Restrictions of n-D behaviours and inverse images of D-modules

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***Abstract*— The problem of characterizing the restriction of the solutions of an n-D system to a subvector space of $\mathbb{R}^n$ has recently been investigated in the literature of multidimensional systems theory. In this paper, we characterize the restriction of an n-D behaviour to an algebraic or analytic submanifold of $\mathbb{R}^n$. To do that, we first use the algebraic analysis approach to multidimensional systems. We then show that the restriction of an n-D behaviour to an algebraic or analytic submanifold can be characterized in terms of the inverse image of the differential module defining the behaviour. Characterization of inverse images of differential modules is investigated. Finally, using the above results, we explain Kashiwara's extension of the Cauchy-Kowalevski theorem for general n-D behaviours and non-characteristic algebraic or analytic submanifolds.**

I. ALGEBRAIC ANALYSIS APPROACH

Let us briefly review the *algebraic analysis approach* to continuous multidimensional (n-D) systems [3], [5], [8], [10]. For more details on algebraic analysis, also called *(algebraic/analytic) D-modules*, see [1], [2], [4].

Let A be a *differential ring* of characteristic 0, namely A is a commutative ring containing $\mathbb{Z}$ equipped with n *commuting derivations* ∂_i, $i = 1, \ldots, n$, namely, maps $\partial_i : A \longrightarrow A$ satisfying the following conditions

$$\forall\, a_1,\, a_2 \in A, \quad \begin{cases} \partial_i(a_1 + a_2) = \partial_i(a_1) + \partial_i(a_2), \\ \partial_i(a_1\, a_2) = \partial_i(a_1)\, a_2 + a_1\, \partial_i(a_2), \end{cases}$$

and $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for all $1 \leq i < j \leq n$. Let $D := A\langle d_1, \ldots, d_n\rangle$ be the (not necessarily commutative) polynomial ring of PD operators in $d_1, \ldots, d_n$ with coefficients in A (i.e., every element of D is of the form $\sum_{0 \leq |\mu| \leq r} a_\mu\, d^\mu$, where $r \in \mathbb{Z}_{\geq 0} := \{0, 1, \ldots\}$, $a_\mu \in A$, $\mu := (\mu_1\ \ldots\ \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and $d^\mu := d_1^{\mu_1} \ldots d_n^{\mu_n}$ is a monomial in the commuting indeterminates $d_1, \ldots, d_n$) satisfying:

$$\forall\, a \in A, \quad d_i\, a = a\, d_i + \partial_i(a). \tag{1}$$

For more details, see [1], [2], [4], [11]. If k is a field (that we shall always suppose to be of characteristic 0) and $A = k[x_1, \ldots, x_n]$ is the commutative polynomial ring in $x_1, \ldots, x_n$ with coefficients in k, then $A\langle d_1, \ldots, d_n\rangle$ is called the *Weyl algebra* and is simply denoted by $A_n(k)$.

We shall assume that D is a *noetherian domain*, i.e., a ring D with no non-zero divisors and such that every left/right ideal of D is finitely generated as a left/right D-module [12].

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Let $R \in D^{q \times p}$ be a $q \times p$-matrix with entries in D and

$$\begin{array}{rccc} .R\colon & D^{1\times q} & \longrightarrow & D^{1\times p} \\ & \lambda & \longmapsto & \lambda\, R, \end{array}$$

the *left D-homomorphism* (i.e., the left D-linear map) defined by the matrix R. If the image of $.R$ is denoted by $D^{1\times q}\, R$, then the *cokernel* of $.R$ is the factor left D-module $M := D^{1\times p}/(D^{1\times q}\, R)$ which is *finitely presented by* R [12]. In order to describe M by means of *generators* and *relations*, let $\{f_j\}_{j=1,\ldots,p}$ be the *standard basis* of $D^{1\times p}$, i.e., f_j is the row vector of length p with 1 at position j and 0 elsewhere. Moreover, let $\pi\colon D^{1\times p} \longrightarrow M$ be the canonical projection onto M, i.e., the left D-homomorphism which maps $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda)$ in M. Then, π is surjective since every $m \in M$ is the class of certain λ's in $D^{1\times p}$, i.e., $m = \pi(\lambda) = \pi(\lambda + \nu\, R)$ for all $\nu \in D^{1\times q}$. If $y_j := \pi(f_j)$ for $j = 1, \ldots, p$, then for every $m \in M$, there exists $\lambda = (\lambda_1\ \ldots\ \lambda_p) \in D^{1\times p}$ such that

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j\, f_j\right) = \sum_{j=1}^{p} \lambda_j\, \pi(f_j) = \sum_{j=1}^{p} \lambda_j\, y_j,$$

which shows that $\{y_j\}_{j=1,\ldots,p}$ is a generating set for M. Let $R_{i\bullet}$ (resp., $R_{\bullet j}$) denotes the i^{th} row (resp., j^{th} column) of R. Then $\{y_j\}_{j=1,\ldots,p}$ satisfies the following relations

$$\sum_{j=1}^{p} R_{ij}\, y_j = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \pi\left(\sum_{j=1}^{p} R_{ij}\, f_j\right) = \pi(R_{i\bullet}) = 0 \tag{2}$$

for all $i = 1, \ldots, q$ since $R_{i\bullet} \in D^{1\times q}\, R$ for $i = 1, \ldots, q$.

Now, let $\mathcal{F}$ be a left D-module, $\mathcal{F}^p := \mathcal{F}^{p\times 1}$, and let

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}$$

be the *linear PD system* or *behaviour* defined by R and $\mathcal{F}$. A remark due to Malgrange is that $\ker_{\mathcal{F}}(R.)$ is isomorphic to the *abelian group* (i.e., $\mathbb{Z}$-module) $\hom_D(M, \mathcal{F})$ formed by the left D-homomorphisms from M to $\mathcal{F}$, i.e.,

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) \tag{3}$$

as abelian groups, where $\cong$ denotes an *isomorphism* (e.g., of abelian groups, left/right modules). This isomorphism can easily be described: if $\phi \in \hom_D(M, \mathcal{F})$, $\eta_j := \phi(y_j)$ for $j = 1, \ldots, p$, and $\eta := (\eta_1\ \ldots\ \eta_p)^T \in \mathcal{F}^p$, then using (2), $R\, \eta = 0$ since for $i = 1, \ldots, q$:

$$\sum_{j=1}^{p} R_{ij}\, \phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \phi\left(\pi\left(R_{i\bullet}\right)\right) = \phi(0) = 0.$$

Conversely, if $\eta \in \ker_{\mathcal{F}}(R.)$, then we can define $\phi_\eta \in \hom_D(M, \mathcal{F})$ by $\phi_\eta(\pi(\lambda)) = \lambda \eta$ for all $\lambda \in D^{1\times p}$. We can prove that the abelian group homomorphism $\chi\colon \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M, \mathcal{F})$ defined by $\chi(\eta) = \phi_\eta$ is bijective. See [1], [3], [4], [11]. Hence, (3) shows that the behaviour $\ker_{\mathcal{F}}(R.)$ can be studied in terms of $\hom_D(M, \mathcal{F})$, and thus by means of the left D-modules M and $\mathcal{F}$.

