Further Results on the Decomposition and Serre's Reduction of Linear Functional Systems

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Abstract: Given a linear functional system (e.g., ordinary/partial differential system, differential time-delay system, difference system), the decomposition problem aims at studying when it can be decomposed as a direct sum of subsystems. This problem was constructively studied in [4] and the corresponding algorithms were implemented in the OreMorphisms package [5]. Using the OreMorphisms package, many classical linear differential time-delay systems were proved to be directly decomposable, which highly simplifies the study of their structural properties. Serre's reduction aims at finding an equivalent linear functional system which contains fewer equations and fewer unknowns. It was constructively studied in [1, 6] and successfully applied to different classical examples of differential time-delay systems. Serre's reduction can be seen as a particular case of the decomposition problem. The goal of the present paper is to explicitly provide the links between these two problems. We illustrate the different results with an explicit example of a differential time-delay system.

1. INTRODUCTION

Algebraic analysis is a mathematical framework initiated in the sixties by Malgrange, Sato, Kashiwara, ... for the study of linear systems of partial differential equations and of integro-differential equations [10]. In the nineties, it was introduced in mathematical systems theory by Oberst [14], Fliess, Pommaret, ... It yields an intrinsic characterization of structural properties of *linear functional systems* (e.g., ordinary/partial differential systems, differential time-delay systems, difference systems) and gives a way to find again and extend Willems' *behavioural approach.* A constructive study of algebraic analysis based on symbolic computation techniques (e.g., Gröbner bases) was initiated in [2, 3, 4, 5] for the study of linear functional systems appearing in engineering sciences and in mathematical physics. For a survey, see [15].

A linear functional system can generally be rewritten as $R\,\eta = 0$, where $R \in D^{q\times p}$ is a $q\times p$ matrix with entries in a *noetherian domain* D [16] (e.g., a non-commutative ring of ordinary/partial differential operators, of differential time-delay operators, of shift operators) and $\eta \in \mathcal{F}^p$, where $\mathcal{F}$ is a left D-module. Malgrange's remark [12] asserts that the linear system or *behavior* defined by

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p := \mathcal{F}^{p\times 1} \mid R\,\eta = 0\}$$

is isomorphic to the abelian group $\hom_D(M, \mathcal{F})$ formed by the left D-homomorphisms (linear maps) from the *finitely presented left* D*-module* $M := D^{1\times p}/(D^{1\times q}\,R)$ to $\mathcal{F}$. This isomorphism is the key for an intrinsic study of the linear system $\ker_{\mathcal{F}}(R.)$ by means of the two left D-modules M and $\mathcal{F}$ using module theory and homological algebra.

An important issue in symbolic computation and in mathematical systems theory consists in simplifying linear functional systems by means of algebraic techniques before investigating their symbolic/numerical integration and studying their structural properties or synthesis problems. In [4], we study the so-called *decomposition problem*, namely, the problem of finding (if they exist) $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$, where $\mathrm{GL}_r(D)$ is defined by

$$\mathrm{GL}_r(D) := \{U \in D^{r\times r} \mid \exists\, V \in D^{r\times r} :\; U\,V = V\,U = I_r\},$$

such that R is equivalent to a block-diagonal matrix $V\,R\,W = \mathrm{diag}(R_1, R_2)$ formed by $R_1 \in D^{q_1\times p_1}$ and $R_2 \in D^{q_2\times p_2}$. Note that this problem yields the direct sum decomposition $M = M_1 \oplus M_2$ of M, where

$$M_1 := D^{1\times p_1}/(D^{1\times q_1}\,R_1), \quad M_2 := D^{1\times p_2}/(D^{1\times q_2}\,R_2),$$

and by Malgrange's remark, the direct sum decomposition $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(R_1.) \oplus \ker_{\mathcal{F}}(R_2.)$, where $\cong$ denotes isomorphic objects. The study of the linear system $\ker_{\mathcal{F}}(R.)$ then reduces to the ones of its two independent subsystems $\ker_{\mathcal{F}}(R_1.)$ and $\ker_{\mathcal{F}}(R_2.)$.

Moreover, in [1] (see also [6]), the authors consider the so-called *Serre's reduction problem* of linear functional systems. It consists in finding an equivalent system which contains fewer unknowns and fewer equations. In some cases, this further provides two matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that $V\,R\,W = \mathrm{diag}(I_r, \overline{R})$ is a block-diagonal matrix having the identity matrix I_r as its first diagonal block. Consequently, $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(\overline{R}.)$. Serre's reduction can be seen as a particular decomposition $(R_1 = I_r)$ and the present paper aims at explicitly giving the relations between these two problems.

2. DECOMPOSITION PROBLEM

In what follows, D is a *left noetherian domain*, namely, the ring D has no zero divisors and is such that every left ideal of D is finitely generated as a left D-module [16].

In this section, we review results obtained in [4].

The matrix $R \in D^{q\times p}$ induces the left D-homomorphism:

$$\begin{array}{rl} .R\colon D^{1\times q} & \longrightarrow D^{1\times p} \\ \lambda & \longmapsto \lambda\, R. \end{array}$$

Then, the *cokernel* of $.R$ is the *factor* left D-module

$$M := D^{1\times p}/(D^{1\times q}\, R),$$

i.e., the left D-module *finitely presented* by R. Now, if $\pi \in \hom_D(D^{1\times p}, M)$ is defined by sending $\lambda \in D^{1\times p}$ onto its *residue class* $\pi(\lambda) \in M$, and $\{f_j\}_{j=1,\ldots,p}$ is the *standard basis* of $D^{1\times p}$ (i.e., $f_j \in D^{1\times p}$ is the vector formed by 1 at the j^{th} position and 0 elsewhere), then one can easily prove that $\{y_j := \pi(f_j)\}_{j=1,\ldots,p}$ is a *family of generators* of M which satisfies the following *left D-linear relations*:

$$\forall\, i = 1,\ldots,q, \quad \sum_{j=1}^{p} R_{ij}\, y_j = 0.$$

For more details, see [2, 4, 15]. Let M', M, and M'' be left D-modules, $f \in \hom_D(M', M)$ and $g \in \hom_D(M, M'')$. If $\ker g = \operatorname{im} f$, then $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called a *exact sequence* [16]. In particular, if $g = 0$, then $\operatorname{im} f = M$, i.e., f is surjective, and if $f = 0$, then $\ker g = 0$, i.e., g is injective. By definition of M, the following exact sequence holds:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0.$$

