Formal Obstruction to the Controllability of Partial Differential Control Systems

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ABSTRACT

In this paper, we study the controllability of ordinary and partial differential control systems with variable coefficients. We recall that controllability is a "built-in" property of these systems and thus only depends on its coefficients. On testing controllability, we have to compute certain ranks and, depending on the vanishing or non vanishing of the corresponding determinants, we are dealing with different cases for the system to be controllable or not. We represent all these conditions by means of a "tree". For ordinary differential control systems with variable coefficients, controllability can be checked in one step by considering its formal adjoint. Thus it only depends on a single tree. Whereas, for partial differential control system, it has to be done in two stages, leading to the construction of a second tree. Many explicit examples will illustrate this new approach.

Keywords: Controllability, elimination theory, formal integrability, commutative algebra, duality theory.

1 Introduction

Controllability is one of the key-concepts of control theory. Its earliest definition and test for time-invariant linear systems go back to Kalman's pioneering work. Recent improvements [1, 4, 5] have shown that it was a "built-in" property of a control system and thus it was not based on any separation of the control variables between inputs and outputs. As a by-product, controllability will depend on the coefficients of the control system. So, we are led to investigate two interesting questions:

- What are the conditions for a control system with variable coefficients to be controllable or not?
- Does an arbitrary small change on the coefficients have an effect on controllability as a structural property?

Recently in [4, 5, 6], it has been suggested to revisit most of the concepts of ordinary differential control theory (OD control theory) within the framework of partial differential control theory (PD control theory), that is, linear or nonlinear input/output relations defined by systems of partial differential equations.

The first question is related to the existence of a constructive test for checking controllability of OD or PD control systems with variable coefficients. Such a test has been found recently in [5]. The new formal integrabillity and duality methods involved in [4, 5, 6] permit a better understanding of the controllability concept. The second question is to know whether controllability is a generic property (almost always verified) for OD and PD control systems with variable coefficients and leads to question about robustness. The generic characterization of controllability property is a question quite developed in the literature. In his book [8], Wonham has demonstrated that it was true for Kalman systems. Friedland gave in [3] some very interesting examples. Let us adapt one of them. Consider the following linear control system describing the motion $(y^1,\ y^2)$ of a pair of masses $m_1 = 1$, $m_2 = 1$ coupled with a spring of constant $k = 1$

$$\left\{ \begin{array}{l} \ddot{y}^1 + y^1 - y^2 + \alpha u = 0 \\ \ddot{y}^2 + y^2 - y^1 - u = 0, \end{array} \right.$$

where u is an external force and α is a constant parameter. What are the conditions on α for this system to be controllable or not? This simple example may be treated through usual tests. Depending on some determinants wich are null or not, we are led to investigate different cases. The test developped in [5, 6], that we shall recall, deals with a more general situation which even allows to treat time-varying linear systems obtained from nonlinear system by means of linearization. In the case of PD control systems, we have to consider successively five PD operators, each determining the next one. We shall see that the two studies of formal integrability in the test will give rise to a tree (PD equations and inequations on the coefficients of the system) of various possibilities. Surprisingly, for linear OD control systems with variable coefficients, controllability only depends on a single tree whereas, strange though it may appear, for PD control systems we have to build a second one. Many examples will illustrate this new approach.

2 Controllability

2.1 Recent Improvements

We will always consider OD or PD algebraic control system with coefficients in a ground *differential ring* A with n *derivations* $\partial_1, \ldots, \partial_n$. We form the ring of linear partial operators with coefficients in k and we denote it by $D = A[\partial_1, \ldots, \partial_n]$. It is in general a non commutative ring satisfying the Ore poperty ($\forall p, q \in D$, $\exists a, b \in D$ such that $ap = bq$). We introduce the *differential indeterminates* y^l where $l = 1, \ldots, m$. The left D-module spanned by the set $y = \{y^l \mid l = 1, \ldots, m\}$ is written $[y]$ and an element of $[y]$ looks like $\sum_{\text{finite}} a_l^\mu \partial_\mu y^l$ where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index with length $\mid \mu \mid = \mu_1 + \ldots + \mu_n$. We form the finitely generated left D-module $[R]$ of linear differential consequences of the system generators and denote $\mathcal{M} = [y]/[R]$ the differential residual module.

