Isomorphisms and Serre's reduction of linear systems

Thomas Cluzeau
University of Limoges ;
CNRS ; XLIM UMR 7252, DMI,
123 avenue Albert Thomas,
87060 Limoges cedex, France
Email: cluzeau@ensil.unilim.fr

Alban Quadrat
Inria Saclay - Île-de-France,
DISCO project, L2S, Supélec,
3 rue Joliot Curie,
91192 Gif-sur-Yvette cedex, France.
Email: alban.quadrat@inria.fr

***Abstract*—In this paper, we give an explicit characterization of isomorphic finitely presented modules in terms of certain inflations of their presentation matrices. In particular cases, this result yields a characterization of isomorphic modules as the completion problem characterizing Serre's reduction, i.e., of the possibility to find a presentation of the module defined by fewer generators and fewer relations. This completion problem is shown to induce different isomorphisms between the modules finitely presented by the matrices defining the inflations. Finally, we show how Serre's reduction implies the existence of a certain idempotent endomorphism of the finitely presented module, i.e., that Serre's reduction implies a particular decomposition, proving the converse of a result obtained in [7].**

I. Introduction

A multidimensional linear system (e.g., a linear system of ordinary differential (OD) equations, partial differential (PD) equations, OD time-delay equations, difference equations) can be written as $R\,\eta = 0$, where R is a $q \times p$ matrix with entries in a (noncommutative) polynomial ring D of functional operators (e.g., OD or PD operators, OD time-delay operators, shift operators, difference operators) and η is a vector of unknown functions. More precisely, if $\mathcal{F}$ is a left D-module, then we can consider the *linear system* or *behavior*:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

The *algebraic analysis approach* to mathematical system theory (see, e.g., [2], [9] and the references therein) is based on the fact that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left D-module $M := D^{1\times p}/(D^{1\times q}\,R)$ *finitely presented by the matrix* R since $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$ (see, e.g., [2], [9]). Module properties of M and $\mathcal{F}$ are then related to system properties of $\ker_{\mathcal{F}}(R.)$. Using constructive homological algebra [10] for (noncommutative) polynomial rings D admitting *Gröbner bases* for admissible term orders [2], one can effectively check some module properties of M (see [2] and references therein). The corresponding algorithms are implemented in packages of computer algebra systems (e.g., OreModules [3], OreMorphisms [4]).

An important issue in mathematical system (resp., module) theory is the *equivalence problem* which consists in testing whether two systems (resp., modules) are isomorphic. The first contribution of the paper, developed in Section II, is to give an explicit characterization of isomorphic finitely presented modules in terms of inflations of their presentation matrices. The classical *Schanuel's lemma* (see, e.g., [10]) on the *syzygy modules* of these modules can then be found again. If the ring D is *stably finite* (e.g., *noetherian*) (see, e.g., [8]) and one of the presentation matrices has full row rank, then this result yields a characterization of isomorphic modules as the unimodular completion problem characterizing *Serre's reduction problem* [1]. This problem aims at finding an equivalent system which contains fewer equations and fewer unknowns [1]. Section III contains the second result of the paper. We show how the completion problem induces different isomorphisms between the modules finitely presented by the matrices defining the inflations. Consequences of this result are given for doubly coprime factorizations. The result is used in Section IV to complete results of [1] on the study of Serre's reduction problem. Serre's reduction is known to be related to the so-called *decomposition problem* [5], [7]. The last contribution of the paper is to prove the converse of a result of [7]. We show how Serre's reduction implies the existence of a particular idempotent endomorphism of M, i.e., that Serre's reduction implies a particular decomposition.

Notation. D will denote a *noetherian ring*, i.e., *a left and a right noetherian ring*, i.e., every left/right ideal of D is finitely generated as a left/right D-module [8], [10]. If M and N are two left/right D-modules, then $\hom_D(M, N)$ is the abelian group formed by the *left/right D-homomorphisms* (i.e., left/right D-linear maps) from M to N. The left D-modules M and N are *isomorphic*, denoted by $M \cong N$, if there exists $\phi \in \hom_D(M, N)$ which is an *isomorphism*, i.e., injective and surjective [8], [10]. If $R \in D^{q\times p}$ is a $q \times p$ matrix with entries in D, then $.R \in \hom_D(D^{1\times q}, D^{1\times p})$ is defined by $(.R)(\lambda) = \lambda\,R$ for all $\lambda \in D^{1\times q}$. Similarly, $R. \in \hom_D(D^{p\times 1}, D^{q\times 1})$ is defined by $(R.)(\eta) = R\,\eta$ for all $\eta \in D^{p\times 1}$. We use the notation $\mathcal{F}^p$ for $\mathcal{F}^{p\times 1}$. Finally, the group of the units of the ring $D^{r\times r}$ is denoted by:

$$\mathrm{GL}_r(D) = \{U \in D^{r\times r} \mid \exists\, V \in D^{r\times r} :\ U\,V = V\,U = I_r\}.$$

II. A characterization of isomorphic modules

Let us first characterize isomorphic finitely presented modules in terms of inflations of their presentation matrices.

Theorem 1: Let $R_1 \in D^{q\times p}$ and $Q_2 \in D^{s\times t}$ be two matrices. Then, the following assertions are equivalent:

1) The left D-modules $M_1 = D^{1\times p}/(D^{1\times q}\,R_1)$ and $M_2 = D^{1\times t}/(D^{1\times s}\,Q_2)$ are isomorphic.
2) There exist matrices
$$R_2 \in D^{q\times s},\ Q_1 \in D^{p\times t},\ S_1 \in D^{p\times q},\ S_2 \in D^{s\times q},$$
$$T_1 \in D^{t\times p},\ T_2 \in D^{t\times s},\ V_1 \in D^{q\times l},\ V_2 \in D^{t\times l},$$
$$W_1 \in D^{p\times m},\ W_2 \in D^{s\times m},\ P_1 \in D^{l\times q},\ P_2 \in D^{m\times s}$$
satisfying the following two identities
$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t} + \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} (P_1 \quad 0), \tag{1}$$
$$\begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_{p+s} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} (0 \quad P_2), \tag{2}$$
where $P_1 \in D^{l\times q}$ and $P_2 \in D^{m\times s}$ are such that:
$$\begin{array}{l} \ker_D(.R_1) := \{\mu \in D^{1\times q} \mid \mu\,R_1 = 0\} \\ = \mathrm{im}_D(.P_1) := D^{1\times l}\,P_1 = \{\xi\,P_1 \mid \xi \in D^{1\times l}\}, \\ \ker_D(.Q_2) = \{\nu \in D^{1\times s} \mid \nu\,Q_2 = 0\} \\ = \mathrm{im}_D(.P_2) = D^{1\times m}\,P_2 = \{\zeta\,P_2 \mid \zeta \in D^{1\times m}\}. \end{array}$$

