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Factoring and decomposing a class of linear functional systems

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This paper is dedicated to Paul A. Fuhrmann for his 70th anniversary. His scientific work, in particular [25–27], has always been an important source of inspiration for the second author and, we believe, for the next generations

Abstract

Within a constructive homological algebra approach, we study the factorization and decomposition problems for a class of linear functional (determined, over-determined, under-determined) systems. Using the concept of Ore algebras of functional operators (e.g., ordinary/partial differential operators, shift operators, time-delay operators), we first concentrate on the computation of morphisms from a finitely presented left module M over an Ore algebra to another one M', where M (resp., M') is a module intrinsically associated with the linear functional system $Ry=0$ (resp., $R'z=0$). These morphisms define applications sending solutions of the system $R'z=0$ to solutions of $R\,y=0$. We explicitly characterize the kernel, image, cokernel and coimage of a general morphism. We then show that the existence of a non-injective endomorphism of the module M is equivalent to the existence of a non-trivial factorization $R=R_2R_1$ of the system matrix R. The corresponding system can then be integrated "in cascade". Under certain conditions, we also show that the system $Ry=0$ is equivalent to a system $R'z=0$, where R' is a block-triangular matrix of the same size as R. We show that the existence of idempotents of the endomorphism ring of the module M allows us to reduce the integration of the system $Ry=0$ to the integration of two independent systems $R_1y_1=0$ and $R_2\,y_2=0$. Furthermore, we prove that, under certain conditions, idempotents provide decompositions of the system $Ry=0$, i.e., they allow us to compute an equivalent system $R'z=0$, where R' is a block-diagonal matrix of the same size as R. Applications of these results in mathematical physics and control theory are given.

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Finally, the different algorithms of the paper are implemented in the Maple package MORPHISMS based on the library OREMODULES.

Keywords: Linear functional systems; Factorization and decomposition problems; Morphisms; Equivalences of systems; Galois transformations; Quadratic first integrals of motion; Quadratic conservation laws; Constructive homological algebra; Module theory; Symbolic computation

1. Introduction

Many systems coming from mathematical physics, applied mathematics and engineering sciences can be described by means of systems of ordinary or partial differential equations (ODEs, PDEs), difference equations, differential time-delay equations... If these systems are linear, they can then be defined by means of matrices with entries in non-commutative algebras of functional operators such as the rings of differential operators, shift operators, time-delay operators... An important class of such non-commutative rings of functional operators is called *Ore algebras* [15]. It is a subclass of the so-called *skew polynomial rings* in several variables (or *Ore extensions*). See [46] for a general exposition on skew polynomial rings and Ore extensions.

Algebraic analysis is a mathematical theory first introduced by Malgrange in his study of linear systems of partial differential equations with constant coefficients [45]. See also [51]. It was then developed by the Japanese school of Sato to treat partial differential equations with variable coefficients. See [10,22,34] and the references therein. Ideas of algebraic analysis have recently been extended to the case of Ore algebras in order to study linear functional systems [16]. The methods of algebraic analysis give us a way to intrinsically study a linear functional system by considering its associated finitely presented left module over an Ore algebra. This idea is natural as the structural properties of the linear functional systems can be studied by handling algebraic manipulations on the system matrix of functional operators, i.e., by performing linear algebra over a ring which is also called *module theory* [37,46,66]. The tools of *homological algebra* have been developed in order to study the properties of modules [66], and thus, the structural properties of the corresponding systems. Using recent developments and implementations of Gröbner and Janet bases over Ore algebras [15,40,41,65], it has been shown in [16,55–59,63,64] how to make effective some of these tools as, for instance, free resolutions, parametrizations, projective dimensions, torsion-free degrees, Hilbert series, extension functors, classification of modules (torsion, torsion-free, reflexive, projective, stably free, free). The corresponding constructive algorithms have been implemented in the library OREMODULES [17]. Applications of these algorithms in multidimensional systems theory have recently been given in [16,17,24,55–59,62–64]. For related works, see [50,52,53,73,76,77]. We also refer the reader to [12,14,48] and the references therein for pioneering works on the development of constructive homological algebra methods in the case of holonomic systems of PDEs [10,22] and to [40,44,70] for their implementations in computer algebra software.

Continuing the development of *constructive homological algebra* for linear systems over Ore algebras and, in particular [58,63,62], the first part of the paper aims at computing effectively morphisms from a left D-module M, finitely presented by a matrix R with entries in a certain Ore algebra D, to a left D-module M' presented by a matrix R'. In particular, a morphism from M to M' defines a transformation sending solutions of the system $R'z = 0$ to solutions of $Ry = 0$. In the case where $R' = R$, the ring $\mathrm{end}_D(M)$ of endomorphisms of M corresponds to the "Galois transformations" of the system $Ry = 0$. In the case of 1-D linear systems or linear systems of PDEs defined by integrable connections, we explain how the endomorphism ring $\mathrm{end}_D(M)$ generalizes the concept of *eigenring* developed in the symbolic computation literature [4,7,11,18,19,61,68,74,75]. The constructive computation of morphisms between differential modules was first studied in [49,71] in the case of holonomic differential modules (see also [41] for morphisms between non-commutative polynomial algebras). As very few systems coming from mathematical physics and systems theory define holonomic differential modules, this paper attempts to study the case of general (determined, over-determined, under-determined) systems. To our knowledge, the only work which goes in the same direction for commutative polynomial rings is [53] following [59]. Algorithms for computing morphisms are given when Gröbner bases exist over the underlying Ore algebra. As an application, we show how to use morphism computations to obtain quadratic first integrals of motion and quadratic conservation laws. Within this new algebraic approach, we find again in a purely algorithmic way the quadratic conservation laws classically studied in mathematical physics (e.g., electromagnetism, hydrodynamics, elasticity theory).

We explicitly characterize the kernel, coimage, image and cokernel of a morphism from M to M' and deduce a heuristic method to check the equivalence of the corresponding systems $Ry = 0$ and $R'z = 0$. In Theorem 3.1, we prove that the existence of a non-injective endomorphism of a left D-module M, finitely presented by a matrix R with entries in an Ore algebra D, corresponds to a factorization of the form $R = R_2 R_1$, where R_1 and R_2 are two matrices with entries in D. As a consequence, the integration of the system $Ry = 0$ is reduced to a cascade of integrations. In Theorem 3.2, under certain conditions on the morphism (freeness), we show that the system $Ry = 0$ is equivalent to a system of the form

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0, \tag{1}$$

where T_1, T_2 and T_3 are three matrices with entries in D such that the matrix defining (1) has the same size as R. One of the main interests of (1) is that the integration of $Ry = 0$ is then reduced to the cascade integration $T_3 z_2 = 0$ and $T_1 z_1 = -T_2 z_2$.

In the fourth part of the paper, we show how to effectively compute some idempotents of $\mathrm{end}_D(M)$ and we prove in Theorem 4.1 that they allow us to decompose the system $Ry = 0$ into two decoupled systems $S_1 y_1 = 0$ and $S_2 y_2 = 0$, where S_1 and S_2 are two matrices with entries in D. Consequently, the integration of the system $Ry = 0$ is then equivalent to the integrations of the two independent systems $S_1 y_1 = 0$ and $S_2 y_2 = 0$. Then, under certain conditions on the idempotents (freeness), we prove in Theorem 4.2 that the system $Ry = 0$ is equivalent to a block-diagonal system of the form

$$\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0, \tag{2}$$

where T_1 and T_2 are two matrices with entries in D such that the matrix defining (2) has the same size as R. In particular, these conditions always hold in the case of a univariate Ore algebra over a field of coefficients (i.e., ordinary differential/difference systems over the field of rational functions) and in the case of a multivariate commutative Ore algebras due to the Quillen-Suslin theorem [43,66] (e.g., linear system of partial differential equations with constant coefficients). Moreover, if some rank conditions on the idempotent are fulfilled, then, using a result due to Stafford [69], we prove that a similar result also holds for the Weyl algebras $A_n(k)$ and $B_n(k)$ over a field k of characteristic 0 (i.e., linear system of partial differential equations with polynomial/rational coefficients). Using recent implementations of both Quillen-Suslin and Stafford theorems in the library OreModules [17,24,64], we obtain a way to find the decomposition (2) of $Ry = 0$ when it exists.

We point out that for all the above-mentioned results, and thus, for the corresponding algorithms, no condition on the system $Ry = 0$ is required such as D-finite, determined, over-determined, under-determined, i.e., this approach handles general linear systems over a certain class of Ore algebras over which Gröbner bases exist. To our knowledge, the problem of factoring or decomposing linear functional systems has been studied only for a few particular cases. For scalar linear ordinary differential operators or linear determined ordinary differential systems, we refer to [4,8,9,13,30,32,33,67,68,72]. Generalizations to linear determined difference and q-difference systems appear in [4,11] and see [42,74,75] for *D-finite* partial differential systems. A more general work in that direction is included in [31]. For similar cases where the base field is of positive characteristic and for modular approaches, see [7,18,19] and the references therein.

All along the paper, we illustrate our results by considering some applications coming from mathematical physics (e.g., quadratic first integrals of motion and quadratic conservation laws, equivalence of systems appearing in linear elasticity, factorization and decomposition of linearized Euler equations and the Dirac equations) and control theory (factorization and decomposition of systems coming from control theory, decoupling of the autonomous and controllable subsystems, parametrizations). See also [21,53,58] for other applications to mathematical systems theory.

The different algorithms presented in the paper have been implemented in the Maple package Morphisms [21] based on the library OreModules [17]. This package is available with a large library of concrete examples which demonstrates the main results of the paper.

Finally, this paper is an extension of the congress paper [20].

2. Morphisms of linear functional systems

2.1. Finitely presented modules and linear functional systems

In this paper, we consider linear functional systems defined by matrices with entries in an *Ore algebra D* and we study them by means of their associated left D-modules. In this first subsection, we gather many useful definitions and properties on these concepts.

Definition 2.1. Let A be a ring, σ an *endomorphism* of A, namely,

$$\forall a, b \in A, \quad \sigma(a + b) = \sigma(a) + \sigma(b), \quad \sigma(ab) = \sigma(a)\sigma(b),$$

and δ a σ*-derivation*, namely, $\delta : A \to A$ satisfies:

$$\forall a, b \in A, \quad \delta(a + b) = \delta(a) + \delta(b), \quad \delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

1. [46] A non-commutative polynomial ring $A[\partial; \sigma, \delta]$ in ∂ is called *skew* if it satisfies the following commutation rule:

   $$\forall a \in A, \quad \partial a = \sigma(a)\partial + \delta(a). \tag{3}$$

   An element P of $A[\partial; \sigma, \delta]$ has the canonical form $P = \sum_{i=0}^{r} a_i \partial^i$, where $a_i \in A$ and $i = 0, \ldots, r \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. If $a_r \neq 0$, then the *order* of P, denoted by $\mathrm{ord}(P)$, is r.
2. [15,46] Let k be a field and A be either k or the commutative polynomial ring $k[x_1, \ldots, x_n]$. The skew polynomial ring $D = A[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m]$ is called an *Ore algebra* if the σ_i's and δ_j's commute for $1 \leqslant i, j \leqslant m$ and satisfy the following conditions:

   $$\sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0, \quad 1 \leqslant j < i \leqslant m,$$

An element P of D has the canonical form $P = \sum_{0 \leqslant |\nu| \leqslant r} a_\nu \partial^\nu$, where $a_\nu \in A$, $r \in \mathbb{Z}_+$, $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}_+^n$ denotes a multi-index of non-negative integers, $|\nu| = \nu_1 + \cdots + \nu_n$ its length, and $\partial^\nu = \partial_1^{\nu_1} \cdots \partial_n^{\nu_n}$. If there exists $\nu \in \mathbb{Z}_+^n$ such that $|\nu| = r$ and $a_\nu \neq 0$, then the (*total*) *order* $\mathrm{ord}(P)$ is r.

We note that the commutation rule (3) must be understood as a generalization of the Leibniz rule for functional operators, namely, for an unknown y, we have

$$\partial(ay) = \sigma(a)\partial(y) + \delta(a)y.$$

Let us give a few examples of Ore algebras which will be important in what follows.

Example 2.1

1. Let k be a field, $A = k$ or $k[n]$, $\sigma : A \to A$ the *forward shift operator*, namely, $\sigma(a(n)) = a(n+1)$, and $\delta = 0$. Then, the skew polynomial ring $A[\partial; \sigma, 0]$ is the ring of shift operators with coefficients in A (i.e., constant or polynomial coefficients).
2. Let k be a field, $A = k$ or $k[t]$, $\sigma = \mathrm{id}_A$ and $\delta : A \to A$ the standard derivation $\frac{\mathrm{d}}{\mathrm{d}t}$. The skew polynomial ring $A[\partial; \mathrm{id}_A, \frac{\mathrm{d}}{\mathrm{d}t}]$ is then the ring of differential operators with coefficients in A (i.e., constant or polynomial coefficients).
3. If k is a field and A is respectively k or $k[x_1, \ldots, x_n]$, then we can consider

   $$\sigma_i = \mathrm{id}_{A[\partial_1;\sigma_1,\delta_1]\cdots[\partial_{i-1};\sigma_{i-1},\delta_{i-1}]}, \quad \forall a \in A, \quad \delta_i(a) = \frac{\partial a}{\partial x_i}$$

   the standard derivation of $a \in A$ with respect to x_i. Then, the Ore algebra

   $$A[\partial_1; \mathrm{id}, \delta_1] \cdots [\partial_n; \mathrm{id}, \delta_n]$$

   is the ring of differential operators with respectively constant or polynomial coefficients. The last algebra is called the *Weyl algebra* and is denoted by

   $$A_n(k) = k[x_1, \ldots, x_n][\partial_1; \mathrm{id}, \delta_1] \cdots [\partial_n; \mathrm{id}, \delta_n].$$
4. Let k be a field, $A = k$ or $k[t]$, and $A[\partial_1; \mathrm{id}_A, \frac{\mathrm{d}}{\mathrm{d}t}]$ the ring of differential operators with coefficients in A. Let $h \in \mathbb{R}_+$ be a positive real and let us denote by $\sigma_2(a(t)) = a(t-h)$ the time-delay operator and $\delta_2(a) = 0$ for all $a \in A$. Then, $A[\partial_1; \mathrm{id}_A, \frac{\mathrm{d}}{\mathrm{d}t}][\partial_2; \sigma_2, 0]$ is the Ore algebra of differential time-delay operators with coefficients in A.

By extension, we can consider the previous Ore algebras with rational coefficients instead of polynomial ones. For instance, the ring of differential operators with rational coefficients $B_n(k) = k(x_1, \ldots, x_n)[\partial_1; \mathrm{id}, \delta_1] \cdots [\partial_n; \mathrm{id}, \delta_n]$, $\delta_i = \frac{\partial}{\partial x_i}$, is also called the Weyl algebra.

We refer the reader to [15] for more examples of functional operators such as, for instance, difference, divided difference, q-difference, q-dilation operators and for their applications in the study of special functions and combinatorics. See also [41] for other polynomial algebras.

Proposition 2.1 [15,36]. *Let k be a computable field (e.g., $k = \mathbb{Q}, \mathbb{F}_p$), $A = k[x_1, \ldots, x_n]$ the polynomial ring with n indeterminates over the field k and $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ an Ore algebra satisfying the conditions*

$$\sigma_i(x_j) = a_{ij} x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij}, \ 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n, \tag{4}$$

for certain $a_{ij} \in k \setminus \{0\}$, $b_{ij} \in k$, $c_{ij} \in A$. Let $\prec$ be an admissible term order, i.e., a total order on the set $B = \{x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_m^{j_m} | (i_1, \ldots, i_n) \in \mathbb{Z}_+^n, (j_1, \ldots, j_m) \in \mathbb{Z}_+^m\}$ with 1 as smallest *element and such that $tu \prec tv$ for all $t \in B$ whenever $u \prec v$ for $u, v \in B$. If the $\prec$-greatest term u in each non-zero c_{ij} satisfies $u \prec x_j \partial_i$, then a non-commutative version of Buchberger's algorithm terminates for this admissible term order and its result is a Gröbner basis with respect to this order.*

We refer the reader to [35,36,41] for more general results. In particular, the existence of Gröbner bases and the generalization of Buchberger's algorithm have been studied in [12,14,40,44,70] for the Weyl algebras $A_n(\mathbb{Q})$ and $B_n(\mathbb{Q})$.

In the rest of the paper, *we shall only consider an Ore algebra D over which the existence of Grobner bases for any admissible term order is ensured.*

We can prove that the hypotheses of Proposition 2.1 implies that the Ore algebra D is a *left noetherian ring*, namely, every left ideal I of D is finitely generated as a left D-module, i.e., there exists a finite family $\{a_i\}_{i=1,\dots,l(I)}$ of elements of D such that $I = D a_1 + \cdots + D a_{l(I)}$ [46]. Moreover, an Ore algebra D satisfying the hypotheses of Proposition 2.1 can be proved to be a *domain*, namely, the product of non-zero elements of D is non-zero [46]. We note that Proposition 2.1 holds for the examples of Ore algebras described in Example 2.1.

Let us recall the well-known concept of *homorphisms* as it will play a central role.

Definition 2.2 [66]. Let M and N be two left D-modules.

1. A *D-homomorphism* or, simply, a *D-morphism* f from M to N is a map satisfying:
   $\forall a_1, a_2 \in D, \forall m_1, m_2 \in M, \quad f(a_1 m_1 + a_2 m_2) = a_1 f(m_1) + a_2 f(m_2).$
2. We denote by $\hom_D(M, M')$ the abelian group of the D-morphisms from M to M'. If M has a D-D' bimodule structure, namely, M is a right D'-module which satisfies $(am)b = a(mb)$ for all a in D and b in D', then $\hom_D(M, M')$ inherits a right D'-module structure. In particular, if D is a commutative ring, then $\hom_D(M, M')$ is a D-module.
3. A D-morphism $f \in \hom_D(M, M')$ is an *isomorphism* if f is injective and surjective.
4. If $M' = M$, then we denote the non-commutative ring of endomorphisms of M by $\mathrm{end}_D(M)$. Moreover, we denote by $\mathrm{aut}_D(M)$ the non-abelian group of *automorphisms* of M, namely, the group of injective and surjective D-morphisms from M to M.

In what follows, we shall assume that a linear functional system (LFS) is defined by means of a matrix of functional operators $R \in D^{q\times p}$, where D is an Ore algebra satisfying the hypotheses of Proposition 2.1. We consider the D-morphism of left D-modules defined by

$$\begin{array}{ccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p}, \\ (\lambda_1, \dots, \lambda_q) & \mapsto & (\lambda_1, \dots, \lambda_q) R = \left(\sum_{i=1}^{q} \lambda_i R_{i1}, \dots, \sum_{i=1}^{q} \lambda_i R_{ip}\right). \end{array} \tag{5}$$

Generalizing an important idea coming from number theory and algebraic geometry, we shall consider the following important left D-module

$$M = D^{1\times p}/(D^{1\times q} R)$$

which is the cokernel of the D-morphism $.R$ defined by (5).

This idea can be traced back to the work of Malgrange [45] on linear systems of PDEs with constant coefficients and it has been extended to the variable coefficients case by Kashiwara [34]. See also [10,22]. We refer to [16] for the extension to linear functional systems.

Before explaining the main interest of the left D-module M, we first recall some basic concepts of homological algebra used in the sequel. We refer the reader to [66] for more details.

Definition 2.3. A sequence $(M_i, d_i : M_i \to M_{i-1})_{i\in\mathbb{Z}}$ of left D-modules M_i and D-morphisms $d_i : M_i \to M_{i-1}$ is said to be:

1. A *complex* if, for all $i \in \mathbb{Z}$, $d_i \circ d_{i+1} = 0$ or, equivalently, $\operatorname{im} d_{i+1} \subseteq \ker d_i$. The complex $(M_i, d_i)_{i\in\mathbb{Z}}$ is then denoted by
$$\cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$
The *defect of exactness* of the complex $(M_i, d_i)_{i\in\mathbb{Z}}$ at M_i is defined by $H(M_i) = \ker d_i / \operatorname{im} d_{i+1}$.
2. *Exact* at M_i if $\ker d_i = \operatorname{im} d_{i+1}$, i.e., $H(M_i) = 0$ and *exact* if $\ker d_i = \operatorname{im} d_{i+1}$, for all $i \in \mathbb{Z}$.
3. *Split exact* if it is exact and there further exist left D-morphisms $s_i : M_{i-1} \to M_i$ satisfying:
$$\forall i \in \mathbb{Z}, s_{i+1} \circ s_i = 0, \quad s_i \circ d_i + d_{i+1} \circ s_{i+1} = id_{M_i}.$$
The complex $(M_{i-1}, s_i)_{i\in\mathbb{Z}}$ is then also exact.

Using (5), we obtain the exact sequence
$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M = D^{1\times p}/(D^{1\times q} R) \to 0, \tag{6}$$
where π denotes the canonical projection of $D^{1\times p}$ onto M that sends an element of $D^{1\times p}$ to its residue class in M. The exact sequence (6) is called a *finite presentation* of M and M is said to be a *finitely presented* left D-module [37,66].

Let us describe M in terms of generators and relations. Let $\{e_i\}_{1\leqslant i\leqslant p}$ (resp., $\{f_j\}_{1\leqslant j\leqslant q}$) be the standard basis of $D^{1\times p}$ (resp., $D^{1\times q}$), namely, the basis of $D^{1\times p}$ formed by the row vectors e_i defined by 1 at the ith position and 0 elsewhere. We denote by y_i the residue class of e_i in M, i.e., $y_i = \pi(e_i)$. Then, $\{y_i\}_{1\leqslant i\leqslant p}$ is a set of generators of M as every element $m \in M$ is the form $\pi(\mu)$, where $\mu = (\mu_1, \dots, \mu_p) \in D^{1\times p}$, and thus, we obtain $m = \pi(\mu) = \sum_{i=1}^p \mu_i \pi(e_i) = \sum_{i=1}^p \mu_i y_i$. The left D-module M is said to be *finitely generated* [37,66]. Now, for $j = 1, \dots, q$, we have

1. $f_j R = (R_{j1}, \dots, R_{jp}) \in (D^{1\times q} R) \Rightarrow \pi(f_j R) = 0$,
2. $\pi(f_j R) = \sum_{k=1}^p R_{jk} \pi(e_k) = \sum_{k=1}^p R_{jk} y_k$.

Hence, the generators $\{y_i\}_{1\leqslant i\leqslant p}$ of M satisfy the relations $\sum_{k=1}^p R_{jk} y_k = 0$ for $j = 1, \dots, q$, or, more compactly, $Ry = 0$ where $y = (y_1, \dots, y_p)^{\mathrm{T}}$.

Example 2.2. Let us consider the equations of a fluid in a tank satisfying Saint-Venant's equations and subjected to a one-dimensional horizontal move developed in [23]:
$$\begin{cases} y_1(t-2h) + y_2(t) - 2\dot{u}(t-h) = 0, \\ y_1(t) + y_2(t-2h) - 2\dot{u}(t-h) = 0. \end{cases} \tag{7}$$

Let $D = \mathbb{Q}[\partial_1; \mathrm{id}_{\mathbb{Q}}, \frac{\mathrm{d}}{\mathrm{d}t}][\partial_2; \sigma_2, 0]$ be the Ore algebra of differential time-delay operators with coefficients in $\mathbb{Q}$ defined in 4 of Example 2.1 and let us consider the system matrix of (7):
$$R = \begin{pmatrix} \partial_2^2 & 1 & -2\partial_1\partial_2 \\ 1 & \partial_2^2 & -2\partial_1\partial_2 \end{pmatrix} \in D^{2\times 3}. \tag{8}$$

The D-module $M = D^{1\times 3}/(D^{1\times 2}R)$ is then defined by the following finite presentation:

$$D^{1\times 2} \xrightarrow{.R} D^{1\times 3} \xrightarrow{\pi} M \to 0.$$

We note that $\ker_D(.R) = \{\lambda \in D^{1\times 2} | \lambda R = 0\} = 0$ as the rows of R are D-linearly independent and we get the following exact sequence $0 \to D^{1\times 2} \xrightarrow{.R} D^{1\times 3} \xrightarrow{\pi} M \to 0$.

