Restrictions of \(n\)-D behaviours and inverse images of \(D\)-modules

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Abstract—The problem of characterizing the restriction of the solutions of an \(n\)-D system to a subvector space of \(\mathbb{R}^n\) has recently been investigated in the literature of multidimensional systems theory. In this paper, we characterize the restriction of an \(n\)-D behaviour to an algebraic or analytic submanifold of \(\mathbb{R}^n\). To do that, we first use the algebraic analysis approach to multidimensional systems. We then show that the restriction of an \(n\)-D behaviour to an algebraic or analytic submanifold can be characterized in terms of the inverse image of the differential module defining the behaviour. Characterization of inverse images of differential modules is investigated. Finally, using the above results, we explain Kashiwara’s extension of the Cauchy-Kowalevski theorem for general \(n\)-D behaviours and non-characteristic algebraic or analytic submanifolds.

I. ALGEBRAIC ANALYSIS APPROACH

Let us briefly review the algebraic analysis approach to continuous multidimensional (\(n\)-D) systems [3], [5], [8], [10]. For more details on algebraic analysis, also called (algebraic/analytic) \(D\)-modules, see [1], [2], [4].

Let \(A\) be a differential ring of characteristic 0, namely \(A\) is a commutative ring containing \(\mathbb{Z}\) equipped with \(n\) commuting derivations \(\partial_i, i = 1, \ldots, n\), namely, maps \(\partial_i : A \rightarrow A\) satisfying the following conditions

\[
\forall a_1, a_2 \in A, \quad \left\{ \begin{array}{l}
\partial_i(a_1 + a_2) = \partial_i(a_1) + \partial_i(a_2), \\
\partial_i(a_1 a_2) = \partial_i(a_1) a_2 + a_1 \partial_i(a_2),
\end{array} \right.
\]

and \(\partial_i \circ \partial_j = \partial_j \circ \partial_i\) for all \(1 \leq i < j \leq n\). Let \(D := A(d_1, \ldots, d_n)\) be the (not necessarily commutative) polynomial ring of \(PD\) operators in \(d_1, \ldots, d_n\) with coefficients in \(A\) (i.e., every element of \(D\) is of the form \(\sum_{0 \leq |\mu| \leq r} a_\mu d^\mu\), where \(r \in \mathbb{Z}_{\geq 0} := \{0, 1, \ldots\}\), \(a_\mu \in A\), \(\mu := (\mu_1 \ldots \mu_n) \in \mathbb{Z}_{\geq 0}^n\) and \(d^\mu := d_1^{\mu_1} \ldots d_n^{\mu_n}\) is a monomial in the commuting indeterminates \(d_1, \ldots, d_n\) satisfying:

\[
\forall a \in A, \quad d_i a = a d_i + \partial_i(a). \quad (1)
\]

For more details, see [1], [2], [4], [11]. If \(k\) is a field (that we shall always suppose to be of characteristic 0), and \(A = k[x_1, \ldots, x_n]\) is the commutative polynomial ring in \(x_1, \ldots, x_n\) with coefficients in \(k\), then \(A(d_1, \ldots, d_n)\) is called the Weyl algebra and is simply denoted by \(A_n(k)\).

We shall assume that \(D\) is a noetherian domain, i.e., a ring \(D\) with no non-zero divisors and such that every left/right ideal of \(D\) is finitely generated as a left/right \(D\)-module [12].

Let \(R \in D^{q \times p}\) be a \(q \times p\)-matrix with entries in \(D\) and

\[
.R : D^{1 \times q} \longrightarrow D^{1 \times p},
\]

the left \(D\)-homomorphism (i.e., the left \(D\)-linear map) defined by the matrix \(R\). If the image of \(R\) is denoted by \(D^{1 \times q}R\), then the cokernel of \(R\) is the factor left \(D\)-module \(M := D^{1 \times q} / (D^{1 \times q}R)\) which is finitely presented by \(R\) [12]. In order to describe \(M\) by means of generators and relations, let \(\{f_j\}_{j=1, \ldots, p}\) be the standard basis of \(D^{1 \times p}\), i.e., \(f_j\) is the row vector of length \(p\) with 1 at position \(j\) and 0 elsewhere. Moreover, let \(\pi : D^{1 \times p} \longrightarrow M\) be the canonical projection onto \(M\), i.e., the left \(D\)-homomorphism which maps \(\lambda \in D^{1 \times p}\) to its residue class \(\pi(\lambda)\) in \(M\). Then, \(\pi\) is surjective since every \(m \in M\) is the class of certain \(\lambda\)'s in \(D^{1 \times p}\), i.e., \(m = \pi(\lambda) = \pi(\lambda + \nu R)\) for all \(\nu \in D^{1 \times q}\). If \(y_j := \pi(f_j)\) for \(j = 1, \ldots, p\), then for every \(m \in M\), there exists \(\lambda = (\lambda_1 \ldots \lambda_p) \in D^{1 \times p}\) such that

\[
m = \pi(\lambda) = \pi \left(\sum_{j=1}^p \lambda_j f_j\right) = \sum_{j=1}^p \lambda_j y_j,
\]

which shows that \(\{y_j\}_{j=1, \ldots, p}\) is a generating set for \(M\). Let \(R_{i*}\) (resp., \(R_{i*}\)) denotes the \(i\)th row (resp., \(j\)th column) of \(R\). Then \(\{y_j\}_{j=1, \ldots, p}\) satisfies the following relations

\[
\sum_{j=1}^p R_{ij} y_j = \sum_{j=1}^p R_{ij} \pi(f_j) = \pi \left(\sum_{j=1}^p R_{ij} f_j\right) = \pi(R_{i*}) = 0 \quad (2)
\]

for all \(i = 1, \ldots, q\) since \(R_{i*} \in D^{1 \times q}R\) for \(i = 1, \ldots, q\).