Proof: Let $\pi_i : D^{1\times r_i} \longrightarrow M_i$ be the canonical projection onto M_i, where $i = 1, 2$, $r_1 = p$ and $r_2 = t$. A left D-homomorphism $f : M_1 \longrightarrow M_2$ is defined by
$$\forall\ \lambda_1 \in D^{1\times p}, \quad f(\pi_1(\lambda_1)) = \pi_2(\lambda_1\,Q_1),$$
for a certain matrix $Q_1 \in D^{p\times t}$ which is such that:
$$\exists\ R_2 \in D^{q\times s} : \quad R_1\,Q_1 = -R_2\,Q_2. \tag{3}$$
For more details, see, e.g., [5]. Similarly, a left D-homomorphism $g : M_2 \longrightarrow M_1$ is defined by
$$\forall\ \lambda_2 \in D^{1\times t}, \quad g(\pi_2(\lambda_2)) = \pi_1(\lambda_2\,T_1),$$
for a certain $T_1 \in D^{t\times p}$ which is such that:
$$\exists\ S_2 \in D^{s\times q} : \quad Q_2\,T_1 = -S_2\,R_1. \tag{4}$$
Hence, $M_1 \cong M_2$ iff $g \circ f = \mathrm{id}_{M_1}$ and $f \circ g = \mathrm{id}_{M_2}$, i.e.:
$$\begin{cases} \pi_1(\lambda_1) = (g\circ f)(\pi_1(\lambda_1)) = g(\pi_2(\lambda_1\,Q_1)) = \pi_1(\lambda_1\,Q_1\,T_1), \\ \pi_2(\lambda_2) = (f\circ g)(\pi_2(\lambda_2)) = f(\pi_1(\lambda_2\,T_1)) = \pi_2(\lambda_2\,T_1\,Q_1), \end{cases}$$
i.e., for all $\lambda_1 \in D^{1\times p}$ and for all $\lambda_2 \in D^{1\times t}$,
$$\pi_1(\lambda_1\,(I_p - Q_1\,T_1)) = 0, \quad \pi_2(\lambda_2\,(I_t - T_1\,Q_1)) = 0,$$
i.e., iff there exist $S_1 \in D^{p\times q}$ and $T_2 \in D^{t\times s}$ such that:
$$I_p - Q_1\,T_1 = S_1\,R_1, \quad I_t - T_1\,Q_1 = T_2\,Q_2. \tag{5}$$
Hence, $M_1 \cong M_2$ is equivalent to the existence of matrices satisfying (3), (4), (5), i.e., satisfying the identities:
$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = \begin{pmatrix} R_1\,S_1 + R_2\,S_2 & 0 \\ T_1\,S_1 + T_2\,S_2 & I_t \end{pmatrix}, \tag{6}$$
$$\begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = \begin{pmatrix} I_p & S_1\,R_2 + Q_1\,T_2 \\ 0 & S_2\,R_2 + Q_2\,T_2 \end{pmatrix}. \tag{7}$$
Let us prove that (1) is equivalent to (6). Let us first compute $R_1\,S_1 + R_2\,S_2$. Using (4), (3), and (5), we obtain:
$$\begin{aligned} (R_1\,S_1 + R_2\,S_2)\,R_1 &= R_1\,S_1\,R_1 - R_2\,Q_2\,T_1 \\ &= R_1\,S_1\,R_1 + R_1\,Q_1\,T_1 \\ &= R_1\,S_1\,R_1 + R_1\,(I_p - S_1\,R_1) = R_1. \end{aligned}$$
Thus, we get $((R_1\,S_1 + R_2\,S_2) - I_q)\,R_1 = 0$, and thus $D^{1\times q}\,((R_1\,S_1 + R_2\,S_2) - I_q) \subseteq \ker_D(.R_1) = D^{1\times l}\,P_1$, which shows that $V_1 \in D^{q\times l}$ exists such that:
$$R_1\,S_1 + R_2\,S_2 = I_q + V_1\,P_1. \tag{8}$$
Let us compute $T_1\,S_1 + T_2\,S_2$. Using (5) and (4), we get
$$\begin{aligned} T_1\,(S_1\,R_1) &= T_1\,(I_p - Q_1\,T_1) = (I_t - T_1\,Q_1)\,T_1 \\ &= T_2\,(Q_2\,T_1) = -T_2\,S_2\,R_1, \end{aligned}$$
i.e., $(T_1\,S_1 + T_2\,S_2)\,R_1 = 0$, i.e., $D^{1\times t}\,(T_1\,S_1 + T_2\,S_2) \subseteq \ker_D(.R_1) = D^{1\times l}\,P_1$, and thus there exists $V_2 \in D^{t\times l}$ such that:
$$T_1\,S_1 + T_2\,S_2 = V_2\,P_1. \tag{9}$$
Using (8) and (9), (6) and (7) is equivalent to (1) and (7).