We call *observable* any linear combination with coefficients in k of the system variables (inputs and outputs) and their derivatives of the control system or, in another words, any element of $\mathcal{M}$. A free observable is an observable which does not satisfy any PD or OD equation. The following definition of the controllability is proposed in [4]:

Definition 1 *A system is controllable if and only if every observable is free.*

A characterization of the controllability in terms of the differential algebraic closure is also given in [4]. The equivalent notion of *torsion-free module* is used in [1]. A *torsion element* m of a module $\mathcal{M}$ over a integral domain D is an element which satisfies:

$$\exists\, a \neq 0 \in D : a\, m = 0,$$

and $\mathcal{M}$ is called *torsion-free* if it has no torsion elements else 0. We denote $\tau(\mathcal{M})$ the submodule of $\mathcal{M}$ made by all the torsion elements and we recall that the module $\mathcal{M}/\tau(\mathcal{M})$ is torsion-free, a result leading to the concept of minimal realization.

Example 1 Let us consider again our example

$$\left\{ \begin{array}{l} \ddot{y}^1 + y^1 - y^2 + \alpha u = 0, \\ \ddot{y}^2 + y^2 - y^1 - u = 0. \end{array} \right.$$

- For $\alpha = -1$, if we substract the first equation from the second, we find an element of torsion τ_1 satisfying

$$\begin{cases} \tau_1 = y^1 - y^2 \\ (\frac{d^2}{dt^2} + 2)\tau_1 = 0 \end{cases}$$

- In the case where $\alpha = 1$, if we add the first equation to the second, we find

$$\begin{cases} \tau_2 = y^1 + y^2, \\ (\frac{d^2}{dt^2})\tau_2 = 0. \end{cases}$$

If the ring D is principal (for example $k[\frac{d}{dt}]$), the module $\mathcal{M}$ is torsion-free if and only if $\mathcal{M}$ is free, that is to say, if there exits a basis of the module (recall that it is not always true for a module). Some authors call this basis *flat outputs* or *linearizing outputs* [2] and compute them by transforming the system into its Brunosky canonical form. However, for non principal rings (for example $k[\partial_1, \dots, \partial_n], n \geq 2$), we have the following module inclusions:

$$\text{free} \subsetneq \text{projective} \subsetneq \text{torsion-free}.$$

Thus for non principal rings, a torsion-free module is no more in general a free module. An D-module $\mathcal{M}$ is *projective* if there exists an D-module $\mathcal{M}'$ such as the direct sum $\mathcal{M} \oplus \mathcal{M}'$ is free. Quillen and Suslin have independently demonstrated in 1976 the Serre conjecture of 1950 claiming that over a polynomial ring $k[\chi_1, \dots, \chi_n]$ where k is a field, any projective module is also a free module [7].

Recently, we find in [6] a formal test permitting to know if a finitely generated D-module $\mathcal{M}$ is torsion-free (i.e. controllable). The test automatically gives a parametrization if the system is controllable, otherwise it exhibits torsion elements. From a geometric point of view, a linear PD control system may be defined by a linear PD operator $\mathcal{D}_1$ acting on the control variables and we define its set of solutions by $\mathcal{D}_1\eta = 0$. We recall the duality of differential operators [4, 5]. If $\mathcal{D}$ is a linear differential operator, we denote its formal adjoint by $\tilde{\mathcal{D}}$ and define it by the following rules:

- The adjoint of a scalar matrix (zero order operator) is the transposed matrix.
- The adjoint of ∂_i is $-\partial_i$.
- For a couple of linear PD operators (P, Q) that can be composed, then $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

We have the relation

$$\mu^t \mathcal{D}\xi = (\tilde{\mathcal{D}}\mu)^t \xi + d(),$$

with d the exterior derivative (divergence in Stokes formula).