Now, let \(\mathcal{F}\) be a left \(D\)-module, \(\mathcal{F}^p := \mathcal{F}^{p \times 1}\), and let

\[
\ker\mathcal{F}(R) := \{\eta \in \mathcal{F}^p \mid R \eta = 0\}
\]

be the linear \(PD\) system or behaviour defined by \(R\) and \(\mathcal{F}\). A remark due to Malgrange is that \(\ker\mathcal{F}(R)\) is isomorphic to the abelian group (i.e., \(\mathbb{Z}\)-module) \(\text{hom}_D(M, \mathcal{F})\) formed by the left \(D\)-homomorphisms from \(M\) to \(\mathcal{F}\), i.e.,

\[
\ker\mathcal{F}(R) \cong \text{hom}_D(M, \mathcal{F}) \quad (3)
\]

as abelian groups, where \(\cong\) denotes an isomorphism (e.g., of abelian groups, left/right modules). This isomorphism can easily be described: if \(\phi \in \text{hom}_D(M, \mathcal{F})\), \(y_j := \phi(y_j)\) for \(j = 1, \ldots, p\), and \(\eta := (\eta_1 \ldots \eta_p)^T \in \mathcal{F}^p\), then using (2), \(R \eta = 0\) since for \(i = 1, \ldots, q\):

\[
\sum_{j=1}^p R_{ij} \phi(y_j) = \phi \left(\sum_{j=1}^p R_{ij} y_j\right) = \phi(\pi(R_{i*})) = \phi(0) = 0.
\]

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Conversely, if $\eta \in \ker_{\mathcal{X}}(R)$, then we can define $\phi_\eta \in \text{Hom}_D(M, F)$ by $\phi_\eta(\pi(\lambda)) = \lambda \eta$ for all $\lambda \in D^{1 \times p}$. We can prove that the abelian group homomorphism $\chi: \ker_{\mathcal{X}}(R) \rightarrow \text{Hom}_D(M, F)$ defined by $\chi(\eta) = \phi_\eta$ is bijective. See [1], [3], [4], [11]. Hence, (3) shows that the behaviour $\ker_{\mathcal{X}}(R)$ can be studied in terms of $\text{Hom}_D(M, F)$, and thus by means of the left $D$-modules $M$ and $F$.

Within the behavioural approach to multidimensional systems, recent investigations have been done in the direction of the restriction of behaviours to subvector spaces of $\mathbb{R}^n$. See [6], [7] and the references therein. The goal of this paper is to shortly explain a possible answer to this problem developed in algebraic analysis or $D$-module theory [1], [2], [4].

In Section II, we introduce the concept of 
inverse images of $D$-modules for linear systems of PD equations with polynomial coefficients. In Section III, we shortly extend this concept to linear systems of PD equations with analytic or holomorphic coefficients. In Section III, we shall show that this concept is the main ingredient for the study of the restriction of linear PD systems to a submanifold.

II. INVERSE IMAGES OF $D$-MODULES

Let $k$ be a field of characteristic $0$ (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $X = k^n$ (resp., $y = k^m$) the affine space of dimension $n$ (resp., $m$) with coordinates $x = (x_1, \ldots, x_n)$ (resp., $y = (y_1, \ldots, y_m)$). Let us consider the following polynomial map

$$f: Y = k^m \longrightarrow X = k^n$$

$$y = (y_1 \ldots y_m) \longmapsto (f(y) \ldots f_n(y)),$$

i.e., the $f_i$’s are elements of $k[Y] := k[y_1, \ldots, y_m]$. Now, if $k[X] := k[x_1, \ldots, x_n]$, then we can define

$$f^*: k[X] \longrightarrow k[Y]$$

$$p \longmapsto p \circ f,$$  

(4)

where $(p \circ f)(y) = p(f_1(y), \ldots, f_n(y)) \in k[Y]$. In particular, $k[Y]$ inherits a $k[X]$-module structure defined by:

$$k[X] \times k[Y] \longrightarrow k[Y]$$

$$(p, q) \longmapsto (p \circ f) \cdot q.$$  

(5)

Let $D_X := k[X]/(d_{m,x}, \ldots, d_{n,x}) = A_m(k)$ and $M$ be a left $D_X$-module. Using the polynomial map $f: Y \longrightarrow X$, we define a left $D_Y := k[Y]/(d_{m,y}, \ldots, d_{n,y}) = A_m(k)$-module $f^*(M)$ called the inverse image of $M$ under $f$ [1], [4].