Similarly, using (5) and (3), we obtain
$$\begin{aligned} Q_1\,(T_2\,Q_2) &= Q_1\,(I_t - T_1\,Q_1) = (I_p - Q_1\,T_1)\,Q_1 \\ &= S_1\,(R_1\,Q_1) = -S_1\,R_2\,Q_2, \end{aligned}$$
i.e., $(S_1\,R_2 + Q_1\,T_2)\,Q_2 = 0$, i.e., $D^{1\times p}\,(S_1\,R_2 + Q_1\,T_2) \subseteq \ker_D(.Q_2) = D^{1\times m}\,P_2$, and thus there exists $W_1 \in D^{p\times m}$ such that:
$$S_1\,R_2 + Q_1\,T_2 = W_1\,P_2. \tag{10}$$
Using (3), (4), and (5), we obtain:
$$\begin{aligned} (S_2\,R_2 + Q_2\,T_2)\,Q_2 &= -S_2\,R_1\,Q_1 + Q_2\,T_2\,Q_2 \\ &= Q_2\,T_1\,Q_1 + Q_2\,T_2\,Q_2 \\ &= Q_2\,(I_t - T_2\,Q_2) + Q_2\,T_2\,Q_2 = Q_2. \end{aligned}$$
Thus, we get $(S_2\,R_2 + Q_2\,T_2 - I_s)\,Q_2 = 0$, and thus $D^{1\times s}\,(S_2\,R_2 + Q_2\,T_2 - I_s) \subseteq \ker_D(.Q_2) = D^{1\times m}\,P_2$, which shows that there exists $W_2 \in D^{s\times m}$ such that:
$$S_2\,R_2 + Q_2\,T_2 = I_s + W_2\,P_2. \tag{11}$$
Hence, (1) and (7) is finally equivalent to (1) and (2). ■

The next corollary of Theorem 1 gives a constructive proof of the standard *Schanuel's lemma* [10] in module theory.

Corollary 1: With the notations and the assumptions of Theorem 1, if we introduce the unimodular matrices
$$P = \begin{pmatrix} I_p & -Q_1 \\ T_1 & I_t - T_1\,Q_1 \end{pmatrix}, \; P^{-1} = \begin{pmatrix} I_p - Q_1\,T_1 & Q_1 \\ -T_1 & I_t \end{pmatrix},$$
then the following left D-homomorphism
$$\begin{array}{rcl} u : D^{1\times q}\,R_1 \oplus D^{1\times t} & \longrightarrow & D^{1\times p} \oplus D^{1\times s}\,Q_2 \\ (\nu_1\,R_1,\,\nu_2) & \longmapsto & (\nu_1\,R_1,\,\nu_2)\,P, \end{array} \tag{12}$$

is an isomorphism and:

$$u^{-1}: D^{1\times p} \oplus D^{1\times s}\, Q_2 \longrightarrow D^{1\times q}\, R_1 \oplus D^{1\times t}$$
$$(\mu_1,\, \mu_2\, Q_2) \longmapsto (\mu_1,\, \mu_2\, Q_2)\, P^{-1}. \quad (13)$$

Proof: The proof of Theorem 1 shows that we have the following *commutative exact diagram* [10]:

$$\begin{array}{ccccccc} D^{1\times l} & \xrightarrow{.P_1} & D^{1\times q} & \xrightarrow{.R_1} & D^{1\times p} & \xrightarrow{\pi_1} & M_1 \longrightarrow 0 \\ & & \downarrow .-R_2 & & \downarrow .Q_1 & & \downarrow f \\ D^{1\times m} & \xrightarrow{.P_2} & D^{1\times s} & \xrightarrow{.Q_2} & D^{1\times t} & \xrightarrow{\pi_2} & M_2 \longrightarrow 0. \end{array}$$

Now, $P_1\, R_1 = 0$ and (3) implies that $P_1\,(R_2\, Q_2) = -P_1\,(R_1\, Q_1) = 0$, i.e., $D^{1\times l}\,(P_1\, R_2) \subseteq \ker_D(.Q_2) = D^{1\times m}\, P_2$, and thus there exists a matrix $X \in D^{l\times m}$ such that $P_1\, R_2 = -X\, P_2$. Similarly, there exists $Y \in D^{m\times l}$ such that $P_2\, S_2 = -Y\, P_1$. With the following notations

$$U := \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \quad U' := \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix}, \quad (14)$$

$(0 \quad P_2)\, U = -Y\,(P_1 \quad 0)$, $(P_1 \quad 0)\, U' = -X\,(0 \quad P_2)$, which yields the following commutative exact diagrams

$$\begin{array}{ccccccc} D^{1\times l} & \xrightarrow{.(P_1 \quad 0)} & D^{1\times (q+t)} & \xrightarrow{\kappa_1} & L_1 & \longrightarrow 0 \\ \uparrow .-Y & & \uparrow .U & & \uparrow g & \\ D^{1\times m} & \xrightarrow{.(0 \quad P_2)} & D^{1\times (p+s)} & \xrightarrow{\kappa_2} & L_2 & \longrightarrow 0, \end{array}$$

$$\begin{array}{ccccccc} D^{1\times l} & \xrightarrow{.(P_1 \quad 0)} & D^{1\times (q+t)} & \xrightarrow{\kappa_1} & L_1 & \longrightarrow 0 \\ \downarrow .-X & & \downarrow .U' & & \downarrow h & \\ D^{1\times m} & \xrightarrow{.(0 \quad P_2)} & D^{1\times (p+s)} & \xrightarrow{\kappa_2} & L_2 & \longrightarrow 0, \end{array}$$

where the left D-modules L_1 and L_2 are defined by

$$L_1 = \operatorname{coker}_D(.(P_1 \quad 0)) := D^{1\times (q+t)}/(D^{1\times l}\,(P_1 \quad 0)),$$
$$L_2 = \operatorname{coker}_D(.(0 \quad P_2)) := D^{1\times (p+s)}/(D^{1\times m}\,(0 \quad P_2)),$$

and the left D-homomorphisms g and h are defined by:

$$g: L_2 \longrightarrow L_1$$
$$\kappa_2((\mu_1 \quad \mu_2)) \longmapsto \kappa_1((\mu_1\, S_1 + \mu_2\, S_2 \quad \mu_1\, Q_1 + \mu_2\, Q_2)),$$
$$h: L_1 \longrightarrow L_2$$
$$\kappa_1((\nu_1 \quad \nu_2)) \longmapsto \kappa_2((\nu_1\, R_1 + \nu_2\, T_1 \quad \nu_1\, R_2 + \nu_2\, T_2)).$$

Then, (1) and (2) show that $g \circ h = \mathrm{id}_{L_1}$ and $h \circ g = \mathrm{id}_{L_2}$, i.e., g is a left D-isomorphism, $h = g^{-1}$ and $L_1 \cong L_2$.