Lemma 1. ([4], Lemma 4.1). Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be the left D-module finitely presented by $R \in D^{q\times p}$ and $R_2 \in D^{r\times q}$ such that $\ker_D(.R) := \{\lambda \in D^{1\times q} \mid \lambda\, R = 0\} = \operatorname{im}_D(.R_2) := D^{1\times r}\, R_2$, i.e., such that the exact sequence $D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$ holds. Then:

(1) A left D-endomorphism $f\colon M \longrightarrow M$ is defined by

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi(\lambda\, P),$$

where $P \in D^{p\times p}$ is a matrix such that the relation $R\, P = Q\, R$ holds for a certain $Q \in D^{q\times q}$. The matrix P is uniquely defined by f *up to homotopy*, namely,

$$\begin{cases} \overline{P} := P + H_1\, R, & \forall\, H_1 \in D^{p\times q}, \\ \overline{Q} := Q + R\, H_1 + H_2\, R_2, & \forall\, H_2 \in D^{q\times r}, \end{cases} \quad (1)$$

satisfy $R\, \overline{P} = \overline{Q}\, R$ and $f(\pi(\lambda)) = \pi(\lambda\, \overline{P})\ \forall\, \lambda \in D^{1\times p}$.

(2) A left D-endomorphism f of M is an *idempotent* of the ring $\operatorname{end}_D(M) := \hom_D(M, M)$, namely, $f^2 = f$, iff there exists a matrix $Z \in D^{p\times q}$ satisfying:

$$P^2 = P + Z\, R.$$

Then, there exists a matrix $Z' \in D^{q\times r}$ such that:

$$Q^2 = Q + R\, Z + Z'\, R_2.$$

If $R \in D^{q\times p}$ has *full row rank*, namely, $\ker_D(.R) = 0$, i.e., $R_2 = 0$, then:

$$Q^2 = Q + R\, Z.$$

Note that a left D-endomorphism f of M induces:

$$\begin{array}{rl} f^\star\colon \ker_{\mathcal{F}}(R.) & \longrightarrow \ker_{\mathcal{F}}(R.) \\ \eta & \longmapsto P\, \eta. \end{array}$$

Hence, the abelian group endomorphism $f^\star$ is a *Galois-like transformation* (i.e., an *internal symmetry*) of $\ker_{\mathcal{F}}(R.)$.

A well-known result in module theory (see, e.g., [16]) asserts that the existence of an idempotent endomorphism of M is equivalent to the existence of a direct sum decomposition $M = M_1 \oplus M_2$ of the left D-module M.

Using the degree of freedom in the choice of the matrix P defining an idempotent f of M (see (1)), if R has full row rank, i.e., $\ker_D(.R) = 0$, then the next lemma gives a sufficient condition for $f \in \operatorname{end}_D(M)$ to be defined by an *idempotent matrix* $\overline{P}$, namely, $\overline{P}^2 = \overline{P}$.

Lemma 2. ([4], Lemma 4.4). Let $R \in D^{q\times p}$ be a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}\, R)$. Let us consider an idempotent $f \in \operatorname{end}_D(M)$ defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying (see Lemma 1):

$$R\, P = Q\, R, \quad P^2 = P + Z\, R, \quad Q^2 = Q + R\, Z.$$

If there exists a solution $\Delta \in D^{p\times q}$ of the following *algebraic Riccati equation*

$$\Delta\, R\, \Delta + (P - I_p)\, \Delta + \Delta\, Q + Z = 0, \quad (2)$$

then the matrices defined by

$$\overline{P} := P + \Delta\, R, \quad \overline{Q} := Q + R\, \Delta,$$

satisfy $R\, \overline{P} = \overline{Q}\, R$, $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$, and:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi(\lambda\, \overline{P}).$$

The interest of defining an idempotent $f \in \operatorname{end}_D(M)$ by means of two idempotents matrices $\overline{P}$ and $\overline{Q}$ is that the finitely generated left D-modules $\ker_D(.\overline{P})$, $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$, and $\operatorname{im}_D(.\overline{Q})$ then satisfy

$$\begin{cases} D^{1\times p} \cong \ker_D(.\overline{P}) \oplus \operatorname{im}_D(.\overline{P}), \\ D^{1\times q} \cong \ker_D(.\overline{Q}) \oplus \operatorname{im}_D(.\overline{Q}), \end{cases}$$

i.e., $\ker_D(.\overline{P})$ and $\operatorname{im}_D(.\overline{P})$ (resp., $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$) are *direct summands* of the free left D-module $D^{1\times p}$ (resp., $D^{1\times q}$), i.e., that they are finitely generated *projective* left D-modules [16].

Theorem 3. ([1], Theorem 3.3). Let M be a finitely generated projective left D-module. Moreover, if

(1) D is a *principal left ideal domain* (e.g., $k[s]$, where k a field, the noncommutative polynomial ring of ordinary differential operators with coefficients in a differential field such as $\mathbb{R}(t)$),
(2) $D = k[x_1, \ldots, x_n]$, where k is a field,
(3) D is the noncommutative polynomial ring of partial differential operators with either polynomial or rational function coefficients over a field of characteristic 0 (the so-called *Weyl algebras*) and $\operatorname{rank}_D(M) \geq 2$,
(4) D is the noncommutative polynomial ring of ordinary differential operators with either formal power series or locally convergent power series (i.e., germs of real analytic/holomorphic functions) and $\operatorname{rank}_D(M) \geq 2$,

then M is a finitely generated free left D-module.