Within the behavioural approach to multidimensional systems, recent investigations have been done in the direction of the *restriction of behaviours* to subvector spaces of $\mathbb{R}^n$. See [6], [7] and the references therein. The goal of this paper is to shortly explain a possible answer to this problem developed in algebraic analysis or D-module theory [1], [2], [4].

In Section II, we introduce the concept of *inverse images* of D-modules for linear systems of PD equations with polynomial coefficients. In Section III, we shortly extend this concept to linear systems of PD equations with analytic or holomorphic coefficients. In Section III, we shall show that this concept is the main ingredient for the study of the restriction of linear PD systems to a submanifold.

II. INVERSE IMAGES OF D-MODULES

Let k be a field of characteristic 0 (e.g., $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$), $X = k^n$ (resp., $Y = k^m$) the *affine space* of dimension n (resp., m) with coordinates $x = (x_1, \ldots, x_n)$ (resp., $y = (y_1, \ldots, y_m)$). Let us consider the following polynomial map

$$\begin{array}{rcl} f : Y = k^m & \longrightarrow & X = k^n \\ y = (y_1 \, \ldots \, y_m) & \longmapsto & (f_1(y) \, \ldots \, f_n(y)), \end{array}$$

i.e., the f_i's are elements of $k[Y] := k[y_1, \ldots, y_m]$. Now, if $k[X] := k[x_1, \ldots, x_n]$, then we can define

$$\begin{array}{rcl} f^\star : k[X] & \longrightarrow & k[Y] \\ p & \longmapsto & p \circ f, \end{array} \tag{4}$$

where $(p \circ f)(y) = p(f_1(y), \ldots, f_n(y)) \in k[Y]$. In particular, $k[Y]$ inherits a $k[X]$-module structure defined by:

$$\begin{array}{rcl} k[X] \times k[Y] & \longrightarrow & k[Y] \\ (p,\, q) & \longmapsto & (p \circ f)\, q. \end{array} \tag{5}$$

Let $D_X := k[X]\langle d_{x_1}, \ldots, d_{x_n}\rangle = A_n(k)$ and M be a left D_X-module. Using the polynomial map $f : Y \longrightarrow X$, we define a left $D_Y := k[Y]\langle d_{y_1}, \ldots, d_{y_m}\rangle = A_m(k)$-module $f^\star(M)$ called the *inverse image of M under f* [1], [4]. The left D_X-module M can be seen as a $k[X]$-module by forgetting the actions of the derivatives d_{x_i}'s. Since $k[Y]$ is a $k[X]$-module, we can define the following $k[Y]$-module $f^\star(M) := k[Y] \otimes_{k[X]} M$ formed by elements of the form

$$n = \sum_{l=1}^{r} q_l(y) \otimes m_l, \quad r \in \mathbb{Z}_{\geq 0}, \quad q_l \in k[Y], \quad m_l \in M,$$

(see, e.g., [12]) which satisfy the following relation:

$$\forall\, p \in k[X], \quad q_l(y) \otimes p(x)\, m_l = q_l(y)\,(p \circ f)(y) \otimes m_l. \tag{6}$$

The $k[Y]$-module structure of $f^\star(M)$ defined by

$$\forall\, q \in k[Y], \quad q\, n = \sum_{l=1}^{r} q(y)\, q_l(y) \otimes m_l,$$

can be extended into a left D_Y-module structure by

$$d_{y_j}\, n := \sum_{l=1}^{r} \left(\partial_{y_j} q_l(y) \otimes m_l + \sum_{i=1}^{n} q_l(y)\, \partial_{y_j} f_i(y) \otimes d_{x_i}\, m_l \right), \tag{7}$$

for $j = 1, \ldots, m$. Let us check that (7) yields a well-defined left D_Y-module structure on $f^\star(M)$. We have

$$\begin{array}{rl} & d_{y_j}(y_k\,(q(y) \otimes m)) = d_{y_j}(y_k\, q(y) \otimes m) \\ = & \partial_{y_j}(y_k\, q(y)) \otimes m + \sum_{i=1}^n y_k\, q(y)\, \partial_{y_j} f_i(y) \otimes d_{x_i}\, m \\ = & (\partial_{y_j} y_k)\, q(y) \otimes m \\ & + y_k\, \left(\partial_{y_j} q(y) \otimes m + \sum_{i=1}^n q(y)\, \partial_{y_j} f_i(y) \otimes d_{x_i}\, m\right) \\ = & \delta_{jk}\, q(y) \otimes m + y_k\, d_{y_j}\,(q(y) \otimes m), \end{array}$$

where $\delta_{jk} = 1$ if $j = k$ or 0 else, i.e., $d_{y_j}\, y_k - y_k\, d_{y_j} = \delta_{jk}$. We can also check that

$$\begin{array}{rl} & (d_{y_j}\, d_{y_k})(q(y) \otimes m) \\ = & \partial_{y_j} \partial_{y_k} q(y) \otimes m + \sum_{l=1} \partial_{y_k} q(y)\, \partial_{y_j} f_l(y) \otimes d_{x_l}\, m \\ & + \sum_{i=1}^n \partial_{y_j} q(y)\, \partial_{y_k} f_i(y) \otimes d_{x_i}\, m \\ & + \sum_{i=1}^n q(y)\, \partial_{y_j} \partial_{y_k} f_i(y) \otimes d_{x_i}\, m \\ & + \sum_{i=1}^n \sum_{l=1}^n q(y)\, \partial_{y_k} f_i(y)\, \partial_{y_j} f_l(y) \otimes d_{x_l}\, d_{x_i}\, m \end{array}$$

is symmetric in j and k, which yields $d_{y_j}\, d_{y_k} = d_{y_k}\, d_{y_j}$. Finally, we prove that (7) is compatible with (6), i.e.:

$$\begin{array}{c} d_{y_j}(q(y) \otimes p(x)\, m) \\ = \partial_{y_j} q(y) \otimes p(x)\, m + \sum_{i=1}^n q(y)\, \partial_{y_j} f_i(y) \otimes d_{x_i}(p(x)\, m) \\ = \partial_{y_j} q(y)\, (p(f(y)) \otimes m \\ + \sum_{i=1}^n q(y)\, \partial_{y_j} f_i(y) \otimes (p(x)\, d_{x_i} + \partial_{x_i} p(x))\, m \\ = (\partial_{y_j} q(y)\, (p(f(y)) + \sum_{i=1}^n q(y)\, \partial_{y_j} f_i(y)\, (\partial_{x_i} p)(f(y))) \otimes m \\ + \sum_{i=1}^n q(y)\, \partial_{y_j} f_i(y)\, p(f(y)) \otimes d_{x_i}\, m \\ = \partial_{y_j}(q(y)\, p(f(y)) \otimes m \\ + \sum_{i=1}^n q(y)\, p(f(y))\, \partial_{y_j} f_i(y) \otimes d_{x_i}\, m \\ = d_{y_j}(q(y)\, p(f(y)) \otimes m). \end{array}$$