We have $\operatorname{coker}_D(.P_1) := D^{1\times q}/(D^{1\times l}\, P_1) \cong D^{1\times q}\, R_1$ and $\operatorname{coker}_D(.P_2) := D^{1\times s}/(D^{1\times m}\, P_2) \cong D^{1\times s}\, Q_2$ and:

$$\begin{cases} L_1 \cong D^{1\times q}/(D^{1\times l}\, P_1) \oplus D^{1\times t}, \\ L_2 \cong D^{1\times p} \oplus D^{1\times s}/(D^{1\times m}\, P_2). \end{cases}$$

Hence, we have the following left D-isomorphism:

$$L_1 \xrightarrow{\alpha} D^{1\times q}\, R_1 \oplus D^{1\times t}$$
$$\kappa_1((\nu_1 \quad \nu_2)) \longmapsto (\nu_1\, R_1,\, \nu_2).$$

Similarly, we have the following left D-isomorphism:

$$L_2 \xrightarrow{\beta} D^{1\times p} \oplus D^{1\times s}\, Q_2$$
$$\kappa_2((\mu_1 \quad \mu_2)) \longmapsto (\mu_1,\, \mu_2\, Q_2).$$

The left D-isomorphism $u = \beta \circ h \circ \alpha^{-1}$ and its inverse $u^{-1} = \alpha \circ g \circ \beta^{-1}$ are then defined by:

$$D^{1\times q}\, R_1 \oplus D^{1\times t} \xrightarrow{u} D^{1\times p} \oplus D^{1\times s}\, Q_2$$
$$(\nu_1\, R_1,\, \nu_2) \longmapsto (\nu_1\, R_1 + \nu_2\, T_1 \quad (\nu_1\, R_2 + \nu_2\, T_2)\, Q_2),$$

$$D^{1\times p} \oplus D^{1\times s}\, Q_2 \xrightarrow{u^{-1}} D^{1\times q}\, R_1 \oplus D^{1\times t}$$
$$(\mu_1,\, \mu_2\, Q_2) \longmapsto ((\mu_1\, S_1 + \mu_2\, S_2)\, R_1 \quad \mu_1\, Q_1 + \mu_2\, Q_2).$$

Using (3) and (5), (5) and (4), we obtain

$$(\nu_1\, R_2 + \nu_2\, T_2)\, Q_2 = -(\nu_1\, R_1)\, Q_1 + \nu_2\,(I_t - T_1\, Q_1),$$
$$(\mu_1\, S_1 + \mu_2\, S_2)\, R_1 = \mu_1\,(I_p - Q_1\, T_1) - (\mu_2\, Q_2)\, T_1,$$

which finally yields (12) and (13). ■

Let us give another corollary of Theorem 1 which connects isomorphisms to the so-called *Serre's reduction* [1].

Corollary 2: With the notations and the assumptions of Theorem 1, let us assume that $q + t = p + s$.

1) Then, we have:

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t}$$
$$\Leftrightarrow \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_{p+s}. \quad (15)$$

2) If either R_1 or Q_2 has *full row rank*, namely, $\ker_D(.R_1) = 0$ or $\ker_D(.Q_2) = 0$, then $M_1 \cong M_2$ is equivalent to the existence of matrices $R_2 \in D^{q\times s}$, $Q_1 \in D^{p\times t}$, $Q_2 \in D^{s\times t}$, $S_1 \in D^{p\times q}$, $S_2 \in D^{s\times q}$, $T_1 \in D^{t\times p}$, and $T_2 \in D^{t\times s}$ such that:

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t}.$$

Proof: 1 is a consequence of $q + t = p + s$ and of the fact that D noetherian ring, and thus a *stably finite ring*, i.e., a ring for which $U\, V = I_r$ for two matrices $U,\, V \in D^{r\times r}$ yields $V\, U = I_r$ [8]. 2 is a direct consequence of Theorem 1 with $P_1 = 0$ or $P_2 = 0$ and of the previous point 1. ■

III. Unimodular completion problem

The next theorem shows that the *unimodular completion problem* induces different isomorphisms between the modules finitely presented by the matrices defining the inflations.

Theorem 2: Let $p,\, q,\, s,\, t \in \mathbb{N}$ satisfy $q + t = p + s$ and $R_1 \in D^{q\times p}$, $R_2 \in D^{q\times s}$, $Q_1 \in D^{p\times t}$, $Q_2 \in D^{s\times t}$, $S_1 \in D^{p\times q}$, $S_2 \in D^{s\times q}$, $T_1 \in D^{t\times p}$, and $T_2 \in D^{t\times s}$ such that:

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t}. \quad (16)$$

Then, we have:

$$\operatorname{coker}_D(.R_1) \cong \operatorname{coker}_D(.Q_2), \quad \ker_D(.R_1) \cong \ker_D(.Q_2),$$
$$\operatorname{coker}_D(.S_1) \cong \operatorname{coker}_D(.T_2), \quad \ker_D(.S_1) \cong \ker_D(.T_2),$$
$$\operatorname{coker}_D(.Q_1) \cong \operatorname{coker}_D(.R_2), \quad \ker_D(.Q_1) \cong \ker_D(.R_2),$$
$$\operatorname{coker}_D(.T_1) \cong \operatorname{coker}_D(.S_2), \quad \ker_D(.T_1) \cong \ker_D(.S_2).$$

Right D-module analogous to the above results hold, i.e.:

$$\operatorname{coker}_D(R_1.) \cong \operatorname{coker}_D(Q_2.),\ \ker_D(R_1.) \cong \ker_D(Q_2.), \ldots$$

Proof: Let us consider the following left D-modules:

$$M_1 := \operatorname{coker}_D(.R_1) = D^{1\times p}/(D^{1\times q} R_1),$$
$$M_2 := \operatorname{coker}_D(.Q_2) = D^{1\times t}/(D^{1\times s} Q_2).$$

By 1 of Corollary 2 (16) yields:

$$\begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_{p+s}. \qquad (17)$$

From (3), we have $R_1 Q_1 = -R_2 Q_2$, which yields the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R_1} & D^{1\times p} & \xrightarrow{\pi_1} & M_1 & \longrightarrow 0 \\ \downarrow .-R_2 & & \downarrow .Q_1 & & \downarrow \alpha_1 & \\ D^{1\times s} & \xrightarrow{.Q_2} & D^{1\times t} & \xrightarrow{\pi_2} & M_2 & \longrightarrow 0, \end{array} \qquad (18)$$

where α_1 is the left D-homomorphism defined by:

$$\begin{array}{rcl} \alpha_1 : M_1 & \longrightarrow & M_2 \\ \pi_1(\lambda_1) & \longmapsto & \pi_2(\lambda_1 Q_1). \end{array}$$