The left $D_X$-module $M$ can be seen as a $k[X]$-module by forgetting the actions of the derivatives $d_{x_i}$’s. Since $k[Y]$ is a $k[X]$-module, we can define the following $k[Y]$-module structure $f^*(M) := k[Y] \otimes_{k[X]} M$ formed by elements of the form

$$n = \sum_{i=1}^r q_i(y) \otimes m_i, \quad r \in \mathbb{Z}_{\geq 0}, \quad q_i \in k[Y], \quad m_i \in M,$$

(see, e.g., [12]) which satisfy the following relation:

$$\forall p \in k[X], \quad q(y) \otimes p(x) m_i = q(y) \cdot (p \circ f)(y) \otimes m_i.$$  

(6)

The $k[Y]$-module structure of $f^*(M)$ defined by

$$\forall q \in k[Y], \quad q n = \sum_{i=1}^r q(y) q_i(y) \otimes m_i,$$

can be extended into a left $D_Y$-module structure by

$$d_{y_j} n := \sum_{i=1}^r \left( \frac{\partial y_j q_i(y)}{\partial x_i} m_i + \sum_{i=1}^n q_i(y) \frac{\partial y_j f_i(y)}{\partial x_i} d_{x_i} m_i \right),$$  

(7)

for $j = 1, \ldots, m$. Let us check that (7) yields a well-defined left $D_Y$-module structure on $f^*(M)$. We have

$$d_{y_j}(y_k \cdot (q(y) \otimes m)) = d_{y_j}(y_k q(y) \otimes m)$$

$$= \frac{\partial y_j y_k (q(y))}{\partial x_i} m + \sum_{i=1}^n y_k q_i(y) \frac{\partial y_j f_i(y)}{\partial x_i} d_{x_i} m$$

$$= (\frac{\partial y_j y_k (q(y))}{\partial x_i} m + \sum_{i=1}^n y_k q_i(y) \frac{\partial y_j f_i(y)}{\partial x_i} d_{x_i} m)$$

$$= \delta_{jk} q(y) \otimes m + y_k d_{y_j}(q(y) \otimes m),$$

where $\delta_{jk} = 1$ if $j = k$ or 0 else, i.e., $d_{y_j} y_k - y_k d_{y_j} = \delta_{jk}$.

We can also check that

$$(d_{y_j} d_{y_k})(q(y) \otimes m)$$

$$= \frac{\partial y_j \partial y_k q(y)}{\partial x_i} m + \sum_{i=1}^n \frac{\partial y_j q_i(y)}{\partial x_i} \frac{\partial y_k f_i(y)}{\partial x_i} m$$

$$+ \sum_{i=1}^n \frac{\partial y_j q_i(y)}{\partial x_i} \frac{\partial y_k f_i(y)}{\partial x_i} d_{x_i} m$$

is symmetric in $j$ and $k$, which yields $d_{y_j} d_{y_k} = d_{y_k} d_{y_j}$. Finally, we prove that (7) is compatible with (6), i.e.,

$$d_{y_j}(q(y) \otimes p(x) m)$$

$$= \frac{\partial y_j q(y)}{\partial x_i} p(x) m + \sum_{i=1}^n q_i(y) \frac{\partial y_j f_i(y)}{\partial x_i} d_{x_i} (p(x) m)$$

$$= \frac{\partial y_j q(y)}{\partial x_i} p(f(y)) \otimes m$$

$$+ \sum_{i=1}^n q_i(y) \frac{\partial y_j f_i(y)}{\partial x_i} p(f(y)) \otimes d_{x_i} m$$

$$+ \sum_{i=1}^n q_i(y) \frac{\partial y_j f_i(y)}{\partial x_i} p(f(y)) \otimes d_{x_i} m$$

$$= \frac{\partial y_j q(y)}{\partial x_i} p(f(y)) \otimes m$$

$$+ \sum_{i=1}^n q_i(y) \frac{\partial y_j f_i(y)}{\partial x_i} p(f(y)) \otimes d_{x_i} m$$

$$= d_{y_j}(q(y) \cdot p(x) \otimes m)).$$

Definition 1 ([11], [2], [4]): Let $f: Y = k^m \longrightarrow X = k^n$ be a polynomial map, $D_X = A_m(k)$, $D_Y = A_m(k)$ and $M$ be a left $D_X$-module, then $f^*(M) := k[Y] \otimes_{k[X]} M$ has a left $D_Y$-module structure defined by (7) and is called the inverse image of $M$ under $f$.

Example 1: Let us consider $M = D_X$ so that $f^*(D_X) = k[Y] \otimes_{k[X]} D_X$, which is usually denoted by $D_{Y \rightarrow X}$ [1], [2], [4]. An element $P$ of $D_{Y \rightarrow X}$ is an operator of the form:

$$P = \sum_{0 \leq |\mu| \leq \lambda} a_\mu(y) \otimes d^\mu_x, \quad a_\mu \in k[Y], \quad d^\mu_x = d^\mu_{x_1} \ldots d^\mu_{x_n}.$$

Note that $D_{Y \rightarrow X}$ has a $D_Y - D_X$-bimodule structure defined by the left $D_X$-module structure given by (7), i.e.,

$$d_{y_j} P = \sum_{0 \leq |\mu| \leq \lambda} (\frac{\partial y_j}{\partial x} a_\mu(y) \otimes d^\mu_x + \sum_{i=1}^\mu a_\mu(y) \frac{\partial y_j f_i(y)}{\partial x} d^\mu_{x_i+1}),$$  

(8)

where $\mu + 1 :: = (\mu_1, \ldots, \mu_i + 1, \ldots, \mu_n)$, and by the natural right $D_X$-module structure of $D_X$. 