Definition 1 ([1], [2], [4]): If $f : Y = k^m \longrightarrow X = k^n$ is a polynomial map, $D_X = A_n(k)$, $D_Y = A_m(k)$ and M a left D_X-module, then $f^\star(M) := k[Y] \otimes_{k[X]} M$ has a left D_Y-module structure defined by (7) and is called the *inverse image of M under f*.

Example 1: Let us consider $M = D_X$ so that $f^\star(D_X) = k[Y] \otimes_{k[X]} D_X$, which is usually denoted by $D_{Y \to X}$ [1], [2], [4]. An element P of $D_{Y \to X}$ is an operator of the form:

$$P = \sum_{0 \leq |\mu| \leq r} a_\mu(y) \otimes d_x^\mu, \quad a_\mu \in k[Y], \quad d_x^\mu = d_{x_1}^{\mu_1} \ldots d_{x_n}^{\mu_n}.$$

Note that $D_{Y \to X}$ has a $D_Y - D_X$-bimodule structure defined by the left D_Y-module structure given by (7), i.e.,

$$\begin{array}{c} d_{y_j}\, P = \\ \sum_{0 \leq |\mu| \leq r} \left(\partial_{y_j}\, a_\mu(y) \otimes d_x^\mu + \sum_{i=1}^n a_\mu(y)\, \partial_{y_j}\, f_i(y) \otimes d_x^{\mu + 1_i}\right), \end{array} \tag{8}$$

where $\mu + 1_i := (\mu_1, \ldots, \mu_i + 1, \ldots, \mu_n)$, and by the natural right D_X-module structure of D_X.

Using $a_\mu(y)\,(1\otimes d_x^\mu)=a_\mu(y)\otimes d_x^\mu$ and (8), the ring $D_{Y\to X}$ is generated by $\{1\otimes d_x^\mu\}_{\mu\in\mathbb{Z}^n_{\geq 0}}$ as a left D_Y-module, i.e., using (8), we have:

$$\left\{\begin{array}{rcl} P & = & \sum_{0\leq|\mu|\leq r} a_\mu(y)(1\otimes d_x^\mu),\\ d_{y_j}\,P & = & \sum_{0\leq|\mu|\leq r}\partial_{y_j}a_\mu(y)(1\otimes d_x^\mu)\\ & & +\sum_{0\leq|\mu|\leq r}\sum_{i=1}^n a_\mu(y)\,\partial_{y_j}f_i(y)(1\otimes d_x^{\mu+1_i}).\end{array}\right.$$

Finally, considering the element $1\otimes 1\in D_{Y\to X}$, (8) yields:

$$\forall\, j=1,\ldots,m,\quad d_{y_j}\,(1\otimes 1)=\sum_{i=1}^n \partial_{y_j}f_i(y)\otimes d_{x_i}.$$

If f is a linear map (see, e.g., [6], [7]), i.e., $f(y)=A\,y$, where $A\in k^{n\times m}$, then, for $j=1,\ldots,m$, we get:

$$d_{y_j}\,(1\otimes 1)=\sum_{i=1}^n A_{ij}\otimes d_{x_i}=1\otimes\sum_{i=1}^n A_{ij}\,d_{x_i}.$$

If $d_x=(d_{x_1}\,\ldots\,d_{x_n})^T$ and $A=(A_{\bullet 1}\,\ldots\,A_{\bullet m})$, where $A_{\bullet i}$ denotes the i^{th} column of A, then $\sum_{i=1}^n A_{ij}\,d_{x_i}=A_{\bullet i}^T\,d_x$.

If M is a left D_X-module, $D_X\otimes_{D_X}M\cong M$ [12] yields

$$\begin{array}{rcl} f^\star(M) & = & k[Y]\otimes_{k[X]}M\cong k[Y]\otimes_{k[X]}(D_X\otimes_{D_X}M)\\ & = & (k[Y]\otimes_{k[X]}D_X)\otimes_{D_X}M\\ & = & D_{Y\to X}\otimes_{D_X}M,\end{array}\tag{9}$$

which shows that the ring $D_{Y\to X}$ has to be studied in detail.

Example 2: Let $m=n+l$, $l\in\mathbb{Z}_{\geq 0}$, $Z=k^l$ and

$$\begin{array}{rcl} f:Y=k^{n+l}=X\times Z & \longrightarrow & X=k^n\\ y=(x,\,z) & \longmapsto & x,\end{array}$$

i.e., f is a projection. If M is a left D_X-module, then:

$$f^\star(M)=k[X,Z]\otimes_{k[X]}M.$$

An element of $f^\star(M)$ is a sum of terms of the form $q(x,z)\otimes m$, where $q\in k[X,Z]$ and $m\in M$. Since $q\in k[X,Z]$ can be rewritten as $q=\sum_{0\leq|\nu|\leq r}q_\nu(x)\,z^\nu$, where $\nu=(\nu_1\,\ldots\,\nu_l)\in\mathbb{Z}^l_{\geq 0}$, we get:

$$q(x,z)\otimes m=\sum_{0\leq|\nu|\leq r}z^\nu\otimes q_\nu(x)\,m.$$

Thus, an element of $f^\star(M)$ can be written as a sum of terms of the form $z^\nu\otimes m'$, where $m'\in M$. Now, we note that we have $k[X,Z]\cong k[Z]\widehat{\otimes}_k k[X]$, where $k[Z]\widehat{\otimes}_k k[X]$ is the k-algebra formed by elements which are sums of terms of the form $p(z)\otimes q(x)$ and with the product defined by

$$(p(z)\otimes q(x))\,(p'(z)\otimes q'(x))=(p(z)\,p'(z))\otimes(q(x)\,q'(x)),$$

for all $p,\,p'\in k[Z]$ and $q,\,q'\in k[X]$. Then, we can define the $k[Z]\widehat{\otimes}_k k[X]$-module $k[Z]\widehat{\otimes}_k M$ formed by elements which are sums of terms of the form $z^\nu\otimes m$ with $\nu\in\mathbb{Z}_{\geq 0}$ and $m\in M$ and endowed with the following product:

$$(p(z)\otimes q(x))\,(z^\nu\otimes m)=p(z)\,z^\nu\otimes q(x)\,m.$$

Then, we have the following $k[X,Z]$-isomorphism:

$$\begin{array}{rcl} f^\star(M) & \longrightarrow & k[Z]\widehat{\otimes}_k M\\ \left(\sum_{0\leq|\nu|\leq r}q_\nu(x)\,z^\nu\right)\otimes m & \longmapsto & \sum_{0\leq|\nu|\leq r}z^\nu\otimes q_\nu(x)\,m.\end{array}\tag{10}$$