The next theorem studies when the presentation matrix R of M is equivalent to a block-diagonal matrix.

Theorem 4. ([4], Theorem 4.2). Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be the left D-module finitely presented by $R \in D^{q\times p}$ and $f \in \operatorname{end}_D(M)$ an idempotent defined by two idempotents matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$, i.e.:

$$R\, P = Q\, R, \quad P^2 = P, \quad Q^2 = Q.$$

If the projective left D-modules $\ker_D(.P)$, $\operatorname{im}_D(.P) = \ker_D(.(I_p - P))$, $\ker_D(.Q)$, and $\operatorname{im}_D(.Q) = \ker_D(.(I_q - Q))$

are free of rank respectively m, $p-m$, l, $q-l$, then there exist full row rank matrices $U_1 \in D^{m \times p}$, $U_2 \in D^{(p-m)\times p}$, $V_1 \in D^{l \times q}$ and $V_2 \in D^{(q-l)\times q}$ such that:

- $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$,
- $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$,
- If $U^{-1} := W = (W_1 \quad W_2)$, where $W_1 \in D^{p\times m}$ and $W_2 \in D^{p \times (p-m)}$, then:

$$V\,R\,W = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix} \in D^{q \times p}. \tag{3}$$

Moreover, the matrix U_1 (resp., U_2, V_1, V_2) defines a basis of the free left D-module $\ker_D(.P)$, (resp., $\mathrm{im}_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q)$) of rank m (resp., $p-m$, l, $q-l$), i.e.:

$$\begin{cases} \ker_D(.P) = D^{1\times m}\,U_1, & \mathrm{im}_D(.P) = D^{1\times(p-m)}\,U_2, \\ \ker_D(.Q) = D^{1\times l}\,V_1, & \mathrm{im}_D(.Q) = D^{1\times(q-l)}\,V_2. \end{cases}$$

Finally, if we note $V^{-1} := (X_1 \quad X_2)$, where $X_1 \in D^{q\times l}$ and $X_2 \in D^{(q-l)\times q}$, then the following *split commutative exact diagram* holds

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D^{1\times l} & \underset{.X_1}{\overset{.V_1}{\rightleftarrows}} & D^{1\times q} & \underset{.V_2}{\overset{.X_2}{\rightleftarrows}} & D^{1\times(q-l)} & \longrightarrow & 0 \\
 & & \downarrow .(V_1\,R\,W_1) & & \downarrow .R & & \downarrow .(V_2\,R\,W_2) & & \\
0 & \longrightarrow & D^{1\times m} & \underset{.W_1}{\overset{.U_1}{\rightleftarrows}} & D^{1\times p} & \underset{.U_2}{\overset{.W_2}{\rightleftarrows}} & D^{1\times(p-m)} & \longrightarrow & 0 \\
 & & \downarrow \tau & & \downarrow \pi & & \downarrow \sigma & & \\
0 & \longrightarrow & \ker f & \underset{\mathrm{id}_M - f}{\overset{i}{\rightleftarrows}} & M & \underset{f_{|\mathrm{im}\, f}}{\overset{f}{\rightleftarrows}} & \mathrm{im}\, f & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4}$$

which shows $M \cong \ker f \oplus \mathrm{im}\, f$, where

$$\begin{cases} \ker f \cong D^{1\times m}/(D^{1\times l}\,(V_1\,R\,W_1)), \\ \mathrm{im}\, f \cong D^{1\times(p-m)}/(D^{1\times(q-l)}\,(V_2\,R\,W_2)), \end{cases}$$

i.e., up to isomorphism of left D-modules, $\ker f$ (resp., $\mathrm{im}\, f$) is finitely presented by the first (resp., second) diagonal block of the matrix $V\,R\,W$.

Conditions of Theorem 4 are fulfilled if D satisfies 1 or 2 of Theorem 3, 3 or 4 of Theorem 3 with the rank conditions.

The matrices appearing in the above results can be computed using the Maple package OreMorphisms [5] developed upon OreModules [3]. For more details, see [4].

3. SERRE'S REDUCTION PROBLEM

The following theorem gathers results of [1] that will be used in what follows.

Theorem 5. ([1], Theorem 4.1, Corollary 4.10). Let $M = D^{1\times p}/(D^{1\times q}\,R)$, be a the left D-module finitely presented by a full row rank matrix $R \in D^{q\times p}$, $0 \le r \le q-1$, and $\Lambda \in D^{q \times (q-r)}$ such that there exists $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying:

$$(R \quad -\Lambda)\,U = (I_q \quad 0).$$

(1) If we note

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & T_2 \end{pmatrix},$$

where $S_1 \in D^{p\times q}$, $S_2 \in D^{(q-r)\times q}$, $Q_1 \in D^{p\times(p-r)}$, $Q_2 \in D^{(q-r)\times(p-r)}$, $T_1 \in D^{(p-r)\times p}$, $T_2 \in D^{(p-r)\times(q-r)}$, then:

$$M = D^{1\times p}/(D^{1\times q}\,R) \cong D^{1\times(p-r)}/(D^{1\times(q-r)}\,Q_2).$$

(2) Moreover, if the matrix $\Lambda \in D^{q\times(q-r)}$ admits a left inverse $\Gamma \in D^{(q-r)\times q}$, i.e., $\Gamma\,\Lambda = I_{q-r}$, then Q_1 admits the left inverse $T_1 - T_2\,\Gamma\,R \in D^{(p-r)\times p}$, the left D-module $\ker_D(.Q_1)$ is such that

$$\ker_D(.Q_1) \oplus D^{1\times(p-r)} \cong D^{1\times p},$$

i.e., $\ker_D(.Q_1)$ is a *stably free*, and thus, a projective left D-module of rank r, and $Q_2 = \Gamma\,R\,Q_1$.

