An elementary proof of the general \( Q \)-parametrization of all stabilizing controllers

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• For more details, see:


The fractional representation approach

• (Vidyasagar) Let $A$ be an algebra of stable SISO plants having the structure of an integral domain, namely, $a b = 0, a \neq 0 \Rightarrow b = 0$.

• Let $K = Q(A)$ be the field of fractions of $A$, i.e.:
  
  
  $K = \{ n/d \mid 0 \neq d, n \in A \}$.

  $K$ corresponds to the class of systems.

• Example: We can consider $A$ to be:

  $\star RH_\infty = \{ n/d \mid 0 \neq d, n \in \mathbb{R}[s], \deg n \leq \deg d, d(s_*) = 0 \Rightarrow \text{Re}(s_*) < 0 \}$,

  $\star H_\infty(\mathbb{C}_+) = \{ f \text{ holomorphic function in } \}
  \mathbb{C}_+ = \{ s \in \mathbb{C} \mid \text{Re } s > 0 \} \text{ bounded w.r.t. } || \cdot ||_\infty \}$,

  $\star \tilde{A} = \{ \mathcal{L}(f)(s) + \sum_{i=0}^{+\infty} a_i e^{-t_is} \mid f \in L_1(\mathbb{R}_+) \\
  (a_i)_{i\geq 0} \in l_1(\mathbb{Z}_+), 0 = t_0 \leq t_1 \leq t_2 \leq \ldots \}$,

  $\star W_+ = \{ \sum_{i=0}^{\infty} a_i z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty \}$ ...

• Example: $p = \frac{e^{-s}}{s-1} \in Q(H_\infty(\mathbb{C}_+))$ as we have:

  $p = \frac{n}{d}, \quad n = \frac{e^{-s}}{(s+1)}, \quad d = \frac{(s-1)}{(s+1)} \in H_\infty(\mathbb{C}_+)$. 
Analysis and synthesis problems

- Let $A$ and $K = Q(A)$ be defined as above.

- **Definition:** 1. A transfer matrix $P \in K^{q \times r}$ is said to admit a **left-coprime factorization** if there exist

\[ D \in A^{q \times q}, \ N \in A^{q \times r}, \ X \in A^{q \times q}, \ Y \in A^{r \times q} \]

such that $P = D^{-1} N$ and $D X - N Y = I_q$.

2. A transfer matrix $P \in K^{q \times r}$ is said to admit a **right-coprime factorization** if there exist

\[ \tilde{D} \in A^{r \times r}, \ \tilde{N} \in A^{q \times r}, \ \tilde{X} \in A^{r \times r}, \ \tilde{Y} \in A^{r \times q} \]

such that $P = \tilde{N} \tilde{D}^{-1}$ and $-\tilde{Y} \tilde{N} + \tilde{X} \tilde{D} = I_r$.

3. A transfer matrix $P \in K^{q \times r}$ is said to admit a **doubly coprime factorization** if $P$ admits a left- and a right-coprime factorization.

4. A plant $P \in K^{q \times r}$ is said to be **internally stabilizable** if there exists a controller $C \in K^{r \times q}$ such that all the entries of the closed-loop transfer matrix

\[
\begin{pmatrix}
I_q & -P \\
-C & I_r
\end{pmatrix}^{-1} = \begin{pmatrix}
(I_q - PC)^{-1} & (I_q - PC)^{-1} P \\
(I_r - CP)^{-1} C & (I_r - CP)^{-1}
\end{pmatrix}
\]

belong to $A$. 
Internal stabilizability

• Proposition 1: $P \in K^{q \times r}$ is internally stabilizable iff one of the following conditions is satisfied:

1. $\exists L = (U^T \ V^T)^T \in A^{(q+r) \times q}$ such that:
   
   a. $LP = \begin{pmatrix} UP \\ VP \end{pmatrix} \in A^{(q+r) \times r}$,
   
   b. $(I_q - P) L = U - PV = I_q$,
   
   c. $\det U \neq 0$.