We call an operator $\mathcal{D}_1$ *parametrizable* if there exists a set of arbitrary functions $\xi = (\xi^1, \dots, \xi^r)$ and a linear operator $\mathcal{D}$ such as all the compatibility conditions of the inhomogeneous system $\mathcal{D}\xi = \eta$ are exactly generated by $\mathcal{D}_1\eta = 0$.

Theorem 1 *A linear PD control system is controllable if and only if it is parametrizable* [4, 5].

We describe the formal test for checking controllability of an operator $\mathcal{D}_1$:

1. Start with $\mathcal{D}_1$
2. Construct its adjoint $\tilde{\mathcal{D}}_1$.
3. Find the compatibility conditions of $\tilde{\mathcal{D}}_1\lambda = \mu$ and denote this operator by $\tilde{\mathcal{D}}$.
4. Construct its adjoint $\mathcal{D}$.
5. Find the compatibility conditions of $\mathcal{D}\xi = \eta$ and let this operator be $\mathcal{D}'_1$.

We are led to two different cases. If $\mathcal{D}'_1 = \mathcal{D}_1$ then the system $\mathcal{D}_1$ is torsion-free (i.e. controllable) and $\mathcal{D}$ is a parametrization of $\mathcal{D}_1$. Else, the operator $\mathcal{D}_1$ is among (not exactly) the compatibility conditions $\mathcal{D}'_1$ of $\mathcal{D}$ and the torsion elements of $\mathcal{M}$ are all the new compatibility conditions modulo the equations $\mathcal{D}_1\eta = 0$. Let use remark that the geometrical duality has nothing to do with the functional duality as Pommaret noticed in [5]. We recall that an operator $\mathcal{D}_1$ is *surjective* if and only if the equations $\mathcal{D}_1\eta = 0$ are differentially independent and *injective* if $\mathcal{D}_1\eta = 0 \Rightarrow \eta = 0$ [4]. We have the following theorem:

Theorem 2 *A surjective linear OD operator is controllable if and only if its adjoint is injective* [4, 5].

Proof In a principal ring, the notion of torsion-free and projective module are equivalent. Thus a linear OD control system is controllable if and only if the module $\mathcal{M}$ is projective. Let $\mathcal{D}_1$ be a surjective operator and $\tilde{\mathcal{D}}_1$ is dual. If $\tilde{\mathcal{D}}_1$ is an injective operator, then bringing it to formal integrability, it provides a left-inverse $\tilde{\mathcal{P}}_1$ of $\tilde{\mathcal{D}}_1$ (*differential lift*). We have $\tilde{\mathcal{P}}_1 \circ \tilde{\mathcal{D}}_1 = \mathrm{id}_\lambda$, then taking the adjoint, we obtain $\mathcal{D}_1 \circ \mathcal{P}_1 = \mathrm{id}_\eta$ and $\mathcal{D}_1$ admits a right-inverse which thus characterizes a projective module [6]. So the system is controllable. Conversely, if $\tilde{\mathcal{D}}_1$ is not injective then we can find a test vector λ which satisfies $\tilde{\mathcal{D}}_1\lambda = 0$. Then $\lambda^t \mathcal{D}_1 \eta$ is a total derivative of an observable which is therefore a torsion element because its derivative is null as soon as η is a solution of the system. Then the system is not controllable.

Example 2 We take our first exemple. First of all, let set $\eta = (\eta^1, \eta^2, \eta^3)$ with $\eta^1 = y^1, \eta^2 = y^2, \eta^3 = u$ in order to mix together inputs and outputs. The system can be rewritten as

$$\begin{cases} \ddot{\eta}^1 + \eta^1 - \eta^2 + \alpha\eta^3 = 0, \\ \ddot{\eta}^2 + \eta^2 - \eta^1 - \eta^3 = 0. \end{cases}$$

