An elementary proof of the general Q-parametrization of all stabilizing controllers

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- **For more details, see:**

$\star$ A. Q., "On a generalization of the Youla-Kučera parametrization. Part I: The fractional ideal approach to SISO systems", Systems & Control Letters, vol. 50 (2003), no. 2, 135-148.

$\star$ A. Q., "On a generalization of the Youla-Kučera parametrization. Part II: The lattice approach to MIMO systems", to appear in Mathematics of Control, Signals, and Systems.

The fractional representation approach

• (Vidyasagar) **Let A be an algebra of stable SISO plants** having the structure of an integral domain, namely, $a\,b = 0, a \neq 0 \Rightarrow b = 0$.

• Let $K = Q(A)$ be the **field of fractions** of A, i.e.:

$$K = \{n/d \mid 0 \neq d,\, n \in A\}\,.$$

K corresponds to the class of systems.

• **Example**: We can consider A to be:

$$\star\; RH_\infty = \{n/d \,|\, 0 \neq d,\, n \in \mathbb{R}[s], \deg n \leq \deg d, \\ d(s_*) = 0 \Rightarrow \mathrm{Re}(s_*) < 0\},$$

$$\star\; H_\infty(\mathbb{C}_+) = \{f \text{ holomorphic function in } \\ \mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 0\} \text{ bounded w.r.t. } \|\cdot\|_\infty\},$$

$$\star\; \hat{\mathcal{A}} = \{\mathcal{L}(f)(s) + \textstyle\sum_{i=0}^{+\infty} a_i\, e^{-t_i\, s} \,|\, f \in L_1(\mathbb{R}_+) \\ (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\; 0 = t_0 \leq t_1 \leq t_2 \leq \ldots\},$$

$$\star\; W_+ = \left\{\textstyle\sum_{i=0}^{\infty} a_i\, z^i \,|\, \sum_{i=0}^{+\infty} |a_i| < +\infty\right\} \ldots$$

• **Example**: $p = \frac{e^{-s}}{s-1} \in Q(H_\infty(\mathbb{C}_+))$ as we have:

$$p = \frac{n}{d}, \quad n = \frac{e^{-s}}{(s+1)},\; d = \frac{(s-1)}{(s+1)} \in H_\infty(\mathbb{C}_+).$$

Analysis and synthesis problems

- Let A and $K = Q(A)$ be defined as above.

- **Definition**: 1. A transfer matrix $P \in K^{q \times r}$ is said to admit a **left-coprime factorization** if there exist

$$D \in A^{q \times q},\, N \in A^{q \times r},\, X \in A^{q \times q},\, Y \in A^{r \times q}$$

such that $P = D^{-1}\, N$ and $D\, X - N\, Y = I_q$.

2. A transfer matrix $P \in K^{q \times r}$ is said to admit a **right-coprime factorization** if there exist

$$\tilde{D} \in A^{r \times r},\, \tilde{N} \in A^{q \times r},\, \tilde{X} \in A^{r \times r},\, \tilde{Y} \in A^{r \times q}$$

such that $P = \tilde{N}\, \tilde{D}^{-1}$ and $-\tilde{Y}\, \tilde{N} + \tilde{X}\, \tilde{D} = I_r$.

3. A transfer matrix $P \in K^{q \times r}$ is said to admit a **doubly coprime factorization** if P admits a left- and a right-coprime factorization.

4. A plant $P \in K^{q \times r}$ is said to be **internally stabilizable** if there exists a controller $C \in K^{r \times q}$ such that **all the entries of** the closed-loop transfer matrix

$$\begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\, C)^{-1} & (I_q - P\, C)^{-1}\, P \\ (I_r - C\, P)^{-1}\, C & (I_r - C\, P)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} I_q + P\,(I_r - C\, P)^{-1}\, C & P\,(I_r - C\, P)^{-1} \\ (I_r - C\, P)^{-1} & (I_r - C\, P)^{-1} \end{pmatrix}$$

belong to A.

