\int \otimes f \otimes \int \mapsto \int f \otimes \int - \int \otimes \int f$
ER	$\mathrm{E} \otimes f \mapsto (\mathrm{E}f)\mathrm{E}$	N	$\sigma_{1,0} \mapsto \varepsilon$
EE	$\mathrm{E} \otimes \mathrm{E} \mapsto \mathrm{E}$	GG	$\sigma_{a,b} \otimes \sigma_{c,d} \mapsto \sigma_{ac,bc+d}$
EI	$\mathrm{E} \otimes \int \mapsto 0$	GR	$\sigma_{a,b} \otimes f \mapsto \sigma_{a,b}f \otimes \sigma_{a,b}$
DE	$\partial \otimes \mathrm{E} \mapsto 0$	GE	$\sigma_{a,b} \otimes \mathrm{E} \mapsto \mathrm{E}$
DI	$\partial \otimes \int \mapsto \varepsilon$	DG	$\partial \otimes \sigma_{a,b} \mapsto a\sigma_{a,b} \otimes \partial$
IE	$\int \otimes \mathrm{E} \mapsto \int 1 \otimes \mathrm{E}$	IG	$\int \otimes \sigma_{a,b} \mapsto a^{-1}(\varepsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int$
ID	$\int \otimes \partial \mapsto \varepsilon - \mathrm{E}$	IRG	$\int \otimes f \otimes \sigma_{a,b} \mapsto a^{-1}(\varepsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int \otimes \sigma_{a,b}^{-1}f$
II	$\int \otimes \int \mapsto \int 1 \otimes \int - \int \otimes \int 1$		

Checking confluence, reducing S-polynomials, and computing with operators are supported by our Mathematica **package `TenReS`** [3]. Using the above reduction system, every $t \in K\langle M\rangle$ has a unique normal form given by a sum of pure tensors

$$f \otimes \mathrm{E} \otimes \sigma_{a,b} \otimes \partial^{\otimes j} \quad \text{and} \quad f \otimes \mathrm{E} \otimes \sigma_{a,b} \otimes \textstyle\int \otimes g$$

where $j \in \mathbb{N}_0$, each of $f, g \in \int R$ and $\sigma_{a,b} \in S \setminus \{\sigma_{1,0}\}$ may be absent.

Example: Variation of constants

Identities in $R\langle\partial, \int, \mathrm{E}, S\rangle$ can be proven using the reduction system. Let $R = C^\infty(\mathbb{R})^{n\times n}$, $L = \partial - A \in R\langle\partial, \int, \mathrm{E}, S\rangle$, and let $\Phi \in R$ be an invertible solution of $Ly = 0$. Then $H := \Phi \otimes \int \otimes \Phi^{-1}$ is a right inverse of L since independent of the size n we have

$$L \otimes H = (\partial - A) \otimes \Phi \otimes \textstyle\int \otimes \Phi^{-1} \to_{r_{\mathsf{DR}}} \Phi \otimes \partial \otimes \int \otimes \Phi^{-1} \to_{r_{\mathsf{DI}}} \Phi \otimes \Phi^{-1} \to_{r_{\mathsf{RR}}} \Phi\Phi^{-1} \to_{r_{\mathsf{K}}} \varepsilon.$$

Artstein's reduction of DTD control systems

Aim [5]: **Algebraically** find Artstein's reduction [1] of a differential time-delay control system

$$x'(t) = A_1(t)x(t) + A_2(t)u(t) + A_3(t)u(t-h)$$

to a differential control system

$$z'(t) = B_1(t)z(t) + B_2(t)v(t).$$

We use our **tensor setting** to find an invertible transformation $\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$, where P_{11} and P_{22} are arbitrary invertible multiplication operators, and make an **ansatz** for

$$P_{12} = C_0 \otimes \sigma_{1,h} \otimes \textstyle\int \otimes C_1 + C_2 \otimes \int \otimes C_3 + C_4 \otimes \sigma_{1,h} + C_5 \in R\langle\partial, \int, \mathrm{E}, S\rangle.$$

We consider the ring R generated by the **symbols** A_i, B_i, C_i, P_{ii} and we can check that only expressions occur in computations that are valid as operators. By plugging the ansatz into

$$(\partial - B_1) \otimes P_{12} - B_2 \otimes P_{22} = P_{11} \otimes (-A_2 - A_3 \otimes \sigma_{1,h})$$

and reducing both sides to normal form using `TenReS`, we obtain conditions for the multiplication operators C_i. Solving these conditions with partial support by our package, we obtain the following transformation

$$v = P_{22}u, \quad z = P_{11}x + \big(\Phi \otimes (\varepsilon - \sigma_{1,h}) \otimes \textstyle\int \otimes (\sigma_{1,-h}\Phi^{-1})\sigma_{1,-h}A_3\big)u,$$

where $P_{11}^{-1}\Phi$ is a fundamental matrix of $y'(t) = A_1(t)y(t)$.

References

[1] Zvi Artstein. Linear systems with delayed controls: A reduction. IEEE Trans. Autom. Control 27, pp. 869–879, 1982.

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