(3) If the left D-module $\ker_D(.Q_1)$ is free of rank r, then there exists a matrix $Q_3 \in D^{p\times r}$ such that:

$$W = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D).$$

If we note $W^{-1} = (Y_3^T \quad Y_1^T)^T$, where $Y_3 \in D^{r\times p}$ and $Y_1 \in D^{(p-r)\times p}$, then $X = (R\,Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$,

$$V := X^{-1} = \begin{pmatrix} Y_3\,S_1 \\ Q_2\,Y_1\,S_1 - S_2 \end{pmatrix},$$

and $R = X\,\mathrm{diag}(I_r, Q_2)\,W^{-1}$, i.e., R is equivalent to:

$$V\,R\,W = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}. \tag{5}$$

Theorem 3 shows that the left D-module $\ker_D(.Q_1)$ is free for different domains D (with the possible condition $r \ge 2$). For more details, see [1, 6]. An implementation of Theorem 5 is under development in the Maple package Serre [8] developed upon OreModules [3].

4. SERRE'S REDUCTION AS A PARTICULAR DECOMPOSITION PROBLEM

The block-diagonal form (5) can be seen as a particular form of (3) where the first diagonal block is an identity matrix. Therefore, Serre's reduction can be viewed as a particular block-diagonal decomposition. This section contains the contributions of this paper: we explain how the results reviewed in Section 2, based on the resolution of algebraic Riccati equations (2), are related to Theorem 5.

Lemma 2 can be applied to the two trivial idempotents of $\mathrm{end}_D(M)$, namely:

(1) $f = \mathrm{id}_M$ defined by $P = I_p$ and $Q = I_q$,
(2) $f = 0$ defined by $P = 0_p$ and $Q = 0_q$.

Then, we respectively obtain the following facts:

(1) If $\Delta \in D^{p\times q}$ is a solution of $\Delta\,R\,\Delta = -\Delta$, then $\overline{P} := I_p + \Delta\,R$ and $\overline{Q} := I_q + R\,\Delta$ satisfy

$$R\,\overline{P} = \overline{Q}\,R, \quad \overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}, \tag{6}$$

and $f(\pi(\lambda)) = \pi(\lambda\,\overline{P}) = \pi(\lambda)$ for all $\lambda \in D^{1\times p}$.

(2) If $\Delta \in D^{p\times q}$ is a solution of $\Delta\,R\,\Delta = \Delta$, then $\overline{P} := \Delta\,R$ and $\overline{Q} := R\,\Delta$ satisfy (6), and $f(\pi(\lambda)) = 0$ for all $\lambda \in D^{1\times p}$.

Note that, if $\Delta \in D^{p\times q}$ satisfies $\Delta\,R\,\Delta = \Delta$, then $\Theta := -\Delta$ satisfies $\Theta\,R\,\Theta = -\Theta$ and conversely. Thus, the idempotent matrices $\overline{P}_1 = \Delta\,R$ and $\overline{Q}_1 = R\,\Delta$ define the idempotent endomorphism 0_M iff the idempotent matrices $\overline{P}_2 = I_p - \Delta\,R$ and $\overline{Q}_2 = I_q - R\,\Delta$ define the idempotent id_M. Moreover, we have $\overline{P}_1 + \overline{P}_2 = I_p$, $\overline{Q}_1 + \overline{Q}_2 = I_q$.

Now, let us assume that R has full row rank and $\ker_D(.\overline{P})$, $\mathrm{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$, $\mathrm{im}_D(.\overline{Q})$ are free of rank respectively m, $p-m$, l, and $q-l$, where $1 \le m \le p$ and $1 \le l \le q$. Then, Theorem 4 holds with $\ker f = 0$ (resp., $\operatorname{im} f = 0$). Since R has full row rank, so are $V\,R\,U^{-1}$, $V_1\,R\,W_1$ and $V_2\,R\,W_2$, i.e., $\ker_D(.(V_1\,R\,W_1)) = 0$ and $\ker_D(.(V_2\,R\,W_2)) = 0$. Thus, (4) provides one of the following two results:

(1) The following short exact sequence holds
$$0 \longrightarrow D^{1\times l} \xrightarrow{.(V_1\,R\,W_1)} D^{1\times m} \longrightarrow 0,$$
which yields $m = l$ and $V_1\,R\,W_1 \in \mathrm{GL}_m(D)$.

(2) The following short exact sequence holds
$$0 \longrightarrow D^{1\times (q-l)} \xrightarrow{.(V_2\,R\,W_2)} D^{1\times (p-m)} \longrightarrow 0,$$
which yields $p-m = q-l$ and $V_2\,R\,W_2 \in \mathrm{GL}_{p-m}(D)$.

We obtain the following corollary of Theorem 4.