Using \( a_\nu(y)(1 \otimes d^\nu_x) = a_\nu(y) \otimes d^\nu_x \) and (8), the ring \( D_{Y \rightarrow X} \) is generated by \( \{1 \otimes d^\nu_x\}_{\nu \in \mathbb{Z}_{\geq 0}} \) as a left \( D_Y \)-module, i.e., using (8), we have:

\[
\begin{align*}
P & = \sum_{0 \leq |\nu| \leq r} a_\nu(y)(1 \otimes d^\nu_x), \\
d_{y_j} P & = \sum_{0 \leq |\nu| \leq r} \partial_{y_j} a_\nu(y)(1 \otimes d^\nu_x) \\
& \quad + \sum_{0 \leq |\nu| \leq s} \sum_{i=1}^n a_\nu(y) \partial_{y_i} f_i(y)(1 \otimes d^{\nu+1}_x).
\end{align*}
\]

Finally, considering the element \( 1 \otimes 1 \in D_{Y \rightarrow X} \), (8) yields:

\[
\forall j = 1, \ldots, m, \quad d_{y_j}(1 \otimes 1) = \sum_{i=1}^n \partial_{y_j} f_i(y) \otimes d_{x_i}.
\]

If \( f \) is a linear map (see, e.g., [6], [7]), i.e., \( f(y) = Ay \), where \( A \in k^{n \times m} \), then, for \( j = 1, \ldots, m \), we get:

\[
d_{y_j}(1 \otimes 1) = \sum_{i=1}^n A_{ij} \otimes d_{x_i} = 1 \otimes \sum_{i=1}^n A_{ij} d_{x_i}.
\]

If \( d_x = (d_{x_1} \ldots d_{x_n})^T \) and \( A = (A_{i1} \ldots A_{im}) \), where \( A_{i*} \) denotes the \( i \)-th column of \( A \), then

\[
\sum_{i=1}^n A_{ij} d_{x_i} = A_{i*} \otimes d_x.
\]

If \( M \) is a left \( D_X \)-module, \( D_X \otimes_{D_X} M \cong M \) [12] yields

\[
\begin{align*}
f^*(M) & = k[Y] \otimes_{k[X]} M \cong k[Y] \otimes_{k[X]} (D_X \otimes_{D_X} M) \cong (k[Y] \otimes_{k[X]} D_X) \otimes_{D_X} M \\
& = D_{Y \rightarrow X} \otimes_{D_X} M,
\end{align*}
\]

which shows that the ring \( D_{Y \rightarrow X} \) has to be studied in detail.

**Example 2:** Let \( m = n + l \), \( l \in \mathbb{Z}_{\geq 0} \), \( Z = k^l \) and

\[
f : Y = k^{n+l} = X \times Z \longrightarrow X = k^n, \quad y = (x, z) \longmapsto x,
\]

i.e., \( f \) is a projection. If \( M \) is a left \( D_X \)-module, then:

\[
f^*(M) = k[X, Z] \otimes_{k[X]} M.
\]

An element of \( f^*(M) \) is a sum of the form \( q(x, z) \otimes m \), where \( q \in k[X, Z] \) and \( m \in M \). Since \( q \in k[X, Z] \), it can be rewritten as \( q = \sum_{0 \leq |\nu| \leq r} q_\nu(x) z^\nu \), where \( \nu = (\nu_1 \ldots \nu_l) \in \mathbb{Z}_{\geq 0}^l \), we get:

\[
q(x, z) \otimes m = \sum_{0 \leq |\nu| \leq r} z^\nu \otimes q_\nu(x) m.
\]

Thus, an element of \( f^*(M) \) can be written as a sum of terms of the form \( z^\nu \otimes m \), where \( m \in M \). Now, we note that we have \( k[X, Z] \cong k[Z] \otimes_k k[X] \), where \( k[Z] \otimes_k k[X] \) is the \( k \)-algebra formed by elements which are sums of terms of the form \( p(z) \otimes q(x) \) and with the product defined by

\[
(p(z) \otimes q(x))(p'(z) \otimes q'(x)) = (p(z)p'(z)) \otimes (q(x)q'(x)),
\]

for all \( p, p' \in k[Z] \) and \( q, q' \in k[X] \). Then, we can define the \( k[Z] \otimes_k k[X] \)-module \( k[Z] \otimes_k M \) formed by elements which are sums of terms of the form \( z^\nu \otimes m \) with \( \nu \in \mathbb{Z}_{\geq 0} \) and \( m \in M \) and endowed with the following product:

\[
(p(z) \otimes q(x))(z^\nu \otimes m) = p(z) z^\nu \otimes q(x) m.
\]

Then, we have the following \( k[X, Z] \)-isomorphism:

\[
f^*(M) \cong k[Z] \otimes_k M \quad \text{(10)}
\]

Now, the left \( D_Y \)-module structure of \( f^*(M) \) is given by

\[
\begin{align*}
d_{z_k}(q_\nu(x) z^\nu \otimes m) & = \partial_{z_k} q_\nu(x) z^\nu \otimes m + q_\nu(x) z^\nu \partial_{z_k} x_j \otimes d_{x_j} m, \\
& = z^\nu \otimes (\partial_{z_k} q_\nu(x) + q_\nu(x) d_{x_j}) m, \\
& = z^\nu \otimes d_{x_j}(q_\nu(x) m), \quad i = 1, \ldots, n,
\end{align*}
\]

Hence, using (10), we obtain that the \( d_z \)'s act only on \( M \) and the \( d_{z_k} \)'s act only on \( k[Z] \), i.e.

\[
\begin{align*}
d_{z_k}(z^\nu \otimes q_\nu(x) m) & = z^\nu \otimes d_{x_k}(q_\nu(x) m), \\
d_{z_k}(z^\nu \otimes q_\nu(x) m) & = \partial_{z_k} z^\nu \otimes q_\nu(x) m. \quad \text{(11)}
\end{align*}
\]

Using the isomorphism \( D_Y \cong D_Z \otimes_{k[X]} D_X \) defined by

\[
D_Y \longrightarrow D_Z \otimes_{k[X]} D_X, \quad \sum_{\mu, \nu} d_\mu^\nu \longrightarrow \sum_{\mu, \nu} (z^\mu z^\nu \otimes \partial_{z_k} z^\nu \otimes d_{x_j} m)
\]

where \( \mu = (\mu_1, \mu_2) \) and \( \nu = (\nu_1, \nu_2) \), then shows that (10) is an isomorphism of \( D_Y \cong D_Z \otimes_{k[X]} D_X \)-modules.