Now, the left D_Y-module structure of $f^\star(M)$ is given by

$$\begin{array}{rl} & d_{x_i}(q_\nu(x)\,z^\nu\otimes m)\\ = & \partial_{x_i}(q_\nu(x)\,z^\nu)\otimes m+\sum_{j=1}^n q_\nu(x)\,z^\nu\,\partial_{x_i}x_j\otimes d_{x_j}\,m,\\ = & \partial_{x_i}q_\nu(x)\,z^\nu\otimes m+q_\nu(x)\,z^\nu\otimes d_{x_i}\,m\\ = & z^\nu\otimes(\partial_{x_i}q_\nu(x)+q_\nu(x)\,d_{x_i})\,m\\ = & z^\nu\otimes d_{x_i}(q_\nu(x)\,m),\quad i=1,\ldots,n,\\ & d_{z_k}(q_\nu(x)\,z^\nu\otimes m)\\ = & \partial_{z_k}(q_\nu(x)\,z^\nu)\otimes m+\sum_{j=1}^n q_\nu(x)\,z^\nu\,\partial_{z_k}x_j\otimes d_{x_j}\,m\\ = & q_\nu(x)\,\partial_{z_k}z^\nu\otimes m,\quad k=1,\ldots,l.\end{array}$$

Hence, using (10), we obtain that the d_{x_i}'s act only on M and the d_{z_k}'s act only on $k[Z]$, i.e.:

$$\left\{\begin{array}{l} d_{x_i}(z^\nu\otimes q_\nu(x)\,m)=z^\nu\otimes d_{x_i}(q_\nu(x)\,m),\\ d_{z_k}(z^\nu\otimes q_\nu(x)\,m)=\partial_{z_k}z^\nu\otimes q_\nu(x)\,m.\end{array}\right.\tag{11}$$

Using the isomorphism $D_Y\cong D_Z\widehat{\otimes}_k D_X$ defined by

$$\begin{array}{rcl} D_Y & \longrightarrow & D_Z\widehat{\otimes}_k D_X\\ \sum a_{\mu\,\nu}\,y^\mu\,d_y^\nu & \longrightarrow & \sum a_{\mu\nu}\,(z^{\mu_2}\,d_z^{\nu_2}\otimes x^{\mu_1}d_x^{\nu_1}),\end{array}$$

where $\mu=(\mu_1,\,\mu_2)$ and $\nu=(\nu_1,\,\nu_2)$, $\mu_1,\,\nu_1\in\mathbb{Z}^n_{\geq 0}$ and $\mu_2,\,\nu_2\in\mathbb{Z}^l_{\geq 0}$, and the $D_Z\widehat{\otimes}_k D_X$-module structure of $k[Z]\widehat{\otimes}_k M$ defined by

$$(z^{\mu_2}\,d_z^{\nu_2}\otimes x^{\mu_1}d_x^{\nu_1})\,(z^\theta\otimes m)=z^{\mu_2}\,\partial_z^{\nu_2}\,z^\theta\otimes x^{\mu_1}d_x^{\nu_1}\,m$$

(11) then shows that (10) is an isomorphism of left $D_Y\cong D_Z\widehat{\otimes}_k D_X$-modules.

If we now consider $M=D_X=A_n(k)$, then we have

$$D_{Y=X\times Z\twoheadrightarrow X}=k[X,Z]\otimes_{k[X]}D_X\cong k[Z]\widehat{\otimes}_k D_X$$

as $D_Y\cong D_Z\widehat{\otimes}_k D_X$-modules. In particular, using (11), we have $d_{x_i}(1\otimes d_x^\mu)=1\otimes d_x^{\mu+1_i}$ and $d_{z_k}(1\otimes d_x^\mu)=0$. Finally, using the left D_Z-isomorphism $k[Z]\cong D_Z/(\sum_{k=1}^l D_Z\,d_{z_k})$ and the following isomorphism of D_Z-D_X-bimodules [2]

$$\left(D_Z/\left(\sum_{k=1}^l D_Z\,d_{z_k}\right)\right)\widehat{\otimes}_k D_X\cong D_Y/\left(\sum_{j=n+1}^m D_Y\,d_{y_j}\right),$$

we get the following isomorphism of D_Z-D_X-bimodules:

$$D_{Y=X\times Z\twoheadrightarrow X}\cong k[Z]\widehat{\otimes}_k D_X\cong D_Y/\left(\sum_{j=n+1}^m D_Y\,d_{y_j}\right).\tag{12}$$

Example 3: Let $n=m+l$, $l\in\mathbb{Z}_{\geq 0}$, $Z=k^l$,

$$\begin{array}{rcl} f:Y=k^m & \longrightarrow & X=k^n=Y\times Z\\ y & \longmapsto & (y,\,0),\end{array}$$

i.e., f is a standard embedding, and a left D_X-module M. If $z_1=x_{m+1},\ldots,z_l=x_n$ and $k[Z]:=k[z_1,\ldots,z_l]$, then we can consider the left D_Y-module $f^\star(M)=k[Y]\otimes_{k[Y,Z]}M$.

Using (4), we get $f^\star(p(y,z)) = p(y,0)$ for all $p \in k[Y,Z]$. We note that $k[Y] \cong k[Y,Z]/(Z)$, where $(Z) = (z_1, \ldots, z_l)$ is the ideal of $k[Y,Z]$ generated by the z_i's. Hence, we get $f^\star(M) \cong k[Y,Z]/(Z) \otimes_{k[Y,Z]} M$ as $k[Y,Z]$-modules.

Using the fact that $D_Y = A_m(k) \subseteq D_X = A_{m+l}(k)$, the D_X-module M can be considered as a left D_Y-module. Any element $P = \sum_{0 \le |\mu| \le r} a_\mu(y)\, d_y^\mu$ of D_Y commutes with all the z_k's, i.e., we have $D_Y\,(Z) = (Z)\,D_Y$ in D_X, which proves that $(Z)\,M := \{p\,m \mid p \in (Z),\, m \in M\}$ is a left D_Y-submodule of M. Then, we can consider the left D_Y-module $M/(Z)\,M$. Let $\sigma : M \longrightarrow M/(Z)\,M$ be the canonical projection. Let us also consider the map:

$$\begin{array}{rcl} \chi : f^\star(M) = k[Y] \otimes_{k[Y,Z]} M & \longrightarrow & M/(Z)\,M \\ q(y) \otimes m & \longmapsto & q(y)\,\sigma(m). \end{array}$$

The map χ is well-defined since $q(y) \otimes m = 1 \otimes q(y)\,m$ and $\chi(q(y) \otimes m) = q(y)\,\sigma(m) = \sigma(q(y)\,m) = \chi(1 \otimes q(y)\,m)$.