Corollary 6. (1) Let $R \in D^{q\times p}$ be a full row rank matrix, $\Delta \in D^{p\times q}$ satisfying $\Delta\,R\,\Delta = -\Delta$, $\overline{P} := I_p + \Delta\,R$ and $\overline{Q} := I_q + R\,\Delta$. If the projective left D-modules $\ker_D(.\overline{P})$, $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$ are free of rank respectively m, $p-m$, l and $q-l$, where $1 \leq m \leq p$ and $1 \leq l \leq q$, then $m = l$ and there exist $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that:
$$V\,R\,W = \begin{pmatrix} I_m & 0 \\ 0 & \overline{R}_2 \end{pmatrix}, \quad \overline{R}_2 \in D^{(q-m)\times(p-m)}.$$

(2) Let $R \in D^{q\times p}$ be a full row rank matrix, $\Delta \in D^{p\times q}$ satisfying $\Delta\,R\,\Delta = \Delta$, $\overline{P} := \Delta\,R$ and $\overline{Q} := R\,\Delta$. If the projective left D-modules $\ker_D(.\overline{P})$, $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$ are free of rank respectively m, $p-m$, l and $q-l$, where $1 \leq m \leq p$ and $1 \leq l \leq q$, then $p-m = q-l$ and there exist $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that:
$$V\,R\,W = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & I_{q-l} \end{pmatrix}, \quad \overline{R}_1 \in D^{l\times m}.$$

Theorem 3 shows that Corollary 6 holds for different domains D interesting in mathematical systems theory.

Example 1. Let us consider the wind tunnel model studied in [13] described by a differential time-delay linear system defined by the following matrix of functional operators
$$R = \begin{pmatrix} \partial + a & k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix},$$
where $\partial\,y(t) = \dot{y}(t)$ is the ordinary differential operator, $\delta\,y(t) = y(t-1)$ is the time-delay operator, and ζ, k, ω, a are constant parameters of the system. We then consider the commutative polynomial ring $D = \mathbb{Q}(\zeta, k, \omega, a)[\partial, \delta]$ of differential time-delay operators with coefficients in the field $\mathbb{Q}(\zeta, k, \omega, a)$. We can check that the matrix
$$\Delta = \frac{1}{\omega^2}\begin{pmatrix} 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 1 & 2\,\zeta\,\omega - a & 1 \end{pmatrix} \in D^{4\times 3}$$
satisfies the algebraic Riccati equation $\Delta\,R\,\Delta = -\Delta$. Then, $\overline{P} := I_4 + \Delta\,R$ and $\overline{Q} := I_3 + R\,\Delta$ defined by
$$\overline{P} = \begin{pmatrix} 1 & -\partial & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \partial & 0 & 0 \\ \omega^{-2}\,(\partial + a) & A & \omega^{-2}\,(\partial + a) & 0 \end{pmatrix},$$
$$\overline{Q} = \begin{pmatrix} 1 & -\partial - a & 0 \\ 0 & 0 & 0 \\ -1 & \partial + a & 1 \end{pmatrix},$$
where $A = \omega^{-2}\,((2\,\zeta\,\omega - a)\,\partial + k\,a\,\delta + \omega^2)$, satisfy the identities (6), and thus they define the trivial idempotent endomorphism $f = \mathrm{id}_M$ of $M = D^{1\times 4}/(D^{1\times 3}\,R)$. Using 2 of Theorem 3 (i.e., the *Quillen-Suslin theorem* [9, 16]), the projective D-modules $\ker_D(.\overline{P})$, $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$, $\operatorname{im}_D(.\overline{Q})$ are free. Using the QuillenSuslin package [9] to compute a basis of these free D-modules, we obtain $\ker_D(.\overline{P}) = D^{1\times 2}\,U_1$, $\operatorname{im}_D(.\overline{P}) = D^{1\times 2}\,U_2$, $\ker_D(.\overline{Q}) = D^{1\times 2}\,V_1$, $\operatorname{im}_D(.\overline{Q}) = D\,V_2$, where:
$$U_1 = \begin{pmatrix} \partial + a & k\,a\,\delta + \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \\ 0 & \partial & -1 & 0 \end{pmatrix} \in D^{2\times 4},$$
$$U_2 = \begin{pmatrix} -1 & \partial & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \in D^{2\times 4},$$
$$V_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in D^{2\times 3}, \quad V_2 = (-1 \;\; \partial + a \;\; 0) \in D^{1\times 3}.$$
Hence, we have
$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_4(D)$, $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_3(D)$,
and Corollary 6 shows that the matrix R is equivalent to:
$$V\,R\,U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \partial + a & k\,a\,\delta \end{pmatrix}.$$

Lemma 7. (1) Let $R \in D^{q\times p}$ and $\Delta \in D^{p\times q}$ satisfy $\Delta\,R\,\Delta = -\Delta$, $\overline{P} := I_p + \Delta\,R$ and $\overline{Q} := I_q + R\,\Delta$. Then, we have:
$$\begin{cases} \ker_D(\overline{P}.) := \{\eta \in D^p \mid \overline{P}\,\eta = 0\} = \operatorname{im}_D(\Delta.), \\ \operatorname{im}_D(.\overline{P}) = \ker_D(.\Delta), \\ \ker_D(.\overline{Q}) = \operatorname{im}_D(.\Delta), \\ \operatorname{im}_D(\overline{Q}.) := \overline{Q}\,D^q = \ker_D(\Delta.). \end{cases}$$
Hence, $\operatorname{im}_D(.\overline{P})$ (resp., $\ker_D(.\overline{Q})$) is a free left D-module iff so is $\ker_D(.\Delta)$ (resp., $\operatorname{im}_D(.\Delta)$).

(2) Let $R \in D^{q\times p}$ and $\Delta \in D^{p\times q}$ satisfy $\Delta\,R\,\Delta = \Delta$, $\overline{P} := \Delta\,R$ and $\overline{Q} := R\,\Delta$. Then, we have:
$$\begin{cases} \ker_D(.\overline{P}) = \ker_D(.\Delta), \\ \operatorname{im}_D(\overline{P}.) = \operatorname{im}_D(\Delta.), \\ \ker_D(\overline{Q}.) = \ker_D(\Delta.), \\ \operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.\Delta). \end{cases}$$
Hence, $\ker_D(.\overline{P})$ (resp., $\operatorname{im}_D(.\overline{Q})$) is a free left D-module iff so is $\ker_D(.\Delta)$ (resp., $\operatorname{im}_D(.\Delta)$).