Then, $C = VU^{-1}$ is a stabilizing controller of $P$,

$U = (I_q - PC)^{-1}, \ V = C(I_q - PC)^{-1}$.

2. $\exists \tilde{L} = (-\tilde{V} \ \tilde{U}) \in A^{r \times (q+r)}$ such that:

   a. $P \tilde{L} = (-P \tilde{V} \ P \tilde{U}) \in A^{q \times (q+r)}$,

   b. $\tilde{L} \begin{pmatrix} P \\ Ir \end{pmatrix} = -\tilde{V}P + \tilde{U} = Ir$,

   c. $\det \tilde{U} \neq 0$.

Then, $C = \tilde{U}^{-1} \tilde{V}$ is a stabilizing controller of $P$,

$\tilde{U} = (Ir - CP)^{-1}, \ \tilde{V} = (Ir - CP)^{-1}C$. 
**Internal stabilizability**

- **Corollary 1**: $P$ is **internally stabilizable** iff there exists $V \in A^{r \times q}$ such that we have:

\[
\begin{cases}
V P \in A^{r \times r}, \\
P V \in A^{q \times q}, \\
(PV + I_q) P = P (VP + I_r) \in A^{q \times r}.
\end{cases}
\]

Then, the controller $C \in K^{r \times q}$ defined by

\[
C = V (PV + I_q)^{-1} = (VP + I_r)^{-1} V
\]

**internally stabilizes** $P$ and we have:

\[
V = C (I_q - PC)^{-1} = (I_r - CP)^{-1} C.
\]

- **Corollary 2**: We have:

1. If $P \in K^{q \times r}$ admits a **left-coprime factorization** $P = D^{-1} N$, $DX - NY = I_q$, $\det X \neq 0$, then $L = ((X D)^T (Y D)^T)^T$ satisfies 1 of Proposition 1 and $C = Y X^{-1}$ **internally stabilizes** $P$.

2. If $P \in K^{q \times r}$ admits a **right-coprime factorization** $P = \tilde{N} \tilde{D}^{-1}$, $-\tilde{Y} \tilde{N} + \tilde{X} \tilde{D} = I_r$, $\det \tilde{X} \neq 0$, then $\tilde{L} = (-\tilde{D} \tilde{Y} \quad \tilde{D} \tilde{X})$ satisfies 2 of Proposition 1 and $C = \tilde{X}^{-1} \tilde{Y}$ **internally stabilizes** $P$. 
Open questions

• The existence of a left-/right-coprime factorization is a necessary but not generally a sufficient condition for internal stabilizability.

• These two concepts are known to be equivalent for the rings $RH_\infty$ and $H_\infty(\mathbb{C}_+)$.

• But, does internal stabilizability imply the existence of doubly coprime factorizations over the rings

\[
\tilde{A} = \{ \mathcal{L}(f)(s) + \sum_{i=0}^{+\infty} a_i e^{-t_i s} \mid f \in L_1(\mathbb{R}_+) \\
(a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+), \ 0 = t_0 \leq t_1 \leq t_2 \leq \ldots \},
\]

(ring of BIBO-stable time-invariant systems)

\[
W_+ = \left\{ \sum_{i=0}^{\infty} a_i z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty \right\},
\]

(ring of BIBO-stable causal digital filters)?