If we now consider \( M = D_X = A_n(k) \), then we have

\[
D_{Y \rightarrow X} \cong k[X, Z] \otimes_{k[X]} D_X \cong k[Z] \otimes_{k[X]} D_X
\]

as \( D_Y \cong D_Z \otimes_{k[X]} D_X \)-modules. In particular, using (11), we have

\[
d_{x_j}(1 \otimes d^\nu_x) = 1 \otimes d_{x_j}^\nu \quad \text{and} \quad d_{z_k}(1 \otimes d^\nu_x) = 0.
\]

Finally, using the left \( D_Z \)-isomorphism \( k[Z] \cong D_Z \otimes_{k[X]} D_X \), we get the following isomorphism of \( D_{Y \rightarrow X} \)-bimodules [2]

\[
\left(D_Y / \left( \sum_{k=1}^l D_Z d_{z_k} \right) \right) \otimes_{k[D_X]} D_Y \cong D_Y / \left( \sum_{j=n+1}^m D_Y y_j \right).
\]

**Example 3:** Let \( m = n + l \), \( l \in \mathbb{Z}_{\geq 0} \), \( Z = k^l \),

\[
f : Y = k^m \longrightarrow X = k^n = Y \times Z \quad y \longmapsto (y, 0),
\]

i.e., \( f \) is a standard embedding, and a left \( D_X \)-module \( M \). If \( z_1 = x_{m+1} \ldots, z_l = x_n \) and \( k[Z] := k[z_1, \ldots, z_l] \) then, we can consider the left \( D_Y \)-module \( f^*(M) = k[Y] \otimes_{k[Y, Z]} M \).
Using (4), we get \( f^*(p(y, z)) = p(y, 0) \) for all \( p \in k[Y, Z] \). We note that \( k[Y] \cong k[Y, Z]/(Z) \), where \( (Z) = (z_1, \ldots, z_l) \) is the ideal of \( k[Y, Z] \) generated by the \( z_i \)'s. Hence, we get \( f^*(M) \cong k[Y, Z]/(Z) \otimes_{k[Y, Z]} M \) as \( k[Y, Z] \)-modules.

Using the fact that \( D_Y = A_m \otimes k \) \( \subseteq D_X = A_{m+1} \otimes k \), the \( D_X \)-module \( M \) can be considered as a left \( D_Y \)-module. Any element \( P = \sum_{0 \leq |p| \leq r} a_p(y) d_p \) of \( D_Y \) commutes with all the \( z_i \)'s, i.e., we have \( D_Y(Z) = (Z) D_Y \) in \( D_X \), which proves that \((Z) M : = \{p M | p \in (Z), m \in M\} \) is a left \( D_Y \)-submodule of \( M \). Then, we can consider the left \( D_Y \)-module \( (Z) M \). Let \( \sigma : M \rightarrow (Z) M \) be the canonical projection. Let us also consider the map:

\[
\chi : f^*(M) = k[Y] \otimes_{k[Y, Z]} M \rightarrow M/(Z) M \quad q(y) \otimes m \mapsto (q(y)) \sigma(m).
\]

The map \( \chi \) is well-defined since \( q(y) \otimes m = 1 \otimes q(y) m \) and \( \chi(q(y) \otimes m) = q(y) \sigma(m) = \sigma(q(y)m) = (1 \otimes q(y)m) \).

Let us now check that \( \chi \) is a \( k[Y] \)-homomorphism:

\[
\forall q, r \in k[Y], \chi(r(q \otimes m)) = (rq \otimes m) = r(q \otimes m) = r \chi(q \otimes m).
\]

Using (7) and the fact that \( \sigma \) is left \( D_Y \)-homomorphism, let us prove that \( \chi \) is a left \( D_Y \)-homomorphism:

\[
\chi(d_{y_j}(q \otimes m)) = \chi(d_{y_j}q \otimes m + \sum_{i=1}^{n} d_{y_j}f_i \otimes d_{x_i}m) = \chi(d_{y_j}q \otimes m) + \sum_{i=1}^{n} \delta \chi(d_{y_j}f_i \otimes d_{x_i}m) = \chi(d_{y_j}q \otimes m) + \sum_{i=1}^{n} \delta \chi(d_{y_j}f_i \otimes d_{x_i}m).
\]

Let us now check that \( \chi \) is an isomorphism of \( D_Y \)-modules. If \( n = \sum_{i=1}^{m} q_i \otimes m_r \in \ker \chi \), i.e., \( \sum_{i=1}^{m} q_i \sigma(m_r) = 0 \), then \( \sigma(\sum_{i=1}^{m} q_i m_r) = 0 \), which shows that there exist \( m_1, \ldots, m_l \in M \) such that \( \sum_{i=1}^{m} q_i m_r = \sum_{k=1}^{l} z_k m_k \), and using (6), we get:

\[
n = 1 \otimes \sum_{r=1}^{m} q_r m_r = 1 \otimes \sum_{k=1}^{l} z_k m_k = \sum_{k=m+1}^{n} f_k \otimes m_k = 0.
\]

Since every element \( \sigma(m) \in M/(Z) M \) is such that \( \chi(1 \otimes m) = \sigma(m) \), which finally proves that \( \chi \) is a left \( D_Y \)-isomorphism, i.e., \( f^*(M) \cong M/(Z) M \).