Moreover, (18) yields the following left D-homomorphism:

$$\begin{array}{rcl} \alpha_1' : \ker_D(.R_1) & \longrightarrow & \ker_D(.Q_2) \\ \mu_1 & \longmapsto & -\mu_1 R_2. \end{array}$$

Similarly, the identity $Q_2 T_1 = -S_2 R_1$ yields the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times s} & \xrightarrow{.Q_2} & D^{1\times t} & \xrightarrow{\pi_2} & M_2 & \longrightarrow 0 \\ \downarrow .-S_2 & & \downarrow .T_1 & & \downarrow \alpha_2 & \\ D^{1\times q} & \xrightarrow{.R_1} & D^{1\times p} & \xrightarrow{\pi_1} & M_1 & \longrightarrow 0, \end{array} \qquad (19)$$

where α_2 is the left D-homomorphism defined by:

$$\begin{array}{rcl} \alpha_2 : M_2 & \longrightarrow & M_1 \\ \pi_2(\nu_2) & \longmapsto & \pi_1(\nu_2 T_1). \end{array}$$

Moreover, (19) yields the following left D-homomorphism:

$$\begin{array}{rcl} \alpha_2' : \ker_D(.Q_2) & \longrightarrow & \ker_D(.R_1) \\ \theta_2 & \longmapsto & -\theta_2 S_2. \end{array}$$

Now, (16) and (17) yield (5) so that

$$\begin{aligned} (\alpha_1 \circ \alpha_2)(\pi_2(\nu_2)) &= \alpha_1(\pi_1(\nu_2 T_1)) = \pi_2(\nu_2 T_1 Q_1) \\ &= \pi_2(\nu_2) - \pi_2((\nu_2 T_2) Q_2) = \pi_2(\nu_2), \end{aligned}$$

$$\begin{aligned} (\alpha_2 \circ \alpha_1)(\pi_1(\lambda_1)) &= \alpha_2(\pi_2(\lambda_1 Q_1)) = \pi_1(\lambda_1 Q_1 T_1) \\ &= \pi_1(\lambda_1) - \pi_1((\lambda_1 S_1) R_1)) = \pi_1(\lambda_1), \end{aligned}$$

i.e., α_1 is a left D-isomorphism, $\alpha_2 = \alpha_1^{-1}$ and $M_1 \cong M_2$.

Now, from (16) and (17), we have $R_2 S_2 = I_q - R_1 S_1$ and $S_2 R_2 = I_s - Q_2 T_2$, which implies that

$$\begin{aligned} (\alpha_1' \circ \alpha_2')(\theta_2) &= -\alpha_1'(\theta_2 S_2) = \theta_2 (S_2 R_2) \\ &= \theta_2 - (\theta_2 Q_2) T_2 = \theta_2, \end{aligned}$$

$$\begin{aligned} (\alpha_2' \circ \alpha_1')(\mu_1) &= -\alpha_2'(\mu_1 R_2) = \mu_1 (R_2 S_2) \\ &= \mu_1 - (\mu_1 R_1) S_1 = \mu_1, \end{aligned}$$

i.e., α_1' is a left D-isomorphism, $\alpha_2' = \alpha_1'^{-1}$ and:

$$\ker_D(.Q_2) \cong \ker_D(.R_1).$$

Since the role played by R_1 (resp., Q_2) in (16) and (17) is symmetric to the one played by S_1 (resp., T_2), we obtain:

$$\operatorname{coker}_D(.S_1) \cong \operatorname{coker}_D(.T_2), \quad \ker_D(.S_1) \cong \ker_D(.T_2).$$

Finally, the other isomorphisms of the theorem can be similarly proved. For instance, the left D-isomorphism $\ker_D(.Q_1) \cong \ker_D(.R_2)$ is defined by:

$$\begin{array}{rcl} \gamma_1' : \ker_D(.Q_1) & \longrightarrow & \ker_D(.R_2) \\ \theta_1 & \longmapsto & -\theta_1 S_1, \end{array} \qquad (20)$$

$$\begin{array}{rcl} \gamma_2' = \gamma_1'^{-1} : \ker_D(.R_2) & \longrightarrow & \ker_D(.Q_1) \\ \mu_2 & \longmapsto & -\mu_2 R_1. \end{array} \qquad (21)$$

■

Theorem 2 generalizes Theorem 4.1 of [1] for a non full row rank R, i.e., $\ker_D(.R)$ is not necessarily reduced to 0.

Let us give an application of Theorem 2 to doubly coprime factorizations. To keep the notations classically used in control theory, the noetherian domain D is denoted by A.

Corollary 3: Let $K := Q(A)$ be the *left and right quotient field* of A [8], $P \in K^{q\times r}$, and $P = D^{-1} N = \widetilde{N} \widetilde{D}^{-1}$ a *doubly coprime factorization* of P, namely, $D \in A^{q\times q}$, $N \in A^{q\times r}$, $\widetilde{D} \in A^{r\times r}$ and $\widetilde{N} \in A^{q\times r}$ satisfy

$$\begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = I_{q+r},$$

for certain matrices $X \in A^{q\times q}$, $Y \in A^{r\times q}$, $\widetilde{X} \in A^{r\times r}$, and $\widetilde{Y} \in A^{r\times q}$. Then, we have:

$$\begin{array}{ll} \operatorname{coker}_A(.D) \cong \operatorname{coker}_A(.\widetilde{D}), & \ker_A(.D) \cong \ker_A(.\widetilde{D}), \\ \operatorname{coker}_A(.X) \cong \operatorname{coker}_A(.\widetilde{X}), & \ker_A(.X) \cong \ker_A(.\widetilde{X}), \\ \operatorname{coker}_A(.N) \cong \operatorname{coker}_A(.\widetilde{N}), & \ker_A(.N) \cong \ker_A(.\widetilde{N}), \\ \operatorname{coker}_A(.Y) \cong \operatorname{coker}_A(.\widetilde{Y}), & \ker_A(.Y) \cong \ker_A(.\widetilde{Y}), \end{array}$$