To develop the relations between the properties of the finitely presented left D-module M defined by (6) and the solutions of the system $Ry = 0$, we need to introduce a few more concepts of module theory (see [37,66] for details).

Definition 2.4

1. A finitely generated left D-module is called *free* if M is isomorphic to a finite power of D, i.e., there exists an injective and surjective D-morphism from M to $D^{1\times r}$, where r is a non-negative integer. r is then called the *rank* of the free D-module M.
2. A finitely generated left D-module M is called *projective* if there exist a left D-module N and a non-negative integer r such that $M \oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules and $P \cong Q$ means that P and Q are isomorphic as left D-modules. Then, N is also a projective left D-module.
3. A *projective resolution* of a left D-module M is an exact sequence of the form
$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0, \tag{9}$$
where the P_i are projective left D-modules. If all the P_i are free left D-modules, then (9) is called a *free resolution* of M. Finally, if there exists a non-negative integer s such that $P_r = 0$ for all $r \geqslant s$ and the P_i's are finitely generated free left D-modules, then (9) is called a *finite free resolution* of M.
4. Let (9) be a projective resolution of a left D-module M. We call *truncated projective resolution of M* the complex defined by $\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$.

Let us suppose that a finitely presented left D-module admits a finite free resolution (we note that it is always the case for the Ore algebras defined in Example 2.1 as it is proved in [16]):

$$0 \to D^{1\times p_l} \xrightarrow{.R_l} \cdots \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \to 0. \tag{10}$$

Let $\mathscr{F}$ be a left D-module. Then, applying the *functor* $\hom(\cdot, \mathscr{F})$ [66] to the following truncated free resolution of M

$$0 \to D^{1\times p_l} \xrightarrow{.R_l} \cdots \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \to 0,$$

and using the isomorphism $\varphi : \hom_D(D^{1\times p_i}, \mathscr{F}) \cong \mathscr{F}^{p_i}$ defined by $\varphi(f) = (f(e_1), \ldots, f(e_{p_i}))^{\mathrm{T}}$, for all $f \in \hom_D(D^{1\times p_i}, \mathscr{F})$, where $\{e_j\}_{1\leqslant j\leqslant p_i}$ denotes the standard basis of the free D-module $D^{1\times p_i}$, we then get the following complex:

$$0 \leftarrow \mathscr{F}^{p_l} \xleftarrow{R_l.} \cdots \xleftarrow{R_2.} \mathscr{F}^{p_1} \xleftarrow{R_1.} \mathscr{F}^{p_0} \leftarrow 0, \tag{11}$$

where, for $i = 1, \ldots, l$, we have

$$R_i. : \mathscr{F}^{p_{i-1}} \to \mathscr{F}^{p_i}$$
$$\zeta = (\zeta_1 \ldots \zeta_{p_{i-1}})^{\mathrm{T}} \mapsto (R_i.)(\zeta) = R_i \zeta.$$

A classical result of homological algebra proves, up to some isomorphisms, the defects of exactness of (11) only depend on M and $\mathscr{F}$ and not on the choice of the finite free resolution (10) of M. See [66] for more details. In particular, these defects of exactness can also be defined by means of projective resolutions of M and not necessarily a finite free resolution of M as we have done for simplicity reasons. These defects of exactness are denoted by

$$\begin{cases} \mathrm{ext}_D^0(M, \mathscr{F}) \cong \ker_{\mathscr{F}}(R_1.) = \{\eta \in \mathscr{F}^{p_0} | R_1 \eta = 0\}, \\ \mathrm{ext}_D^i(M, \mathscr{F}) \cong \ker_{\mathscr{F}}(R_{i+1}.)/\mathrm{im}_{\mathscr{F}}(R_i.), \quad i \geqslant 1. \end{cases}$$

It is quite easy to show that

$$\mathrm{ext}_D^0(M, \mathscr{F}) = \hom_D(M, \mathscr{F})$$

(see, e.g., [66]) which proves that the abelian group $\ker_{\mathscr{F}}(R_1.)$ of $\mathscr{F}$-solutions of the linear functional system $R_1\eta = 0$ is isomorphic to $\hom_D(M, \mathscr{F})$. We refer to [34,45,51] for more details. The abelian group $\ker_{\mathscr{F}}(R_1.)$ is sometimes called the *behaviour* of the left D-module $M = D^{1\times p_0}/(D^{1\times p_1} R_1)$ [50,52,54,60,73,76]. Moreover, if we want to solve the inhomogeneous system $R_1\eta = \zeta$, where $\zeta \in \mathscr{F}^{p_1}$ is fixed, then, using the fact that (10) is exact, we obtain that a necessary condition for the existence of a solution $\eta \in \mathscr{F}^{p_0}$ is given by $R_2\zeta = 0$ as we have

$$R_1\eta = \zeta \Rightarrow R_2(R_1\eta) = R_2\zeta \Rightarrow R_2\zeta = 0.$$

In order to understand if the compatibility condition $R_2\zeta = 0$ is also sufficient, we need to investigate the residue class of ζ in $\mathrm{ext}_D^1(M, \mathscr{F}) = \ker_{\mathscr{F}}(R_2.)/(R_1\mathscr{F}^{p_0})$. If its residue class is reduced to 0, then it means that $\zeta \in \mathscr{F}^{p_1}$ satisfying $R_2\zeta = 0$ is such that $\zeta \in (R_1\mathscr{F}^{p_0})$, i.e., there exists $\gamma \in \mathscr{F}^{p_0}$ such that $R_1\gamma = \zeta$. The solution η is generally not unique as we can add any element of $\ker_{\mathscr{F}}(R_1.) = \{\eta \in \mathscr{F}^{p_0} | R_1\eta = 0\}$ to γ.

Definition 2.5 [37,66].

1. A left D-module $\mathscr{F}$ is called *injective* if, for every left D-module M, we have $\mathrm{ext}_D^i(M, \mathscr{F}) = 0$ for $i \geqslant 1$.
2. A left D-module $\mathscr{F}$ is called *cogenerator* if $\hom_D(M, \mathscr{F}) = 0$ implies $M = 0$.

If $\mathscr{F}$ is an injective left D-module, then $R_2\zeta = 0$ is a necessary and sufficient condition for the existence of $\eta \in \mathscr{F}^{p_0}$ satisfying $R_1\eta = \zeta$. Moreover, if $\mathscr{F}$ is a cogenerator left D-module and M is not reduced to the trivial module 0, then $\hom_D(M, \mathscr{F}) \neq 0$, which means that the system $R_1\eta = 0$ admits at least one solution in $\mathscr{F}^{p_0}$. Finally, if $\mathscr{F}$ is an injective cogenerator left D-module, then we can prove that any complex of the form

$$0 \leftarrow \mathscr{F}^{p_l} \xleftarrow{R_l.} \cdots \xleftarrow{R_2.} \mathscr{F}^{p_1} \xleftarrow{R_1.} \mathscr{F}^{p_0} \leftarrow \hom_D(M, \mathscr{F}) \leftarrow 0$$

is exact if and only if the corresponding complex (10) is exact [50].

Proposition 2.2 [66]. *For every ring D, there exists an injective cogenerator left D-module $\mathscr{F}$.*

In some interesting situations, explicit injective cogenerators are known.

Example 2.3

1. If Ω is a convex open subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathscr{D}'(\Omega)$) of smooth functions (resp., distributions) on Ω is an injective cogenerator module over the commutative ring $\mathbb{R}[\partial_1; \mathrm{id}, \delta_1]\cdots[\partial_n; \mathrm{id}, \delta_n]$ of partial differential operators with coefficients in $\mathbb{R}$ (see, e.g., Corollary 7.8.4 of [51] and also [45,50]).

2. If $\mathscr{F}$ is the set of all functions that are smooth on $\mathbb{R}$ except for a finite number of points, then $\mathscr{F}$ is an injective cogenerator left $\mathbb{R}(t)[\partial; \mathrm{id}_{\mathbb{R}(t)}, \frac{\mathrm{d}}{\mathrm{d}t}]$-module. See [77] for more details.

To finish, let us recall two classical results of homological algebra. See [66] for more information.

Proposition 2.3

1. *Let us consider the following short exact sequence of left D-modules*:

   $$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

   If M'' is a projective left D-module, then the previous exact sequence splits.
2. *Let $\mathscr{F}$ be a left D-module. Then, the contravariant left exact functor* $\hom_D(\cdot, \mathscr{F})$ *transforms split exact sequences of left D-modules into split exact sequences of abelian groups.*

2.2. Morphisms of finitely presented modules

2.2.1. Definitions and results

Let us first introduce a few definitions of homological algebra. See [66] for more details.

Definition 2.6

1. Let $(P_i, d_i)_{i\in\mathbb{Z}}$ and $(P'_i, d'_i)_{i\in\mathbb{Z}}$ be two complexes of left D-modules. A *morphism of complexes* $f : (P_i, d_i)_{i\in\mathbb{Z}} \to (P'_i, d'_i)_{i\in\mathbb{Z}}$ is a set of D-morphisms $f_i : P_i \to P'_i$ satisfying

   $$\forall i \in \mathbb{Z}, \quad d'_i \circ f_i = f_{i-1} \circ d_i,$$

   i.e., such that we have the following commutative diagram:

   $$\begin{array}{ccccccccc} \ldots \xrightarrow{d_{i+2}} & P_{i+1} & \xrightarrow{d_{i+1}} & P_i & \xrightarrow{d_i} & P_{i-1} & \xrightarrow{d_{i-2}} \ldots \\ & \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} & \\ \ldots \xrightarrow{d'_{i+2}} & P'_{i+1} & \xrightarrow{d'_{i+1}} & P'_i & \xrightarrow{d'_i} & P'_{i-1} & \xrightarrow{d'_{i-2}} \ldots \end{array}$$
2. A morphism of complexes $f : (P_i, d_i)_{i\in\mathbb{Z}} \to (P'_i, d'_i)_{i\in\mathbb{Z}}$ is said to be *homotopic to zero* if there exist D-morphisms $s_i : P_i \to P'_{i+1}$ such that:

   $$\forall i \in \mathbb{Z}, \quad f_i = d'_{i+1} \circ s_i + s_{i-1} \circ d_i.$$

   By extension, two morphisms of complexes $f, f' : (P_i, d_i)_{i\in\mathbb{Z}} \to (P'_i, d'_i)_{i\in\mathbb{Z}}$ are *homotopic* if $f - f'$ is homotopic to zero.
3. A morphism of complexes $f : (P_i, d_i)_{i\in\mathbb{Z}} \to (P'_i, d'_i)_{i\in\mathbb{Z}}$ is called a *homotopy equivalence* or a *homotopism* if there exists a morphism of complexes $g : (P'_i, d'_i)_{i\in\mathbb{Z}} \to (P_i, d_i)_{i\in\mathbb{Z}}$ such that $f \circ g - \mathrm{id}_{P'}$ and $g \circ f - \mathrm{id}_P$ are homotopic to zero, where $\mathrm{id}_P = (P_i, \mathrm{id}_{P_i})_{i\in\mathbb{Z}}$. The complexes $(P_i, d_i)_{i\in\mathbb{Z}}$ and $(P'_i, d'_i)_{i\in\mathbb{Z}}$ are then said to be *homotopy equivalent*.

We have the following important result. See [66] for a proof.

Proposition 2.4. *Let $(P_i, d_i)_{i\in\mathbb{Z}}$ (resp., $(P'_i, d'_i)_{i\in\mathbb{Z}}$) be a truncated projective resolution of a left D-module M (resp., M'). Then, a morphism $f : M \to M'$ induces a morphism of complexes $\tilde{f} : (P_i, d_i)_{i\in\mathbb{Z}} \to (P'_i, d'_i)_{i\in\mathbb{Z}}$ defined uniquely up to a homotopy equivalence.*

Conversely, a morphism of complexes $\tilde{f} : (P_i, d_i)_{i\in\mathbb{Z}} \to (P'_i, d'_i)_{i\in\mathbb{Z}}$ *from a truncated projective resolution* $(P_i, d_i)_{i\in\mathbb{Z}}$ *of* M *to a truncated projective resolution* $(P'_i, d'_i)_{i\in\mathbb{Z}}$ *of* M' *induces a morphism* $f : M \to M'$.

From Proposition 2.4, we deduce the following interesting corollary.

Corollary 2.1. *Let us consider the following finite presentations of respectively* M *and* M':

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \to 0,$$

$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \to 0.$$

1. *The existence of a morphism* $f : M \to M'$ *is equivalent to the existence of two matrices* $P \in D^{p\times p'}$ *and* $Q \in D^{q\times q'}$ *satisfying the relation*:

   $$RP = QR'. \tag{12}$$

   We then have the following commutative exact diagram

   $$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \to 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \to 0, \end{array} \tag{13}$$

   where the D*-morphism* $f \in \hom_D(M, M')$ *is defined by*

   $$\forall \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda P). \tag{14}$$

2. *Let us denote by* $R'_2 \in D^{q'_2\times q'}$ *a matrix satisfying* $\ker_D(.R') = D^{1\times q'_2} R'_2$ *and let* $P \in D^{p\times p'}$ *and* $Q \in D^{q\times q'}$ *be two matrices satisfying* $RP = QR'$. *Then, the matrices defined by*

   $$\begin{cases} \overline{P} = P + Z_1 R', \\ \overline{Q} = Q + R Z_1 + Z_2 R'_2, \end{cases}$$

   where $Z_1 \in D^{p\times q'}$ *and* $Z_2 \in D^{q\times q'_2}$ *are two arbitrary matrices, satisfy* $R\overline{P} = \overline{Q}R'$ *and*:

   $$\forall \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda P) = \pi'(\lambda \overline{P}).$$

We note that a D-morphism $f \in \hom_D(M, M')$ is in fact defined by a matrix $P \in D^{p\times p}$ satisfying that $(D^{1\times q} R)P \subseteq (D^{1\times q'} R')$. Hence, the matrix P plays a more important role than Q, a fact which is also clear from (14).

In the particular case where $R' = R$, from Corollary 2.1, we obtain that the existence of an endomorphism f of M is equivalent to the existence of two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying the following relation:

$$RP = QR. \tag{15}$$

Example 2.4. Let us consider the left D-module $M = D/I$, where $I = \sum_{i=1}^{q} DR_i$ is a finitely generated left ideal of D. Then, the left D-module M admits the following presentation:

$$D^{1\times q} \xrightarrow{.R} D \xrightarrow{\pi} M \to 0, \quad R = (R_1, \ldots, R_q)^{\mathrm{T}} \in D^{q\times 1}.$$

By Corollary 2.1, $f \in \mathrm{end}_D(M)$ is defined by $P \in D$ satisfying $(D^{1\times q} R)P \subseteq (D^{1\times q} R)$, i.e., by $P \in D$ such that $IP \subseteq I$. The set $\mathscr{E}(I) = \{P \in D \mid IP \subseteq I\}$ is called the *idealizer* of the left ideal I of D. It is the largest subring of D which contains I as a two-sided ideal [46]. The *eigenring* of I is then the ring defined by $\mathscr{E}(I)/I$. We have the ring isomorphism $\sigma : \mathrm{end}_D(D/I) \cong \mathscr{E}(I)/I$ defined by $\sigma(f) = \kappa(P)$, where $P \in \mathscr{E}(I)$ satisfies that, for all $\lambda \in D$, $f(\pi(\lambda)) = \pi(\lambda P)$ and $\kappa : \mathscr{E}(I) \to \mathscr{E}(I)/I$ denotes the canonical projection [46].

If D is a commutative ring, we then have $\mathscr{E}(I) = D$ and $\mathrm{end}_D(D/I) \cong D/I$. For instance, if we consider the commutative ring $D = \mathbb{Q}[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}][\partial_x; \mathrm{id}, \frac{\partial}{\partial x}]$ of differential operators with rational constant coefficients and $I = D(\partial_t - \partial_x^2)$ the ideal of D generated by the differential operator defining the heat equation, then $\mathrm{end}_D(D/I) \cong D/I \cong \mathbb{Q}[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}]$. Finally, if we consider the ring $D = \mathbb{Q}[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}][\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}][\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}]$ and the D-module $M = D/(D\partial_1 + D\partial_2 + D\partial_3)$ associated with the gradient operator in $\mathbb{R}^3$, we then get $\mathrm{end}_D(M) \cong M \cong \mathbb{Q}$.

Before illustrating Corollary 2.1, let us give a direct consequence of this corollary which shows one interest of computing morphisms between finitely presented left D-modules.

Corollary 2.2. *With the same hypotheses and notations as in Corollary* 2.1, *if $\mathscr{F}$ is a left D-module, then the morphism*

$$\begin{aligned} P. : \mathscr{F}^{p'} &\to \mathscr{F}^p \\ \zeta &\mapsto (P.)(\zeta) = P\zeta, \end{aligned}$$

sends elements of $\ker_{\mathscr{F}}(R'.)$ *to elements of* $\ker_{\mathscr{F}}(R.)$, *i.e.*, *$\mathscr{F}$-solutions of the system $R'\zeta = 0$ to $\mathscr{F}$-solutions of the system $R\eta = 0$.*

Proof. Applying the contravariant left exact functor $\mathrm{hom}_D(\cdot, \mathscr{F})$ to the commutative exact diagram (13), we obtain the following commutative exact diagram (see [66] for more details):

$$\begin{array}{ccccccc} \mathscr{F}^q & \xleftarrow{R.} & \mathscr{F}^p & \xleftarrow{\pi^\star} & \mathrm{hom}_D(M, \mathscr{F}) & \leftarrow 0 \\ \uparrow Q. & & \uparrow P. & & \uparrow f^\star & \\ \mathscr{F}^{q'} & \xleftarrow{R'.} & \mathscr{F}^{p'} & \xleftarrow{(\pi')^\star} & \mathrm{hom}_D(M', \mathscr{F}) & \leftarrow 0. \end{array}$$

Up to an isomorphism, at the end of the previous subsection, we have seen that we can identify $\mathrm{hom}_D(M, \mathscr{F})$ (resp., $\mathrm{hom}_D(M', \mathscr{F})$) with $\ker_{\mathscr{F}}(R.)$ (resp., $\ker_{\mathscr{F}}(R'.)$). An easy chase in the previous exact diagram proves that, for all $\zeta \in \ker_{\mathscr{F}}(R'.)$, we have $f^\star(\zeta) = P\zeta \in \ker_{\mathscr{F}}(R.)$. □

Remark 2.1. From Corollary 2.2, we see that the computation of morphisms from a finitely presented left D-module M to a finitely presented left D-module M' gives some kind of "Galois transformations" which send solutions of the second system to solutions of the first one. This fact is particularly clear when $R = R'$: we then send a solution of the system to another one.

As an example, we now apply Corollary 2.1 to a particular case and recover in a unified way the *eigenring* introduced in the literature (see [4,11,18,19,61,68,74,75] for more details).

Example 2.5. Let us consider a skew polynomial ring $D = A[\partial; \sigma, \delta]$ over a (non-commutative) ring A and two matrices $E, F \in A^{p\times p}$. Let us consider $R = (\partial I_p - E) \in D^{p\times p}$ (resp., $R' = (\partial I_p - F) \in D^{p\times p}$) and the finitely presented left D-module $M = D^{1\times p}/(D^{1\times p}R)$ (resp., $M' = D^{1\times p}/(D^{1\times p}R')$). Let π (resp., π') be the canonical projection of $D^{1\times p}$ onto

M (resp., M') and $\{e_i\}_{1\leqslant i\leqslant p}$ the standard basis of $D^{1\times p}$. As we have seen in Section 2.1, $\{y_i=\pi(e_i)\}_{1\leqslant i\leqslant p}$ and $\{z_i=\pi'(e_i)\}_{1\leqslant i\leqslant p}$ satisfy the equations:

$$\begin{aligned} \partial y_i &= \sum_{j=1}^{p} E_{ij}\, y_j, \quad i=1,\dots,p,\\ \partial z_i &= \sum_{j=1}^{p} F_{ij}\, z_j, \quad i=1,\dots,p. \end{aligned} \tag{16}$$

Let f be a morphism from M to M'. Then, there exists a matrix $P=(P_{ij})_{1\leqslant i,j\leqslant p}\in D^{p\times p}$ such that $f(y_i)=\sum_{j=1}^{p} P_{ij}\, z_j$. Using (16), we easily check that we can always suppose that all the P_{ij} belong to A, i.e., $P\in A^{p\times p}$. By Corollary 2.1, there exists $Q\in D^{p\times p}$ satisfying (12).

f is the zero morphism if and only if there exists a matrix $Z\in D^{p\times p}$ satisfying $P=ZR'$. As the order of P is 0 in ∂ and that of R' is 1 in ∂, we get that $Z=0$, i.e., $P=0$ and $Q=0$.

Now, let us suppose that P and Q are different from zero. As both the orders of RP and R' in ∂ are 1, we deduce that the order of Q must be 0, i.e., $Q\in A^{p\times p}$. Then, we get

$$\begin{aligned} (12) \ &\Leftrightarrow (\partial I_p - E)P = Q(\partial I_p - F)\\ &\Leftrightarrow \sigma(P)\partial + \delta(P) - EP = Q\partial - QF\\ &\Leftrightarrow (\sigma(P)-Q)\partial + (\delta(P) - EP + QF) = 0. \end{aligned} \tag{17}$$

The first order polynomial matrix in the left-hand side of (17) must be equal to 0 so that:

$$(17) \Leftrightarrow \begin{cases} Q=\sigma(P),\\ \delta(P) = EP - \sigma(P)F. \end{cases} \tag{18}$$

We then obtain the following commutative exact diagram:

$$\begin{array}{ccccccccc} 0 & \rightarrow & D^{1\times p} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \rightarrow 0\\ & & \downarrow .\sigma(P) & & \downarrow .P & & \downarrow f & \\ 0 & \rightarrow & D^{1\times p} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{\pi'} & M' & \rightarrow 0. \end{array} \tag{19}$$

Conversely, if there exist $P\in A^{p\times p}$ and $Q\in A^{p\times p}$ which satisfy (18), we can check that (12) holds, i.e., the commutative exact diagram (19) where the morphism $f: M\rightarrow M'$ is defined by

$$\forall m=\pi(\lambda)\in M, \quad \lambda\in D^{1\times p}, \quad f(m)=\pi'(\lambda P).$$

The previous results prove that we have

$$\hom_D(M,M') = \left\{ f: M\rightarrow M' \mid f(y_i)=\sum_{j=1}^{p} P_{ij}\, z_j,\ 1\leqslant i\leqslant p,\ P\in A^{p\times p},\ \delta(P) = EP - \sigma(P)F \right\},$$

$$\operatorname{end}_D(M) = \left\{ f: M\rightarrow M \mid f(y_i)=\sum_{j=1}^{p} P_{ij}\, y_j,\ 1\leqslant i\leqslant p,\ P\in A^{p\times p},\ \delta(P) = EP - \sigma(P)E \right\}.$$

For instance, if we consider the ring $A=k[t]$ or $k(t)$ and $D=A[\partial;\mathrm{id}_A,\frac{\mathrm{d}}{\mathrm{d}t}]$, then (18) becomes:

$$\begin{cases} Q(t) = P(t), \\ \dot{P}(t) = E(t)P(t) - P(t)F(t). \end{cases} \tag{20}$$

If we consider $A = k[n]$ or $k(n)$ and $D = A[\partial; \sigma, 0]$, where $\sigma(a(n)) = a(n+1)$, then (18) gives:

$$\begin{cases} Q_n = \sigma(P_n) = P_{n+1}, \\ E_n P_n - \sigma(P_n) F_n = E_n P_n - P_{n+1} F_n = 0. \end{cases} \tag{21}$$

We find again in a unified way known results concerning the eigenring of a linear system.