Internal stabilizability

• **Proposition 1**: $P \in K^{q\times r}$ is **internally stabilizable** iff one of the following conditions is satisfied:

1. $\exists\, L = (U^T \quad V^T)^T \in A^{(q+r)\times q}$ such that:

$$\boxed{\begin{array}{ll} a. & L\,P = \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{(q+r)\times r}, \\ b. & (I_q \quad -P)\,L = U - P\,V = I_q, \\ c. & \det U \neq 0. \end{array}}$$

Then, $C = V\,U^{-1}$ is a **stabilizing controller of** P,

$$U = (I_q - P\,C)^{-1}, \quad V = C\,(I_q - P\,C)^{-1}.$$

2. $\exists\, \tilde{L} = (-\tilde{V} \quad \tilde{U}) \in A^{r\times(q+r)}$ such that:

$$\boxed{\begin{array}{ll} a. & P\,\tilde{L} = (-P\,\tilde{V} \quad P\,\tilde{U}) \in A^{q\times(q+r)}, \\ b. & \tilde{L}\begin{pmatrix} P \\ I_r \end{pmatrix} = -\tilde{V}\,P + \tilde{U} = I_r, \\ c. & \det \tilde{U} \neq 0. \end{array}}$$

Then, $C = \tilde{U}^{-1}\,\tilde{V}$ is a **stabilizing controller of** P,

$$\tilde{U} = (I_r - C\,P)^{-1}, \quad \tilde{V} = (I_r - C\,P)^{-1}\,C.$$

Internal stabilizability

• **Corollary 1**: P is **internally stabilizable** iff there exists $V \in A^{r\times q}$ such that we have:

$$\boxed{\left\{\begin{array}{l} V\,P \in A^{r\times r}, \\ P\,V \in A^{q\times q}, \\ (P\,V + I_q)\,P = P\,(V\,P + I_r) \in A^{q\times r}. \end{array}\right.}$$

Then, the controller $C \in K^{r\times q}$ defined by

$$\boxed{C = V\,(P\,V + I_q)^{-1} = (V\,P + I_r)^{-1}\,V}$$

internally stabilizes P and we have:

$$V = C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C.$$

• **Corollary 2**: We have:

1. If $P \in K^{q\times r}$ admits a **left-coprime factorization** $P = D^{-1}\,N,\ D\,X - N\,Y = I_q,\ \det X \neq 0$, then $L = ((X\,D)^T \quad (Y\,D)^T)^T$ satisfies 1 of Proposition 1 and $C = Y\,X^{-1}$ **internally stabilizes** P.

2. If $P \in K^{q\times r}$ admits a **right-coprime factorization** $P = \tilde{N}\,\tilde{D}^{-1},\ -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r,\ \det \tilde{X} \neq 0$, then $\tilde{L} = (-\tilde{D}\,\tilde{Y} \quad \tilde{D}\,\tilde{X})$ satisfies 2 of Proposition 1 and $C = \tilde{X}^{-1}\,\tilde{Y}$ **internally stabilizes** P.

Open questions

• **The existence of a left-/right-coprime factorization is a necessary but not generally a sufficient condition for internal stabilizability.**

• These two concepts are known to be **equivalent** for the rings RH_∞ and $H_\infty(\mathbb{C}_+)$.

• But, does internal stabilizability imply the existence of doubly coprime factorizations over the rings

$$\hat{\mathcal{A}} = \{\mathcal{L}(f)(s) + \textstyle\sum_{i=0}^{+\infty} a_i\, e^{-t_i\, s} \mid f \in L_1(\mathbb{R}_+)$$
$$(a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\ 0 = t_0 \leq t_1 \leq t_2 \leq \ldots\},$$

(ring of BIBO-stable time-invariant systems)

$$W_+ = \left\{\textstyle\sum_{i=0}^{\infty} a_i\, z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty\right\},$$

(ring of BIBO-stable causal digital filters)?

• If it is not the case:

Is it possible to parametrize all stabilizing controllers of a stabilizable plant which does not admit doubly coprime factorizations?

• We now solve this last question.