Let us now check that χ is a $k[Y]$-homomorphism:

$$\begin{array}{l} \forall\, q,\, r \in k[Y],\ \chi(r\,(q \otimes m)) = \chi(r\,q \otimes m) = (r\,q)\,\sigma(m) \\ \qquad\qquad\qquad\qquad = r\,\sigma(q\,m) = r\,\chi(q \otimes m). \end{array}$$

Using (7) and the fact that σ is left D_Y-homomorphism, let us prove that χ is a left D_Y-homomorphism:

$$\begin{array}{l} \chi(d_{y_j}(q \otimes m)) = \chi(\partial_{y_j} q \otimes m + \sum_{i=1}^n q\,\partial_{y_j} f_i \otimes d_{x_i}\, m) \\ = \chi(\partial_{y_j} q \otimes m + \sum_{i=1}^m q\,\partial_{y_j} y_i \otimes d_{y_i}\, m) \\ = \chi\left(\partial_{y_j} q \otimes m + q \otimes d_{y_j}\, m\right) = \partial_{y_j} q\,\sigma(m) + q\,\sigma(d_{y_j}\, m) \\ = \left(\partial_{y_j} q + q\,d_{y_j}\right)\,\sigma(m) = d_{y_j}\,(q\,\sigma(m)) = d_{y_j}\,\chi(q \otimes m). \end{array}$$

Let us now check that χ is an isomorphism of D_Y-modules. If $n = \sum_{r=1}^s q_r \otimes m_r \in \ker \chi$, i.e., $\sum_{r=1}^s q_r\,\sigma(m_r) = 0$, then $\sigma\left(\sum_{r=1}^s q_r\,m_r\right) = 0$, which shows that there exist $m_1, \ldots, m_l \in M$ such that $\sum_{r=1}^s q_r\,m_r = \sum_{k=1}^l z_k\,m_k$, and using (6), we get:

$$n = 1 \otimes \sum_{r=1}^s q_r\,m_r = 1 \otimes \sum_{k=1}^l z_k\,m_k = \sum_{k=m+1}^n f_k \otimes m_k = 0.$$

χ is surjective since every element $\sigma(m) \in M/(Z)\,M$ is such that $\chi(1 \otimes m) = \sigma(m)$, which finally proves that χ is a left D_Y-isomorphism, i.e., $f^\star(M) \cong M/(Z)\,M$.

Using the above result, we obtain:

$$D_{Y \rightarrowtail X = Y \times Z} \cong D_X/(Z)\,D_X. \tag{13}$$

Proposition 1: [1], [2], [4] Let X, Y and Z be three affine spaces, $f : Y \longrightarrow X$ and $g : Z \longrightarrow Y$ two polynomial maps and M a left D_X-module. Then, we have:

$$(f \circ g)^\star(M) \cong g^\star(f^\star(M)).$$

In particular, if $g = f^{-1}$, then $M \cong f^{-1\star}(f^\star(M))$.

Let $f : Y = k^m \longrightarrow X = k^n$ be a polynomial map and:

$$\begin{array}{rcl} g : Y & \longrightarrow & Y \times X \\ y & \longmapsto & (y, f(y)), \end{array} \qquad \begin{array}{rcl} w : Y \times X & \longrightarrow & X \\ (y,\, x) & \longmapsto & x. \end{array} \tag{14}$$

Then, we have $f = w \circ g$, i.e., any map f is the composition of an embedding and a projection. g is not a standard embedding but g can be written as $g = v \circ u$, where

$$\begin{array}{rcl} u : Y & \longrightarrow & Y \times X \\ y & \longmapsto & (y,\, 0), \end{array} \qquad \begin{array}{rcl} v : Y \times X & \longrightarrow & Y \times X \\ (y,\, x) & \longmapsto & (y,\, x + f(y)). \end{array} \tag{15}$$

The polynomial map u is a standard embedding and v is an invertible polynomial map. Then, using $f = w \circ v \circ u$ and Proposition 1, we get $f^\star(M) = u^\star(v^\star(w^\star(M)))$ for all left D_X-modules. Using (9), we obtain:

$$\begin{array}{l} f^\star(M) \cong D_{Y \overset{u}{\rightarrowtail} Y \times X} \otimes_{D_{Y \times X}} D_{Y \times X \overset{v}{\longrightarrow} Y \times X} \\ \qquad\qquad \otimes_{D_{Y \times X}} D_{Y \times X \overset{w}{\twoheadrightarrow} X} \otimes_{D_X} M. \end{array} \tag{16}$$

Example 4: Let us consider the above polynomial map v and a left $D_{Y \times X} = A_{m+n}(k)$-module N. Using (4), we have $(p \circ v)((y,\, x)) = p(y, x + f(y))$ for all $p \in k[Y,X]$. Then, $v^\star(N) = k[Y,X] \otimes_{k[Y,X]} N$, where the left $D_{Y \times X}$-module structure defined by:

$$\begin{array}{rl} & d_{y_j}(q(y,x) \otimes n) \\ = & \partial_{y_j} q(y,x) \otimes n + \sum_{i=1}^m q(y,x)\,\partial_{y_j} y_i \otimes d_{y_i}\, n \\ & + \sum_{i=1}^n q(y,x)\,\partial_{y_j}(x_i + f_i(y)) \otimes d_{x_i}\, n \\ = & \partial_{y_j} q(y,x) \otimes n + q(y,x) \otimes d_{y_j}\, n \\ & + \sum_{i=1}^n q(y,x)\,\partial_{y_j} f_i(y) \otimes d_{x_i}\, n, \\ & d_{x_j}(q(y,x) \otimes n) \\ = & \partial_{x_j} q(y,x) \otimes n + \sum_{i=1}^m q(y,x)\,\partial_{x_j} y_i \otimes d_{y_i}\, n \\ & + \sum_{i=1}^n q(y,x)\,\partial_{x_j}(x_i + f_i(y)) \otimes d_{x_i}\, n \\ = & \partial_{x_j} q(y,x) \otimes n + q(y,x) \otimes d_{x_j}\, n. \end{array} \tag{17}$$