Finally, if $\mathscr{F}$ is a left D-module, then applying the contravariant left exact functor $\hom_D(\cdot, \mathscr{F})$ to the commutative exact diagram (19), we then obtain the following commutative exact diagram:

$$\begin{array}{ccccccc} \mathscr{F}^p & \overset{R.}{\leftarrow} & \mathscr{F}^p & \leftarrow & \hom_D(M, \mathscr{F}) & \leftarrow & 0 \\ \uparrow \sigma(P). & & \uparrow P. & & \uparrow f^\star & & \\ \mathscr{F}^p & \overset{R'.}{\leftarrow} & \mathscr{F}^p & \leftarrow & \hom_D(M', \mathscr{F}) & \leftarrow & 0. \end{array}$$

If $\eta \in \hom_D(M', \mathscr{F})$, i.e., $\eta \in \mathscr{F}^p$ is a solution of the system $\partial \eta = F\eta$, then an easy chase in the previous commutative exact diagram shows that $\zeta = P\eta$ is then a solution of $\partial \zeta = E\zeta$, that is to say, $\zeta = f^\star(\eta) \in \hom_D(M, \mathscr{F})$. This last result can also be proved as follows:

$$\partial \zeta - E\zeta = \partial(P\eta) - E(P\eta) = \sigma(P)\partial\eta + \delta(P)\eta - (EP)\eta = \sigma(P)(\partial\eta - F\eta) = 0.$$

For instance, if $D = A[\partial; \mathrm{id}_A, \frac{\mathrm{d}}{\mathrm{d}t}]$, using (20), we then obtain

$$\begin{aligned} \partial\zeta(t) - E(t)\zeta(t) &= \partial(P(t)\eta(t)) - (E(t)P(t))\eta(t) = P(t)\partial\eta(t) - \dot{P}(t)\eta(t) - (EP)\eta(t) \\ &= P(t)(\partial\eta(t) - F\eta(t)) = 0. \end{aligned}$$

If we now consider $D = A[\partial; \sigma, 0]$, using (21), then we get

$$\zeta_{n+1} - E_n \zeta_n = P_{n+1}\eta_{n+1} - E_n P_n \eta_n = P_{n+1}(\eta_{n+1} - F_n \eta_n) = 0.$$

Let us introduce the definition of an *integrable connection*.

Definition 2.7. Let us consider the Weyl algebra $D = B_n(k)$, where k is a field of characteristic 0, and n matrices $E_i \in k(x_1, \ldots, x_n)^{p \times p}$. A *connection* is a linear system of PDEs of the form:

$$\begin{cases} \partial_1 y - E_1 y = 0, \\ \quad\vdots \\ \partial_n y - E_n y = 0. \end{cases} \tag{22}$$

Let us denote by $\nabla_i = \partial_i I_p - E_i \in D^{p \times p}$, $i = 1, \ldots, n$. Then, the connection (22) is said to be *integrable* if the following *integrability conditions* hold:

$$1 \leqslant i < j \leqslant n, \quad \nabla_i \nabla_j - \nabla_j \nabla_i = \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} + E_i E_j - E_j E_i = 0. \tag{23}$$

Using a Gröbner basis computation, we can always transform a *D-finite system* of PDEs [15], namely, a linear system of PDEs whose formal power series solutions at non-singular points only depend on a finite number of arbitrary constants, into an integrable connection.

Let us characterize the endomorphism ring of the left D-module M associated with the integrable connection (22).

Proposition 2.5. *Let $D = B_n(k)$ be the Weyl algebra, where k is a field of characteristic* 0, *n matrices $E_1, \ldots, E_n \in k(x_1, \ldots, x_n)^{p\times p}$ satisfying* (23), *the matrix of functional operators*

$$R = ((\partial_1 I_p - E_1)^{\mathrm{T}} \cdots (\partial_n I_p - E_n)^{\mathrm{T}})^{\mathrm{T}} \in D^{np\times p},$$

and the finitely presented left D-module $M = D^{1\times p}/(D^{1\times np} R)$ associated with (22).

Then, any D-endomorphism f of M can be defined by a pair of matrices $P \in k(x_1, \ldots, x_n)^{p\times p}$ and $Q \in k(x_1, \ldots, x_n)^{np\times np}$ satisfying the following relations

$$\begin{cases} \dfrac{\partial P}{\partial x_i} + P E_i - E_i P = 0, \quad i = 1, \ldots, n, \\ Q = \mathrm{diag}(P, \ldots, P), \end{cases} \tag{24}$$

where $\mathrm{diag}(P, \ldots, P)$ *denotes the diagonal matrix with n matrices P on the diagonal.*

Proof. Using the fact that the integrability conditions (23) are fulfilled, we can easily check that (22) forms a Janet basis with x_1 as a multiplicative variable for the equations $\partial_1 y - E_1 y = 0$, x_1 and x_2 as multiplicative variables for the equations $\partial_2 y - E_2 y = 0$, ..., and $x_1, \ldots, x_n$ as multiplicative variables for the equations $\partial_n y - E_n y = 0$ [65]. Let f be an endomorphism of M defined by a pair of matrices $P \in D^{p\times p}$ and $Q \in D^{np\times np}$. Using the special first-order form of (22), we can assume without a loss of generality that $P \in k(x_1, \ldots, x_n)^{p\times p}$. Then, the matrix $P \in k(x_1, \ldots, x_n)^{p\times p}$ defines an endomorphism of M if and only if we have $RP \in (D^{np\times np} R)$, i.e., if and only if the rows of $(\partial_i - E_i)P = P\partial_i + \partial P/\partial x_i - E_i P$ are reduced to zero with respect to the Janet basis (22) for $i = 1, \ldots, n$. Hence, we obtain that the zero-order equations $\partial P/\partial x_i + P E_i - E_i P = 0, i = 1, \ldots, n$, must be satisfied and $Q = \mathrm{diag}(P, \ldots, P)$. $\square$

In [7,61,75], the eigenring of the connection (22) is defined by

$$\mathscr{E} = \left\{ P \in k(x_1, \ldots, x_n)^{n\times n} \,\middle|\, \frac{\partial P}{\partial x_i} + P E_i - E_i P = 0, i = 1, \ldots, n \right\}.$$

We refer to [21] for a generalization of the previous result to linear functional systems.

The previous results show that the concept of endomorphism ring $\mathrm{end}_D(M)$ generalizes the concept of the eigenring of a 1-D linear system or of an integrable connection.

Finally, we refer the reader to [20,21] for relations between the concepts of endomorphism rings, eingerings and Lax pairs developed in the integrability theory of Hamiltonian systems and evolution equations.

2.2.2. Algorithms

Before giving two algorithms for the computation of morphisms between two finitely presented left modules, we first recall the notion of the *Kronecker product* of two matrices.

Definition 2.8. Let $E \in D^{q\times p}$ and $F \in D^{r\times s}$ be two matrices with entries in a ring D. The *Kronecker product* of E and F, denoted by $E \otimes F$, is the matrix defined by

$$E \otimes F = \begin{pmatrix} E_{11}F & \ldots & E_{1p}F \\ \vdots & \vdots & \vdots \\ E_{q1}F & \ldots & E_{qp}F \end{pmatrix} \in D^{(qr)\times(ps)}.$$

The next result is very classical.

Lemma 2.1. *Let D be a commutative ring, $E \in D^{r\times q}$, $F \in D^{q\times p}$ and $G \in D^{p\times m}$ three matrices. If we denote by* $\operatorname{row}(F) = (F_{1\bullet}, \dots, F_{q\bullet}) \in D^{1\times qp}$ *the row vector obtained by stacking the rows of F one after the other, then the product of the three matrices can be obtained by*

$$EFG = \operatorname{row}(F)(E^{\mathrm{T}} \otimes G).$$

We point out that Lemma 2.1 is only valid for commutative rings.

Let us consider a commutative ring D and the matrices $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$, $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$. Then, from the previous lemma, we get

$$\begin{cases} RP = RPI_{p'} = \operatorname{row}(P)(R^{\mathrm{T}} \otimes I_{p'}), \\ QR' = I_q QR' = \operatorname{row}(Q)(I_q \otimes R'), \end{cases}$$

which implies that (12) is equivalent to:

$$(\operatorname{row}(P) \quad -\operatorname{row}(Q)) \begin{pmatrix} R^{\mathrm{T}} \otimes I_{p'} \\ I_q \otimes R' \end{pmatrix} = 0.$$

This leads to an algorithm for computing matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying (12) in the case of a commutative polynomial ring D.

Algorithm 2.1

- **Input**: A commutative Ore algebra D, $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$.
- **Output**: A finite family of generators $\{f_i\}_{i\in I}$ of the D-module $\hom_D(M, M')$, where

  $M = D^{1\times p}/(D^{1\times q} R), \quad M' = D^{1\times p'}/(D^{1\times q'} R'),$

  and each f_i is defined by means of two matrices $\overline{P_i}$ and $\overline{Q_i}$ satisfying the relation (12), i.e.:

  $\forall \lambda \in D^{1\times p}: \quad f_i(\pi(\lambda)) = \pi'(\lambda \overline{P_i}), \quad i \in I.$

1. Form the following matrix with entries in D:

$$K = \begin{pmatrix} R^{\mathrm{T}} \otimes I_{p'} \\ I_q \otimes R' \end{pmatrix} \in D^{(pp'+qq')\times qp'}. \tag{25}$$

2. Compute $\ker_D(.K)$, i.e., the first syzygy left D-module of $D^{1\times(pp'+qq')}K$ using, for instance, a computation of a Gröbner basis for an elimination order (see [16])) or a more efficient method developed in the symbolic computation literature. We obtain $L \in D^{s\times(pp'+qq')}$ satisfying $\ker_D(.K) = D^{1\times s}L$.
3. For $i = 1, \dots, s$, construct the following matrices

$$\begin{cases} P_i(j,k) = r_i(L)(1, (j-1)p' + k), \\ Q_i(l,m) = -r_i(L)(1, pp' + (l-1)q' + m), \end{cases}$$

   where $r_i(L)$ denotes the ith row of L, $E(i,j)$ the $i \times j$ entry of the matrix E, $j = 1, \dots, p$, $k = 1, \dots, p'$, $l = 1, \dots, q$ and $m = 1, \dots, q'$. We then have

   $RP_i = Q_i R', \quad i = 1, \dots, s.$

4. Compute a Gröbner basis G of the rows of R' for a total degree order.
5. For $i = 1, \dots, s$, reduce the rows of P_i with respect to G by computing their normal forms with respect to G. We obtain the matrices $\overline{P_i}$ which satisfy $\overline{P_i} = P_i + Z_i R'$, for certain matrices $Z_i \in D^{p\times q'}$ which can be obtained by means of factorizations.

6. For $i = 1, \ldots, s$, define the following matrices $\overline{Q_i} = Q_i + R Z_i$. The pair $(\overline{P_i}, \overline{Q_i})$ then satisfies the relation $R\overline{P}_i = \overline{Q_i} R'$.

Remark 2.2. If we denote by $R_2' \in D^{q_2' \times q'}$ a matrix satisfying $\ker_D(.R') = D^{1\times q_2'} R_2'$, we then note that any matrix of the form $\overline{Q}_i = Q_i + R Z_i + Z_i' R_2'$, where $Z_i' \in D^{q \times q_2'}$ is an arbitrary matrix, also satisfies the relation $R\overline{P}_i = \overline{Q}_i R'$.

Remark 2.3. As D is a commutative ring, we know that $\hom_D(M, M')$ has a D-module structure. Let us prove that the family $\{f_i\}_{i\in I}$ obtained in the output of Algorithm 2.1 generates $\hom_D(M, M')$. Let us consider $f \in \hom_D(M, M')$. By Corollary 2.1, we know that there exist $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ such that $RP = QR'$, i.e., $(\mathrm{row}(P) \quad -\mathrm{row}(Q))K = 0$, where K is defined by (25). Using the fact that the matrix L, defined in Step 2 of Algorithm 2.1, generates $\ker_D(.K)$, we obtain that $(\mathrm{row}(P) \quad -\mathrm{row}(Q)) \in (D^{1\times s} L)$, i.e., there exists $(\alpha_1, \ldots, \alpha_s) \in D^{1\times s}$ such that

$$(\mathrm{row}(P) \quad -\mathrm{row}(Q)) = (\alpha_1, \ldots, \alpha_s) L \Rightarrow \begin{cases} P = \sum_{i=1}^{s} \alpha_i P_i, \\ Q = \sum_{i=1}^{s} \alpha_i Q_i, \end{cases}$$

where the matrices P_i and Q_i are defined in Step 3 of Algorithm 2.1. Using the definitions of $\overline{P}_i$ and $\overline{Q}_i$ defined in Steps 5 and 6 of Algorithm 2.1, we then get

$$\begin{cases} P = \sum_{i=1}^{s} \alpha_i \overline{P_i} - \left(\sum_{i=1}^{s} \alpha_i Z_i\right) R', \\ Q = \sum_{i=1}^{s} \alpha_i \overline{Q_i} - R\left(\sum_{i=1}^{s} \alpha_i Z_i\right). \end{cases}$$

Using 2 of Corollary 2.1, we obtain $f = \sum_{i=1}^{s} \alpha_i f_i$, which proves the result.

Hence, if $\{f_i\}_{i\in I}$ is a family of morphisms obtained by Algorithm 2.1 defined by the pairs of matrices $(\overline{P}_i, \overline{Q}_i)$, then any element $f \in \hom_D(M, M')$ has the form $f = \sum_{i\in I} \alpha_i f_i$, $\alpha_i \in D$ for $i \in I$, and, up to a homotopy equivalence, f can be defined by $(\sum_{i\in I} \alpha_i \overline{P}_i, \sum_{i\in I} \alpha_i \overline{Q}_i)$.

Finally, let us explain how to compute the relations between the generators $\{f_i\}_{i=1,\ldots,s}$ of the D-module $\hom_D(M, M')$. We first define the matrices $U = (\mathrm{row}(\overline{P}_1)^{\mathrm{T}} \ldots \mathrm{row}(\overline{P}_s)^{\mathrm{T}})^{\mathrm{T}} \in D^{s\times pp'}$, $V = I_p \otimes R' \in D^{pq'\times pp'}$ and $W = (U^{\mathrm{T}} \quad V^{\mathrm{T}})^{\mathrm{T}} \in D^{(s+pq')\times pp'}$. Computing $\ker_D(.W)$, we get $(X \quad -Y) \in D^{l\times(s+pq')}$, $X \in D^{l\times s}$ and $Y \in D^{l\times pq'}$, satisfying $\ker_D(.W) = D^{1\times l}(X \quad -Y)$. If we denote by

$$Z_i = \begin{pmatrix} Y_{(i,1)} & \ldots & Y_{(i,q')} \\ Y_{(i,q'+1)} & \ldots & Y_{(i,2q')} \\ \vdots & \vdots & \vdots \\ Y_{(i,(p-1)q'+1)} & \ldots & Y_{(i,pq')} \end{pmatrix} \in D^{p\times q'}, \quad i = 1, \ldots, l,$$

then, for $i = 1, \ldots, l$, we obtain the relations $\sum_{j=1}^{s} X_{ij} \overline{P}_j = Z_i R'$, and thus, $\sum_{j=1}^{s} X_{ij} f_j = 0$.

Example 2.6. Let us consider again Example 2.2. Applying Algorithm 2.1 to the matrix R defined by (8), we obtain that the D-endomorphisms of M are generated by the endomorphisms f_{e_1}, f_{e_2}, f_{e_3} and f_{e_4} defined by $f_\alpha(\pi(\lambda)) = \pi(\lambda P_\alpha)$, for all $\lambda \in D^{1\times 3}$, where

$$P_\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & 2\alpha_3\partial_1\partial_2 \\ \alpha_2 + 2\alpha_4\partial_1 & \alpha_1 - 2\alpha_4\partial_1 & 2\alpha_3\partial_1\partial_2 \\ \alpha_4\partial_2 & -\alpha_4\partial_2 & \alpha_1 + \alpha_2 + \alpha_3(\partial_2^2 + 1) \end{pmatrix},$$
$$Q_\alpha = \begin{pmatrix} \alpha_1 - 2\alpha_4\partial_1 & \alpha_2 + 2\alpha_4\partial_1 \\ \alpha_2 & \alpha_1 \end{pmatrix},$$

$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D^{1\times 4}$ and $\{e_i\}_{1\leqslant i\leqslant 4}$ denotes the standard basis of $D^{1\times 4}$. Finally, we can check that the generators $\{f_{e_i}\}_{1\leqslant i\leqslant 4}$ of the D-module $\mathrm{end}_D(M)$ satisfy the following relations:

$$(\partial_2^2 - 1) f_{e_4} = 0, \quad \partial_2^2 f_{e_1} + f_{e_2} - f_{e_3} = 0, \quad f_{e_1} + \partial_2^2 f_{e_2} - f_{e_3} = 0.$$

Let $D = A[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m]$ be an Ore algebra, where $A = k$ or $k[x_1, \ldots, x_n]$, or $D = k(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m]$, where k is a *field of constants*, namely:

$$k = \{a \in A | \delta_i(a) = 0, i = 1, \ldots, m\}.$$

The next algorithm computes the morphisms of $\hom_D(M, M')$ which can be defined by means of a matrix P with a fixed total order in the functional operators ∂_i and a fixed degree in x_i for the numerators and denominators of the polynomial/rational coefficients.

Algorithm 2.2

- **Input**: An Ore algebra D satisfying the hypotheses of Proposition 2.1, two matrices $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ and three non-negative integers α, β, γ.
- **Output**: A family of pairs $(\overline{P}_i, \overline{Q}_i)_{i\in I}$ satisfying:

  $$\begin{cases} R\overline{P}_i = \overline{Q}_i R', \\ \mathrm{ord}_\partial(\overline{P}_i) \leqslant \alpha, \text{ i.e., } \overline{P}_i = \sum\limits_{0\leqslant|\nu|\leqslant\alpha} a_\nu^{(i)} \partial^\nu, \\ \text{and } \forall \nu \in \mathbb{Z}_+^n, 0 \leqslant |\nu| \leqslant \alpha, a_\nu^{(i)} \in A^{p\times p} \text{ satisfies:} \\ \deg_x(\mathrm{num}(a_\nu^{(i)})) \leqslant \beta, \\ \deg_x(\mathrm{denom}(a_\nu^{(i)})) \leqslant \gamma, \end{cases}$$

  where $\mathrm{ord}_\partial(\overline{P}_i)$ denotes the maximal of the total orders of the entries of $\overline{P}_i$, $\deg_x(\mathrm{num}(a_\nu^{(i)}))$ (resp., $\deg_x(\mathrm{denom}(a_\nu^{(i)}))$) the maximal of the degrees of the numerators (resp., denominators) of $a_\nu^{(i)}$. For all $i \in I$, $f_i \in \hom_D(M, M')$ is then defined by $f_i(\pi(\lambda)) = \pi'(\lambda\overline{P}_i)$.

1. Take an ansatz for P satisfying the input
   $$P(i, j) = \sum_{0\leqslant|\nu|\leqslant\alpha} b_\nu^{(i,j)} \partial^\nu, \quad 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant p',$$
   where $b_\nu^{(i,j)}$ is a rational function whose numerator (resp., denominator) has a total degree β (resp., γ).
2. Compute RP and denote the result by F.
3. Compute a Gröbner basis G of the rows of R' for a total degree order.
4. Compute the normal forms of the rows of F with respect to G.
5. Solve the system for the coefficients of $b_\nu^{(i,j)}$ so that all the normal forms vanish.
6. Substitute the solutions into the matrix P. Denote the set of solutions by $\{P_i\}_{i\in I}$.
7. For $i \in I$, compute the normal forms $\overline{P}_i$ of the rows of P_i with respect to G.

8. Using $r_j(R\overline{P}_i) \in (D^{1\times q'} R')$, $j = 1, \ldots, q$, where $r_j(R\overline{P}_i)$ denotes the jth row of the matrix $R\overline{P}_i \in D^{q\times p'}$, compute a matrix $\overline{Q}_i \in D^{q\times q'}$ satisfying $R\overline{P}_i = \overline{Q}_i R$, $i \in I$, by reducing to 0 the row $r_j(R\overline{P}_i)$ with respect to the Gröbner basis G.

Remark 2.4. If we search for morphisms with only polynomial coefficients, i.e., $\gamma = 0$, then we note that the algebraic system in the coefficients $b_\nu^{(i,j)}$ that we need to solve in Step 5 of Algorithm 2.2 is linear. Hence, the solutions of this system belong to field k. However, if we look for morphisms with rational coefficients, we then have to solve a non-linear algebraic system in the coefficients $b_\nu^{(i,j)}$, meaning that its solutions generally belong to the algebraic closure $\overline{k}$ of k.

If M and M' are two finite-dimensional k-vector spaces (e.g., the linear systems defined in Example 2.5 where A is a field k or $k(t)$, integrable connections, D-finite modules [15]) or holonomic modules over the Weyl algebra $A_n(\mathbb{Q})$ [49,71], using Algorithm 2.2, we can then compute a basis of the finite-dimensional k-vector space $\hom_D(M, M')$. In order to apply this algorithm, we need to know some bounds on the orders and degrees of the entries of solutions of (12). In other words, we need to find some bounds for the inputs α, β and γ of Algorithm 2.2 . In some cases, such bounds are known. Let us recall some known results.

In Example 2.5, we saw that if $D = A[\partial; \sigma, \delta]$ was a skew polynomial ring over a commutative ring A, $E, F \in A^{p\times p}$ and $R = (\partial I_p - E)$, $R' = (\partial I_p - F)$, the morphisms from $M = D^{1\times p}/(D^{1\times p} R)$ to $M' = D^{1\times p}/(D^{1\times p} R)$ were defined by matrices $P \in A^{p\times p}$ satisfying:

$$\delta(P) = EP - \sigma(P)F. \tag{26}$$

Hence, we need to solve (26). There are two main cases:

1. If $A = k[t]$ or $k(t)$ and $D = A[\partial; \mathrm{id}_A, \frac{\mathrm{d}}{\mathrm{d}t}]$, then (26) becomes $\dot{P}(t) = E(t)P(t) - P(t)F(t)$. A direct method to solve the previous linear system of ODEs is developed in [8]. Another method, based on the fact that the entries of the matrices E, F and P belong to a commutative ring A, uses the equivalence of the previous system with the following first order linear system of ODEs
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{row}(P)) = \mathrm{row}(P)((E^{\mathrm{T}} \otimes I_p) - (I_p \otimes F)), \tag{27}$$
where $\otimes$ denotes the Kronecker product (see Definition 2.8). Hence, computing $\hom_D(M, M')$ is equivalent to computing the A-solutions of the auxiliary linear differential system (27) (see for example [8,18,19,68]). Consequently, we can use the bounds appearing in [2,5] on the degrees of numerators (and denominators) of polynomial (rational) solutions to deduce bounds on the entries of P. These bounds only depend on the valuations and degrees of the entries of the two matrices E and F.
2. If we consider the ring $A = k[n]$ or $A = k(n)$ and $D = A[\partial; \sigma, 0]$ with $\sigma(a)(n) = a(n + 1)$, then (26) becomes $P_{n+1}F_n = E_n P_n$. A direct method to solve the previous linear difference system is developed in [6]. Another one, based again on the fact that the entries of the matrices E_n, F_n and P_n belong to a commutative ring A, uses the equivalence of the previous system with the following first order linear discrete system:
$$\mathrm{row}(P_{n+1})(E_n^{\mathrm{T}} \otimes I_p) = \mathrm{row}(P_n)(I_p \otimes F_n). \tag{28}$$
Moreover, if $E \in \mathrm{GL}_p(A)$, i.e., the matrix E is invertible over A, then (28) becomes
$$\mathrm{row}(P_{n+1}) = \mathrm{row}(P_n)((I_p \otimes F)(E_n^{\mathrm{T}} \otimes I_p)^{-1}).$$

Some bounds exist on the degrees of numerators (and denominators) of polynomial (rational) solutions of the previous system (see [1,6]), and thus, for the matrices P and Q.