Multiplying it by a row vector $\lambda = (\lambda_1, \lambda_2)$ and integrating by part, we obtain:

$$\begin{cases} \eta^1 \Rightarrow \ddot{\lambda}_1 + \lambda_1 - \lambda_2 = \mu_1, \\ \eta^2 \Rightarrow \ddot{\lambda}_2 + \lambda_2 - \lambda_1 = \mu_2, \\ \eta^3 \Rightarrow -\lambda_2 + \alpha\lambda_1 = \mu_3. \end{cases}$$

We now study the formal integrability of the corresponding homogenous system. Differentiating the zero-order equation and substituting, we obtain

$$(\alpha + 1)(\alpha - 1)\lambda_1 = 0,$$

and thus the operator $\tilde{\mathcal{D}}_1$ is injective if and only if

$$\begin{cases} \alpha \neq -1, \\ \alpha \neq 1. \end{cases}$$

We can verify that the torsion elements are exactly those of example 1.

2.2 Controllability of Systems with Variable Coefficients

We have seen a formal test for checking controllability of both OD and PD linear control systems. Thanks to theorem 1, for surjective OD linear control systems, we only have to study formal integrability of the dual $\tilde{\mathcal{D}}_1$ and thus controllability only depends on one tree of formal integrability conditions.

Example 3 Consider the following OD control system:

$$\ddot{y} + \alpha(t)\dot{y} + \dot{\alpha}(t)y = \ddot{u} - \beta u,$$

where α is time-varying parameter and β is a constant. Thus $\mathcal{D}_1$ is defined by

$$\ddot{\eta}^1 + \alpha(t)\dot{\eta}^1 + \dot{\alpha}(t)\eta^1 - \ddot{\eta}^2 + \beta\eta^2 = 0.$$

We propose to find the conditions on α and β for $\mathcal{D}_1$ to be controllable or not. We construct the operator $\tilde{\mathcal{D}}_1$ (be careful, the adjoint of $\alpha\dot{y}$ is $-\alpha\dot{\lambda} - \dot{\alpha}\lambda$)

$$\begin{cases} \ddot{\lambda} - \beta\lambda = \mu_2, \\ \ddot{\lambda} - \alpha(t)\dot{\lambda} = \mu_1, \end{cases}$$

and investiguate whether it is injective. So let us put $(\mu_1, \mu_2) = (0, 0)$ and rewrite the corresponding system into the following form

$$\Pi \begin{cases} \ddot{\lambda} - \beta\lambda = 0, \\ \alpha(t)\dot{\lambda} - \beta\lambda = 0. \end{cases}$$

Let us bring Π to formal integrability:

1. If $\alpha(t) = 0$ then Π is defined by

$$\begin{cases} \ddot{\lambda} - \beta\lambda = 0, \\ \beta\lambda = 0. \end{cases}$$

   (a) If $\beta = 0$ then Π is equal to $\ddot{\lambda} = 0$ and thus $\tilde{\mathcal{D}}_1$ is not injective. For these values of the parameters $\mathcal{D}_1$ is not controllable.

   (b) If $\beta \neq 0$ then Π is equal to $\lambda = 0$ and thus $\tilde{\mathcal{D}}_1$ is injective and $\mathcal{D}_1$ is controllable for these values of the parameters.

2. If $\alpha(t) \neq 0$ then differentiating the second equation of Π and substituting it, we obtain

$$\left\{\begin{array}{l} \ddot{\lambda} - \beta\lambda = 0, \\ \alpha(t)\dot{\lambda} - \beta\lambda = 0, \\ \beta(\dot{\alpha(t)} + \alpha(t)^2 - \beta)\lambda = 0. \end{array}\right.$$