Let us consider the following automorphism of $D_{Y \times X}$

$$\begin{array}{rcl} \alpha : D_{Y \times X} & \longrightarrow & D_{Y \times X} \\ y_j & \longmapsto & y'_j = y_j, \\ x_i & \longmapsto & x'_i = x_i - f_i(y), \\ d_{y_j} & \longrightarrow & d_{y'_j} = d_{y_j} + \sum_{i=1}^n \partial_{y_j} f_i(y)\, d_{x_i}, \\ d_{x_i} & \longmapsto & d_{x'_i} = d_{x_i}, \end{array} \tag{18}$$

for $j = 1, \ldots, m$ and $i = 1, \ldots, n$. We can easily check that:

$$d_{y'_i}\, y'_j - y'_j\, d_{y'_i} = \delta_{ij}, \quad d_{x'_i}\, x'_j - x'_i\, d_{x'_j} = \delta_{ij}.$$

Let us now introduce the left $D_{Y \times X}$-module M_α defined by M as an abelian group but equipped with the new left $D_{Y \times X}$-module structure defined by:

$$\forall\, d \in D_{Y \times X}, \quad \forall\, m \in M_\alpha, \quad d \bullet m := \alpha(d)\, m.$$

Let us also define the following map:

$$\begin{array}{rcl} \iota : v^\star(M) & \longrightarrow & M_\alpha \\ q \otimes m & \longmapsto & q \bullet m = \alpha(q)\, m. \end{array}$$

Let us now prove that ι is a left $D_{Y \times X}$-homomorphism, i.e.:

$$\left\{ \begin{array}{l} \iota(d_{y_j}\,(q \otimes m)) = d_{y_j} \bullet \iota(q \otimes m) = \alpha(d_{y_j})\,\iota(q \otimes m), \\ \iota(d_{x_i}\,(q \otimes m)) = d_{x_i} \bullet \iota(q \otimes m) = \alpha(d_{x_i})\,\iota(q \otimes m). \end{array} \right.$$

Applying ι to the first identity of (17), we obtain:

$$\begin{array}{rl} & \iota(d_{y_j}(q(y,x)\otimes m)) \\ = & \iota\left(\partial_{y_j} q(y,x)\otimes m\right)+\iota(q(y,x)\otimes d_{y_j}\, m) \\ & +\sum_{i=1}^n \iota\left(q(y,x)\,\partial_{y_j} f_i(y)\otimes d_{x_i}\, m\right) \\ = & \alpha\left(\partial_{y_j} q(y,x)\right)\, m+\alpha(q(y,x))\, d_{y_j}\, m \\ & +\sum_{i=1}^n \alpha\left(q(y,x)\,\partial_{y_j} f_i(y)\right)\, d_{x_i}\, m \\ = & \alpha\left(\partial_{y_j} q(y,x)\right)\, m+\alpha(q(y,x))\, d_{y_j}\, m \\ & +\sum_{i=1}^n \alpha\left(q(y,x)\right)\, \partial_{y_j} f_i(y)\, d_{x_i}\, m \\ = & \alpha(q(y,x))\, \left(d_{y_j}+\sum_{i=1}^n \partial_{y_j} f_i(y)\, d_{x_i}\right)\, m \\ & +\alpha\left(\partial_{y_j} q(y,x)\right)\, m \\ = & \left(\alpha(q(y,x))\,\alpha(d_{y_j})+\alpha\left(\partial_{y_j} q(y,x)\right)\right)\, m. \end{array}$$

Applying α to $q(y,x)\, d_{y_j}+\partial_{y_j} q(y,x) = d_{y_j}\, q(y,x)$ (see (1)), we obtain $\alpha(q(y,x))\,\alpha(d_{y_j})+\alpha\left(\partial_{y_j} q(y,x)\right)=\alpha(d_{y_j})\,\alpha(q(y,x))$, which then yields:

$$\begin{aligned} \iota(d_{y_j}(q(y,x)\otimes m)) &= \alpha(d_{y_j})\,\alpha(q(y,x))\, m \\ &= \alpha(d_{y_j})\,\iota(q(y,x)\otimes m). \end{aligned}$$

Now, applying ι to the second identity of (17), we get:

$$\begin{array}{rl} & \iota(d_{x_j}(q(y,x)\otimes m)) \\ = & \iota\left(\partial_{x_j} q(y,x)\otimes m\right)+\iota(q(y,x)\otimes d_{x_j}\, m) \\ = & \alpha\left(\partial_{x_j} q(y,x)\right)\, m+\alpha(q(y,x))\, d_{x_j}\, m \\ = & (\alpha(\partial_{x_j} q(y,x))+\alpha(q(y,x))\,\alpha(d_{x_j}))\, m. \end{array}$$

Applying α to $q(y,x)\, d_{x_j}+\partial_{x_j} q(y,x)=d_{x_j}\, q(y,x)$ (see (1)), we obtain

$$\alpha(q(y,x))\,\alpha(d_{x_j})+\alpha\left(\partial_{x_j} q(y,x)\right)=\alpha(d_{x_j})\,\alpha(q(y,x)),$$

which then yields:

$$\begin{aligned} \iota(d_{x_j}\,(q(y,x)\otimes m)) &= \alpha(d_{x_j})\,\alpha(q(y,x))\, m \\ &= \alpha(d_{x_j})\,\iota(q(y,x)\otimes m). \end{aligned}$$

ι is surjective since $\iota(1\otimes m)=m$. Using (6), we have:

$$\begin{aligned} 1\otimes q(y,x)\, m = (q\circ v)(y,x)\otimes m &= q(y,x+f(y))\otimes m \\ &= \alpha^{-1}(q)\otimes m. \end{aligned}$$

Hence, $q\otimes m\in\ker\iota$ iff $\alpha(q)\, m=0$, which finally yields $q\otimes m=1\otimes\alpha(q)\, m=0$, i.e., ι is injective and thus ι is an isomorphism of left $D_{Y\times X}$-modules, i.e., $v^\star(M)\cong M_\alpha$.