It was shown in [49] how to compute the space of polynomial or rational solutions of a holonomic system of PDEs [10,22]. In the case where M and M' are two holonomic modules over the Weyl algebras $A_n(k)$, this result was used in [71] to compute bases of the finite-dimensional k-vector space $\hom_D(M, M')$. See [44,70] for implementations of these results in computer algebra systems. The result developed in [49] can be used to compute the eigenring of a linear system of PDEs defined by means of an integrable connection (see Proposition 2.5). See also [74,75].

Finding bounds in more general situations is a subject for future researches.

If $\hom_D(M, M')$ is an infinite-dimensional k-vector space, we can then only consider the morphisms of $\hom_D(M, M')$ which can be defined by means of a matrix P with a fixed total order in the functional operators ∂_i's and a fixed degree in the x_j's for the numerators and denominators of the polynomial/rational coefficients (a kind of "filtration of $\hom_D(M, M')$"). Hence, giving bounds α, β and γ, Algorithm 2.2 computes those matrices P.

Let us illustrate Algorithm 2.2 by means of an example.

Example 2.7. We consider the *Euler-Tricomi equation* $\partial_1^2 u(x_1, x_2) - x_1 \partial_2^2 u(x_1, x_2) = 0$ which appears in the study of transonic flow. Let $D = A_2(\mathbb{Q})$ be the Weyl algebra with coefficients in $\mathbb{Q}$, $R = (\partial_1^2 - x_1 \partial_2^2) \in D$ and $M = D/(DR)$ the associated left D-module. We can easily prove that M is not a holonomic left D-module [10,22] and $\mathrm{end}_D(M)$ is an infinite-dimensional k-vector space. However, using Algorithm 2.2, we can compute the D-endomorphisms of M defined by $P \in D$ with a fixed total order in the ∂_i's and a fixed total degree in the x_j's. We denote by $\mathrm{end}_D(M)_{\alpha,\beta}$ the $\mathbb{Q}$-vector space of all the elements of $\mathrm{end}_D(M)$ defined by differential operators whose total orders (resp., degrees) in the ∂_i's (resp., x_j's) are less or equal to α (resp., β), where α and β are two non-negative integers. Below is a list of some of these $\mathbb{Q}$-vector spaces obtained by means of Algorithm 2.2:

- $\mathrm{end}_D(M)_{0,0}$ is defined by $P = Q = a$, $a \in \mathbb{Q}$.
- $\mathrm{end}_D(M)_{1,1}$ is defined by $P = a_1 + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1$, where $Q = P + 2a_3$, $a_1, a_2, a_3 \in \mathbb{Q}$.
- $\mathrm{end}_D(M)_{2,0}$ is defined by $P = Q = a_1 + a_2 \partial_2 + a_3 \partial_2^2$, where $a_1, a_2, a_3 \in \mathbb{Q}$.
- $\mathrm{end}_D(M)_{2,1}$ is defined by $(a_1, \ldots, a_5 \in \mathbb{Q})$:

$$\begin{cases} P = a_1 + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1 + a_4 \partial_2^2 + \frac{3}{2} a_5 x_2 \partial_2^2 + a_5 x_1 \partial_1 \partial_2, \\ Q = P + 2a_3 + 2a_5 \partial_2. \end{cases}$$

2.2.3. Applications: Quadratic first integrals of motion and conservation laws

We illustrate the interest of the computation of morphisms in the search of quadratic first integrals of motion of linear systems of ODEs and quadratic conservation laws of linear systems of PDEs.

We consider the Ore algebra $D = A[\partial; \mathrm{id}_A, \frac{\mathrm{d}}{\mathrm{d}t}]$ of ordinary differential operators with coefficients in a commutative k-algebra A (e.g., $A = k[t]$, $k(t)$), where k is a field of characteristic 0, $E \in A^{p \times p}$ and the matrix $R = (\partial I_p - E) \in D^{p \times p}$. Using (20), we can check that any solution $P \in A^{p \times p}$ of the following *Liapunov equation*

$$\dot{P}(t) + E^{\mathrm{T}}(t) P(t) + P(t) E(t) = 0$$

defines a morphism from the finitely presented left D-module $\widetilde{N} = D^{1\times p}/(D^{1\times p}\widetilde{R})$ to the finitely presented left D-module $M = D^{1\times p}/(D^{1\times p}R)$, where $\widetilde{R} = (-(\partial I_p + E^{\mathrm{T}})) \in D^{p\times p}$ denotes the *formal adjoint* of R [56,57]. As we have $D^{1\times p}\widetilde{R} = D^{1\times p}(\partial I_p + E^{\mathrm{T}})$, we can also use the matrix $(\partial I_p + E^{\mathrm{T}})$ instead of $\widetilde{R}$ in the definition of $\widetilde{N}$.

We recall that the *formal adjoint* $\widetilde{R}$ of a matrix R of differential operators is obtained by contracting the column vector $R\eta$ by a row vector λ^{T} and integrating the result by parts to get

$$\lambda^{\mathrm{T}}(R\eta) = \eta^{\mathrm{T}}(\widetilde{R}\lambda) + \partial(\Phi(\lambda, \eta)), \tag{29}$$

where Φ denotes the boundary terms of the integration by parts (see [56–58]). We note that Φ is a bilinear application in (η, λ) constructed from the difference of the two bilinear applications $(\eta, \lambda) \mapsto \lambda^{\mathrm{T}}(R\eta)$ and $(\eta, \lambda) \mapsto \eta^{\mathrm{T}}(\widetilde{R}\lambda)$.

In particular, in our case, we have

$$\lambda^{\mathrm{T}}(\partial\eta - E\eta) = -(\partial\lambda^{\mathrm{T}} + \lambda^{\mathrm{T}}E)\eta + \partial(\lambda^{\mathrm{T}}\eta) = \eta^{\mathrm{T}}(-(\partial\lambda + E^{\mathrm{T}}\lambda)) + \partial(\eta^{\mathrm{T}}\lambda). \tag{30}$$

If $\mathscr{F}$ is a left D-module and $\eta \in \mathscr{F}^p$ satisfies the system $\partial\eta - E\eta = 0$, then, following the results obtained in Example 2.5, $\lambda = P\eta$ is a solution of $\partial\lambda + E^{\mathrm{T}}\lambda = 0$. Hence, using (30), we then get

$$\partial(\eta^{\mathrm{T}}\lambda) = \partial(\eta^{\mathrm{T}}P\eta) = 0,$$

which proves that the quadratic form $V = \eta^{\mathrm{T}}P\eta$ is a first integral of the motion of the system $\partial\eta - E\eta = 0$. Hence, we obtain that there exists a one-to-one correspondence between the quadratic first integrals of the motion of the form $V = \eta^{\mathrm{T}}P\eta$, $P \in A^{q\times p}$, of $\partial\eta - E\eta = 0$ and the morphisms between the left D-modules $\widetilde{N}$ and M, i.e., the elements of $\hom_D(\widetilde{N}, M)$.

We note that if E is a *skew-symmetric matrix*, namely, $E^{\mathrm{T}} = -E$, then we get

$$-\widetilde{R} = (\partial I_p + E^{\mathrm{T}}) = (\partial I_p - E) = R,$$

$\widetilde{N} = M$ and $\hom_D(\widetilde{N}, M) = \mathrm{end}_D(M)$. Such a particular case usually appears in mechanics.

Let us illustrate the previous result.

Example 2.8. Let us consider the example of a linear system of ODEs defined in page 117 of [38] and let us compute its quadratic first integrals. In order to do that, let us introduce the matrix

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & \alpha \end{pmatrix},$$

where ω and α are two real constants, the ring $D = \mathbb{Q}(\omega, \alpha)[\partial; \mathrm{id}, \frac{\mathrm{d}}{\mathrm{d}t}]$ of differential operators, the matrix $R = (\partial I_4 - E) \in D^{4\times 4}$ of differential operators and the D-module $M = D^{1\times 4}/(D^{1\times 4}R)$. Then, we have $\widetilde{R} = -(\partial I_4 + E^{\mathrm{T}})$ and $\widetilde{N} = D^{1\times 4}/(D^{1\times 4}\widetilde{R})$. Using Algorithm 2.1, we obtain that an element of the D-module $\hom_D(\widetilde{N}, M)$ can be defined by means of the matrix

$$P = \begin{pmatrix} c_1\omega^4 & c_2\omega^2 & -\omega^2(c_1\alpha + c_2) & c_1\omega^2 \\ -c_2\omega^2 & c_1\omega^2 & -c_1\omega^2 + c_2\alpha & -c_2 \\ -\omega^2(c_1\alpha - c_2) & -c_1\omega^2 - c_2\alpha & c_1(\alpha^2 + \omega^2) & -c_1\alpha + c_2 \\ c_1\omega^2 & c_2 & -c_1\alpha - c_2 & c_1 \end{pmatrix},$$

where c_1 and c_2 are two arbitrary elements of $\mathbb{Q}$, which leads to the quadratic first integral:

$$
\begin{aligned}
V(x) &= x^{\mathrm{T}} P x \\
&= c_1\omega^4 x_1(t)^2 - 2c_1\alpha\omega^2 x_1(t)x_3(t) + 2c_1\omega^2 x_1(t)x_4(t) + c_1\omega^2 x_2(t)^2 \\
&\quad - 2c_1\omega^2 x_2(t)x_3(t) + c_1\alpha^2 x_3(t)^2 + c_1\omega^2 x_3(t)^2 \\
&\quad - 2c_1\alpha x_3(t)x_4(t) + c_1 x_4(t)^2.
\end{aligned}
$$

More generally, let us consider a matrix $R \in D^{q\times p}$ of differential operators, $\widetilde{R} \in D^{p\times q}$ its formal adjoint and the finitely presented left D-modules $M = D^{1\times p}/(D^{1\times q}R)$ and $\widetilde{N} = D^{1\times q}/(D^{1\times p}\widetilde{R})$. Let us suppose that there exists a morphism f from $\widetilde{N}$ to M defined by $P \in D^{q\times p}$ and $Q \in D^{p\times q}$, i.e., we have the commutative exact diagram:

$$
\begin{array}{ccccccc}
D^{1\times p} & \xrightarrow{.\widetilde{R}} & D^{1\times q} & \xrightarrow{\pi'} & \widetilde{N} & \to & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow f & & \\
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \to & 0.
\end{array}
$$

Applying the left exact contravariant functor $\hom_D(\cdot, \mathscr{F})$ to the previous commutative exact diagram, we then obtain the following commutative exact diagram

$$
\begin{array}{ccccccc}
\mathscr{F}^p & \xleftarrow{\widetilde{R}.} & \mathscr{F}^q & \leftarrow & \ker_{\mathscr{F}}(\widetilde{R}.) & \leftarrow & 0 \\
\uparrow Q. & & \uparrow P. & & \uparrow f^\star & & \\
\mathscr{F}^q & \xleftarrow{R.} & \mathscr{F}^p & \leftarrow & \ker_{\mathscr{F}}(R.) & \leftarrow & 0,
\end{array}
$$

where $f^\star(\eta) = P\eta$. If $\eta \in \mathscr{F}^p$ is a solution of $R\eta = 0$, then $\lambda = P\eta$ is a solution of $\widetilde{R}\lambda = 0$ as

$$\widetilde{R}(P\eta) = Q(R\eta) = 0.$$

Therefore, using (29), we obtain that $V = \Phi(P\eta, \eta)$ is a quadratic first integral of the motion of system $R\eta = 0$, i.e., V satisfies $\partial V(t) = 0$.

An extension of the previous ideas exists for the computation of quadratic conservation laws of linear system of PDEs, namely, a vector $\Phi = (\Phi_1, \ldots, \Phi_n)^{\mathrm{T}}$ of quadratic functions of the system variables and their derivatives which satisfies $\operatorname{div}\Phi = \sum_{i=1}^n \partial_i \Phi_i = 0$, where n denotes the number of independent variables.

Let us give a simple example as the general theory exactly follows the same lines.

Example 2.9. The movement of an incompressible rotating fluid with a rotation axis lying along the x_3 axis and a small velocity is defined by

$$
\begin{cases}
\rho_0 \dfrac{\partial u_1}{\partial t} - 2\rho_0\Omega_0 u_2 + \dfrac{\partial p}{\partial x_1} = 0, \\
\rho_0 \dfrac{\partial u_2}{\partial t} + 2\rho_0\Omega_0 u_1 + \dfrac{\partial p}{\partial x_2} = 0, \\
\rho_0 \dfrac{\partial u_3}{\partial t} + \dfrac{\partial p}{\partial x_3} = 0, \\
\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3} = 0,
\end{cases}
\tag{31}
$$

where $u = (u_1, u_2, u_3)^{\mathrm{T}}$ denotes the local rate of velocity, p the pressure, ρ_0 the constant fluid density and Ω_0 the constant angle speed. See page 62 of [39] for more details.

Let us denote by $D = \mathbb{Q}(\rho_0, \Omega_0)[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}][\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}][\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}][\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}]$ the ring of differential operators, the system matrix

$$R=\begin{pmatrix}\rho_0\partial_t & -2\rho_0\Omega_0 & 0 & \partial_1\\ 2\rho_0\Omega_0 & \rho_0\partial_t & 0 & \partial_2\\ 0 & 0 & \rho_0\partial_t & \partial_3\\ \partial_1 & \partial_2 & \partial_3 & 0\end{pmatrix},$$

and the D-module $M = D^{1\times 4}/(D^{1\times 4}R)$ associated with (31).

If we denote by $\xi = (u_1, u_2, u_2, p)^{\mathrm{T}}$, we then have the following identity

$$(\lambda, R\xi) = (\xi, \widetilde{R}\lambda) + \begin{pmatrix}\dfrac{\partial}{\partial t} & \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\end{pmatrix}\begin{pmatrix}\rho_0(\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3)\\ \lambda_1 p+\lambda_4 u_1\\ \lambda_2 p+\lambda_4 u_2\\ \lambda_3 p+\lambda_4 u_3\end{pmatrix}, \tag{32}$$

where $\widetilde{R} = -R$. Using the fact that R is skew-symmetric, we get that $\widetilde{N} = D^{1\times 4}/(D^{1\times 4}\widetilde{R}) = M$ and $\hom_D(\widetilde{N}, M) = \mathrm{end}_D(M)$. Hence, if $(\vec{u}, p)$ is a solution of (31), then $\lambda_1 = u_1$, $\lambda_2 = u_2$, $\lambda_3 = u_3$ and $\lambda_4 = p$ is a solution of $\widetilde{R}\lambda = 0$. Hence, if we take $\lambda = \xi$, using (32), we then obtain

$$\frac{\partial}{\partial t}(\rho_0(u_1^2+u_2^2+u_3^2)) + \frac{\partial}{\partial x_1}(2pu_1) + \frac{\partial}{\partial x_2}(2pu_2) + \frac{\partial}{\partial x_3}(2pu_3) = 0,$$

i.e., we obtain the following conservation of law of (31):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 \parallel \vec{u} \parallel^2\right) + \mathrm{div}\,(p\vec{u}) = 0.$$

We refer the reader to [21] for examples coming from electromagnetism and elasticity theory.

3. Reducible modules and factorizations

We recall that D denotes an Ore algebra satisfying the hypotheses of Proposition 2.1.

3.1. Modules associated with a morphism and equivalences

Let $f : M \to M'$ be a morphism between two left D-modules. Then, we can define the following left D-modules:

$$\begin{cases}\ker f = \{m\in M \mid f(m)=0\},\\ \mathrm{im}\, f = \{m'\in M' \mid \exists\, m\in M : m' = f(m)\},\\ \mathrm{coim}\, f = M/\ker f,\\ \mathrm{coker}\, f = M'/\mathrm{im}\, f.\end{cases}$$

Let us explicitly characterize the above-mentioned kernel, image, coimage and cokernel of a morphism $f : M \to M'$ between two finitely presented left D-modules M and M'.

Proposition 3.1. *Let $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$, $M = D^{1\times p}/(D^{1\times q}R)$ and $M' = D^{1\times p'}/(D^{1\times q'}R')$. Let $f : M \to M'$ be a morphism defined by means of the two matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying* (12)*. Then, we have*:

1. $\ker f = (D^{1\times r}S)/(D^{1\times q}R)$, *where $S \in D^{r\times p}$ is the matrix defined by*

$$\ker_D\left(.\begin{pmatrix}P\\ R'\end{pmatrix}\right) = D^{1\times r}(S \quad -T), \quad T\in D^{r\times q'}. \tag{33}$$

2. $\operatorname{coim} f = D^{1\times p}/(D^{1\times r} S)$,
3. $\operatorname{im} f = \left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right) \Big/ (D^{1\times q'} R')$,
4. $\operatorname{coker} f = D^{1\times p'} \Big/ \left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right)$.

Proof. 1. Let $m \in \ker f$ and write $m = \pi(\lambda)$ for a certain $\lambda \in D^{1\times p}$. Then, $f(m) = \pi'(\lambda P) = 0$ implies that $\lambda P \in (D^{1\times q'} R')$, i.e., there exists $\mu \in D^{1\times q'}$ satisfying $\lambda P = \mu R'$. Hence, $m = \pi(\lambda) \in \ker f$ implies that there exists $\mu \in D^{1\times q'}$ such that $\lambda P = \mu R'$. Conversely, we can easily check that any element $(\lambda \quad -\mu) \in \ker_D(.(P^{\mathrm{T}} \quad R'^{\mathrm{T}})^{\mathrm{T}})$ gives $m = \pi(\lambda) \in \ker f$.

2. Using the canonical short exact sequence $0 \to \ker f \xrightarrow{i} M \xrightarrow{\rho} \operatorname{coim} f \to 0$, where i (resp., ρ) denotes the canonical injection (resp., surjection), $M = D^{1\times p}/(D^{1\times q} R)$ and the fact that $\ker f = (D^{1\times r} S)/(D^{1\times q} R)$, we obtain the following exact sequence

$$0 \to (D^{1\times r} S)/(D^{1\times q} R) \xrightarrow{i} D^{1\times p}/(D^{1\times q} R) \xrightarrow{\rho} \operatorname{coim} f \to 0,$$

which proves that $\operatorname{coim} f = D^{1\times p}/(D^{1\times r} S)$ by the *third isomorphism theorem* (see, e.g., [66]).

3. For all $\lambda \in D^{1\times p}$, we have $f(\pi(\lambda)) = \pi'(\lambda P)$, which clearly proves that we have

$$\operatorname{im} f = \left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right) \Big/ (D^{1\times q'} R').$$

4. Using the canonical short exact sequence $0 \to \operatorname{im} f \xrightarrow{j} M' \xrightarrow{\sigma} \operatorname{coker} f \to 0$, where j (resp., σ) denotes the canonical injection (resp., surjection), $M' = D^{1\times p'}/(D^{1\times q'} R')$ and the fact that $\operatorname{im} f = (D^{1\times p} P + D^{1\times q'} R')/(D^{1\times q'} R')$, we then obtain the following exact sequence

$$0 \to \left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right) \Big/ (D^{1\times q'} R') \xrightarrow{j} D^{1\times p'}/(D^{1\times q'} R') \xrightarrow{\sigma} \operatorname{coker} f \to 0,$$

and thus, $\operatorname{coker} f = D^{1\times p'}/\left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right)$ by the third isomorphism theorem [66]. □

Let us state the first main result of the paper.

Theorem 3.1. *With the notations of Proposition 3.1, any non-zero morphism $f : M \to M'$ leads to a factorization of $R \in D^{q\times p}$ of the form $R = LS$, where $L \in D^{q\times r}$ and $S \in D^{r\times p}$ satisfies* $\operatorname{coim} f = D^{1\times p}/(D^{1\times r} S)$.

Proof. Using (33) and the fact that $RP = QR'$, we obtain that

$$(D^{1\times q}(R \quad -Q)) \subseteq \ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = D^{1\times r}(S \quad -T),$$

and thus, there exists a matrix $L \in D^{q\times r}$ satisfying

$$\begin{cases} R = LS, \\ Q = LT. \end{cases} \tag{34}$$

We then have the following commutative exact diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & 0 & & \ker f & & \\ & & \downarrow & & \downarrow & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \to 0 & \\ \downarrow .L & & \| & & \downarrow \rho & & \\ D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & \operatorname{coim} f & \to 0, & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{35}$$

where $\rho : M \to \operatorname{coim} f$ denotes the canonical projection onto $\operatorname{coim} f = M/\ker f$. $\square$

We point out that the results of Proposition 3.1 and Theorem 3.1 are constructive as Gröbner bases exist for the class of Ore algebras we are considering (see Proposition 2.1), a fact allowing us to compute syzygy modules and factorizations of matrices. We note that the problem of factoring R by S can be reduced to a membership problem as we need to check that every row of R belongs to the left D-module $D^{1\times r} S$. We refer to the library OreModules [17] for more details.

Definition 3.1

1. A non-zero left D-module M is called *simple* if it only admits 0 and M as left D-submodules.
2. A factorization $R = LS$ of the matrix $R \in D^{q\times p}$, where $L \in D^{q\times r}$ and $S \in D^{r\times p}$, is said to be *non-trivial* if we have $(D^{1\times q} R) \subsetneq (D^{1\times r} S)$.

If $R = LS$ is a trivial factorization of a full row rank matrix R, namely, $\ker_D(.R) = 0$, using the equality $(D^{1\times r} S) \subseteq (D^{1\times q} R)$, then there exist a matrix $L' \in D^{r\times q}$ such that $S = L'R$. Hence, we get $(I_q - LL')R = 0$ and, using the fact that $\ker_D(.R) = 0$, we obtain that $LL' = I_q$, i.e., L admits a right-inverse over D. Moreover, if R is a square matrix, i.e., $p = q$, we then get that $L'L = I_p$, showing that L is an invertible matrix over D, which explains the name of non-trivial factorization.

Using 1 of Proposition 3.1, we obtain the following corollary of Theorem 3.1.

Corollary 3.1. *With the notations of Proposition* 3.1, *we obtain that the existence of a non-injective endomorphism of M defines a non-trivial factorization of R and proves that M is not a simple left D-module.*

If M is a simple left D-module, Corollary 3.1 then shows that any non-zero endomorphism f of M is injective. Moreover, as $\operatorname{im} f$ is a non-zero left D-submodule of M and M is a simple left D-module, we obtain that $\operatorname{im} f = M$. Hence, any element $0 \neq f \in \operatorname{end}_D(M)$ is a D-automorphism of M, i.e., $f \in \operatorname{aut}_D(M)$. This is the classical Schur's lemma saying that the endormorphism ring $\operatorname{end}_D(M)$ of a simple left D-module M is a *division ring* (see, e.g., [46]).

Let us illustrate Theorem 3.1 and Corollary 3.1 on an example.