Using the above result, we obtain:

\[
D_{Y \rightarrow X \rightarrow Y \times Z} \cong D_X/(Z) D_X. \tag{13}
\]

**Proposition 1:** [1], [2], [4] Let \( X, Y \) and \( Z \) be three affine spaces, \( f : Y \rightarrow X \) and \( g : Z \rightarrow Y \) two polynomial maps and \( M \) a left \( D_X \)-module. Then, we have:

\[
(f \circ g)^*(M) \cong g^*(f^*(M)).
\]

In particular, if \( g = f^{-1} \), then \( M \cong f^{-1}(M) \).

Let \( f : Y = k^m \rightarrow X = k^n \) be a polynomial map and:

\[
g : Y \rightarrow Y \times X \quad y \mapsto (y, f(y)) \quad (y, x) \mapsto x. \tag{14}
\]

Then, we have \( f = w \circ q \), i.e., any map \( f \) is the composition of an embedding and a projection. \( g \) is not a standard embedding but \( g \) can be written as \( g = v \circ u \), where

\[
u : Y \rightarrow Y \times X \quad v : Y \times X \rightarrow Y \times X \quad y \mapsto (y, 0) \quad (y, x) \mapsto (y, x + f(y)). \tag{15}
\]

The polynomial map \( u \) is a standard embedding and \( v \) is an invertible polynomial map. Then, using \( f = w \circ v \circ u \) and Proposition 1, we get \( f^*(M) = u^*(v^*(w^*(M))) \) for all left \( D_X \)-modules. Using (9), we obtain:

\[
f^*(M) \cong D_{Y \rightarrow Y \times X \rightarrow Y \times X} \cong D_{Y \times X} \cong D_{Y \times X} \cong D_X M. \tag{16}
\]

**Example 4:** Let us consider the above polynomial map \( v \) and a left \( D_{Y \times X} = A_{m+n}(k) \)-module \( N \). Using (4), we have \( (p \circ v)((y, x)) = p(y, x + f(y)) \) for all \( p \in k[Y, X] \). Then, \( v^*(N) = k[Y, X] \otimes_{k[Y, X]} N \), where the left \( D_{Y \times X} \)-module structure defined by:

\[
d_{y_j}(q(y, x) \otimes n) = \partial_{y_j} q(y, x) \otimes n + \sum_{i=1}^{m} q(y, x) \partial_{y_j} f_i \otimes d_{x_i} n + \sum_{i=1}^{n} q(y, x) \partial_{y_j} f_i \otimes d_{x_i} n + \partial_{y_j} q(y, x) \otimes n + \partial_{y_j} q(y, x) \otimes d_{x_i} n + \partial_{y_j} q(y, x) \otimes d_{x_i} n
\]

\[
= \partial_{y_j} q(y, x) \otimes n + \sum_{i=1}^{m} q(y, x) \partial_{y_j} f_i \otimes d_{x_i} n + \sum_{i=1}^{n} q(y, x) \partial_{y_j} f_i \otimes d_{x_i} n + \partial_{y_j} q(y, x) \otimes n + \partial_{y_j} q(y, x) \otimes d_{x_i} n + \partial_{y_j} q(y, x) \otimes d_{x_i} n.
\]

Let us consider the following automorphism of \( D_{Y \times X} \):

\[
\alpha : D_{Y \times X} \rightarrow D_{Y \times X} \quad y_j \mapsto y'_j = y_j, \quad x_i \mapsto x'_i = x_i - f_i(y), \quad d_{y_j} \mapsto d_{y'_j} = d_{y_j} + \sum_{i=1}^{n} \partial_{y_j} f_i(y) d_{x_i}, \quad d_{x_i} \mapsto d_{x'_i} = d_{x_i}. \tag{18}
\]

For \( j = 1, \ldots, m \) and \( i = 1, \ldots, n \). We can easily check that:

\[
d_{y'_j} y'_j - y_j d_{y'_j} = \delta_{ij}, \quad d_{x'_i} x'_i - d_{x_i} \delta_{ij} = \delta_{ij}.
\]

Let us now introduce the left \( D_{Y \times X} \)-module \( M_\alpha \) defined by \( M \) as an abelian group but equipped with the new left \( D_{Y \times X} \)-module structure defined by:

\[
\forall d \in D_{Y \times X}, \forall m \in M_\alpha, \quad d \bullet m := \alpha(d) m.
\]

Let us also define the following map:

\[
i : v^*(M) \rightarrow M_\alpha \quad q \otimes m \mapsto q \otimes m = \alpha(q) m.
\]

Let us now prove that \( i \) is a left \( D_{Y \times X} \)-homomorphism, i.e.:

\[
i(d_{y_j}(q \otimes m)) = d_{y_j} \bullet i(q \otimes m) = \alpha(d_{y_j}) \iota(q \otimes m), \quad i(d_{x_i}(q \otimes m)) = d_{x_i} \bullet i(q \otimes m) = \alpha(d_{x_i}) \iota(q \otimes m).
\]
Applying \( \iota \) to the first identity of (17), we obtain:
\[
\iota(d_{y_j}(q(y,x) \otimes m)) = \iota\bigl(\partial_{y_j}q(y,x) \otimes m + \iota(q(y,x) \otimes d_{y_j}m)\bigr) + \sum_{i=1}^n \iota\bigl(q(y,x) \partial_{y_j}f_i(y) \otimes d_{x_i}m\bigr)
\]
\[
= \alpha(\partial_{y_j}q(y,x))m + \alpha(q(y,x))d_{y_j}m + \sum_{i=1}^n \alpha(q(y,x))\partial_{y_j}f_i(y)d_{x_i}m
\]
\[
= \alpha(q(y,x))(d_{y_j} + \sum_{i=1}^n \partial_{y_j}f_i(y)d_{x_i})m + \alpha(q(y,x))d_{y_j}m
\]
\[
= \alpha(q(y,x))d_{y_j}m \quad \text{(18).}
\]