Example 5: Let $Y=k^m$, $X=k^n$, $f: Y\longrightarrow X$ be a polynomial map, the map g defined by (14)

$$\begin{array}{rcl} g: Y=k^m & \longrightarrow & Z=k^{m+n}=Y\times X \\ y & \longmapsto & (y,\ f(y)), \end{array}$$

and a left D_Z-module M. Using (4), we have $(p\circ g)(y)=p(y,f(y))$ for all $p\in k[Y,X]$. Following Example 3, we get that $k[Y]\cong k[Y,X]/(x_1-f_1(y),\dots,x_n-f_n(y))$ and

$$\begin{aligned} g^\star(M) &= k[Y]\otimes_{k[Y,X]} M \\ &\cong k[Y,X]/(x_1-f_1(y),\dots,x_n-f_n(y))\otimes_{k[Y,X]} M, \end{aligned}$$

where the left D_Z-module structure defined by:

$$\begin{aligned} d_{y_j}(q(y)\otimes m) &= \partial_{y_j} q(y)\otimes m+\sum_{i=1}^m q(y)\,\partial_{y_j} y_i\otimes d_{y_i}\, m \\ &\quad +\sum_{i=1}^n q(y)\,\partial_{y_j} f_i(y)\otimes d_{x_i}\, m \\ &= \partial_{y_j} q(y)\otimes m+q(y)\otimes d_{y_j}\, m+\sum_{i=1}^n q(y)\,\partial_{y_j} f_i(y)\otimes d_{x_i}\, m. \end{aligned}$$

Using (15), we have $g=v\circ u$ which by Proposition 1 yields $g^\star(M)=u^\star(v^\star(M))$:

$$g^\star(M)=D_{Y\xrightarrow{u} Y\times X}\otimes_{D_Z} D_{Z\xrightarrow{v} Z}\otimes_{D_Z} M.$$

Using Example 4, we get $v^\star(M)\cong D_{Z\xrightarrow{v} Z}\otimes_{D_Z} M\cong M_\alpha$, where α is the automorphism of $D_Z=A_{m+n}(k)$ defined by (18). If $(X')=(x_1',\dots,x_n')=(x_1-f_1(y),\dots,x_n-f_n(y))$, then using Example 3, we obtain:

$$g^\star(M)=u^\star(v^\star(M))\cong M_\alpha/(X')\, M_\alpha\cong D_{Y\xrightarrow{g} Z}\otimes_{D_Z} M.$$

Finally, if $f: Y\longrightarrow X$ is a general polynomial map and $D_Z := D_{Y\times X}=A_{m+n}(k)$, then using (14) and (16), we obtain $f^\star(M)\cong D_{Y\xrightarrow{f} X}\otimes_{D_Z} M$, where:

$$D_{Y\xrightarrow{f} X}\cong D_Z/\left(\left(\sum_{j=1}^m D_Z\, d_{y_j}\right)+\sum_{i=1}^n (x_i-f_i(y))\, D_Z\right).$$

III. Cauchy-Kowalevski-Kashiwara Theorem

Let X (resp., Y) be a manifold of dimension n (resp., m) with local coordinates $x=(x_1,\dots,x_n)$ (resp., $y=(y_1,\dots,y_m)$). Let TX (resp., $T^\star X$) denotes the *tangent bundle* (resp., *cotangent bundle*) of X, i.e., the *disjoint union* of the tangent spaces of X (resp., the *dual bundle* of TX)

$$\begin{array}{l} \bigcup_{x\in X}\{x\}\times T_x X=\bigcup_{x\in X}\{(x,t)\mid t\in T_x X\}, \\ \left(\bigcup_{x\in X}\{x\}\times T_x^\star X=\bigcup_{x\in X}\{(x,s)\mid s\in T_x^\star X\}\right). \end{array}$$

In the context of analytic D-modules [4], all the results developed in Sections I and II can be extended to linear PD systems with analytic or holomorphic coefficients. Analytic D-module theory is usually more complicated than algebraic D-module theory since it uses *sheaf theory* [12]. The ring D of PD operators with coefficients in the differential ring A has to be replaced by the *sheaf $\mathcal{D}_X$ of rings of PD operators* on a complex manifold X of dimension n. The *stalk* $\mathcal{D}_x$ at a point $x\in X$ is defined by elements of the form $\sum_{0\leq|\mu|\leq r} a_\mu(x)\, d_x^\mu$, where the $a_\mu(x)$'s are *germs* of holomorphic functions at x. Moreover, the finitely generated left D-module M is replaced by a *coherent sheaf* left $\mathcal{D}_X$-module $\mathcal{M}$ [12], i.e., for any $x\in X$, there exists a neighbourhood U of x in which $\mathcal{M}$ admits a finite presentation:

$$\mathcal{D}_U^q \xrightarrow{.\mathcal{R}} \mathcal{D}_U^p \xrightarrow{\pi} \mathcal{M}_U \longrightarrow 0.$$

If $\mathcal{O}_X$ is the *sheaf of germs of holomorphic functions* over X, then the sheaf $hom_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)$ corresponds to the sheaf of holomorphic solutions of the linear PD system defined by

$\mathcal{M}$, i.e., locally in a neighbourhood U of x, by a linear PD system of the form $\mathcal{R}\,\eta = 0$, where $\mathcal{R} \in \mathcal{D}_U^{q\times p}$.

We can define an *increasing filtration* $\{F_r(\mathcal{D}_X)\}_{r\geq -1}$ of the ring $\mathcal{D}_X$ by $F_{-1}(\mathcal{D}_X) := 0$, $F_0(\mathcal{D}_X) := \mathcal{O}_X$ and

$$F_r(\mathcal{D}_X) := \{P \in \mathrm{end}_{\mathbb{C}}(\mathcal{O}_X) \mid \forall\, f \in \mathcal{O}_X : [P, f] \in F_{r-1}(\mathcal{D}_X)\},$$

where $[P, f_1] := P\, f_1 - f_1\, P$, and the associated *graded ring*:

$$\mathrm{gr}(\mathcal{D}_X) := \bigoplus_{r\in\mathbb{Z}_{\geq 0}} F_r(\mathcal{D}_X)/F_{r-1}(\mathcal{D}_X).$$

We note that $F_1(\mathcal{D}_X) = \mathcal{O}_X \oplus \Theta_X$, where Θ_X is the *sheaf of vector fields* on X, i.e., locally, $\Theta_X = \bigoplus_{i=1}^n \mathcal{O}_X\, d_{x_i}$. If $P = \sum_{0\leq|\mu|\leq r} a_\mu(x)\, d_x^\mu \in F_r(\mathcal{D}_X) \setminus F_{r-1}(\mathcal{D}_X)$, then $\sigma_r(P) := \sum_{|\mu|=r} a_\mu(x)\, \chi^\mu$ is called the *principal symbol* of P. If $\chi_i := \sigma_1(d_{x_i})$, $i = 1, \ldots, n$, then we can prove that $\mathrm{gr}(\mathcal{D}_X) = \mathcal{O}_X[\chi_1, \ldots, \chi_n]$ and that the χ_i's are the coordinate system of the *cotangent space* $\bigoplus_{i=1}^n \mathbb{C}\, dx_i$ a fact showing that an element of $\mathrm{gr}(\mathcal{D}_X)$ is a function on $T^\star X$ which is analytic in the x_i's and polynomial in the χ's [4].