Example 3.1. We consider the linearized Euler equations for an incompressible fluid (see page 356 of [39]) defined by

$$\begin{cases} \operatorname{div} \vec{v}(x,t) = 0, \\ \dfrac{\partial \vec{v}(x,t)}{\partial t} + \operatorname{grad} p(x,t) = 0, \end{cases} \tag{36}$$

where $x = (x_1, x_2, x_3)$ and $\vec{v} = (v_1, v_2, v_3)^{\mathrm{T}}$ (resp., p) denotes the perturbations of the speed (resp., pressure). Let us denote by $D = \mathbb{Q}[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}][\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}][\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}][\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}]$ the ring of differential operators with rational constant coefficients, the system matrix corresponding to (36) is then defined by

$$R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} \in D^{4\times 4}.$$

Let $M = D^{1\times 4}/(D^{1\times 4}R)$ be the D-module associated with the system (36). Using Algorithm 2.1, we obtain that an endomorphism f of M is defined by the following two matrices:

$$P = \begin{pmatrix} 0 & \partial_3 & -\partial_2 & 0 \\ -\partial_3 & 0 & \partial_1 & 0 \\ \partial_2 & -\partial_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_3 & -\partial_2 \\ 0 & -\partial_3 & 0 & \partial_1 \\ 0 & \partial_2 & -\partial_1 & 0 \end{pmatrix}.$$

We then obtain the following factorization $R = LS$ of R where

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & -1 & \partial_3 \end{pmatrix}, \quad S = \begin{pmatrix} -\partial_t & 0 & 0 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can check that $\ker f = (D^{1\times 5}S)/(D^{1\times 4}R) \neq 0$, which shows that $R = LS$ is a non-trivial factorization of R and M is not a simple D-module as $\operatorname{coim} f = D^{1\times 4}/(D^{1\times 5}S)$ is a non-trivial D-submodule of M. If we consider $\mathscr{F} = C^{\infty}(\Omega)$, where Ω is an open convex subset of $\mathbb{R}^4$, we easily check that all $\mathscr{F}$-solutions of $S\eta = 0$ satisfy

$$\begin{cases} \vec{v}(x,t) = \vec{v}(x), \\ \operatorname{div} \vec{v}(x) = 0, \\ p(x,t) = 0, \end{cases} \Leftrightarrow \begin{cases} \vec{v}(x,t) = \operatorname{curl} \vec{\psi}(x), \\ p(x,t) = 0, \end{cases}$$

where $\vec{\psi} = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}}$ is any vector of smooth functions on $\Omega \cap \mathbb{R}^3$ and curl denotes the standard curl operator (see, e.g., [38]). Finally, we can easily check that the solutions of the system $S\eta = 0$ are particular solutions of (36).

Let us state a useful lemma.

Lemma 3.1. *Let $R \in D^{q\times p}$, $R' \in D^{q'\times p}$, $R'' \in D^{q\times q'}$ be three matrices satisfying the relation $R = R''R'$ and let $T' \in D^{r'\times q'}$ be such that $\ker_D(.R') = D^{1\times r'}T'$. Let us also consider the canonical projections $\pi_1 : D^{1\times q'}R' \to M_1 = (D^{1\times q'}R')/(D^{1\times q}R)$ and*:

$$\pi_2 : D^{1\times q'} \to M_2 = D^{1\times q'}/(D^{1\times q}R'' + D^{1\times r'}T').$$

Then, the morphism ψ defined by

$$\begin{aligned} \psi &: M_2 \to M_1 \\ m_2 = \pi_2(\lambda) &\mapsto \psi(m_2) = \pi_1(\lambda R'), \end{aligned}$$

is an isomorphism and its inverse ϕ *is defined by*

$$\begin{aligned}&\phi : M_1 \to M_2\\ &m_1 = \pi_1(\lambda R') \mapsto \phi(m_1) = \pi_2(\lambda).\end{aligned}$$

In other words, we have the following isomorphism of left D-modules:

$$(D^{1\times q'} R')/(D^{1\times q} R) \cong D^{1\times q'}/(D^{1\times q} R'' + D^{1\times r'} T'). \tag{37}$$

Proof. Let us prove that ψ is a well-defined morphism. Let us assume that $m_2 = \pi_2(\lambda) = \pi_2(\lambda')$, where $\lambda, \lambda' \in D^{1\times q'}$. Then, we have $\pi_2(\lambda - \lambda') = 0$, i.e., $\lambda - \lambda' \in (D^{1\times q} R'' + D^{1\times r'} T')$ so that there exist $\mu \in D^{1\times q}$ and $\nu \in D^{1\times r'}$ such that $\lambda - \lambda' = \mu R'' + \nu T'$. We then have

$$\begin{aligned}(\lambda - \lambda')R' = (\mu R'' + \nu T')R' = \mu R \;&\Rightarrow \pi_1((\lambda - \lambda')R') = \pi_1(\mu R) = 0\\ &\Rightarrow \pi_1(\lambda' R') = \pi_1(\lambda R') = \psi(m_2).\end{aligned}$$

Let us prove that ϕ is also well-defined. Let us suppose that $m_1 = \pi_1(\lambda R') = \pi_1(\lambda' R')$, where $\lambda, \lambda' \in D^{1\times q'}$. Then, we get $\pi_1((\lambda - \lambda')R') = 0$, i.e., $(\lambda - \lambda')R' \in (D^{1\times q} R)$, and thus, there exists $\mu \in D^{1\times q}$ such that $(\lambda - \lambda')R' = \mu R$. Now, using the factorization $R = R''R'$, we then get $(\lambda - \lambda' - \mu R'')R' = 0$ so that $\lambda - \lambda' - \mu R'' \in \ker_D(.R') = D^{1\times r'} T'$. Therefore, there exists $\nu \in D^{1\times r'}$ such that $\lambda - \lambda' = \mu R'' + \nu T'$ and then

$$\pi_2(\lambda) - \pi_2(\lambda') = \pi_2(\lambda - \lambda') = \pi_2(\mu R'' + \nu T') = 0.$$

Finally, for all $m_1 = \pi_1(\lambda R') \in M_1$ and $m_2 = \pi_2(\lambda) \in M_2$, where $\lambda \in D^{1\times q'}$, we have

$$\begin{cases}(\psi \circ \phi)(m_1) = \psi(\pi_2(\lambda)) = \pi_1(\lambda R') = m_1,\\ (\phi \circ \psi)(m_2) = \phi(\pi_1(\lambda R')) = \pi_2(\lambda) = m_2,\end{cases}$$

which proves that $\psi \circ \phi = \mathrm{id}_{M_1}$, $\phi \circ \psi = \mathrm{id}_{M_2}$ and we thus obtain (37). □

We deduce the following corollary of Lemma 3.1 and Proposition 3.1.

Corollary 3.2. *With the notations of Proposition* 3.1, *if* $L \in D^{q\times r}$ *denotes a matrix satisfying* $R = LS$ *and* $\ker_D(.S) = D^{1\times r_2} S_2$, $S_2 \in D^{r_2\times r}$, *we then have*:

$$\ker f \cong D^{1\times r}\Big/\left(D^{1\times(q+r_2)}\begin{pmatrix} L\\ S_2\end{pmatrix}\right).$$

We recall that the *first isomorphism theorem* says that we have $\mathrm{coim}\, f = M/\ker f \cong \mathrm{im}\, f$ (see, e.g., [66]). This result can also be checked again as follows. Using the following two facts

$$R' = (0 \quad I_{q'})\begin{pmatrix} P\\ R'\end{pmatrix}, \quad \ker_D\left(.\begin{pmatrix} P\\ R'\end{pmatrix}\right) = D^{1\times r}(S \quad -T),$$

where $S \in D^{r\times p}$, $T \in D^{r\times q'}$, and applying Lemma 3.1 to 3 of Proposition 3.1, we then get:

$$\mathrm{im}\, f \cong D^{1\times(p+q')}\Big/\left(D^{1\times(q'+r)}\begin{pmatrix} 0 & I_{q'}\\ S & -T\end{pmatrix}\right) \cong D^{1\times p}/(D^{1\times r} S) = \mathrm{coim}\, f.$$

We give a corollary of Proposition 3.1 and Corollary 3.2.

Corollary 3.3. *With the notations of Corollary 3.2 and Proposition 3.1*, $f \in \hom_D(M, M')$ *is*:

1. *The zero morphism* ($f = 0$) *if and only if one of the following conditions holds*:
   (a) *There exists a matrix* $Z \in D^{p \times q'}$ *such that* $P = ZR'$. *Then, there exists* $Z' \in D^{q \times q'_2}$ *such that* $Q = RZ + Z'R'_2$, *where* $R'_2 \in D^{q'_2 \times q'}$ *satisfies* $\ker_D(.R') = D^{1\times q'_2} R'_2$.
   (b) *The matrix* S *admits a left-inverse over* D.
2. *Injective if and only if one of the following conditions holds*:
   (a) *There exists a matrix* $F \in D^{r \times q}$ *such that* $S = FR$.
   (b) *The matrix* $(L^{\mathrm{T}} \quad S_2^{\mathrm{T}})^{\mathrm{T}}$ *admits a left-inverse over* D.
3. *Surjective if and only if* $(P^{\mathrm{T}} \quad R'^{\mathrm{T}})^{\mathrm{T}}$ *admits a left-inverse over* D.
4. *An Isomorphism if the matrices* $(L^{\mathrm{T}} \quad S_2^{\mathrm{T}})^{\mathrm{T}}$ *and* $(P^{\mathrm{T}} \quad R'^{\mathrm{T}})^{\mathrm{T}}$ *admit left-inverses over* D.

Proof. 1. Using 3 of Proposition 3.1, $\operatorname{im} f = 0$ if and only if we have $D^{1\times p}P + D^{1\times q'}R' = D^{1\times q'}R'$, that is, if and only if $(D^{1\times p}P) \subseteq (D^{1\times q'}R')$ which is equivalent to the existence of a matrix $Z \in D^{p\times q'}$ such that $P = ZR'$. Now, substituting $P = ZR'$ into (12), we then get

$$RZR' = QR' \Rightarrow (Q - RZ)R' = 0.$$

Thus, there exists $Z' \in D^{q\times q'_2}$ satisfying $Q - RZ = Z'R'_2$, which proves the result. We also note that 1.a is a trivial consequence of Corollary 2.1.

Let us prove 1.b. Using the canonical isomorphism $\epsilon : \operatorname{coim} f \to \operatorname{im} f$, defined by

$$\forall m \in M : \epsilon(\rho(m)) = f(m),$$

where $\rho : M \to \operatorname{coim} f$ denotes the canonical projection, we obtain that $\operatorname{im} f = 0$ if and only if

$$\operatorname{coim} f = D^{1\times p}/(D^{1\times r}S) = 0 \Leftrightarrow D^{1\times r}S = D^{1\times p},$$

i.e., if and only if S admits a left-inverse over D.

2. From 1 of Proposition 3.1, $\ker f = 0$ if and only if $D^{1\times r}S = D^{1\times q}R$, i.e., if and only if there exists $F \in D^{r\times q}$ satisfying $S = FR$.

Moreover, using Corollary 3.2, we have $\ker f = 0$ if and only if $D^{1\times q}L + D^{1\times r_2}S_2 = D^{1\times r}$, i.e., if and only if the matrix $(L^{\mathrm{T}} \quad S_2^{\mathrm{T}})^{\mathrm{T}}$ admits a left-inverse over D.

3. f is surjective if and only if $\operatorname{coker} f = 0$, i.e., from 4 of Proposition 3.1, if and only if $D^{1\times p}P + D^{1\times q'}R' = D^{1\times p}$, which is equivalent to the fact that the matrix $(P^{\mathrm{T}} \quad R'^{\mathrm{T}})^{\mathrm{T}}$ admits a left-inverse over D.

4. The result is a direct consequence of 2.b and 3. □

We note that the results of Corollary 3.3 can algorithmically be checked as factoring matrices and deciding the existence of left-inverses have been made constructive and implemented in the library OreModules [17]. Moreover, we can sometimes use the previous results in order to check that two given modules are isomorphic, i.e., that the two corresponding systems are equivalent (see also [53] for a commutative polynomial ring).

Let us illustrate Corollary 3.3.

Example 3.2. We consider two systems of PDEs appearing in the theory of elasticity (see [55]): one half of the so-called *Killing operator*, namely, the *Lie derivative* of the euclidean metric defined by $\omega_{ij} = 1$ for $i = j$ and 0 otherwise ($1 \leqslant i, j \leqslant 2$) and *the Spencer operator* of the Killing operator:

$$\begin{cases} \partial_1 \xi_1 = 0, \\ \frac{1}{2}(\partial_2 \xi_1 + \partial_1 \xi_2) = 0, \\ \partial_2 \xi_2 = 0, \end{cases} \quad \begin{cases} \partial_1 z_1 = 0, \\ \partial_2 z_1 - z_2 = 0, \\ \partial_1 z_2 = 0, \\ \partial_1 z_3 + z_2 = 0, \\ \partial_2 z_3 = 0, \\ \partial_2 z_2 = 0. \end{cases}$$

Let $D = \mathbb{Q}[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}][\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}]$ be the ring of differential operators with rational constant coefficients and let us define the following two matrices:

$$R = \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \in D^{3\times 2}, \quad R' = \begin{pmatrix} \partial_1 & \partial_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & \partial_1 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & 0 \end{pmatrix}^{\mathrm{T}} \in D^{6\times 3},$$

and the associated finitely presented D-modules $M = D^{1\times 2}/(D^{1\times 3} R)$ and $M' = D^{1\times 3}/(D^{1\times 6} R')$. Using Algorithm 2.1, we find that the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

satisfy the relation $RP = QR'$, i.e., define a morphism $f : M \to M'$ given by $f(\xi_1) = z_1$ and $f(\xi_2) = z_3$. The morphism f is injective as the matrix

$$S = \begin{pmatrix} \partial_2 & \partial_1 & \partial_2^2 & 0 \\ \partial_1 & 0 & 0 & \partial_2 \end{pmatrix}^{\mathrm{T}}$$

defined by Proposition 3.1, satisfies the relation $S = FR$, where

$$F = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 2\partial_2 & -\partial_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, f is surjective as the matrix $(P^{\mathrm{T}} \quad R'^{\mathrm{T}})^{\mathrm{T}}$ admits the following left-inverse over D:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This proves that f is a D-isomorphism and $M \cong M'$.

3.2. *Reducible modules and block-triangular matrices*

In what follows, we shall denote by $\mathrm{GL}_p(D)$ the *general linear group* over D, namely,

$$\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists V \in D^{p\times p} : UV = VU = I_p\},$$

where I_p denotes the unit of $\mathrm{GL}_p(D)$, i.e., the identity $p \times p$ matrix. An element of $\mathrm{GL}_p(D)$ will be called a *unimodular matrix*.

The next proposition will play an important role in what follows.

Proposition 3.2. *Let us consider a matrix $P \in D^{p\times p}$. The following assertions are equivalent*:

1. *The left D-modules* $\ker_D(.P)$ *and* $\mathrm{coim}_D(.P)$ *are free of rank respectively m and $p-m$.*
2. *There exists a matrix $U \in \mathrm{GL}_p(D)$ and a matrix $J \in D^{p\times p}$ of the form*

   $$J = \begin{pmatrix} 0 & 0 \\ J_1 & J_2 \end{pmatrix}, \quad J_1 \in D^{(p-m)\times m},\ J_2 \in D^{(p-m)\times(p-m)},$$

   where $(J_1 \quad J_2)$ has full row rank, i.e., $\ker_D(.(J_1 \quad J_2)) = 0$, *satisfying the relation*:

   $$UP = JU. \tag{38}$$

   The matrix U has then the form

   $$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{39}$$

where the matrix $U_1 \in D^{m\times p}$ defines a basis of $\ker_D(.P)$, *i.e., U_1 is a full row rank matrix satisfying* $\ker_D(.P) = D^{1\times m}U_1$, *and $U_2 \in D^{(p-m)\times p}$ defines a basis of* $\mathrm{coim}_D(.P) = D^{1\times p}/(D^{1\times m}U_1)$, *i.e., U_2 is a full row rank matrix such that we have the following split exact sequence*

$$0 \to D^{1\times m} \underset{\overset{.W_1}{\leftarrow}}{\overset{.U_1}{\rightarrow}} D^{1\times p} \underset{\overset{.U_2}{\leftarrow}}{\overset{.W_2}{\rightarrow}} D^{1\times(p-m)} \to 0,$$

for certain matrices $W_1 \in D^{p\times m}$ and $W_2 \in D^{p\times(p-m)}$.

In particular, we have the relations:

$$U_1 P = 0, \quad U_2 P = J_1 U_1 + J_2 U_2.$$

Proof. ($1 \Rightarrow 2$). Let us suppose that $\ker_D(.P)$ and $\mathrm{coim}_D(.P)$ are two free left D-modules of rank respectively m and $p-m$. Let $U_1 \in D^{m\times p}$ be a basis of $\ker_D(.P)$, i.e., the full row rank matrix U_1 satisfies $\ker_D(.P) = D^{1\times m}U_1$. Using the fact that we have the short exact sequence

$$0 \to \ker_D(.P) \to D^{1\times p} \xrightarrow{\kappa} \mathrm{coim}_D(.P) \to 0$$

and $\ker_D(.P) = D^{1\times m}U_1$, we then obtain the following short exact sequence:

$$0 \to D^{1\times m} \xrightarrow{.U_1} D^{1\times p} \xrightarrow{\kappa} \mathrm{coim}_D(.P) \to 0.$$

If we denote by $N = D^{1\times p}/(D^{1\times m}U_1)$, then we get $\mathrm{coim}(.P) = D^{1\times p}/\ker_D(.P) = N$. Using the fact that N is a free left D-module of rank $p-m$ and denoting by $\phi : N \to D^{1\times(p-m)}$ the associated isomorphism, $\kappa : D^{1\times p} \to N$ the canonical projection and $W_2 \in D^{p\times(p-m)}$ the matrix corresponding to the D-morphism $\phi \circ \kappa$ in the canonical bases of $D^{1\times p}$ and $D^{1\times(p-m)}$, we then obtain the short exact sequence:

$$0 \to D^{1\times m} \xrightarrow{.U_1} D^{1\times p} \xrightarrow{.W_2} D^{1\times(p-m)} \to 0.$$

Using the fact that $D^{1\times(p-m)}$ is a free left D-module, by 1 of Proposition 2.3, the previous short exact sequence splits, and thus, there exist two matrices $W_1 \in D^{p\times m}$ and $U_2 \in D^{(p-m)\times p}$ such that we have the Bézout identities:

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} (W_1 \quad W_2) = I_p, \quad (W_1 \quad W_2) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = I_p.$$

Using the fact that $U^{-1} = (W_1 \quad W_2) \in D^{p \times p}$, we have

$$UP = \begin{pmatrix} U_1 P \\ U_2 P \end{pmatrix} = \begin{pmatrix} 0 \\ (U_2 P U^{-1}) U \end{pmatrix} = \begin{pmatrix} 0 \\ U_2 P U^{-1} \end{pmatrix} U,$$

which proves a part of the result with the notations:

$$(J_1 \quad J_2) = U_2 P U^{-1}, \quad J = \begin{pmatrix} 0 \\ U_2 P U^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ J_1 & J_2 \end{pmatrix} \in D^{p \times p}.$$

Finally, if $\lambda \in \ker_D(.(U_2 P U^{-1}))$, we then have

$$\begin{aligned} \lambda(U_2 P U^{-1}) = 0 &\Leftrightarrow (\lambda U_2) P = 0 \Leftrightarrow \lambda U_2 \in \ker_D(.P) = D^{1 \times m} U_1 \\ &\Leftrightarrow \exists \mu \in D^{1 \times m} : \lambda U_2 = \mu U_1 \\ &\Leftrightarrow \exists \mu \in D^{1 \times m} : (-\mu, \lambda) \in \ker_D(.U) = 0, \end{aligned}$$

which proves that $\lambda = 0$ as $U \in \mathrm{GL}_p(D)$, i.e., $\ker_D(.(U_2 P U^{-1})) = 0$, and thus, the matrix $(J_1 \quad J_2)$ has full row rank.

$(2 \Rightarrow 1)$. Using the relation (38) and the fact that U is a unimodular matrix, we have the commutative exact diagram

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \uparrow & & \uparrow \\ 0 \to \ker_D(.P) & \to & D^{1 \times p} & \xrightarrow{.P} & D^{1 \times p} \\ & & \uparrow .U & & \uparrow .U \\ 0 \to \ker_D(.J) & \to & D^{1 \times p} & \xrightarrow{.J} & D^{1 \times p}, \\ & & \uparrow & & \uparrow \\ & & 0 & & 0 \end{array}$$

which shows that $\ker_D(.P) \cong \ker_D(.J)$ (more precisely, $\ker_D(.P) = (\ker_D(.J))U$). Let us characterize $\ker_D(.J)$. Let us consider $(\lambda_1, \lambda_2) \in \ker_D(.J)$. We then have $\lambda_2 (J_1 \quad J_2) = 0$ and using the fact that $(J_1 \quad J_2)$ has full row rank, we obtain that $\lambda_2 = 0$ and λ_1 is any arbitrary element of $D^{1 \times m}$, which proves that $\ker_D(.J) = D^{1 \times m}$ and $\ker_D(.P)$ is a free left D-module of rank m.

Similarly, we have $\mathrm{im}_D(.P) = (\mathrm{im}_D(.J))U$ as U is a unimodular matrix and:

$$\forall \lambda, \mu \in D^{1 \times p}, \quad \begin{cases} \lambda P = ((\lambda U^{-1}) J) U, \\ (\mu J) U = (\mu U) P. \end{cases}$$

Therefore, we have $\mathrm{im}_D(.P) \cong \mathrm{im}_D(.J) = D^{1 \times (p-m)} (J_1 \quad J_2)$. Using the fact that the matrix $(J_1 \quad J_2)$ has full row rank, we obtain that $D^{1 \times (p-m)} (J_1 \quad J_2) \cong D^{1 \times (p-m)}$, which proves that $\mathrm{coim}_D(.P) \cong \mathrm{im}_D(.P)$ is a free left D-module of rank $p - m$. $\square$

We note that (38) is equivalent to $P = U^{-1} J U$, which means that P and J are *similar*.

Remark 3.1. We refer to [29,64] (resp., [24,43]) for constructive algorithms for computing bases of free left modules over the Weyl algebras $A_n(\mathbb{Q})$ and $B_n(\mathbb{Q})$ (resp., over a commutative polynomial ring with coefficients in $\mathbb{Q}$). These algorithms and heuristic methods have been implemented in the packages QuillenSuslin and Stafford of the library OreModules [17,24,64].

The constructive algorithms developed in [24,29,43,64] compute bases of finitely presented free left D-modules, i.e., modules of the form $D^{1 \times l} / (D^{1 \times m} L)$, where $L \in D^{m \times l}$. Hence, if we want to compute a basis of the free left D-module $\ker_D(.P)$, we first need to compute the beginning of the free resolution of $\ker_D(.P)$:

$$D^{1\times l_1} \xrightarrow{.L_1} D^{1\times l_0} \xrightarrow{.L_0} \ker_D(.P) = D^{1\times l_0} L_0 \to 0.$$

If $L_1 = 0$, then we get $\ker_D(.P) = D^{1\times l_0} L_0 \cong D^{1\times l_0}$ as the rows of L_0 are left D-linearly independent. Hence, we can take $m = l_0$ and $U_1 = L_0$. Otherwise, we need to use the constructive algorithms or the heuristic methods developed in [16,24,29,43,56,57,64] to compute two matrices $K_1 \in D^{l_0\times m}$ and $S_1 \in D^{m\times l_0}$ such that we have the split exact sequence:

$$D^{1\times l_1} \xrightarrow{.L_1} D^{1\times l_0} \underset{\xleftarrow{.S_1}}{\xrightarrow{.K_1}} D^{1\times m} \to 0.$$

A basis of the free left D-module $\ker_D(.P)$ is then defined by the rows of the matrix $S_1 L_0$.