• If it is not the case:

**Is it possible to parametrize all stabilizing controllers of a stabilizable plant which does not admit doubly coprime factorizations?**

• We now solve this last question.
Theorem: Let $P \in K^{q \times r}$ be a stabilizable plant. All stabilizing controllers of $P$ have the form

$$C(Q) = (V + Q)(U + PQ)^{-1} = (\tilde{U} + QP)^{-1}(\tilde{V} + Q),$$

where $C_*$ is a stabilizing controller of $P$,

$$U = (I_q - PC_*)^{-1} \in A^{q \times q},$$

$$V = C_*(I_q - PC_*)^{-1} \in A^{r \times q},$$

$$\tilde{U} = (I_r - CP)^{-1} \in A^{r \times q},$$

$$\tilde{V} = (I_r - CP)^{-1} C_* \in A^{r \times r}.$$ 

and $Q$ is any matrix which belongs to

$$\Omega = \{ L \in A^{r \times q} \mid LP \in A^{r \times r}, PL \in A^{q \times q}, PLP \in A^{q \times r} \}$$

satisfying $\det(U + PQ) \neq 0$, $\det(\tilde{U} + QP) \neq 0$.

- The general $Q$-parametrization is a linear fractional transformation in the free parameter $Q \in \Omega$.

- We only need the knowledge of a stabilizing controller $C_*$ of $P$. The existence of a doubly coprime factorization is not required.
Set of free parameters

• Proposition: Let \( P \in K^{q \times r} \) be a internally stabilizable plant, \( C_* \in K^{r \times q} \) a stabilizing controller and:

\[
\begin{align*}
L &= \begin{pmatrix} (I_q - PC)^{-1} \\ C_*(I_q - PC^*)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\
\tilde{L} &= \begin{pmatrix} -(I_r - C_*P)^{-1} C_* \\ (I_r - C_*P)^{-1} \end{pmatrix} \in A^{r \times (q+r)}.
\end{align*}
\]

The set of free parameters of the parametrization (*)

\[\Omega = \{ L \in A^{r \times q} | LP \in A^{r \times r}, PL \in A^{q \times q}, PLP \in A^{q \times r} \}\]

then satisfies:

\[\Omega = \tilde{L} A^{(q+r)\times(q+r)} L.\]

Equivalently, if we denote by \( \tilde{L}_i \) the \( i \)th column of \( \tilde{L} \) and \( L^j \) the \( j \)th row of \( L \), we then have

\[\Omega = \sum_{i,j=1}^{q+r} A (\tilde{L}_i L^j)\]

showing that \( \{ \tilde{L}_i L^j \}_{i,j=1,\ldots,q+r} \) is a set of generators of the \( A \)-module \( \Omega \).
Youla-Kučera parametrization

- **Corollary**: Let \( P \in Q(A)^{q \times r} \) be a plant which admits a **doubly coprime factorization**:

\[
\begin{align*}
\begin{cases}
P = D^{-1} N = \tilde{N} \tilde{D}^{-1}, \\
\begin{pmatrix}
D & -N \\
-\tilde{Y} & \tilde{X}
\end{pmatrix}
\begin{pmatrix}
X & \tilde{N} \\
Y & \tilde{D}
\end{pmatrix} = I_{q+r}.
\end{cases}
\end{align*}
\]

Then, the set \( \Omega \) of free parameters of \( (\star) \) satisfies:

\[
\Omega = \tilde{D} A^{r \times q} \tilde{D}.
\]

Hence, subsituting \( Q = \tilde{D} \Lambda \tilde{D} \) and (see Corollary 2)

\[
U = XD, \quad V = YD, \quad \tilde{V} = \tilde{D} \tilde{Y}, \quad \tilde{U} = \tilde{D} \tilde{X},
\]

into the general \( Q \)-parametrization \( (\star) \), we obtain that **all stabilizing controllers of** \( P \) **have the form**

\[
C(Q) = (Y + \tilde{D} \Lambda) (X + \tilde{N} \Lambda)^{-1} = (\tilde{X} + \Lambda \tilde{N})^{-1} (\tilde{Y} + \Lambda D),
\]

where \( \Lambda \) is **any element** of \( A^{r \times q} \) satisfying:

\[
\det(X + \tilde{N} \Lambda) \neq 0, \quad \det(\tilde{X} + \Lambda \tilde{N}) \neq 0.
\]