Similarly, right A-module analogous to the above results hold, i.e., $\operatorname{coker}_A(D.) \cong \operatorname{coker}_A(\widetilde{D}.), \ldots$

Corollary 4: With the notations and the hypotheses of Theorem 2, we have:

1) R_1 has full row rank iff so is Q_2.
2) R_2 admits a left inverse iff so is Q_1. More precisely:
   a) If $Z_2 \in D^{s\times q}$ is a left inverse of R_2, then Q_1 admits the left inverse $T_1 - T_2 Z_2 R_1$.
   b) If $Y_1 \in D^{t\times p}$ is a left inverse of Q_1, then R_2 admits the left inverse $S_2 - Q_2 Y_1 S_1$.
3) If R_2 or Q_1 admits a left inverse, then $\ker_D(.R_2) \cong \ker_D(.Q_1)$ is *stably free* of rank $q - s = p - t$, i.e.:
$$\ker_D(.R_2) \oplus D^{1\times s} \cong D^{1\times q}.$$
4) $\ker_D(.R_2)$ is a free left D-module of rank r iff so is $\ker_D(.Q_1)$. More precisely, we have:
   a) If $B_2 \in D^{r\times q}$ is a *basis* of $\ker_D(.R_2)$, i.e., the matrix B_2 has full row rank and satisfies

$\ker_D(.R_2) = D^{1\times r} B_2$, then $C_2 := B_2 R_1$ is a basis of $\ker_D(.Q_1)$, i.e., $C_2 \in D^{r\times p}$ has full row rank and satisfies $\ker_D(.Q_1) = D^{1\times r} C_2$.

b) If $C_1 \in D^{r\times p}$ is a basis of $\ker_D(.Q_1)$, i.e., the matrix C_1 has full row rank and satisfies $\ker_D(.Q_1) = D^{1\times r} C_1$, then $B_1 := C_1 S_1$ is a basis of $\ker_D(.Q_1)$, i.e., $B_1 \in D^{r\times q}$ has full row rank and satisfies $\ker_D(.Q_1) = D^{1\times r} B_1$.

Proof: 1. By Theorem 2, $\ker_D(.Q_2) \cong \ker_D(.R_1) = 0$.

2. $\mathrm{coker}_D(.R_2) = 0$ iff $D^{1\times s} = D^{1\times q} R_2$, i.e., iff R_2 admits a left inverse $Z_2 \in D^{s\times q}$, i.e., $Z_2 R_2 = I_s$. Similarly for $\mathrm{coker}_D(.Q_1)$. The first result follows from $\mathrm{coker}_D(.R_2) \cong \mathrm{coker}_D(.Q_1)$ by Theorem 2. If $Z_2 \in D^{s\times q}$ is such that $Z_2 R_2 = I_s$, then using (16), we get $R_1 Q_1 = -R_2 Q_2$ and $T_1 Q_1 + T_2 Q_2 = I_t$, and thus $(T_1 - T_2 Z_2 R_1) Q_1 = T_1 Q_1 + T_2 (Z_2 R_2) Q_2 = T_1 Q_1 + T_2 Q_2 = I_t$. Now, if $Y_1 \in D^{t\times p}$ is a left inverse of Q_1, i.e., $Y_1 Q_1 = I_t$, then using (17), we get $S_1 R_2 = -Q_1 T_2$ and $S_2 R_2 + Q_2 T_2 = I_s$, and thus $(S_2 - Q_2 Y_1\ S_1) R_2 = S_2 R_2 + Q_2 (Y_1 Q_1) T_2 = S_2 R_2 + Q_2 T_2 = I_s$.

3. If $Z_2 \in D^{s\times q}$ is a left inverse of R_2, then the matrix $\Pi := R_2 Z_2$ is an *idempotent* of the ring $D^{q\times q}$, i.e., $\Pi^2 = \Pi$, and thus $D^{1\times q} = \ker_D(.\Pi) \oplus \mathrm{im}_D(.\Pi)$. Now, since $.Z_2 \in \hom_D(D^{1\times s}, D^{1\times q})$ is injective, $\ker_D(.\Pi) = \ker_D(.R_2)$. Moreover, since $.R_2 \in \hom_D(D^{1\times q}, D^{1\times s})$ is surjective, $\mathrm{im}_D(.\Pi) = D^{1\times q}\,\Pi = D^{1\times s} Z_2$, which shows that $\ker_D(.R_2) \oplus D^{1\times s} Z_2 = D^{1\times q}$, i.e., $\ker_D(.R_2)$ is a stably free left D-module of rank $q - s = p - t$. The result follows from $\ker_D(.Q_1) \cong \ker_D(.R_2)$ by Theorem 2.

4. The first point follows from $\ker_D(.Q_1) \cong \ker_D(.R_2)$ by Theorem 2. Now, if $\ker_D(.R_2)$ is a free left D-module of rank r and the full row rank $B_2 \in D^{r\times q}$ defines a basis of $\ker_D(.R_2)$, i.e., $\ker_D(.R_2) = D^{1\times r} B_2$, then the left D-isomorphism γ_2' defined by (21) sends a basis of $\ker_D(.R_2)$ to a basis of $\ker_D(.Q_1)$, which shows that the full row rank matrix $C_2 := B_2 R_1 \in D^{r\times p}$ defines a basis of $\ker_D(.Q_1)$, i.e., $\ker_D(.Q_1) = D^{1\times r} C_2$. The last point can be similarly proved using $\gamma_1' = \gamma_2'^{-1}$ defined by (20). ■

IV. Serre's reduction

Let us state a necessary and sufficient condition for a linear system to be equivalent to a linear system defined by fewer unknowns and equations (the so-called *Serre's reduction*).

Theorem 3: Let $R \in D^{q\times p}$ (not necessarily full row rank). Then the following assertions are equivalent:

1) There exist $\overline{R} \in D^{(q-r)\times(p-r)}$, where $0 \le r \le q-1$, and $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that:

$$V\,R\,W = \begin{pmatrix} I_r & 0 \\ 0 & \overline{R} \end{pmatrix}.$$

2) There exists $\Lambda \in D^{q\times(q-r)}$ such that:

a) a matrix $U \in \mathrm{GL}_{p+q-r}(D)$ exists such that:

$$(R \quad -\Lambda)\,U = (I_q \quad 0),$$

b) a matrix $\Gamma \in D^{(q-r)\times q}$ exists such that $\Gamma\,\Lambda = I_q$,

c) the stably free left D-module $\ker_D(.\Lambda)$ is free of rank r, i.e., there exists a full row rank matrix $B \in D^{r\times q}$ such that $\ker_D(.\Lambda) = D^{1\times r} B$.