Proof. 1. Let us first prove $\ker_D(\overline{P}.) = \operatorname{im}_D(\Delta.)$. If $\lambda \in \ker_D(\overline{P}.)$, then $\lambda = \Delta\,(-R\,\lambda) \in \operatorname{im}_D(\Delta.)$, which proves $\ker_D(\overline{P}.) \subseteq \operatorname{im}_D(\Delta.)$. Conversely, if $\mu \in \operatorname{im}_D(\Delta.)$, then there exists $\nu \in D^q$ such that $\mu = \Delta\,\nu$, and thus $\overline{P}\,\mu = (\Delta + \Delta\,R\,\Delta)\,\nu = 0$, i.e., $\mu \in \ker_D(\overline{P}.)$, which proves $\operatorname{im}_D(\Delta.) \subseteq \ker_D(\overline{P}.)$ and the result. The equality $\ker_D(.\overline{Q}) = \operatorname{im}_D(.\Delta)$ can be proved similarly.

Let us prove $\operatorname{im}_D(.\overline{P}) = \ker_D(.\Delta)$. If $\lambda \in \operatorname{im}_D(.\overline{P})$, then there exists $\mu \in D^{1\times p}$ such that $\lambda = \mu\,\overline{P}$, and thus $\lambda\,\Delta = \mu\,(\Delta + \Delta\,R\,\Delta) = 0$, i.e., $\lambda \in \ker_D(.\Delta)$, which proves the inclusion $\operatorname{im}_D(.\overline{P}) \subseteq \ker_D(.\Delta)$. Conversely, if $\lambda \in \ker_D(.\Delta)$, then $\lambda = \lambda\,(I_p + \Delta\,R) = \lambda\,\overline{P} \in \operatorname{im}_D(.\overline{P})$, which proves $\ker_D(.\Delta) \subseteq \operatorname{im}_D(.\overline{P})$ and the result. The equality $\operatorname{im}_D(\overline{Q}.) = \ker_D(\Delta.)$ can be proved similarly.

2. Let us prove that $\ker_D(.\overline{P}) = \ker_D(.\Delta)$. If we consider $\lambda \in \ker_D(.\overline{P})$, then post-multiplying $\lambda\,\overline{P} = 0$ by Δ,

we get $\lambda\,\overline{P}\,\Delta = \lambda\,\Delta\,R\,\Delta = \lambda\,\Delta = 0$, which proves $\ker_D(.\overline{P}) \subseteq \ker_D(.\Delta)$. Conversely, if $\mu \in \ker_D(.\Delta)$, then, post-multiplying $\mu\,\Delta = 0$ by R, we get $\mu\,\overline{P} = 0$, which proves $\ker_D(.\Delta) \subseteq \ker_D(.\overline{P})$ and the result. The equality $\ker_D(\overline{Q}.) = \ker_D(\Delta.)$ can be proved similarly.

Let us prove that $\operatorname{im}_D(\overline{P}.) = \operatorname{im}_D(\Delta.)$. If $\lambda \in \operatorname{im}_D(\overline{P}.)$, then there exists $\mu \in D^p$ such that $\lambda = \overline{P}\,\mu$, i.e., $\lambda = \Delta\,(R\,\mu)$, and thus $\lambda \in \operatorname{im}_D(\Delta.)$, i.e., $\operatorname{im}_D(\overline{P}.) \subseteq \operatorname{im}_D(\Delta.)$. Conversely, if $\mu \in \operatorname{im}_D(\Delta.)$, i.e., $\mu = \Delta\,\nu$ with $\nu \in D^q$, then $\mu = (\Delta\,R)\,(\Delta\,\nu) = \overline{P}\,(\Delta\,\nu)$, i.e., $\mu \in \operatorname{im}_D(\overline{P}.)$, which proves $\operatorname{im}_D(\Delta.) \subseteq \operatorname{im}_D(\overline{P}.)$ and the result. The identity $\operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.\Delta)$ can be proved similarly.

Remark 1. If we want to find a presentation matrix of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ of minimal size, using the equality $m = l$ (resp., $p - m = q - l$) of 1 (resp., 2) of Corollary 6, we then have to seek for the solutions $\Delta \in D^{p\times q}$ of the equation $\Delta\,R\,\Delta = -\Delta$ (resp., $\Delta\,R\,\Delta = \Delta$) which are such that the projective left D-modules $\operatorname{im}_D(.\Delta)$ (resp., $\ker_D(.\Delta)$) are free with maximal (resp., minimal) rank.

Lemma 8. With the notations of 1 of Lemma 7, if we note $\Omega := R\,\Delta\,R \in D^{q\times p}$, then we have:

(1) The left D-homomorphism
$$\phi:\ \ker_D(.\overline{Q}) \longrightarrow \ker_D(.\overline{P}),\qquad \mu \longmapsto \mu\,\Omega,$$
is an isomorphism and:
$$\phi^{-1}:\ \ker_D(.\overline{P}) \longrightarrow \ker_D(.\overline{Q}),\qquad \lambda \longmapsto \lambda\,\Delta.$$
In particular, $\ker_D(.\overline{P}) \cong \ker_D(.\overline{Q}) = \operatorname{im}_D(.\Delta)$.

(2) The projective left D-module $\ker_D(.\overline{P})$ is free iff the projective left D-module $\ker_D(.\overline{Q})$ is free. Moreover, if $V_1 \in D^{m\times q}$ (resp., $U_1 \in D^{m\times p}$) is a full row rank matrix such that $\ker_D(.\overline{Q}) = D^{1\times m}\,V_1$ (resp., $\ker_D(.\overline{P}) = D^{1\times m}\,U_1$), then $\ker_D(.\overline{P}) = D^{1\times m}\,(V_1\,\Omega)$ (resp., $\ker_D(.\overline{Q}) = D^{1\times m}\,(U_1\,\Delta)$).