General Q-parametrization

• **Theorem**: Let $P \in K^{q \times r}$ be a **stabilizable plant. All stabilizing controllers of P have the form**

$$\boxed{\begin{aligned} C(Q) &= (V + Q)\,(U + P\,Q)^{-1} \\ &= (\tilde{U} + Q\,P)^{-1}\,(\tilde{V} + Q), \end{aligned} \qquad (\star)}$$

where C_* is **a stabilizing controller of** P,

$$\begin{cases} U = (I_q - P\,C_*)^{-1} \in A^{q \times q}, \\ V = C_*\,(I_q - P\,C_*)^{-1} \in A^{r \times q}, \\ \tilde{U} = (I_r - C_*\,P)^{-1} \in A^{r \times q}, \\ \tilde{V} = (I_r - C_*\,P)^{-1}\,C_* \in A^{r \times r}. \end{cases}$$

and Q is **any matrix which belongs to**

$$\boxed{\begin{aligned} \Omega = \{L \in A^{r \times q} \mid \ & L\,P \in A^{r \times r},\ P\,L \in A^{q \times q}, \\ & P\,L\,P \in A^{q \times r}\} \end{aligned}}$$

satisfying $\det(U + P\,Q) \neq 0,\ \det(\tilde{U} + Q\,P) \neq 0$.

• The general Q-parametrization is **a linear fractional transformation** in the free parameter $Q \in \Omega$.

• We only need the knowledge of a stabilizing contoller $C_\star$ of P. **The existence of a doubly coprime factorization is not required**.

Set of free parameters

• **Proposition**: Let $P \in K^{q\times r}$ be a internally stabilizable plant, $C_\star \in K^{r\times q}$ a stabilizing controller and:

$$\begin{cases} L = \begin{pmatrix} (I_q - P\,C)^{-1} \\ C_\star\,(I_q - P\,C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \tilde{L} = (-(I_r - C_\star\,P)^{-1}\,C_\star \quad (I_r - C_\star\,P)^{-1}) \in A^{r\times(q+r)}. \end{cases}$$

The **set of free parameters** of the parametrization ($\star$)

$$\Omega = \{L \in A^{r\times q}\ |L\,P \in A^{r\times r},\ P\,L \in A^{q\times q},\ P\,L\,P \in A^{q\times r}\}$$

then satisfies:

$$\boxed{\Omega = \tilde{L}\,A^{(q+r)\times(q+r)}\,L.}$$

Equivalently, if we denote by $\tilde{L}_i$ **the** i^{th} **column** of $\tilde{L}$ and L^j **the** j^{th} **row of** L, we then have

$$\boxed{\Omega = \sum_{i,j=1}^{q+r} A\,(\tilde{L}_i\,L^j)}$$

showing that $\{\tilde{L}_i\,L^j\}_{i,j=1,\ldots,q+r}$ is a **set of generators of the A-module Ω.**

Youla-Kučera parametrization

• **Corollary**: Let $P \in Q(A)^{q\times r}$ be a plant which admits a **doubly coprime factorization**:

$$\left\{\begin{array}{l} P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}, \\ \left(\begin{array}{cc} D & -N \\ -\tilde{Y} & \tilde{X} \end{array}\right)\left(\begin{array}{cc} X & \tilde{N} \\ Y & \tilde{D} \end{array}\right) = I_{q+r}. \end{array}\right.$$

Then, the set Ω of free parameters of $(\star)$ satisfies:

$$\boxed{\Omega = \tilde{D}\, A^{r\times q}\, D.}$$

Hence, subsituting $Q = \tilde{D}\Lambda\, D$ and (see Corollary 2)

$$U = X\, D, \quad V = Y\, D, \quad \tilde{V} = \tilde{D}\, \tilde{Y}, \quad \tilde{U} = \tilde{D}\, \tilde{X},$$

into the general Q-parametrization $(\star)$, we obtain that **all stabilizing controllers of P have the form**

$$\boxed{\begin{array}{ll} C(Q) & = (Y + \tilde{D}\,\Lambda)\,(X + \tilde{N}\,\Lambda)^{-1} \\ & = (\tilde{X} + \Lambda\, N)^{-1}\,(\tilde{Y} + \Lambda\, D), \end{array}}$$

where Λ is **any element of $A^{r\times q}$** satisfying:

$$\det(X + \tilde{N}\,\Lambda) \neq 0, \quad \det(\tilde{X} + \Lambda\, N) \neq 0.$$