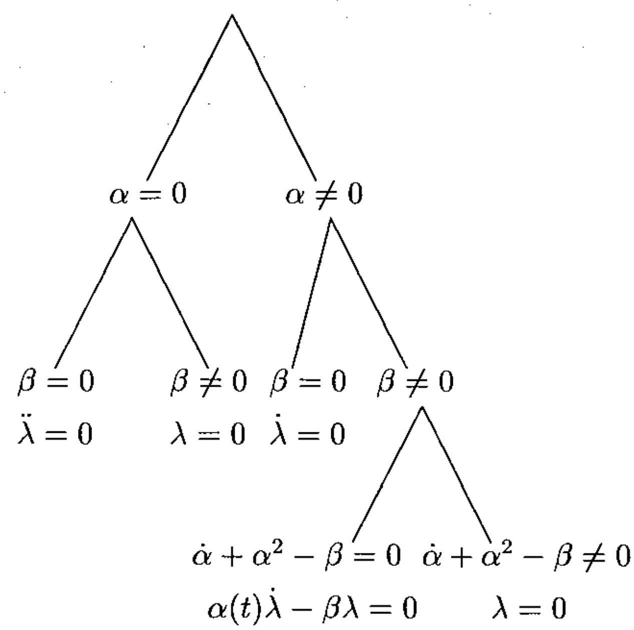
   (a) If $\beta = 0$ then Π is equal to $\dot{\lambda} = 0$ and thus $\tilde{\mathcal{D}}_1$ is not injective. $\mathcal{D}_1$ is not controllable for these values of parameters.

   (b) If $\beta \neq 0$ then

      i. If $\dot{\alpha(t)} + \alpha(t)^2 - \beta = 0$ then Π is equal to $\alpha(t)\dot{\lambda} - \beta\lambda = 0$ and thus $\mathcal{D}_1$ is not controllable for these values of parameters.

      ii. If $\dot{\alpha(t)} + \alpha(t)^2 - \beta \neq 0$ then Π is equal to $\lambda = 0$ and thus $\mathcal{D}_1$ is controllable.

We obtain the following tree of integrability conditions:



We remark that in this example, the system is controllable for the leaf corresponding to the most generic situation $\alpha \neq 0, \beta \neq 0, \dot{\alpha} + \alpha^2 - \beta \neq 0$ as well as for a less generic one $\alpha = 0, \beta \neq 0$. Thus only these two leaves are robust in the sense of the second question of the introduction.

For linear PD control systems, controllability depends on two studies of formal integrability ($\tilde{\mathcal{D}}_1$ and $\mathcal{D}$). Thus it depends on two trees of integrability conditions. Let us give a example (such examples are very rare !).

Example 4 Let us consider the finite transformation $y = f(x)$ satisfying the Pfaffian system:

$$dy^3 - a(y)dy^1 = \rho(x)(dx^3 - a(x)dx^1),$$

where $a(x)$ only depends on x^2. Linearizing such a transformation around the identity by setting $y = x + t\xi(x) + \ldots$ and making $t \to 0$, after eliminating $\rho(x)$, we discover easily that infinitesimal transformations are defined by the kernel of the differential system $\mathcal{D}\xi = \eta$ as follows [4]:

$$\left\{\begin{array}{lcl} -a(x)\partial_1\xi^1 + \partial_1\xi^3 + \frac{1}{2}a(x)(\partial_1\xi^1 + \partial_2\xi^2 & & \\ +\partial_3\xi^3) - \xi^2\partial_2 a(x) & = & \eta^1, \\ -a(x)\partial_2\xi^1 + \partial_2\xi^3 & = & \eta^2, \\ -a(x)\partial_3\xi^1 + \partial_3\xi^3 - \frac{1}{2}(\partial_1\xi^1 + \partial_2\xi^2 & & \\ +\partial_3\xi^3) & = & \eta^3. \end{array}\right.$$

From the theory of Lie pseudogroups, we can prove [4] that the PD system $\mathcal{D}\xi = 0$ is formally integrable if and only if $\partial_2 a(x) = c = cst$, the "classical case" of *contact transformations* corresponding to $a(x) = x^2$ ($\Rightarrow c = 1$). It follows that the only compatibility condition $\mathcal{D}_1\eta = 0$ is

$$-a(x)(\partial_2\eta^3 - \partial_3\eta^2) + \partial_1\eta^2 - \partial_2\eta^1 + \partial_2 a(x)\eta^3 = 0,$$

and the operator $\mathcal{D}_1$ is surjective. The adjoint operator $\tilde{\mathcal{D}}_1$ is defined by:

$$\left\{\begin{array}{lcl} \eta^1 \to \partial_2\lambda & = & \mu_1, \\ \eta^2 \to -a(x)\partial_3\lambda - \partial_1\lambda & = & \mu_2, \\ \eta^3 \to a(x)\partial_2\lambda + 2c\lambda & = & \mu_3. \end{array}\right.$$