Proof. 1. Note first that ϕ is well-defined since:
$$\Omega\,\overline{P} = (R\,\Delta\,R)\,(I_p + \Delta\,R) = R\,\Delta\,R + R\,\Delta\,R\,\Delta\,R = R\,\Delta\,R - R\,\Delta\,R = 0.$$
Let us now prove that ϕ is injective: if $\mu \in \ker_D(.\overline{Q})$, i.e., $\mu = -\mu\,R\,\Delta$, is such that $\mu\,\Omega = 0$, i.e., $\mu\,R\,\Delta\,R = 0$, then $\mu\,R = -\mu\,R\,\Delta\,R = 0$, which yields $\mu = -(\mu\,R)\,\Delta = 0$. Let us now prove that ϕ is surjective. If $\nu \in \ker_D(.\overline{P})$, i.e., $\nu = -\nu\,\Delta\,R$, then $\nu = \nu\,\Delta\,R\,\Delta\,R = (\nu\,\Delta)\,\Omega = \phi(\nu\,\Delta)$. Now, $(\nu\,\Delta)\,\overline{Q} = \nu\,\Delta + \nu\,\Delta\,R\,\Delta = \nu\,\Delta - \nu\,\Delta = 0$, i.e., $\nu\,\Delta \in \ker_D(.\overline{Q})$, which proves that ϕ is surjective, and thus that ϕ is an isomorphism.

Now, let us consider the following left D-homomorphism:
$$\varphi:\ \ker_D(.\overline{P}) \longrightarrow \ker_D(.\overline{Q}),\qquad \lambda \longmapsto \lambda\,\Delta.$$
Then, φ is well-defined since:
$$\Delta\,\overline{Q} = \Delta\,(I_q + R\,\Delta) = \Delta + \Delta\,R\,\Delta = 0.$$
Using the identity $\Omega\,\Delta = R\,\Delta\,R\,\Delta = -R\,\Delta$, we get
$$(\varphi \circ \phi)(\mu) = \varphi(\mu\,\Omega) = \mu\,\Omega\,\Delta = -\mu\,R\,\Delta$$
for all $\mu \in \ker_D(.\overline{Q})$, i.e., for all $\mu \in D^{1\times q}$ satisfying $-\mu\,R\,\Delta = \mu$, which proves that $\varphi \circ \phi = \mathrm{id}_{\ker_D(.\overline{Q})}$. Finally, using the identity $\Delta\,\Omega = \Delta\,R\,\Delta\,R = -\Delta\,R$, we get $(\phi \circ \varphi)(\lambda) = \phi(\lambda\,\Delta) = \lambda\,\Delta\,\Omega = -\lambda\,\Delta\,R$ for all $\lambda \in \ker_D(.\overline{P})$, i.e., for all $\lambda \in D^{1\times p}$ satisfying $-\lambda\,\Delta\,R = \lambda$, which yields $\phi \circ \varphi = \mathrm{id}_{\ker_D(.\overline{P})}$ and proves $\phi^{-1} = \varphi$.

2. By 1 of Corollary 6, if the full row rank matrices $U_1 \in D^{m\times p}$ and $V_1 \in D^{m\times q}$ are such that $\ker_D(.\overline{P}) = D^{1\times m}\,U_1$ and $\ker_D(.\overline{Q}) = D^{1\times m}\,V_1$, then 1 shows that $\ker_D(.\overline{Q}) = D^{1\times m}\,(U_1\,\Delta)$ and $\ker_D(.\overline{P}) = D^{1\times m}\,(V_1\,\Omega)$, and thus, there exist two matrices $Z_1 \in \mathrm{GL}_m(D)$ and $Z_2 \in \mathrm{GL}_m(D)$ such that $U_1 = Z_1\,V_1\,\Omega$ and $V_1 = Z_2\,U_1\,\Delta$.

Example 2. Let us consider again Example 1. Let:
$$\Omega := R\,\Delta\,R = \begin{pmatrix} 0 & -(\partial + a)\,\partial & \partial + a & 0 \\ 0 & -\partial & 1 & 0 \\ -\partial - a & \partial^2 + a\,\partial - k\,a\,\delta - \omega^2 & -2\,\partial - 2\,\zeta\,\omega - a & \omega^2 \end{pmatrix}.$$
Then, we can easily check that the matrices U_1 and V_1 defined in Example 1 satisfy $U_1 = -V_1\,\Omega$ and $V_1 = -U_1\,\Delta$.

Similarly, we can prove the following lemma.

Lemma 9. With the notations of 1 of Lemma 7, if we note $\Omega := R\,\Delta\,R \in D^{q\times q}$, then we have: The following right D-homomorphism
$$\gamma:\ \ker_D(\overline{P}.) \longrightarrow \ker_D(\overline{Q}.),\qquad \nu \longmapsto \Omega\,\nu,$$
is an isomorphism and:
$$\gamma^{-1}:\ \ker_D(\overline{Q}.) \longrightarrow \ker_D(\overline{P}.),\qquad \omega \longmapsto \Delta\,\omega.$$
In particular, $\ker_D(\overline{Q}.) \cong \ker_D(\overline{P}.) = \operatorname{im}_D(\Delta.)$ so that the finitely generated projective right D-module $\ker_D(\overline{Q}.)$ is free iff the finitely generated projective right D-module $\ker_D(\overline{P}.)$ is free.

We now state the main result of the paper providing the relation between Corollary 6 and Theorem 5.