If $\mathcal{M}$ is a coherent left $\mathcal{D}_X$-module, then $\mathcal{M}$ is locally generated by $\{y_j\}_{j=1,\ldots,p}$ and we can consider the increasing filtration $F_r(\mathcal{M}) := \sum_{j=1}^p F_r(\mathcal{D}_X)\, y_j$ of $\mathcal{M}$ and

$$\mathrm{gr}(\mathcal{M}) := \bigoplus_{r\in\mathbb{Z}_{\geq 0}} F_r(\mathcal{M})/F_{r-1}(\mathcal{M})$$

the *graded* $\mathrm{gr}(\mathcal{D}_X)$*-module* $\mathrm{gr}(\mathcal{M})$ associated with $\mathcal{M}$. The *characteristic ideal* $J(\mathcal{M})$ of $\mathcal{M}$ is then defined by:

$$J(\mathcal{M}) = \sqrt{\mathrm{ann}_{\mathrm{gr}(\mathcal{D}_X)}(\mathcal{M})}$$
$$:= \{a \in \mathrm{gr}(\mathcal{D}_X) \mid \exists\, l \in \mathbb{Z}_{\geq 0} : \; \forall\, u \in \mathrm{gr}(\mathcal{M}),\; a^l\, u = 0\}.$$

Definition 2: [4] The *characteristic variety* $\mathrm{char}(\mathcal{M})$ is the conic analytic subset of $T^\star X$ defined by:

$$\mathrm{char}(\mathcal{M}) := \{(x, \chi) \in T^\star X \mid \forall\, a \in J(\mathcal{M}) : \; a((x, \chi)) = 0\}.$$

Definition 3: [4] A submanifold Y of X is called *non-characteristic* for $\mathcal{M}$ if for every $(x, \chi) \in \mathrm{char}(\mathcal{M})$ such that $\chi_{|Y} = 0$, we then have $\chi = 0$, where $\chi_{|Y}$ denotes the restriction of χ to Y, i.e., if there is no non-trivial element of $\mathrm{char}(\mathcal{M})$ which reduces to 0 on Y.

Example 6: Let $X = \mathbb{C}^2$, $P := d_{x_1} - d_{x_2} \in \mathcal{D}_X$ and $\mathcal{M} = \mathcal{D}_X/(\mathcal{D}_X\, P)$ be the coherent left $\mathcal{D}_X$-module defined by $\partial_{x_1} u(x_1, x_2) - \partial_{x_2} u(x_1, x_2) = 0$. Then, we have:

$$\begin{aligned}\mathrm{char}(\mathcal{M}) &= \{((x_1, x_2), \chi = \chi_1\, dx_1 + \chi_2\, dx_2) \mid \chi_1 = \chi_2\} \\ &= \{((x_1, x_2), \chi = \chi_1\, (dx_1 + dx_2))\}.\end{aligned}$$

Let Y be a submanifold of X defined by a smooth curve $s \in \mathbb{R} \longmapsto (x_1 = \phi_1(s), x_2 = \phi_2(s)) \in X$ and the 1-form $\omega = dx_1 + dx_2 \in \mathrm{char}(\mathcal{M})$. Then, $dx_1 = \dot{\phi}_1(s)\, ds$ and $dx_2 = \dot{\phi}_2(s)\, ds$, which yields $\omega_{|Y} = (\dot{\phi}_1 + \dot{\phi}_2)\, ds$. Thus, $\omega_{|Y} = 0$ iff $\dot{\phi}_1 + \dot{\phi}_2 = 0$, i.e., $\phi_1 + \phi_2 = c \in \mathbb{R}$, which yields $(x_1 = \phi_1(s), x_2 = c - \phi_1(s))$, i.e., $x_1 + x_2 = c$. Hence, $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = c\}$ is characteristic for $\mathcal{M}$.

Example 7: Let $P = \sum_{0\leq|\mu|\leq r} a_\mu(x)\, \partial^\mu$ be a differential operator of order r and $\mathcal{M} = \mathcal{D}_X/(\mathcal{D}_X\, P)$. Then, we have

$$\begin{aligned}\mathrm{char}(\mathcal{M}) = \Big\{&(x, \chi = \textstyle\sum_{i=1}^n \chi_i\, dx_i) \\ &\mid \sigma_r(P)(x, \chi) = \textstyle\sum_{|\mu|=r} a_\mu(x)\, \chi_1^{\mu_1}\, \ldots\, \chi_n^{\mu_n} = 0\Big\}.\end{aligned}$$

Let $Y = \{x \in X \mid x_1 = 0\}$ be a submanifold of X. Then, $\chi_{|Y} = \chi_2\, dx_2 + \ldots + \chi_n\, dx_n = 0$ yields $\chi_i = 0$ for $i = 2, \ldots, n$. If we note $z := (0, x_2, \ldots, x_n) \in X$, then we get:

$$\sigma_r(P)(z, \chi_1\, dx_1) = a_{(r,0,\ldots,0)}(z)\, \chi_1^r.$$

Thus, Y is non-characteristic for $\mathcal{M}$ iff $a_{(r,0,\ldots,0)}(z) \neq 0$, i.e., $\sigma_r(P)(z, dx_1) \neq 0$.

Let $f : Y \longrightarrow X$ be holomorphic map of holomorphic manifolds and $\mathcal{F}_X$ a sheaf on X. Then, $f^{-1}\mathcal{F}_X$ is the sheaf on Y defined by $(f^{-1}\mathcal{F}_X)(V) := \lim_{\longrightarrow} \mathcal{F}_X(U)$ for all open sets V of Y, where $\lim_{\longrightarrow}$ denotes the *inductive limit* over the set $\{U$ open set of $X \mid f(V) \subseteq U\}$ [12].

Theorem 1: [4] Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module, Y a submanifold of X which is non-characteristic for $\mathcal{M}$, $i : Y \rightarrowtail X$ the embedding and $\mathcal{D}_{Y\to X} := \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X$. Then, the inverse image of $\mathcal{M}$ under i, i.e.,

$$\mathcal{M}_Y := i^\star(\mathcal{M}) = \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{M} \cong \mathcal{D}_{Y\to X} \otimes_{i^{-1}\mathcal{D}_X} i^{-1}\mathcal{M}$$

is a coherent left $\mathcal{D}_Y$-module, and the canonical map

$$i^{-1}(hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \longrightarrow hom_{\mathcal{D}_Y}(i^\star(\mathcal{M}), i^\star(\mathcal{O}_X)),$$

that is to say $hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{|Y} \longrightarrow hom_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$, is an isomorphism. In other words, we have:

$$hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{|Y} \cong hom_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Theorem 1 shows that the restriction of the behaviour $hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ to the non-characteristic submanifold Y of X is the behaviour $hom_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$ defined by the inverse image $\mathcal{M}_Y = i^\star(\mathcal{M})$ of $\mathcal{M}$ under $i : Y \rightarrowtail X$.

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