Applying \( \alpha \) to \( q(y,x) \partial_{y_j}q(y,x) = d_{y_j}, q(y,x) \) (see (1)), we obtain \( \alpha(q(y,x))\alpha(d_{y_j}) + \alpha(\partial_{y_j}q(y,x)) = \alpha(d_{y_j}) \alpha(q(y,x)) \), which then yields:
\[
\iota(d_{y_j}(q(y,x) \otimes m)) = \alpha(d_{y_j})\alpha(q(y,x))m
\]
\[
= \alpha(d_{y_j})\iota(q(y,x) \otimes m).
\]

Now, applying \( \iota \) to the second identity of (17), we get:
\[
\iota(d_{x_j}(q(y,x) \otimes m))
\]
\[
= \iota\bigl(\partial_{x_j}q(y,x) \otimes m + \iota(q(y,x) \otimes d_{x_j}m)\bigr) + \sum_{i=1}^n \iota\bigl(q(y,x) \partial_{x_j}f_i(y) \otimes d_{x_i}m\bigr)
\]
\[
= \alpha(\partial_{x_j}q(y,x))m + \alpha(q(y,x))d_{x_j}m + \sum_{i=1}^n \alpha(q(y,x))\partial_{x_j}f_i(y)d_{x_i}m
\]
\[
= \alpha(q(y,x))(d_{x_j} + \sum_{i=1}^n \partial_{x_j}f_i(y)d_{x_i})m + \alpha(q(y,x))d_{x_j}m
\]
\[
= \alpha(q(y,x))d_{x_j}m.
\]

Applying \( \alpha \) to \( q(y,x) \partial_{x_j}q(y,x) = d_{x_j}, q(y,x) \) (see (1)), we obtain:
\[
\alpha(q(y,x))\alpha(d_{x_j}) + \alpha(\partial_{x_j}q(y,x)) = \alpha(d_{x_j})\alpha(q(y,x)),
\]
which then yields:
\[
\iota(d_{x_j}(q(y,x) \otimes m)) = \alpha(d_{x_j})\alpha(q(y,x))m
\]
\[
= \alpha(d_{x_j})\iota(q(y,x) \otimes m).
\]

\( \iota \) is surjective since \( \iota(1 \otimes m) = m \). Using (6), we have:
\[
1 \otimes q(y,x) m = (q \circ v)(y,x) \otimes m = q(y,x + f(y)) \otimes m = \alpha^{-1}(q) \otimes m.
\]

Hence, \( q \otimes m \in \ker \iota \) iff \( \alpha(q)m = 0 \), which finally yields \( q \otimes m = 1 \otimes \alpha(q)m = 0 \), i.e., \( \iota \) is injective and thus \( \iota \) is an isomorphism of left \( D_{Y \times X} \)-modules, i.e., \( v^*(M) \cong M \).

**Example 5:** Let \( Y = k^m, X = k^n, f : Y \rightarrow X \) be a polynomial map, the map \( g \) defined by (14)
\[
g : Y = k^m \rightarrow Z = k^{m+n} = Y \times X
\]
\[
y \mapsto (y, f(y)),
\]
and a left \( D_Z \)-module \( M \). Using (4), we have \( (p \circ g)(y) = p(y, f(y)) \) for all \( p \in k[Y,X] \). Following Example 3, we get
\[
k[Y] \cong k[Y,X] / (x_1 - f_1(y), \ldots, x_m - f_m(y))
\]
\[
g^*(M) = k[Y] \otimes_{k[Y,X]} M
\]
\[
\cong k[Y,X] / (x_1 - f_1(y), \ldots, x_m - f_m(y)) \otimes_{k[Y,X]} M,
\]
where the left \( D_Z \)-module structure defined by:
\[
d_{y_j}(q(y) \otimes m) = \partial_{y_j}q(y) \otimes m + \sum_{i=1}^n q(y) \partial_{y_j}f_i(y) \otimes d_{x_i}m
\]
\[
+ \sum_{i=1}^n q(y) \partial_{y_j}f_i(y) \otimes d_{x_i}m
\]
\[
= \partial_{y_j}q(y) \otimes m + q(y) \otimes d_{y_j}m + \sum_{i=1}^n q(y) \partial_{y_j}f_i(y) \otimes d_{x_i}m.
\]

Using (15), we have \( g = v \circ u \) which by Proposition 1 yields \( g^*(M) = u^*(v^*(M)) \):
\[
g^*(M) = D_{Y \times X} \otimes_{D_Z} D_{Z \times Y} \otimes D_Z M.
\]

Using Example 4, we get \( v^*(M) \cong D_Z \otimes_{D_Z} M \cong M \), where \( \alpha \) is the automorphism of \( D_Z = A_{m+n}(k) \) defined by (18). If \( X' = (x_1', \ldots, x_m') = (x_1-f_1(y), \ldots, x_m-f_m(y)) \), then using Example 3, we obtain:
\[
g^*(M) = u^*(v^*(M)) \cong M / (X') M \cong D_{Y \times X} \otimes_{D_Z} M.
\]