Theorem 10. Let $R \in D^{q\times p}$ be a full row rank matrix, $\Delta \in D^{p\times q}$ a matrix satisfying $\Delta\,R\,\Delta = -\Delta$ and such that the projective left D-modules $\operatorname{im}_D(.\overline{P}) = \ker_D(.\Delta)$, $\ker_D(.\overline{Q}) = \operatorname{im}_D(.\Delta)$, and $\operatorname{im}_D(.\overline{Q})$ are free of rank respectively $p - m$, $l = m$, and $q - l = q - m$, with $1 \leq m \leq q$, where $\overline{P} := I_p + \Delta\,R$ and $\overline{Q} := I_q + R\,\Delta$. Let $\Omega := R\,\Delta\,R$ and the full row rank matrix $U_2 \in D^{(p-m)\times p}$ (resp., $V_1 \in D^{m\times q}$ and $V_2 \in D^{(q-m)\times q}$) define a basis of $\operatorname{im}_D(.\overline{P})$ (resp., $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$), i.e.,
$$\begin{cases} \operatorname{im}_D(.\overline{P}) = D^{1\times(p-m)}\,U_2, \\ \ker_D(.\overline{Q}) = D^{1\times m}\,V_1, \\ \operatorname{im}_D(.\overline{Q}) = D^{1\times(q-m)}\,V_2, \end{cases}$$
and $U_1 = V_1\,\Omega$. Then, we have
$$U := (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D),\ V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D),$$
and if we denote by
$$\begin{cases} U^{-1} := W = (W_1 \quad W_2),\ W_1 \in D^{p\times m},\ W_2 \in D^{p\times(p-m)}, \\ V^{-1} := X = (X_1 \quad X_2),\ \ X_1 \in D^{q\times m},\ X_2 \in D^{q\times(q-m)}, \end{cases} \tag{7}$$
then $V_1\,R\,W_1 \in \mathrm{GL}_m(D)$. Moreover, we have
$$\begin{pmatrix} R & -X_2 \\ U_2 & 0 \end{pmatrix} \begin{pmatrix} W_1\,(V_1\,R\,W_1)^{-1}\,V_1 & W_2 \\ -V_2 & V_2\,R\,W_2 \end{pmatrix} = I_{q+p-m}, \tag{8}$$

which shows that Theorem 5 holds with the matrix

$$\Lambda = X_2 \in D^{q\times(q-m)}$$

which admits the left inverse $V_2 \in D^{(q-m)\times q}$, i.e.:

$$V\,R\,W = \begin{pmatrix} I_m & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix}.$$

Proof. The fact that $U := (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$ and $V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$ was proved in [4, Proposition 4.3]. Now, using (7), Corollary 6 yields

$$V\,R\,W = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix},$$

or equivalently,

$$R\,W = X \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix},$$

with $V_1\,R\,W_1 \in \mathrm{GL}_m(D)$, which is equivalent to:

$$R\,W_1 = X_1\,(V_1\,R\,W_1), \quad R\,W_2 = X_2\,(V_2\,R\,W_2). \qquad (9)$$

The first equation of (9) yields $X_1 = R\,W_1\,(V_1\,R\,W_1)^{-1}$, which combined with the identity $X_1\,V_1 + X_2\,V_2 = I_q$ gives:

$$R\,(W_1\,(V_1\,R\,W_1)^{-1}\,V_1) - X_2\,(-V_2) = I_q. \qquad (10)$$

Moreover, the identity $U\,W = I_p$ yields:

$$U_2\,W_1 = 0, \quad U_2\,W_2 = I_{p-m}. \qquad (11)$$

Hence, combining (10), the second identity of (9) and (11), we get (8). Now, since D is a noetherian domain, it is *stably finite*, namely, for any $r \in \mathbb{N}$ and for all $A, B \in D^{r\times r}$ satisfying $A\,B = I_r$, we have $B\,A = I_r$, i.e., $A \in \mathrm{GL}_r(D)$ and $B = A^{-1}$ [11], and thus the second matrix in the left-hand side of (8) belongs to $\mathrm{GL}_{q+p-m}(D)$, which shows that Theorem 5 holds with the matrix $\Lambda = X_2 \in D^{q\times(q-m)}$ and the identity $V\,X = I_q$ yields $V_2\,X_2 = V_2\,\Lambda = I_{q-m}$.

A similar result holds for the idempotent 0_M of $\mathrm{end}_D(M)$ defined by $\overline{P} := \Delta\,R$ and $\overline{Q} := R\,\Delta$, where $\Delta\,R\,\Delta = \Delta$.

Example 3. Let us consider again Example 1. Computing the inverses U^{-1} and V^{-1}, we obtain:

$$W_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ -\omega^{-2} & B \end{pmatrix}, \quad W_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\partial \\ -\omega^{-2}\,(\partial + a) & C \end{pmatrix},$$

$$B = \omega^{-2}\,(a - 2\,\zeta\,\omega), \quad C = -\omega^{-2}\,(\partial^2 + 2\,\zeta\,\omega\,\partial + k\,a\,\delta + \omega^2),$$

$$X_1 = \begin{pmatrix} 0 & \partial + a \\ 0 & 1 \\ 1 & -(\partial + a) \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Then, (8) yields the following identity:

$$\begin{pmatrix} \partial + a & k\,a\,\delta & 0 & 0 & 1 \\ 0 & \partial & -1 & 0 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 & -1 \\ -1 & \partial & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -\partial \\ -\omega^{-2} & \omega^{-2}\,(a - 2\,\zeta\,\omega) & -\omega^{-2} & -\omega^{-2}\,(\partial + a) & E \\ 1 & -(\partial + a) & 0 & \partial + a & k\,a\,\delta \end{pmatrix} = I_5,$$

where $E = -\omega^{-2}\,(\partial^2 + 2\,\omega\,\zeta\,\partial + k\,a\,\delta + \omega^2)$.

All the computations can be performed using the packages OreModules [3], OreMorphisms [5], and Serre [8].

Further results and applications of the decomposition and Serre's reduction problems for mathematical systems theory will be developed in the forthcoming paper [7].

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