As $\mu_3 - a(x)\mu_1 = 2c\lambda$, the operator $\tilde{\mathcal{D}}_1$ is injective if and only if $c \neq 0$. In that case, the two independent compatibility conditions can be written:

$$\left\{\begin{array}{lcl} \partial_2\mu_3 - a(x)\partial_2\mu_1 - 3c\mu_1 & = & 2\nu_2, \\ -a(x)\partial_3(\mu_3 - a(x)\mu_1) - \partial_1(\mu_3 & & \\ -a(x)\mu_1) - 2c\mu_2 & = & -2(\nu_1 + a(x)\nu_3), \end{array}\right.$$

after introducting the adjoint $\tilde{\mathcal{D}}$ of $\mathcal{D}$ as follows:

$$\left\{\begin{array}{lcl} \frac{1}{2}a(x)\partial_1\mu_1 + \frac{1}{2}\partial_1\mu_3 + a(x)\partial_3\mu_3 & & \\ +a(x)\partial_2\mu_2 + \partial_2 a(x)\mu_2 & = & \nu_1, \\ -\frac{1}{2}a(x)\partial_2\mu_1 + \frac{1}{2}\partial_2\mu_3 - \frac{3}{2}\partial_2 a(x)\mu_1 & = & \nu_2, \\ -\partial_1\mu_1 - \frac{1}{2}a(x)\partial_3\mu_1 - \partial_2\mu_2 - \frac{1}{2}\partial_3\mu_3 & = & \nu_3. \end{array}\right.$$

In order to point out the link with the double set of obstructions to controllability in the PD situation, let us start with the operator $\tilde{\mathcal{D}}$ depending on the arbitrary function $a(x)$ and let us question about its controllability. According to the general test, we must construct the adjoint of $\tilde{\mathcal{D}}$ which is $\mathcal{D}$ and look for its compatibility conditions $\mathcal{D}_1$, a result bringing out the condition $\partial_2 a(x) = c$, where c is an arbitary constant. When $c = 0$, we should find the zero order compatibility condition $\mu_3 - a(x)\mu_1 = 0$ which is not a consequence of $\tilde{\mathcal{D}}$. When $c \neq 0$, the adjoint $\tilde{\mathcal{D}}_1$ admits the compatibility condition expressed by $\tilde{\mathcal{D}}$ because we have in that case:

$$a(x)\partial_3\nu_2 - \partial_2\nu_1 + \partial_1\nu_2 - a(x)\partial_2\nu_3 - 2c\,\nu_3 = 0,$$

which gives:

$$c\,\nu_3 = (\partial_1 + a(x)\partial_3)\nu_2 - \partial_2(\nu_1 + a(x)\nu_3).$$

Once again, controllability arises in the most generic situation. To conclude with this example, we notice that a similar but more difficult computation can be achieved with an arbitrary 1-form $\omega^i(x)dx_i$, our situation being $\omega^1(x) = -a(x), \omega^2(x) = 0, \omega^3(x) = 1$.

3 Conclusion

We have seen that controllability of linear OD or PD control systems with variable coefficients depends at most two problems of formal integrability. Thus, it depends at least on two trees of integrability conditions. These methods are interesting, not only because they allow to simplify the calculations as the reader can check by himself but mostly because they bring the calculations closer to basic concepts of geometry (operators, differential sequences, lifts, ...) or algebra (torsion, projective and free modules, ...). In respect of this purpose, these methods are the only ones, we know, allowing for an intrinsic study of control systems.

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