Finally, if \( f : X \rightarrow Y \) is a generic polynomial map and \( D_Z := D_{Y \times X} = A_{m+n}(k) \), then using (14) and (16), we obtain \( f^*(M) = D_Y \otimes_{D_Z} D_Z M \), where:
\[
D_Y \otimes_{D_Z} D_Z = D_Y / \left( \sum_{j=1}^m D_Z d_{y_j} + \sum_{i=1}^n (x_i-f_i(y)) D_Z \right).
\]

**III. CAUCHY-KOWALEVSKI-KASHIWARA**

**THEOREM**

Let \( X \) (resp., \( Y \)) be a manifold of dimension \( n \) (resp., \( m \)) with local coordinates \( x = (x_1, \ldots, x_n) \) (resp., \( y = (y_1, \ldots, y_m) \)). Let \( TX \) (resp., \( TX^* \)) denote the tangent bundle (resp., cotangent bundle) of \( X \), i.e., the disjoint union of the tangent spaces of \( X \) (resp., the dual bundle of \( TX \)):
\[
\bigcup_{x \in X} \{x\} \times TX = \bigcup_{x \in X} \{x\} \times TX, \quad t \in TX
\]
\[
\bigcup_{x \in X} \{x\} \times TX = \bigcup_{x \in X} \{x\} \times TX.
\]

In the context of analytic \( D \)-modules [4], all the results developed in Sections I and II can be extended to linear PD systems with analytic or holomorphic coefficients. Analytic \( D \)-module theory is usually more complicated than algebraic \( D \)-module theory since it uses sheaf theory [12]. The ring \( D \) of PD operators with coefficients in the differential ring \( A \) has to be replaced by the sheaf \( DX \) of rings of PD operators on a complex manifold \( X \) of dimension \( n \). The stalk \( DX \) at a point \( x \in X \) is defined by elements of the form \( \sum_{|\mu| \leq r} a_\mu(x) d_{y_\mu} \), where the \( a_\mu(x) \)'s are germs of holomorphic functions at \( x \). Moreover, the finitely generated left \( D \)-module \( M \) is replaced by a coherent sheaf \( DX \)-module \( M \) [12], i.e., for any \( x \in X \), there exists a neighbourhood \( U \) of \( x \) in \( X \) which \( M \) admits a finite presentation:
\[
D_U^q \rightarrow D_U^p \rightarrow \cdots \rightarrow 0.
\]

If \( O_X \) is the sheaf of germs of holomorphic functions over \( X \), then the sheaf \( hom_{DX}(M, O_X) \) corresponds to the sheaf of holomorphic solutions of the linear PD system defined by
Let $Y = \{ x \in X \mid x_1 = 0 \}$ be a submanifold of $X$. Then, $\chi_{|Y} = \chi_2 dx_2 + \ldots + \chi_n dx_n = 0$ yields $\chi_i = 0$ for $i = 2, \ldots, n$. If we note $z := (0, x_2, \ldots, x_n) \in X$, then we get:

$$
\sigma_r(P)(z, \chi_1 dx_1) = \alpha(r, 0, 0, \ldots, 0)(z) \chi_1^r.
$$

Thus, $Y$ is non-characteristic for $M$ iff $\alpha(r, 0, 0, \ldots, 0)(z) \neq 0$, i.e., $\sigma_r(P)(z, dx_1) \neq 0$.

Let $f : Y \rightarrow X$ be holomorphic map of holomorphic manifolds and $F_X$ a sheaf on $X$. Then, $f^{-1}F_X$ is the sheaf on $Y$ defined by $(f^{-1}F_X)(V) := \lim_{\longrightarrow} F_X(U)$ for all open sets $V$ of $Y$, where $\lim_{\longrightarrow}$ denotes the inductive limit over the set $\{ U \text{ open set of } X \mid f(V) \subseteq U \}$.[12]

**Theorem 1**: [4] Let $M$ be a coherent left $D_X$-module, $Y$ a submanifold of $X$ which is non-characteristic for $M$, $i : Y \rightarrow X$ the embedding and $D_{Y \leftarrow X} := \mathcal{O}_Y \otimes_{\mathcal{O}_X} i^{-1}D_X$. Then, the inverse image of $M$ under $i$, i.e.,

$$
M_Y := i^*(M) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} i^{-1}M \cong D_{Y \leftarrow X} \otimes_{\mathcal{O}_Y} i^{-1}M
$$

is a coherent left $D_Y$-module, and the canonical map

$$
i^{-1}(\text{hom}_{D_X}(M, \mathcal{O}_X)) \rightarrow \text{hom}_{D_Y}(i^*(M), i^*(\mathcal{O}_Y)),
$$

that is to say $\text{hom}_{D_X}(M, \mathcal{O}_X)|_Y \rightarrow \text{hom}_{D_Y}(M_Y, \mathcal{O}_Y)$, is an isomorphism. In other words, we have:

$$
\text{hom}_{D_X}(M, \mathcal{O}_X)|_Y \cong \text{hom}_{D_Y}(M_Y, \mathcal{O}_Y).
$$

Theorem 1 shows that the restriction of the behaviour $\text{hom}_{D_X}(M, \mathcal{O}_X)$ to the non-characteristic submanifold $Y$ of $X$ is the behaviour $\text{hom}_{D_Y}(M_Y, \mathcal{O}_Y)$ defined by the inverse image $M_Y = i^*(M)$ of $M$ under $i : Y \rightarrow X$.

**REFERENCES**