Moreover, we have $\operatorname{coim}_D(.P) = D^{1\times p}/\ker_D(.P) = D^{1\times p}/(D^{1\times l_0} L_0)$ and we can use the constructive algorithms developed in [24,29,43,64] to obtain the matrices $K_0 \in D^{p\times(p-m)}$ and $S_0 \in D^{(p-m)\times p}$ such that we have the split exact sequence:

$$D^{1\times l_0} \xrightarrow{.L_0} D^{1\times p} \underset{\xleftarrow{.S_0}}{\xrightarrow{.K_0}} D^{1\times(p-m)} \to 0.$$

Then, a basis of $\operatorname{coim}_D(.P) = D^{1\times p}/(D^{1\times l_0} L_0)$ is defined by the residue classes of the rows of the matrix $S_0 \in D^{(p-m)\times p}$ in $\operatorname{coim}_D(.P)$. Hence, we can take $U_2 = S_0$.

We shall need the next two lemmas.

Lemma 3.2. *Let $R \in D^{q\times p}$, $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ be three matrices satisfying* (15). *Assume further that there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that*

$$\begin{cases} U P = J_P U, \\ V Q = J_Q V, \end{cases} \tag{40}$$

for certain matrices $J_P \in D^{p\times p}$ and $J_Q \in D^{q\times q}$. Then, we have the following equality:

$$(V R U^{-1}) J_P = J_Q (V R U^{-1}). \tag{41}$$

Proof. We can easily check that we have the following commutative diagram

$$\begin{tikzcd}
& D^{1\times q} \arrow[rr, ".(V R U^{-1})"] \arrow[dl, ".V"'] \arrow[dd, ".J_Q"] & & D^{1\times p} \arrow[dl, ".U"'] \arrow[dd, ".J_p"] \\
D^{1\times q} \arrow[rr, ".R"] \arrow[dd, ".Q"'] & & D^{1\times p} \arrow[dd, ".P"] & \\
& D^{1\times q} \arrow[rr, ".(V R U^{-1})"] \arrow[dl, ".V"'] & & D^{1\times p} \arrow[dl, ".U"'] \\
D^{1\times q} \arrow[rr, ".R"] & & D^{1\times p} &
\end{tikzcd}$$

from which we obtain (41). Let us give the corresponding explicit computations. Starting with the second equation of (40) and post-multiplying it by R and using (15), we obtain

$$J_Q V R = V Q R = V R P = (V R U^{-1})(U P).$$

Using the first equation of (40), we get $J_Q V R = (V R U^{-1})(J_P U)$ and post-multiplying the previous equality by U^{-1}, we finally have $J_Q (V R U^{-1}) = (V R U^{-1}) J_P$, which proves (41). □

Lemma 3.3. *Let us consider two matrices of the form*

$$\begin{cases} J_P = \begin{pmatrix} 0 & 0 \\ J_1 & J_2 \end{pmatrix}, \\ J_Q = \begin{pmatrix} 0 & 0 \\ J_3 & J_4 \end{pmatrix}, \end{cases} \tag{42}$$

with the notations $1 \leqslant m \leqslant p$, $1 \leqslant l \leqslant q$ *and*:

$$J_1 \in D^{(p-m)\times m}, \quad J_2 \in D^{(p-m)\times(p-m)}, \quad J_3 \in D^{(q-l)\times l}, \quad J_4 \in D^{(q-l)\times(q-l)}.$$

Moreover, let us suppose that the matrix $(J_1 \quad J_2)$ *has full row rank. If the matrix* $\overline{R} \in D^{q\times p}$ *satisfies the relation*

$$\overline{R} J_P = J_Q \overline{R},$$

then there exist three matrices $\overline{R}_1 \in D^{l\times m}$, $\overline{R_2} \in D^{l\times(p-m)}$, $\overline{R_3} \in D^{(q-l)\times(p-m)}$ *such that*:

$$\overline{R} = \begin{pmatrix} \overline{R}_1 & 0 \\ \overline{R}_2 & \overline{R}_3 \end{pmatrix}. \tag{43}$$

Proof. Let us write

$$\overline{R} = \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix},$$

where $\overline{R}_{11} \in D^{l\times m}$, $\overline{R}_{12} \in D^{l\times(p-m)}$, $\overline{R}_{21} \in D^{(q-l)\times m}$, $\overline{R}_{22} \in D^{(q-l)\times(p-m)}$. Then, we have

$$\overline{R} J_P = \begin{pmatrix} \overline{R}_{12} J_1 & \overline{R}_{12} J_2 \\ \overline{R}_{22} J_1 & \overline{R}_{22} J_2 \end{pmatrix}, \quad J_Q \overline{R} = \begin{pmatrix} 0 & 0 \\ J_3 \overline{R}_{11} + J_4 \overline{R}_{21} & J_3 \overline{R}_{12} + J_4 \overline{R}_{21} \end{pmatrix}.$$

Therefore, we obtain $\overline{R}_{12}(J_1 \quad J_2) = 0$. Using the fact that $(J_1 \quad J_2)$ has full row rank, we then get $\overline{R}_{12} = 0$, which proves the result. $\square$

Let us state the second main result of the paper (the fairy's first theorem).

Theorem 3.2. *Let us consider* $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q} R)$ *and* $f : M \to M$ *an endomorphism defined by two matrices* $P \in D^{p\times p}$ *and* $Q \in D^{q\times q}$ *satisfying* (15). *If the left D-modules* $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ *and* $\mathrm{coim}_D(.Q)$ *are free of rank respectively* m, $p-m$, l *and* $q-l$, *where* $1 \leqslant m \leqslant p$ *and* $1 \leqslant l \leqslant q$, *then the following results hold*:

1. *There exist* $U \in \mathrm{GL}_p(D)$ *and* $V \in \mathrm{GL}_q(D)$ *satisfying the relations*

   $$\begin{cases} P = U^{-1} J_P U, \\ Q = V^{-1} J_Q V, \end{cases}$$

   where J_P *and* J_Q *are the matrices defined by* (42). *In particular,* U *and* V *are defined by*

   $$\begin{cases} U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad U_1 \in D^{m\times p}, \quad U_2 \in D^{(p-m)\times p}, \\ V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad V_1 \in D^{l\times q}, \quad V_2 \in D^{(q-l)\times q}, \end{cases}$$

   where the full row rank matrices U_1 *and* V_1 *respectively define bases of the free left D-modules* $\ker_D(.P)$ *and* $\ker_D(.Q)$, *i.e.*,

   $$\begin{cases} \ker_D(.P) = D^{1\times m} U_1, \\ \ker_D(.Q) = D^{1\times l} V_1, \end{cases}$$

and the full row rank matrices U_2 and V_2 respectively define bases of the free left D-modules:

$$\operatorname{coim}_D(.P) = D^{1\times p}/(D^{1\times m} U_1), \quad \operatorname{coim}_D(.Q) = D^{1\times q}/(D^{1\times l} V_1).$$

2. *The matrix R is equivalent to $\overline{R} = V R U^{-1}$.*
3. *If we denote by $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$, $W_2 \in D^{p\times(p-m)}$, we then have*:

$$\overline{R} = \begin{pmatrix} V_1 R W_1 & 0 \\ V_2 R W_1 & V_2 R W_2 \end{pmatrix} \in D^{q\times p}.$$

Proof. 1. The result directly follows from 2 of Proposition 3.2.

2. Using the fact that U and V are unimodular, we obtain $R = V^{-1}\overline{R}U$.

3. From Lemma 3.2, the matrix $\overline{R} = V R U^{-1}$ satisfies (41). Then, applying Lemma 3.3 to the matrix $\overline{R}$, we obtain that $\overline{R}$ has the triangular form (43), where $\overline{R}_1 \in D^{l\times m}$, $\overline{R}_2 \in D^{l\times(p-m)}$ and $\overline{R}_3 \in D^{(q-l)\times(p-m)}$. Finally, we have

$$\overline{R} = V R U^{-1} = \begin{pmatrix} V_1 R W_1 & V_1 R W_2 \\ V_2 R W_1 & V_2 R W_2 \end{pmatrix} \in D^{q\times p},$$

where $V_1 R W_1 \in D^{l\times m}$, $V_2 R W_1 \in D^{(p-l)\times m}$ and $V_1 R W_2 \in D^{l\times(p-m)}$, $V_2 R W_2 \in D^{(p-l)\times(p-m)}$. □

We refer to Remark 3.1 for more details on the computation of the unimodular matrices U and V defined in Theorem 3.2.

Remark 3.2. Let us consider a skew polynomial ring $D = A[\partial; \sigma, \delta]$ over a ring A, the matrix $R = (\partial I_p - E) \in D^{p\times p}$ and $M = D^{1\times p}/(D^{1\times q} R)$ the left D-module associated with the linear functional system $\partial y = E y$. Using the results proved in Example 2.5, we know that any endomorphism f can be defined by means of two matrices $P \in A^{p\times p}$ and $Q \in A^{q\times q}$. If A is a field (e.g., $A = k(t)$, $k(n)$), then we can use standard linear algebra techniques to compute the bases of the A-vector spaces $\ker_A(.P)$, $\operatorname{coim}_A(.P)$, $\ker_A(.Q)$ and $\operatorname{coim}_A(.Q)$, i.e., to compute the matrices $U_1 \in A^{m\times p}$, $U_2 \in A^{(p-m)\times p}$, $V_1 \in A^{l\times q}$ and $V_2 \in A^{(q-l)\times q}$ defined in Theorem 3.2 as we have

$$\ker_D(.P) = D \otimes_A \ker_A(.P), \quad \operatorname{coim}_D(.P) = D \otimes_A \operatorname{coim}_A(.P),$$

and similarly for $\ker_D(.Q) = D \otimes_A \ker_A(.Q)$ and $\operatorname{coim}_D(.Q) = D \otimes_A \operatorname{coim}_A(.Q)$, where $D \otimes_A \cdot$ denotes the tensor product of A-modules (see, e.g., [66]). Finally, if A is a (left) principal ideal domain (e.g., $\mathbb{Z}$, $k[t]$, k is a field), then we can compute bases of $\ker_A(.P)$, $\operatorname{coim}_A(.P)$, $\ker_A(.Q)$ and $\operatorname{coim}_A(.Q)$ by means of a (Jacobson) Smith form (see, e.g., [24]).

Example 3.3. Let us consider the Weyl algebra $D = A_1(\mathbb{Q})$, the following matrix

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}, \tag{44}$$

and the finitely presented left D-module $M = D^{1\times 4}/(D^{1\times 4} R)$. Using Algorithm 2.2 with $\alpha = 0$, $\beta = 1$ and $\gamma = 0$, we obtain that an endomorphism f of M is defined by means of the two matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in k^{4\times 4}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix} \in k[t]^{4\times 4}, \tag{45}$$

i.e., $RP = QR$. Using linear algebra techniques and Smith form computations, we can prove that the left D-modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$ are free of rank 2 with bases:

$$\begin{cases} U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{cases} \quad \begin{cases} V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t-1 & -t \end{pmatrix}, \\ V_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \end{cases}$$

If we denote by $U = (U_1^{\mathrm{T}} \quad U_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_4(D)$ and $V = (V_1^{\mathrm{T}} \quad V_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_4(D)$, then we obtain that R is equivalent to the following block-triangular matrix:

$$\overline{R} = VRU^{-1} = \begin{pmatrix} -\partial & 1 & 0 & 0 \\ t\partial - t & -\partial - t & 0 & 0 \\ \partial + t & \partial - 1 & \partial & -t \\ -\partial & 1 & 0 & \partial \end{pmatrix}.$$

Example 3.4. Let us consider the following four complex matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The *Dirac equation* for a particle without a mass has the form

$$\sum_{j=1}^{4} \gamma^j \frac{\partial \psi(x)}{\partial x_j} = 0, \tag{46}$$

where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^{\mathrm{T}}$ is *Dirac spinor* and $x = (x_1, x_2, x_3, x_4)$ is the space-time coordinates.

Let us consider the ring $D = \mathbb{Q}(i)[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}][\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}][\partial_3, \mathrm{id}; \frac{\partial}{\partial x_3}][\partial_4; \mathrm{id}, \frac{\partial}{\partial x_4}]$ of differential operators, the system matrix associated with (46)

$$R = \begin{pmatrix} \partial_4 & 0 & -i\partial_3 & -(i\partial_1 + \partial_2) \\ 0 & \partial_4 & -i\partial_1 + \partial_2 & i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & -\partial_4 & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & -\partial_4 \end{pmatrix} \in D^{4\times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 4}R)$.

Using Algorithm 2.1, we obtain that an endomorphism f of M is defined by the matrices:

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

As the entries of P and Q belong to $\mathbb{Q}$, using linear techniques, we can easily compute bases of the free $\mathbb{Q}$-modules $\ker_{\mathbb{Q}}(.P)$, $\mathrm{coim}_{\mathbb{Q}}(.P)$, $\ker_{\mathbb{Q}}(.Q)$ and $\mathrm{coim}_{\mathbb{Q}}(.Q)$, i.e., bases of the free D-modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$:

$$\begin{cases} U_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \\ U_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{cases} \quad \begin{cases} V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \\ V_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{cases}$$

Forming the matrices $U = (U_1^{\mathrm{T}} \quad U_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_4(D)$ and $V = (V_1^{\mathrm{T}} \quad V_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_4(D)$, we then obtain that the matrix R is equivalent to the following block-triangular one:

$$\overline{R} = VRU^{-1} = \begin{pmatrix} \partial_4 - i\partial_3 & \partial_2 + i\partial_1 & 0 & 0 \\ \partial_2 - i\partial_1 & -(\partial_4 + i\partial_3) & 0 & 0 \\ i\partial_3 & -(\partial_2 + i\partial_1) & -(\partial_4 + i\partial_3) & \partial_2 + i\partial_1 \\ \partial_2 - i\partial_1 & -i\partial_3 & -(\partial_2 - i\partial_1) & -(\partial_4 - i\partial_3) \end{pmatrix}.$$

Example 3.5. Let us consider again the equation of the tank subjected to a one dimensional horizontal move defined by (7). Using Algorithm 2.1, we obtain that an endomorphism f of the D-module $M = D^{1\times 3}/(D^{1\times 2}R)$, defined in Example 2.2, can be generated by the pair of matrices:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 2\partial_1\partial_2 & -2\partial_1\partial_2 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 2\partial_1\partial_2 & -2\partial_1\partial_2 \end{pmatrix}.$$

Using algorithms developed in [16,24,56,64], we can prove that $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$ are free D-modules with bases

$$\begin{cases} U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\partial_1\partial_2 \end{pmatrix}, \\ U_2 = (0 \quad 0 \quad 1), \end{cases} \quad \begin{cases} V_1 = (1 \quad 0), \\ V_2 = (0 \quad 1), \end{cases}$$

Finally, if we form $U = (U_1^{\mathrm{T}} \quad U_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_3(D)$ and $V = (V_1^{\mathrm{T}} \quad V_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_2(D)$, we then obtain that the matrix R is equivalent to the following matrix:

$$\overline{R} = VRU^{-1} = \begin{pmatrix} \partial_1^2 & -1 & 0 \\ 1 & -\partial_1^2 & 2\partial_1\partial_2(\partial_1^2 - 1) \end{pmatrix}.$$

We refer the reader to the library of examples of Morphisms [21] for more difficult examples.

4. Idempotents and decompositions

We recall that D denotes an Ore algebra which satisfies the hypotheses of Proposition 2.1.

4.1. *Idempotents of* $\mathrm{end}_D(M)$ *and solution space decompositions*

Let us introduce the definition of an *idempotent* of a ring.

Definition 4.1. An element a of a ring A satisfying $a^2 = a$ is called an *idempotent* of A.

We give a lemma which characterizes the idempotents of $\mathrm{end}_D(M)$ and we deduce an algorithm for computing them.

Lemma 4.1. *Let us consider the beginning of a finite free resolution of a left D-module M*

$$D^{1\times q_2} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \to 0,$$

and a morphism $f : M \to M$ defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying (15). *Then, f is an idempotent of* $\mathrm{end}_D(M)$ *if and only if there exists a matrix $Z \in D^{p\times q}$ satisfying*:

$$P^2 = P + ZR. \tag{47}$$

Then, there exists $Z' \in D^{q\times q_2}$ such that:

$$Q^2 = Q + RZ + Z'R_2. \tag{48}$$

In particular, if $R \in D^{q\times p}$ has full row rank, namely, $R_2 = 0$, we then have:

$$Q^2 = Q + RZ. \tag{49}$$

Proof. Post-multiplying (15) by P, we obtain $RP^2 = QRP$ and using again (15), we get $RP^2 = Q^2R$, which shows that $f^2 : M \to M$ can be defined by the matrices P^2 and Q^2. By 1 of Corollary 3.3, the morphism $f^2 - f$ is 0 if and only if there exist two matrices $Z \in D^{p\times q}$ and $Z' \in D^{q\times q_2}$ satisfying (47) and (48) holds. The end of the lemma is straightforward. □

From this lemma, we deduce an algorithm which computes idempotents of $\mathrm{end}_D(M)$ defined by matrices P with a fixed total order in the ∂_i's and a fixed degree in the x_j's for the numerators and denominators of the polynomial/rational coefficients.

Algorithm 4.1

- **Input**: An Ore algebra D, a matrix $R \in D^{q\times p}$ and the output of Algorithm 2.2 for fixed α, β and γ.
- **Output**: A family of pairs $(\overline{P}_i, \overline{Q}_i)_{i\in I}$ and a set of matrices $\{Z_i\}_{i\in I}$ satisfying

$$\begin{cases} R\overline{P}_i = \overline{Q}_i R, \\ \overline{P}_i^2 = \overline{P}_i + \overline{Z}_i R, \ \text{for certain } Z_i \in D^{p\times q}, \\ \mathrm{ord}_\partial(\overline{P}_i) \leqslant \alpha, \ \text{i.e., } \overline{P}_i = \sum_{0\leqslant|\nu|\leqslant\alpha} a_\nu^{(i)} \partial^\nu, \\ \text{and } \forall\, 0 \leqslant |\nu| \leqslant \alpha, a_\nu^{(i)} \in A^{p\times p} \text{ satisfies:} \\ \deg_x(\mathrm{num}(a_\nu^{(i)})) \leqslant \beta, \\ \deg_x(\mathrm{denom}(a_\nu^{(i)})) \leqslant \gamma, \end{cases}$$

  where $\mathrm{ord}_\partial(\overline{P}_i)$ denotes the maximal of the total orders of the entries of $\overline{P}_i$, $\deg_x(\mathrm{num}(a_\nu^{(i)}))$ (resp., $\deg_x(\mathrm{denom}(a_\nu^{(i)}))$) the maximal of the degrees of the numerators (resp., denominators) of $a_\nu^{(i)}$. The morphisms f_i are defined by $f_i(\pi(\lambda)) = \pi(\lambda\overline{P}_i)$, for all $\lambda \in D^{1\times p}$, $i \in I$.

1. Consider a generic element $P = \sum_{i\in I} c_i P_i$ of the output of Algorithm 2.2 for fixed α, β and γ, where $c_i \in \overline{k}$ for $i \in I$ and $\overline{k}$ denotes an algebraic closure of k.
2. Compute $P^2 - P$ and denote the result by F.
3. Compute a Gröbner basis G of the rows of R for a total degree order.
4. Computing the normal forms of the rows of F with respect to the Gröbner basis G.

5. Solve the system on the coefficients of c_i so that all the normal forms vanish.
6. Substitute the solutions into the matrix P. Denote the set of solutions by $\{P_j\}_{j\in J}$.
7. For $j \in J$, compute the normal forms $\overline{P}_j$ of the rows of P_j with respect to G.
8. Using $r_k(\overline{P}_j^2 - \overline{P}_j) \in (D^{1\times q} R)$, $k = 1, \ldots, p$, where $r_k(\overline{P}_j^2 - \overline{P}_j)$ denotes the kth row of $\overline{P}_j^2 - \overline{P}_j$, compute a matrix $\overline{Z}_j \in D^{p\times q}$ satisfying $\overline{P}_j^2 - \overline{P}_j = \overline{Z}_j R$, for $j \in J$, by reducing to 0 the row $r_k(\overline{P}_j^2 - \overline{P}_j)$ with respect to the Gröbner basis G.

We shall say that an idempotent f of $\mathrm{end}_D(M)$ is *trivial* if either $f = 0$ or $f = \mathrm{id}_M$. We note that the trivial endomorphisms $f = 0$, defined by $P = 0$ and $Q = 0$, and $f = \mathrm{id}_M$, defined by $P = I_p$ and $Q = I_q$, are always outputs of Algorithm 4.1.

In the forthcoming Theorem 4.1, we shall show how the knowledge of an idempotent f of $\mathrm{end}_D(M)$, namely, an element $f \in \mathrm{end}_D(M)$ satisfying $f^2 = f$, can be used to decompose the system $Ry = 0$ into two decoupled systems $S_1 y_1 = 0$ and $S_2 y_2 = 0$ or, in other words, how to decompose the left D-module M into two direct summands M_1 and M_2, namely, $M \cong M_1 \oplus M_2$. In order to do that, we first need to start with the following lemma.

Lemma 4.2. *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q} R)$ and $f \in \mathrm{end}_D(M)$ be an idempotent.*

1. *We have the following split exact sequence:*

$$0 \to \ker f \underset{\mathrm{id}_M - f}{\overset{i}{\rightleftarrows}} M \underset{f^\sharp}{\overset{\rho}{\rightleftarrows}} \mathrm{coim}\, f \to 0,$$

where $f^\sharp : \mathrm{coim}\, f \to M$ is defined by

$$\forall m \in M, \quad f^\sharp(\rho(m)) = f(m). \tag{50}$$

2. *We have the following isomorphism:*

$$\varphi : \ker f \to \mathrm{coker}\, f$$
$$m \mapsto \sigma(m),$$

whose inverse is defined by

$$\psi : \mathrm{coker}\, f \to \ker f$$
$$\sigma(m) \mapsto m - f(m),$$

where $\sigma : M \to \mathrm{coker}\, f$ denotes the canonical projection onto $\mathrm{coker}\, f = M/\mathrm{im}\, f$.

Proof. 1. For all $\rho(m) \in \mathrm{coim}\, f$, we have

$$((\mathrm{id}_M - f) \circ f^\sharp)(\rho(m)) = f(m) - f^2(m) = 0,$$

i.e., $(\mathrm{id}_M - f) \circ f^\sharp = 0$. Moreover, we can easily check that $(\mathrm{id}_M - f) \circ i = \mathrm{id}_{\ker f}$. Now, for all $m \in M$, we have

$$(i \circ (\mathrm{id}_M - f) + f^\sharp \circ \rho)(m) = m - f(m) + f(m) = m,$$

i.e., $(i \circ (\mathrm{id}_M - f)) + f^\sharp \circ \rho = \mathrm{id}_M$. Composing the last identity by ρ on the left and using the fact that $\rho \circ i = 0$, we get $\rho \circ f^\sharp \circ \rho = \rho$ which proves $\rho \circ f^\sharp = \mathrm{id}_{\mathrm{coim}\, f}$ and the result.

2. Let us check that ψ is well-defined. We first note that $m - f(m) \in \ker f$. Let us consider $\sigma(m) = \sigma(m')$ and let us prove that $\psi(\rho(m)) = \psi(\rho(m'))$. The fact that we have $\sigma(m) = \sigma(m')$ implies that $\sigma(m - m') = 0$, i.e., $m - m' \in \operatorname{im} f$, and thus, there exists $n \in M$ such that $m - m' = f(n)$. Then, we get $\psi(\rho(m)) - \psi(\rho(m')) = \psi(\rho(m - m')) = \psi(\rho(f(n)) = 0$ as $\rho(f(n)) = 0$, which proves that ψ is a well-defined morphism.