If 2 of Theorem 3 holds, then we have

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & -T_2 \end{pmatrix}, \tag{22}$$

$$Q_1 \in D^{p\times(p-r)},\ Q_2 \in D^{(q-r)\times(p-r)},\ S_1 \in D^{p\times q},$$
$$S_2 \in D^{(q-r)\times q},\ T_1 \in D^{(p-r)\times p},\ T_2 \in D^{(p-r)\times(q-r)},$$

i.e., we are in the position of Theorem 2 with $R_2 = -\Lambda$, $s = q - r$, $t = p - r$ and T_2 has been changed into $-T_2$ to follow the notations used in [1].

Remark 1: Theorem 3 is proved in Corollaries 4.10 and 4.14 of [1], where 2.b of Theorem 3 is replaced by $\ker_D(.Q_1)$ is a free left D-module of rank r. These conditions are equivalent since $\ker_D(.\Lambda) \cong \ker_D(.Q_1)$ by Theorem 2. The hypothesis that R has full row rank is not used in the proofs of Corollaries 4.10 and 4.14 of [1].

Corollary 5: 1) If 1 of Theorem 3 holds, then the matrices of 2 of Theorem 3 can be chosen as follows

$$\Lambda = X_2, \quad \Gamma = V_2, \quad B = V_1,$$
$$U = \begin{pmatrix} W_1 V_1 & W_2 \\ -V_2 & \overline{R} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} R & -\Lambda \\ Z_2 & 0 \end{pmatrix},$$

with the following notations:

$$\begin{cases} V = (V_1^T \quad V_2^T)^T,\ V_1 \in D^{r\times q},\ V_2 \in D^{(q-r)\times q}, \\ V^{-1} = (X_1 \quad X_2),\ X_1 \in D^{q\times r},\ X_2 \in D^{q\times(q-r)}, \\ W = (W_1 \quad W_2),\ W_1 \in D^{p\times r},\ W_2 \in D^{p\times(p-r)}, \\ W^{-1} = (Z_1^T \quad Z_2^T)^T,\ Z_1 \in D^{r\times p},\ Z_2 \in D^{(p-r)\times p}. \end{cases}$$

2) If 2 of Theorem 3 holds, then, with the notations (22), the matrices defined in 1 of Theorem 3 can be taken as $\overline{R} = Q_2$ and

$$\begin{array}{ll} V_1 = B, & V_2 = \Gamma - S_2\,F\,B, \\ W_1 = (S_1 + Q_1\,T_2\,\Gamma)\,F, & W_2 = Q_1, \\ X_1 = R\,W_1, & X_2 = \Lambda, \\ Z_1 = B\,R, & Z_2 = T_1 - T_2\,\Gamma\,R, \end{array} \tag{23}$$

where $F \in D^{q\times r}$ is such that:

$$I_q - \Lambda\,\Gamma = F\,B. \tag{24}$$

Proof: 1 is proved in Corollary 4.14 of [1] up to the characterization of B. By (20), $\ker_D(.\Lambda) = \ker_D(.Q_1)\,S_1$, where $Q_1 = W_2$ and $S_1 = W_1 V_1$. Now, the identity $W^{-1} W = I_p$ (resp., $W\,W^{-1} = I_p$) yields $Z_1 W_2 = 0$ (resp., $W_1 Z_1 + W_2 Z_2 = I_p$), i.e., $D^{1\times r} Z_1 \subseteq \ker_D(.W_2)$. Now, if $\lambda \in \ker_D(.W_2)$, then $W_1 Z_1 + W_2 Z_2 = I_p$ yields $\lambda = (\lambda\,W_1)\,Z_1 \in D^{1\times r} Z_1$, i.e., $\ker_D(.W_2) \subseteq D^{1\times r} Z_1$, which shows that $\ker_D(.Q_1) = \ker_D(.W_2) = D^{1\times r} Z_1$. Using the identity $Z_1 W_1 = I_r$ coming from $W^{-1} W = I_p$, we get $\ker_D(.\Lambda) = D^{1\times r}(Z_1 W_1 V_1) = D^{1\times r} V_1$. Finally,

$V\,V^{-1}=I_q$ yields $V_1\,X_1=I_r$, which shows that V_1 has full row rank ($\nu\in\ker_D(.V_1)$ yields $\nu=(\nu\,V_1)\,X_1=0$), and thus $\ker_D(.\Lambda)$ is a free left D-module of rank r.

2. Let us define the following matrices

$$X=(X_1\quad X_2)\in D^{q\times q},\quad V=(V_1^T\quad V_2^T)^T\in D^{q\times q},$$
$$W=(W_1\quad W_2)\in D^{p\times p},\quad Z=(Z_1^T\quad Z_2^T)^T\in D^{p\times p},$$

with the notations (23). Then, using (22), we get:

$$R\,W=(R\,W_1\quad R\,Q_1)=(R\,W_1\quad \Lambda\,Q_2)$$
$$=(R\,W_1\quad \Lambda)\begin{pmatrix} I_r & 0\\ 0 & Q_2\end{pmatrix}=X\begin{pmatrix} I_r & 0\\ 0 & \overline{R}\end{pmatrix}.$$