For all $m \in \ker f$, we have $(\psi \circ \varphi)(m) = \psi(\sigma(m)) = m - f(m) = m$, i.e., $\psi \circ \varphi = \mathrm{id}_{\ker f}$.

Finally, for all $\sigma(m) \in \operatorname{coker} f$, we have $(\varphi \circ \psi)(\sigma(m)) = \varphi(m - f(m)) = \sigma(m)$, which proves $\varphi \circ \psi = \mathrm{id}_{\operatorname{coker} f}$ and the result. □

Let us introduce the definition of a *decomposable* left D-module.

Definition 4.2. A non-zero left D-module M is said to be *decomposable* if it can be written as a direct sum of two proper left D-submodules. A left D-module M which is not decomposable, i.e., which is not the direct sum of two proper left D-submodules, is said to be *indecomposable*.

By Lemma 4.2, we obtain that the existence of a non-trivial idempotent f of $\mathrm{end}_D(M)$ implies that we have $M \cong \ker f \oplus \operatorname{coim} f$, i.e., M is a decomposable left D-module. Conversely, if there exist two left D-modules M_1 and M_2 such that M is isomorphic to $M_1 \oplus M_2$ and if we denote this isomorphism by $\phi : M \to M_1 \oplus M_2$ and $p_1 : M_1 \oplus M_2 \to M_1 \oplus 0$ the canonical projection (i.e., $p_1^2 = p_1$), then $p = \phi^{-1} \circ p_1 \circ \phi$ is an idempotent of $\mathrm{end}_D(M)$. We obtain the following well-known corollary of Lemma 4.2 (see, e.g., [46,37]).

Corollary 4.1. *A left D-module M is decomposable if and only if* $\mathrm{end}_D(M)$ *admits a non-trivial idempotent.*

If we consider again the example of the heat equation defined in Example 2.4, we proved that the endomorphism ring $\mathrm{end}_D(M)$ of the corresponding D-module M is isomorphic to a univariate commutative polynomial ring $\mathbb{Q}\left[\partial_x; \mathrm{id}_{\mathbb{Q}}, \frac{\partial}{\partial x}\right]$. Hence, we obtain that M is an indecomposable D-module. A similar result holds for the gradient operator in $\mathbb{R}^3$ defined in Example 2.4.

Checking whether or not a finitely presented left D-module M is decomposable is generally a difficult issue. Indeed, we can easily check that the set of idempotents of $\mathrm{end}_D(M)$ has no algebraic structure (the only main thing that we can say is that if f is an idempotent of $\mathrm{end}_D(M)$ then so is $\mathrm{id}_M - f$). Hence, Algorithm 4.1 only gives a heuristic method for checking that a left D-module M is decomposable. However, in [21], it was shown to be quite efficient on different classical linear functional systems appearing in engineering sciences, control theory and mathematical physics.

The next proposition gives a necessary and sufficient condition for $f \in \mathrm{end}_D(M)$ to be an idempotent. This result will play an important role for decomposing the solution space $\ker_{\mathcal{F}}(R.)$.

Proposition 4.1. *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q} R)$ and $f : M \to M$ be an endomorphism of M defined by a pair of matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying $RP = QR$. Then, the following results are equivalent*:

1. *f is an idempotent of* $\mathrm{end}_D(M)$, *namely, $f^2 = f$.*
2. *There exists $X \in D^{p\times r}$ satisfying*

$$P = I_p - XS, \tag{51}$$

where $S \in D^{r \times p}$ *is the matrix defined in Proposition* 3.1, *i.e.*, $\operatorname{coim} f = D^{1 \times p}/(D^{1 \times r} S)$.

Finally, we have the following commutative exact diagram where $f^\sharp$ *is defined by* (50)

$$\begin{array}{ccccccc} & \overset{.X}{\leftarrow} & & & 0 & & \\ & & & & \downarrow & & \\ D^{1\times r} & \overset{.S}{\to} & D^{1\times p} & \overset{\kappa}{\to} & \operatorname{coim} f & \to & 0 \\ \downarrow .T & & \downarrow .P & & \downarrow f^\sharp & & \\ D^{1\times q} & \overset{.R}{\to} & D^{1\times p} & \overset{\pi}{\to} & M & \to & 0. \\ & & & & \downarrow & & \\ & & & & \ker f & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Proof. $(1 \Rightarrow 2)$. By 1 of Lemma 4.2, the morphism $f^\sharp$ defined by (50) satisfies $\rho \circ f^\sharp = \mathrm{id}_{\operatorname{coim} f}$, and thus, we have $M = i(\ker f) \oplus f^\sharp(\operatorname{coim} f)$, where i denotes the canonical injection of $\ker f$ into M. Using the relation $SP = TR$, we obtain that $f^\sharp$ induces the morphism of complexes:

$$\begin{array}{ccccccc} D^{1\times r} & \overset{.S}{\to} & D^{1\times p} & \overset{\kappa}{\to} & \operatorname{coim} f & \to & 0 \\ \downarrow .T & & \downarrow .P & & \downarrow f^\sharp & & \\ D^{1\times q} & \overset{.R}{\to} & D^{1\times p} & \overset{\pi}{\to} & M & \to & 0. \end{array}$$

Composing the morphisms of complexes corresponding to ρ (see Theorem 3.1) and $f^\sharp$, we obtain that the morphism $\mathrm{id} - \rho \circ f^\sharp = 0$ is defined by the following morphism of complexes

$$\begin{array}{ccccc} D^{1\times r_2} & \overset{.S_2}{\to} & D^{1\times r} & \overset{.S}{\to} & D^{1\times p} \\ & & \downarrow .(I_r - TL) & & \downarrow .(I_p - P) \\ D^{1\times r_2} & \overset{.S_2}{\to} & D^{1\times r} & \overset{.S}{\to} & D^{1\times p} \end{array}$$

which must be homotopic to zero. Thus, there exist $X \in D^{p \times r}$ and $X_2 \in D^{r \times r_2}$ such that:

$$\begin{cases} I_p - P = XS, \\ I_r - TL = SX + X_2 S_2. \end{cases} \tag{52}$$

$(2 \Rightarrow 1)$. Using (51) and $SP = TR$, we get $P^2 = (I_p - XS)P = P - XSP = P - (XT)R$, which proves that f is an idempotent of $\operatorname{end}_D(M)$ by Lemma 4.1. □

We note that, substituting (51) into $SP = TR$, we obtain

$$S(I_p - XS) = TR \Leftrightarrow S - SXS = TR.$$

We give a necessary and sufficient condition for a left D-module M to be of the form

$$M \cong N \oplus P$$

for a given left D-module N.

Proposition 4.2. *Let* $R \in D^{q \times p}$ *and* $S \in D^{r \times p}$ *be two matrices satisfying* $(D^{1\times q} R) \subseteq (D^{1\times r} S)$. *Then, the left* D*-module* $M' = D^{1\times p}/(D^{1\times r} S)$ *is isomorphic to a direct summand of the left* D*-module* $M = D^{1\times p}/(D^{1\times q} R)$, *i.e., we have*

$$M \cong M' \oplus \ker \rho, \tag{53}$$

where $\rho : M \to M'$ *is defined by* $\rho(\pi(\lambda)) = \kappa(\lambda)$, *for all* $\lambda \in D^{1\times p}$, *and* $\kappa : D^{1\times p} \to M'$ *denotes the canonical projection onto* M', *if and only if there exist two matrices* $X \in D^{p\times r}$ *and* $T \in D^{r\times q}$ *satisfying the relation*:

$$S - SXS = TR. \tag{54}$$

Proof. ($\Rightarrow$). The isomorphism (53) is equivalent to the existence of a morphism $g : M' \to M$ which satisfies $\rho \circ g = \mathrm{id}_{M'}$ (see, e.g., [66]). Following the same techniques as the ones used in the proof of Proposition 4.1, (53) is then equivalent to the existence of three matrices $P \in D^{p\times p}$, $T \in D^{r\times q}$ and $X \in D^{p\times r}$ satisfying the relations:

$$\begin{cases} SP = TR, \\ I_p - P = XS, \end{cases} \Rightarrow S - SXS = TR.$$

($\Leftarrow$). From (54), we get $S(I_p - XS) = TR$, and, if we set $P = I_p - XS$, then we have the following commutative diagram

$$\begin{array}{ccccccc} D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & M' & \to & 0 \\ \downarrow .T & & \downarrow .P & & & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \to & 0, \end{array}$$

which induces a morphism $g : M' \to M$ defined by $g(\kappa(\lambda)) = \pi(\lambda P)$, for all $\lambda \in D^{1\times p}$. Using the fact that $\kappa = \rho \circ \pi$, for all $\lambda \in D^{1\times p}$, we then get

$$(\rho \circ g)(\kappa(\lambda)) = \rho(\pi(\lambda P)) = \kappa(\lambda P) = \kappa(\lambda) - \kappa((\lambda X)S) = \kappa(\lambda),$$

i.e., $\rho \circ g = \mathrm{id}_{M'}$, which proves that the exact sequence $0 \to \ker \rho \xrightarrow{i} M \xrightarrow{\rho} M' \to 0$ splits, which implies that $M = \ker \rho \oplus g(M')$, i.e., $M \cong M' \oplus \ker \rho$ as g is an injective D-morphism (i.e., $g(m) = 0 \Rightarrow m = \rho(g(m)) = 0$). $\square$

Remark 4.1. If S has full row rank, i.e., $\ker_D(.S) = 0$, the second equation of (52) becomes:

$$SX + TL = I_r. \tag{55}$$

Note that the factorization $R = LS$ satisfying (55) is nothing else than the generalization for matrices and non-commutative rings of the classical decomposition of a commutative polynomial into coprime factors. Indeed, if R belongs to a commutative polynomial ring $D = k[x_1, \ldots, x_n]$, where k is a field, then (55) becomes $XS + TL = 1$ (Bézout identity), i.e., the ideal of D generated by S and L is D and we obtain that $R = LS$ is a factorization of R into coprime factors L and S.

We have the following corollary of Proposition 4.1.

Corollary 4.2. *With the hypotheses and notations of Proposition* 4.1, *we have the equality*:

$$D^{1\times r} S = D^{1\times (p+q)} \begin{pmatrix} I_p - P \\ R \end{pmatrix}.$$

Proof. Using the factorization $R = LS$ and (51), we obtain the following equality:

$$\begin{pmatrix} I_p - P \\ R \end{pmatrix} = \begin{pmatrix} X \\ L \end{pmatrix} S,$$

which proves the first inclusion. The second one is a direct consequence of (54) as we have $XS = I_p - P$ and:

$$S = SXS + TR = (S \quad T)\begin{pmatrix} XS \\ R \end{pmatrix} = (S \quad T)\begin{pmatrix} I_p - P \\ R \end{pmatrix}. \qquad \square$$

Let us state the third main result of the paper.

Theorem 4.1. *Let $R \in D^{q\times p}$ and let us assume that the left D-module $M = D^{1\times p}/(D^{1\times q} R)$ is such that there exists a non-trivial idempotent f of $\mathrm{end}_D(M)$. Moreover, let us denote by $S \in D^{r\times p}$, $L \in D^{q\times r}$, $X \in D^{p\times r}$ and $S_2 \in D^{r_2\times s}$ the matrices defined by*

$$\begin{cases} \mathrm{coim}\, f = D^{1\times p}/(D^{1\times r} S), \\ R = LS, \\ I_p - P = XS, \\ \ker_D(.S) = D^{1\times r_2} S_2. \end{cases}$$

If $\mathscr{F}$ is an injective left D-module, then we obtain that a solution $\eta \in \mathscr{F}^p$ of $R\eta = 0$ has the form $\eta = \zeta + X\tau$, where $\zeta \in \mathscr{F}^p$ is a solution of $S\zeta = 0$ and $\tau \in \mathscr{F}^r$ is a solution of the system:

$$\begin{cases} L\tau = 0, \\ S_2\tau = 0. \end{cases} \tag{56}$$

The integration of the system $R\eta = 0$ is then equivalent to the integration of the two independent systems $S\zeta = 0$ and (56).

Proof. Applying the functor $\hom_D(\cdot, \mathscr{F})$ to the commutative exact diagram (35) and using the fact that $\mathscr{F}$ is an injective left D-module, we obtain the following commutative exact diagram:

$$\begin{array}{ccccccccc} & & \mathscr{F}^q & \xleftarrow{R.} & \mathscr{F}^p & \leftarrow & \ker_{\mathscr{F}}(R.) & \leftarrow & 0 \\ & & \uparrow L. & & \| & & \uparrow \rho^\star & & \\ \mathscr{F}^{r_2} & \xleftarrow{S_2.} & \mathscr{F}^r & \xleftarrow{S.} & \mathscr{F}^p & \leftarrow & \ker_{\mathscr{F}}(S.) & \leftarrow & 0. \end{array}$$

Let us first prove that any element of the form $\eta = \zeta + X\tau$, where $\zeta \in \mathscr{F}^p$ (resp., $\tau \in \mathscr{F}^r$) satisfies $S\zeta = 0$ (resp., (56)), is a solution of $R\eta = 0$. Using $R = LS$ and $S\zeta = 0$, we get

$$R\eta = R\zeta + R(X\tau) = L(S\zeta) + R(X\tau) = R(X\tau).$$

Using the fact that τ satisfies the second equation of (56) and the exactness of the last horizontal sequence of the previous commutative exact diagram, there exists $\overline{\eta} \in \mathscr{F}^p$ satisfying $\tau = S\overline{\eta}$. Substituting this relation into the first equation of (56), we obtain

$$L\tau = L(S\overline{\eta}) = R\overline{\eta} = 0.$$

Then, using (51) and the relation $RP = QR$, we obtain

$$\overline{\eta} - P\overline{\eta} = XS\overline{\eta} = X\tau \Rightarrow R\overline{\eta} - RP\overline{\eta} = R(X\tau) \Rightarrow R(X\tau) = R\overline{\eta} - QR\overline{\eta} = 0.$$

This last result proves that $R\eta = 0$, i.e., $\eta = \zeta + X\tau$ is a solution of the system $R\eta = 0$.

Secondly, let us prove that any solution $\eta \in \mathscr{F}^p$ of $R\eta = 0$ has the form of $\eta = \zeta + X\tau$, where $\zeta \in \mathscr{F}^p$ satisfies $S\zeta = 0$ and $\tau \in \mathscr{F}^r$ satisfies (56). Let us consider $\eta \in \mathscr{F}^p$ satisfying $R\eta = 0$, i.e., $(LS)\eta = 0$. Using the previous commutative exact diagram, we obtain that the element $\tau \in \mathscr{F}^r$ defined by $\tau = S\eta$ satisfies (56). Then, from (54), we obtain

$$S\eta - S(X(S\eta)) = T(R\eta) = 0 \Rightarrow S(X\tau) = \tau.$$

All the solutions of the inhomogeneous system $S\eta = \tau$ are defined as the sum of a particular solution of $S\overline{\eta} = \tau$ and any of solution of $S\zeta = 0$, i.e., we have $\eta = \zeta + X\tau$. □

Let us suppose that an idempotent $f \in \mathrm{end}_D(M)$ defined by the pair of matrices (P, Q) is obtained by Algorithm 4.1. Then, the matrices S, S_2 and L defined in Theorem 4.1 can easily be obtained by means of Gröbner bases computations. We refer to [16,17] for more details.

The previous result has already been obtained in [62] in the particular case where

$$M \cong t(M) \oplus (M/t(M)), \tag{57}$$

where the *torsion submodule* $t(M)$ is defined by $t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : Pm = 0\}$.

We recall that the previous decomposition is generally only true over a *left hereditary ring* D [46] (e.g., $D = A_1(k)$) or over a left principal ideal domain (e.g., $D = B_1(k)$). In [62], we constructively characterize when the exact sequence $0 \to t(M) \to M \to M/t(M) \to 0$ splits, i.e., when we have (57). In control theory, the previous result gives a general answer to the question of knowing when a behaviour $\hom_D(M, \mathscr{F})$ can be split into the *autonomous behaviour* $\hom_D(t(M), \mathscr{F})$ and the *parametrizable behaviour* $\hom_D(M/t(M), \mathscr{F})$ (see, e.g., [52,56,57,73,76]). We refer the reader to [62,63] for more details and examples.

Let us illustrate Theorem 4.1.

Example 4.1. Let D be the Weyl algebra $A_1(\mathbb{Q}) = \mathbb{Q}[t][\partial; \mathrm{id}_{k[t]}, \frac{\mathrm{d}}{\mathrm{d}t}]$ and the finitely presented left D-module $M = D^{1\times 4}/(D^{1\times 4}R)$ defined in Example 3.3, where $R \in D^{4\times 4}$ is defined by (44). We consider again the endomorphism f of M defined by the pair of matrices (P, Q) given in (45). We can check that $P^2 = P$ which implies that $f^2 = f$, i.e., f is a non-trivial idempotent of $\mathrm{end}_D(M)$. With the notations used in this section, we obtain the following matrices:

$$S = \begin{pmatrix} \partial & -t & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & t & \partial \\ 1 & t & \partial & -1 \\ 1 & 0 & \partial + t & \partial - 1 \\ 1 & 1 & t & \partial \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can also verify that $\ker_D(.S) = 0$ which implies $S_2 = 0$. Theorem 4.1 then asserts that the integration of $R\eta = 0$ is equivalent to both the integration of $S\zeta = 0$, which easily gives

$$\zeta_1 = \frac{1}{2} C_1 t^2 + C_2, \quad \zeta_2 = C_1, \quad \zeta_3 = 0, \quad \zeta_4 = 0,$$

where C_1 and C_2 are two arbitrary constants, and the integration of $L\tau = 0$, i.e.:

$$\begin{cases} \tau_1 = 0, \\ \tau_2 = 0, \\ t\tau_3 + \partial\tau_4 = 0, \\ \partial\tau_3 - \tau_4 = 0. \end{cases} \Leftrightarrow \begin{cases} \tau_1 = 0, \\ \tau_2 = 0, \\ \partial^2\tau_3 + t\tau_3 = 0, \\ \tau_4 = \partial\tau_3, \end{cases} \Leftrightarrow \begin{cases} \tau_1 = 0, \\ \tau_2 = 0, \\ \tau_3(t) = C_3\mathrm{Ai}(t) + C_4\mathrm{Bi}(t), \\ \tau_4(t) = C_3\partial\mathrm{Ai}(t) + C_4\partial\mathrm{Bi}(t), \end{cases}$$

where Ai and Bi denote the two independent solutions of $\partial^2 y(t) - t y(t) = 0$ called the *Airy functions* and C_3 and C_4 are two constants. The general solution of $R\eta = 0$ is then given by

$$\eta = \zeta + X\tau = \begin{pmatrix} \frac{1}{2}C_1 t^2 + C_2 \\ C_1 \\ C_3\mathrm{Ai}(t) + C_4\mathrm{Bi}(t) \\ C_3\partial\mathrm{Ai}(t) + C_4\partial\mathrm{Bi}(t) \end{pmatrix}, \tag{58}$$

where C_1, C_2, C_3 and C_4 are four arbitrary constants.

4.2. Idempotents of $D^{p\times p}$ and block-diagonal decompositions

We recall that $P \in D^{p\times p}$ is an idempotent of the ring $D^{p\times p}$ if it satisfies $P^2 = P$.

We are now going further by proving that, under certain conditions, the existence of idempotents P of $D^{p\times p}$ defining $f \in \mathrm{end}_D(M)$ allows us to obtain a system $\overline{R}z = 0$ equivalent to $Ry = 0$, where $\overline{R}$ is a block-diagonal matrix of the same size as R.

We shall need the following three lemmas.

Lemma 4.3. *Let $R \in D^{q\times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $P \in D^{p\times p}$, $Q \in D^{q\times q}$ be two matrices satisfying (15). Then, if P is an idempotent of $D^{p\times p}$, namely $P^2 = P$, so is Q, i.e., $Q^2 = Q$.*

Proof. Post-multiplying (15) by P, we obtain $RP^2 = QRP$. Using again (15), we get $RP^2 = Q^2R$. Then, the relation $P^2 = P$ implies $RP = Q^2R$, and using again (15), we obtain $Q^2R = QR$, i.e., $(Q^2 - Q)R = 0$. Finally, the fact that R has full row rank implies $Q^2 = Q$. □

Example 4.2. Let $D = A[\partial; \sigma, \delta]$ be a skew polynomial ring over a ring A, a matrix $E \in A^{p\times p}$, $R = (\partial I_p - E) \in D^{p\times p}$ and $M = D^{1\times p}/(D^{1\times p}R)$ the left D-module associated with the linear functional system $\partial y = Ey$. In Example 2.5, we proved that we could always suppose without any restriction that $f \in \mathrm{end}_D(M)$ is defined by $P \in A^{p\times p}$ and $Q \in A^{q\times q}$ satisfying (18) where $F = E$. By Lemma 4.1, we obtain that any idempotent f of $\mathrm{end}_D(M)$ is defined by a matrix $P \in A^{p\times p}$ satisfying $P^2 = P + ZR$, where $Z \in D^{p\times q}$. Using the fact that R is a first order matrix in ∂ and P is a zero order matrix in ∂, we obtain that $Z = 0$, i.e., $P^2 = P$. Now, using the fact that R has full row rank, i.e., $\ker_D(.R) = 0$, by Lemma 4.3, we obtain that $Q^2 = Q$.

Lemma 4.4. *Let $R \in D^{q\times p}$ be a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}R)$. Let us consider an idempotent $f : M \to M$ defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying $RP = QR$, $P^2 = P + ZR$ and $Q^2 = Q + RZ$ (see Lemma 4.1). If there exists a solution $\Lambda \in D^{p\times q}$ of the algebraic Riccati equation*

$$\Lambda R\Lambda + (P - I_p)\Lambda + \Lambda Q + Z = 0, \tag{59}$$

then the matrices defined by

$$\begin{cases} \overline{P} = P + \Lambda R, \\ \overline{Q} = Q + R\Lambda, \end{cases} \tag{60}$$

satisfy the following relations:

$$R\overline{P} = \overline{Q}R, \quad \overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}.$$

Proof. By hypothesis, the matrices P and Q satisfy (47) and (49). Let us define $\overline{P} = P + \Lambda R$ for a certain matrix $\Lambda \in D^{p\times q}$. Then, we have

$$\overline{P}^2 = (P + \Lambda R)(P + \Lambda R) = P^2 + P\Lambda R + \Lambda RP + \Lambda R\Lambda R.$$

Using (15), we get $\overline{P}^2 = P^2 + (P\Lambda + \Lambda Q + \Lambda R\Lambda)R$. From (47) and $\overline{P} = P + \Lambda R$, we then obtain

$$\overline{P}^2 = \overline{P} + (Z - \Lambda + P\Lambda + \Lambda Q + \Lambda R\Lambda)R.$$

Hence, we have $\overline{P}^2 = \overline{P}$ if and only if Λ satisfies the following equation:

$$(Z - \Lambda + P\Lambda + \Lambda Q + \Lambda R \Lambda) R = 0,$$

i.e., since R has full row rank, if and on if Λ satisfies the Riccati equation (59).