We note that $(I_q-\Lambda\,\Gamma)\,\Lambda=0$ since $\Gamma\,\Lambda=I_{q-r}$, and thus $D^{1\times q}\,(I_q-\Lambda\,\Gamma)\subseteq\ker_D(.\Lambda)=D^{1\times r}\,B$, which shows that $F\in D^{q\times r}$ exists such that $I_q-\Lambda\,\Gamma=F\,B$. Then, we have $\Gamma\,F\,B=\Gamma\,(I_q-\Lambda\,\Gamma)=\Gamma-(\Gamma\,\Lambda)\,\Gamma=0$ since $\Gamma\,\Lambda=I_{q-r}$. Then, using (22), $\Gamma\,F\,B=0$ and (24), we obtain:

$$X\,V=(R\,W_1\quad\Lambda)\begin{pmatrix} B\\ \Gamma-S_2\,F\,B\end{pmatrix}$$
$$=R\,(S_1+Q_1\,T_2\,\Gamma)\,F\,B+\Lambda\,(\Gamma-S_2\,F\,B)$$
$$=(R\,S_1)\,(F\,B)+\Lambda\,\Gamma-\Lambda\,S_2\,F\,B$$
$$=(I_q+\Lambda\,S_2)\,F\,B+\Lambda\,\Gamma-\Lambda\,S_2\,F\,B=F\,B+\Lambda\,\Gamma=I_q,$$

which yields $X\in\mathrm{GL}_q(D)$ and $V=X^{-1}\in\mathrm{GL}_q(D)$. Finally, using (22), $\Gamma\,F\,B=0$ and (24), we obtain

$$W\,Z=((S_1+Q_1\,T_2\,\Gamma)\,F\quad Q_1)\begin{pmatrix} B\,R\\ T_1-T_2\,\Gamma\,R\end{pmatrix}$$
$$=S_1\,(F\,B)\,R+Q_1\,T_1-(Q_1\,T_2)\,\Gamma\,R$$
$$=S_1\,(I_q-\Lambda\,\Gamma)\,R+Q_1\,T_1+S_1\,\Lambda\,\Gamma\,R$$
$$=S_1\,R+Q_1\,T_1=I_p,$$

which yields $W\in\mathrm{GL}_p(D)$ and $Z=W^{-1}\in\mathrm{GL}_p(D)$. ■

V. From Serre's reduction to decomposition

Theorem 4: Let $R\in D^{q\times p}$ and $\Lambda\in D^{q\times(q-r)}$ satisfy the conditions of 2 of Theorem 3. If $\Gamma\in D^{(q-r)\times q}$ is a left inverse of Λ and $B\in D^{r\times q}$ a basis of the free left D-module $\ker_D(.\Lambda)$ of rank r, then, with the notations (22) and (23), we have:

1) $\Delta:=S_1\,F\,B\in D^{p\times q}$ satisfies $\Delta\,R\,\Delta=\Delta$.
2) The matrices $P:=W_2\,Z_2=I_p-\Delta\,R\in D^{p\times p}$ and $Q:=I_q-R\,\Delta\in D^{q\times q}$ are *idempotents*, i.e., $P^2=P$ and $Q^2=Q$, and satisfy $R\,P=Q\,R$.
3) With the notations (23), the left D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q)$ are defined by

$$\ker_D(.P)=D^{1\times r}\,Z_1,\quad \mathrm{im}_D(.P)=D^{1\times(p-r)}\,Y_1,$$
$$\ker_D(.Q)=D^{1\times r}\,V_1,\quad \mathrm{im}_D(.Q)=D^{1\times(q-r)}\,V_2,$$

i.e., are free of rank respectively r, $p-r$, r and $q-r$.

Hence, Theorem 4.2 of [5] holds with the above P and Q.

Proof: 1. We first note that (24) and $B\,\Lambda=0$ yield $B\,F\,B+B\,\Lambda\,\Gamma=B$, i.e., $(B\,F-I_r)\,B=0$, i.e., $B\,F=I_r$ since B has full row rank. Now, using (22), $B\,\Lambda=0$ and $B\,F=I_r$, we get:

$$\Delta\,R\,\Delta=S_1\,F\,B\,(R\,S_1)\,F\,B=S_1\,F\,B\,(I_q+\Lambda\,S_2)\,F\,B$$
$$=S_1\,F\,(B\,F)\,B=S_1\,F\,B=\Delta.$$

2. Since $W_1\,Z_1+W_2\,Z_2=I_p$, we get

$$P:=W_2\,Z_2=I_p-W_1\,Z_1$$
$$=I_p-((S_1+Q_1\,T_2\,\Gamma)\,F\,B)\,R,$$

where, using (24) and $\Gamma\,(I_q-\Lambda\,\Gamma)=0$, we obtain

$$(S_1+Q_1\,T_2\,\Gamma)\,(F\,B)=(S_1+Q_1\,T_2\,\Gamma)\,(I_q-\Lambda\,\Gamma)$$
$$=S_1\,(I_q-\Lambda\,\Gamma)=S_1\,F\,B=\Delta,$$

i.e., $P=I_p-\Delta\,R$. Now, $\Delta\,R\,\Delta=\Delta$ yields $P^2=P$, $Q^2=Q$ and $R\,P=Q\,R$. Using (22) and (24), we get:

$$Q=I_q-(R\,S_1)\,F\,B=I_q-(I_q+\Lambda\,S_2)\,F\,B$$
$$=(I_q-F\,B)-\Lambda\,S_2\,F\,B=\Lambda\,(\Gamma-S_2\,F\,B)=X_2\,V_2.$$

The identity $Z\,W=I_p$ yields $\ker_D(.W_2)=D^{1\times r}\,Z_1$ and $Z_2\,W_2=I_{p-r}$. The latter identity implies that $.Z_2$ is injective, $.W_2$ is surjective and Z_2 has full row rank. Thus,

$$\ker_D(.P)=\ker_D(.(W_2\,Z_2))=\ker_D(.W_2)=D^{1\times r}\,(B\,R),$$
$$\mathrm{im}_D(.P)=\mathrm{im}_D(.(W_2\,Z_2))=\mathrm{im}_D(.Z_2)=D^{1\times(p-r)}\,Z_2,$$

which proves that $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are free left D-modules of rank respectively r and $p-r$.

The identity $V\,X=I_q$ yields $\ker_D(.X_2)=D^{1\times r}\,V_1$ and $V_2\,X_2=I_{q-r}$. The latter identity implies that $.V_2$ is injective, $.X_2$ is surjective and V_2 has full row rank. Thus,

$$\ker_D(.Q)=\ker_D(.(X_2\,V_2))=\ker_D(.X_2)=D^{1\times r}\,B,$$
$$\mathrm{im}_D(.Q)=\mathrm{im}_D(.(X_2\,V_2))=\mathrm{im}_D(.V_2)=D^{1\times(q-r)}\,V_2,$$

which proves that $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free left D-modules of rank respectively r and $q-r$. ■

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