Finally, we have

$$\overline{Q}^2 = (Q + R\Lambda)(Q + R\Lambda) = Q^2 + QR\Lambda + R\Lambda Q + R\Lambda R \Lambda.$$

Using (15), we get $\overline{Q}^2 = Q^2 + R(P\Lambda + \Lambda Q + \Lambda R \Lambda)$, and using (49) and $\overline{Q} = Q + R\Lambda$, we obtain

$$\overline{Q}^2 = \overline{Q} + R(Z - \Lambda + P\Lambda + \Lambda Q + \Lambda R \Lambda) = \overline{Q}. \qquad \square$$

Remark 4.2. Determining whether or not the algebraic Riccati equation (59) admits a solution seems to be a difficult issue. This problem will be studied in detail in the future.

For instance, if we consider the trivial projector $f = 0$ defined by the matrices $P = 0$, $Q = 0$ and $Z = 0$, we then obtain the equation $\Lambda R \Lambda = \Lambda$, i.e., we need to determine when a given matrix R is a generalized inverse of a certain matrix Λ and, if so, compute it. It does not seem that this problem has been studied in the algebra contrary to the problem of determining whether or not a matrix Λ admits a generalized inverse. This first problem plays an important role in recognizing when a matrix R is equivalent to its Smith form (see [21]).

We can compute a solution Λ of (59) with fixed order and fixed degrees for numerators and denominators by substituting an ansatz in (59) and solving the quadratic algebraic system obtained on its coefficients. Similarly, we can compute some idempotent matrices P of $D^{p \times p}$ with fixed order and fixed degrees for numerators and denominators by solving a quadratic algebraic system in the unknowns of the linear k-combinations of the $\overline{P}_i$ given in the output of Algorithm 2.2.

Example 4.3. Let us consider $D = A_1(\mathbb{Q})$, $R = (\partial^2 \quad -t\partial - 1)$ and $M = D^{1\times 2}/(DR)$. Searching for idempotents of $\mathrm{end}_D(M)$ defined by matrices P and Q of total order 1 and total degree 2, Algorithm 4.1 gives $P_1 = Q_1 = 0$, $P_2 = Q_2 = I_2$ and

$$\begin{cases} P_3 = \begin{pmatrix} -(t+a)\partial + 1 & t^2 + at \\ 0 & 1 \end{pmatrix}, \\ Q_3 = -((t+a)\partial + 1), \end{cases} \qquad \begin{cases} P_4 = \begin{pmatrix} (t-a)\partial & -t^2 + at \\ 0 & 0 \end{pmatrix}, \\ Q_4 = (t-a)\partial + 2, \end{cases}$$

where a is an arbitrary constant of $\mathbb{Q}$. We can check that $P_i^2 = P_i + Z_i R$, $i = 3, 4$, where

$$Z_3 = ((t+a)^2 \quad 0)^{\mathrm{T}}, \quad Z_4 = ((t-a)^2 \quad 0)^{\mathrm{T}}.$$

Using Remark 4.2, we obtain that (59) admits respectively the following solutions:

$$\Lambda_3 = (at \quad a\partial - 1)^{\mathrm{T}}, \quad \Lambda_4 = (at \quad a\partial + 1)^{\mathrm{T}}.$$

The matrices (60) are then defined by

$$\begin{cases} \overline{P}_3 = \begin{pmatrix} at\partial^2 - (t+a)\partial + 1 & t^2(1 - a\partial) \\ (a\partial - 1)\partial^2 & -at\partial^2 + (t - 2a)\partial + 2 \end{pmatrix}, \\ \overline{Q}_3 = 0, \end{cases}$$

$$\begin{cases} \overline{P}_4 = \begin{pmatrix} at\partial^2 + (t-a)\partial & -t^2(1 + a\partial) \\ (a\partial + 1)\partial^2 & -at\partial^2 - (t + 2a)\partial - 1 \end{pmatrix}, \\ \overline{Q}_4 = 1, \end{cases}$$

and we can easily check that we have $\overline{P}_i^2 = \overline{P}_i$, $\overline{Q}_i^2 = \overline{Q}_i$, for $i = 3, 4$.

The next lemma characterizes the kernel and the image of an idempotent P of $D^{p\times p}$.

Lemma 4.5. *Let $P \in D^{p\times p}$ be an idempotent, i.e., $P^2 = P$. Then, we have the following results:*

1. $\ker_D(.P)$ *and* $\operatorname{im}_D(.P)$ *are two projective left D-modules of rank respectively m and $p - m$, with $0 \leqslant m \leqslant p$.*
2. *We have the following equalities:*

$$\begin{cases} \operatorname{im}_D(.P) = \ker_D(.(I_p - P)), \\ \operatorname{im}_D(.(I_p - P)) = \ker_D(.P). \end{cases}$$

Proof. 1. We have the following short exact sequence:

$$0 \to \ker_D(.P) \to D^{1\times p} \xrightarrow{.P} \operatorname{im}_D(.P) \to 0.$$

Let us define the D-morphism $i : \operatorname{im}_D(.P) \to D^{1\times p}$ by $i(m) = m$, for all $m \in \operatorname{im}_D(.P)$. For every element $m \in \operatorname{im}_D(.P)$, there exists $\lambda \in D^{1\times p}$ such that $m = \lambda P$. Therefore, we have $((.P) \circ i)(m) = mP = \lambda P^2$ and, using the fact that $P^2 = P$, we get $((.P) \circ i)(m) = \lambda P = m$, i.e., $((.P) \circ i) = \mathrm{id}_{\operatorname{im}_D(.P)}$, which shows that the previous exact sequence splits, and thus, we obtain

$$D^{1\times p} = \ker_D(.P) \oplus \operatorname{im}_D(.P). \tag{61}$$

This proves that $\ker_D(.P)$ and $\operatorname{im}_D(.P)$ are two finitely generated projective left D-modules. Finally, from the previous short exact sequence, we get (see, e.g., [66])

$$\operatorname{rank}_D(D^{1\times p}) = \operatorname{rank}_D(\ker_D(.P)) + \operatorname{rank}_D(\operatorname{im}_D(.P)),$$

and using the fact that, by hypothesis, D is a left noetherian ring, and thus, D has *the Invariant Basis Number* (IBN) [37,46], we finally get $\operatorname{rank}_D(D^{1\times p}) = p$, which proves the first result.

2. The fact that $P^2 = P$ implies that $P(I_p - P) = 0$, i.e., $\operatorname{im}_D(.P) \subseteq \ker_D(.(I_p - P))$. Now, let $\lambda \in \ker_D(.(I_p - P))$ and let us prove that $\lambda \in \operatorname{im}_D(.P)$. Applying λ on the left of the identity $I_p = P + (I_p - P)$, we obtain $\lambda = \lambda P$, which proves $\ker_D(.(I_p - P)) \subseteq \operatorname{im}_D(.P)$ and the equality.

The second result can be proved similarly. □

We note that if $P = 0$ (resp., $P = I_p$) is the trivial idempotent, then we have $\ker_D(.P) = D^{1\times p}$ and $\operatorname{im}_D(.P) = 0$ (resp., $\ker_D(.P) = 0$, $\operatorname{im}_D(.P) = D^{1\times p}$), i.e., $\ker_D(.P)$ and $\operatorname{im}_D(.P)$ are two trivial free left D-modules, namely, defined as the images of the trivial matrices I_p and 0. We are going to show that the case where $\ker_D(.P)$ and $\operatorname{im}_D(.P)$ are two non-trivial free left D-modules plays an important role in the block-diagonal decomposition problem.

Proposition 4.3. *Let $P \in D^{p\times p}$ be an idempotent, i.e., $P^2 = P$. The following assertions are equivalent:*

1. *The left D-modules* $\ker_D(.P)$ *and* $\operatorname{im}_D(.P)$ *are free of rank respectively m and $p - m$.*
2. *There exists a unimodular matrix $U \in D^{p\times p}$, i.e., $U \in \mathrm{GL}_p(D)$, and a matrix $J_P \in D^{p\times p}$ of the form*

$$J_P = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix},$$

which satisfy the relation:

$$U P = J_P U. \tag{62}$$

The matrix U has then the form

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{63}$$

where the matrices $U_1 \in D^{m \times p}$ and $U_2 \in D^{(p-m)\times p}$ have full row ranks and satisfy:

$$\begin{cases} \ker_D(.P) = D^{1\times m} U_1, \\ \operatorname{im}_D(.P) = D^{1\times(p-m)} U_2. \end{cases} \tag{64}$$

In particular, we have the relations $U_1 P = 0$ and $U_2 P = U_2$.

Proof. ($1 \Rightarrow 2$). Let us suppose that $\ker_D(.P)$ (resp., $\operatorname{im}_D(.P)$) is a free left D-module of rank m (resp., $p - m$) and let $U_1 \in D^{m\times p}$ (resp., $U_2 \in D^{(p-m)\times p}$) be a basis of $\ker_D(.P)$ (resp., $\operatorname{im}_D(.P)$), i.e., (64) holds. Let us form the matrix U defined by (63).

Using (61), for all $\lambda \in D^{1\times p}$, there exist unique $\lambda_1 \in \ker_D(.P)$ and $\lambda_2 \in \operatorname{im}(.P)$ such that $\lambda = \lambda_1 + \lambda_2$. Then, there exist unique $\mu_1 \in D^{1\times m}$ and $\mu_2 \in D^{1\times(p-m)}$ such that $\lambda_1 = \mu_1 U_1$ and $\lambda_2 = \mu_2 U_2$, and thus, a unique $\mu = (\mu_1, \mu_2) \in D^{1\times p}$ satisfying $\lambda = \mu U$. Hence, using the standard basis $\{e_i\}_{1\leqslant i\leqslant p}$ of $D^{1\times p}$, for $i = 1, \ldots, p$, there exists a unique $V_i \in D^{1\times p}$ such that $e_i = V_i U$. The matrix $V = (V_1^{\mathrm T}, \ldots, V_p^{\mathrm T})^{\mathrm T}$ is then a left-inverse of U. By hypothesis, D is a left noetherian ring, and thus, *stably finite* [37], which implies that $UV = I_p$, i.e., $U \in \mathrm{GL}_p(D)$.

Finally, for all $\mu \in D^{1\times p}$, we have $\mu U_2 \in \operatorname{im}_D(.P)$, and thus, there exists $\nu \in D^{1\times p}$ such that $\mu U_2 = \nu P$. Using the fact that $P^2 = P$, we get

$$\mu U_2 P = \nu P^2 = \nu P = \mu U_2.$$

Hence, for all $\mu \in D^{1\times p}$, we have $\mu(U_2 P - U_2) = 0$, which proves that $U_2 P = U_2$. Using $U_1 P = 0$, we finally obtain

$$U P = \begin{pmatrix} U_1 P \\ U_2 P \end{pmatrix} = \begin{pmatrix} 0 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix} U.$$

($2 \Rightarrow 1$). Using the relation (62) and the fact that U is a unimodular matrix, we get the commutative exact diagram

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \uparrow & & \uparrow \\ 0 \to \ker_D(.P) \to & & D^{1\times p} & \xrightarrow{.P} & D^{1\times p} \\ & & \uparrow .U & & \uparrow .U \\ 0 \to \ker_D(.J_P) \to & & D^{1\times p} & \xrightarrow{.J_P} & D^{1\times p}, \\ & & \uparrow & & \uparrow \\ & & 0 & & 0 \end{array}$$

which shows that $\ker_D(.P) \cong \ker_D(.J_P)$ (more precisely, $\ker_D(.P) = \ker_D(.J_P)U$). Using the fact that we have trivially $\ker_D(.J_P) = D^{1\times m}$, we obtain that $\ker_D(.P)$ is a free left D-module of rank m. Similarly, we have $\operatorname{im}_D(.P) = \operatorname{im}_D(.J_P)U$ as U is a unimodular matrix and:

$$\forall \lambda, \mu \in D^{1\times p}, \begin{cases} \lambda P = ((\lambda U^{-1}) J_p) U, \\ (\mu J_P) U = (\mu U) P. \end{cases}$$

Therefore, we have $\operatorname{im}_D(.P) \cong \operatorname{im}_D(.J_P)$. We can easily check that $\operatorname{im}_D(.J_P) = D^{1\times(p-m)}$, which proves that $\operatorname{im}_D(.P)$ is a free left D-module of rank $p-m$. □

We note that (62) is equivalent to $P = U^{-1} J_P U$, which means that P and J_P are similar. We shall need the next lemma.

Lemma 4.6. *Let us consider the following two matrices:*

$$\begin{cases} J_P = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix} \in D^{p\times p}, \\ J_Q = \begin{pmatrix} 0 & 0 \\ 0 & I_{q-l} \end{pmatrix} \in D^{q\times q}, \end{cases} \tag{65}$$

where $1 \leqslant m \leqslant p$ and $1 \leqslant l \leqslant q$, and a matrix $\overline{R} \in D^{q\times p}$ satisfying the following relation:

$$\overline{R} J_P = J_Q \overline{R}. \tag{66}$$

Then, there exist $\overline{R}_1 \in D^{l\times m}$ and $\overline{R}_2 \in D^{(q-l)\times(p-m)}$ such that:

$$\overline{R} = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & \overline{R}_2 \end{pmatrix}. \tag{67}$$

Proof. If we write

$$\overline{R} = \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix},$$

where $\overline{R}_{11} \in D^{l\times m}$, $\overline{R}_{12} \in D^{l\times(p-m)}$, $\overline{R}_{21} \in D^{(q-l)\times m}$, $\overline{R}_{22} \in D^{(q-l)\times(p-m)}$, then, we have

$$\overline{R} J_P = \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix} = \begin{pmatrix} 0 & \overline{R}_{12} \\ 0 & \overline{R}_{22} \end{pmatrix},$$

$$J_Q \overline{R} = \begin{pmatrix} 0 & 0 \\ 0 & I_{q-l} \end{pmatrix} \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix}.$$

Therefore, (66) implies that $\overline{R}_{12} = 0$ and $\overline{R}_{21} = 0$, which proves the result. □

We are now in position to state the last main result of the paper (the fairy's second theorem).

Theorem 4.2. *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q} R)$ and $f : M \to M$ be an idempotent defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying* (12) *and*:

$$P^2 = P, \quad Q^2 = Q.$$

If the left D-modules $\ker_D(.P)$, $\operatorname{im}_D(.P)$, $\ker_D(.Q)$, $\operatorname{im}_D(.Q)$ are free of rank respectively m, $p-m$, l and $q-l$, where $1 \leqslant m \leqslant p$ and $1 \leqslant l \leqslant q$, then the following results hold:

1. *There exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ satisfying the relations*

$$\begin{cases} P = U^{-1} J_P U, \\ Q = V^{-1} J_Q V, \end{cases}$$

where J_P and J_Q are the matrices defined by (65).

In particular, the matrices U and V are defined by

$$\begin{cases} U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad U_1 \in D^{m\times p}, \quad U_2 \in D^{(p-m)\times p}, \\ V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad V_1 \in D^{l\times q}, \quad V_2 \in D^{(q-l)\times q}, \end{cases}$$

where the full row rank matrices U_1, U_2, V_1 and V_2 are respectively bases of the free left D-modules $\ker_D(.P)$, $\operatorname{im}_D(.P)$, $\ker_D(.Q)$ *and* $\operatorname{im}_D(.Q)$, *i.e.*:

$$\begin{cases} \ker_D(.P) = D^{1\times m} U_1, \\ \operatorname{im}_D(.P) = D^{1\times (p-m)} U_2, \\ \ker_D(.Q) = D^{1\times l} V_1, \\ \operatorname{im}_D(.Q) = D^{1\times (q-l)} V_2. \end{cases}$$

2. *The matrix R is equivalent to $\overline{R} = V R U^{-1}$.*
3. *If we denote by $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$, $W_2 \in D^{p\times (p-m)}$, we then have*:

$$\overline{R} = \begin{pmatrix} V_1 R W_1 & 0 \\ 0 & V_2 R W_2 \end{pmatrix} \in D^{q\times p}. \tag{68}$$

Proof. 1. The result directly follows from 2 of Proposition 4.3.

2. Using the fact that the matrices U and V are unimodular, we obtain $R = V^{-1}\overline{R}U$.

3. From Lemma 3.2, the matrix $\overline{R} = V R U^{-1}$ satisfies the relation (66). Then, applying Lemma 4.6 to $\overline{R}$, we obtain that $\overline{R}$ has the block-diagonal form (67), where $\overline{R}_1 \in D^{l\times m}$ and $\overline{R}_2 \in D^{(q-l)\times(p-m)}$. Finally, we have

$$\overline{R} = V R U^{-1} = \begin{pmatrix} V_1 R W_1 & V_1 R W_2 \\ V_2 R W_1 & V_2 R W_2 \end{pmatrix} \in D^{q\times p},$$

where $V_1 R W_1 \in D^{l\times m}$, $V_2 R W_1 \in D^{(p-l)\times m}$, $V_1 R W_2 \in D^{l\times (p-m)}$, $V_2 R W_2 \in D^{(p-l)\times(p-m)}$. $\square$

Remark 4.3. Using 2 of Lemma 4.5 and Remark 3.1, we can compute a basis of the free left D-module $\operatorname{im}_D(.P) = \ker_D(.(I_p - P))$.

Let us illustrate Theorem 4.2 on two examples.

Example 4.4. Let us consider again the Dirac equations studied in Example 3.4. We can check that the matrices P and Q defined in Example 3.4 are idempotents of $D^{4\times 4}$, i.e., $P^2 = P$ and $Q^2 = Q$. As the entries of P and Q belong to $\mathbb{Q}$, by linear algebra, we know that the D-modules $\ker_D(.P)$, $\operatorname{im}_D(.P)$, $\ker_D(.Q)$ and $\operatorname{im}_D(.Q)$ are free. Hence, by Theorem 4.2, the system matrix R of the Dirac equations defined in Example 3.4 is equivalent to a block-diagonal matrix. In order to compute this equivalent form, we only need to compute a basis of the free D-modules $\operatorname{im}_D(.P)$ and $\operatorname{im}_D(.Q)$ instead of a basis of the free D-modules $\operatorname{coim}_D(.P)$ and $\operatorname{coim}_D(.Q)$. Using linear algebra techniques, we obtain $\operatorname{im}_D(.P) = D^{1\times 2}U_2'$ and $\operatorname{im}_D(.Q) = D^{1\times 2}V_2'$, where

$$U_2' = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad V_2' = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}.$$

Hence, if we define by $U' = (U_1^{\mathrm{T}} \quad U_2'^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_4(D)$ and $V' = (V_1^{\mathrm{T}} \quad V_2'^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_4(D)$, where the matrices U_1 and V_1 are defined in Example 3.4, we then obtain

$$\overline{\overline{R}} = V' R U'^{-1} = \begin{pmatrix} i\partial_3 - \partial_4 & -i\partial_1 - \partial_2 & 0 & 0 \\ i\partial_1 - \partial_2 & i\partial_3 + \partial_4 & 0 & 0 \\ 0 & 0 & i\partial_3 + \partial_4 & i\partial_1 + \partial_2 \\ 0 & 0 & i\partial_1 - \partial_2 & -i\partial_3 + \partial_4 \end{pmatrix}.$$

Example 4.5. Let us consider again system (7) defined in Example 2.2. Using Algorithm 4.1, we obtain that the matrices

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

define an idempotent $f \in \mathrm{end}_D(M)$. Moreover, we have $P^2 = P$ and $Q^2 = Q$. As P and Q are two matrices with rational coefficients, we obtain that $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free D-modules. Using linear algebra techniques, we then get

$$\begin{cases} \ker_D(.P) = D U_1, & U_1 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \\ \mathrm{im}_D(.P) = D^{1\times 2} U_2, & U_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{cases} \quad \begin{cases} \ker_D(.Q) = D V_1, & V_1 = \begin{pmatrix} 1 & -1 \end{pmatrix}, \\ \mathrm{im}_D(.Q) = D V_2, & V_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}. \end{cases}$$

If we define the matrices $U = (U_1^{\mathrm{T}} \quad U_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_3(D)$ and $V = (V_1^{\mathrm{T}} \quad V_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_2(D)$, by Theorem 4.2, we obtain that the matrix R is equivalent to the following block-diagonal matrix:

$$\overline{R} = V R U^{-1} = \begin{pmatrix} \partial_2^2 - 1 & 0 & 0 \\ 0 & \partial_2^2 + 1 & -4\partial_1\partial_2 \end{pmatrix}.$$

Hence, we get the following isomorphism:

$$M \cong D/(D(\partial_2^2 - 1)) \oplus D^{1\times 2}/(D(\partial_2^2 + 1 \quad -4\partial_1\partial_2)).$$

In particular, we have $t(M) \cong D/(D(\partial_2^2 - 1))$ and $M/t(M) \cong D^{1\times 2}/(D(\partial_2^2 + 1 \quad -4\partial_1\partial_2))$ as the second direct summand of M is torsion-free because the greatest common divisor of $\partial_2^2 + 1$ and $-4\partial_1\partial_2$ is 1 and D is a greatest common divisor domain (see, e.g., [16,56,57] for algorithms testing torsion-freeness). With the notation $(z_1, z_2, v)^{\mathrm{T}} = U(y_1, y_2, u)^{\mathrm{T}}$ and within a system theoretic language [16,52,57,73,76], we obtain that the first scalar diagonal block corresponds to the *autonomous subsystem*

$$\begin{cases} z_1(t) = y_1(t) - y_2(t), \\ z_1(t - 2h) - z_1(t) = 0, \end{cases}$$

i.e., z_1 is a $2h$-periodic function, and the second one corresponds to the *parametrizable subsystem*

$$\begin{cases} z_2(t) = y_1(t) + y_2(t), \\ v(t) = u(t), \\ z_2(t) + z_2(t - 2h) - 4\dot{v}(t - h) = 0, \end{cases}$$

of the system $R(y_1, y_2, u)^{\mathrm{T}} = 0$. We note that the parametrizable subsystem is not *flat* [16,24,47] as the corresponding D-module $D^{1\times 2}/(D(\partial_2^2 + 1 \quad -4\partial_1\partial_2))$ is torsion-free but not free (see [16,24,56] for more details). The previous decomposition can be seen as a generalization of the Kalman decomposition of state-space control systems [54] for multidimensional systems.

Finally, if $\mathscr{F} = C^\infty(\mathbb{R})$ and ψ is any smooth $2h$-periodic function, then

$$\forall \xi \in \mathscr{F}, \begin{cases} z_1(t) = \psi(t), \\ z_2(t) = 4\partial_1\partial_2\xi(t) = 4\dot{\xi}(t - h), \\ v(t) = (\partial_2^2 + 1)\xi(t) = \xi(t - 2h) + \xi(t), \end{cases}$$

is a solution of the system $\overline{R}z = 0$. Hence, we obtain that

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} z_1(t) \\ z_2(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\psi(t) + 2\dot{\xi}(t-h) \\ -\frac{1}{2}\psi(t) + 2\dot{\xi}(t-h) \\ \xi(t-2h) + \xi(t) \end{pmatrix}$$

is a solution of (7) for any smooth function ξ and any smooth $2h$-periodic function ψ [23].

We have the following important corollary of Theorem 4.2.

Corollary 4.3. *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}R)$ and $f : M \to M$ be an idempotent defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying* (15) *and*:

$$P^2 = P, \quad Q^2 = Q.$$

Assume further that one of the following conditions holds:

1. *$D = A[\partial; \sigma, \delta]$ is a skew polynomial ring over a division ring A (e.g., A is a field) and σ is injective, as, e.g., the ring $D = k(t)[\partial; \mathrm{id}_{k(t)}, \frac{\mathrm{d}}{\mathrm{d}t}]$ of differential operators with rational coefficients or the ring $D = k(n)[\partial; \sigma, 0]$ of shift operators with rational coefficients*,
2. *$D = A[\partial_1; \sigma_1, \delta_1] \dots [\partial_n; \sigma_n, \delta_n]$ is a commutative Ore algebra where A is a field k or a principal ideal domain as, e.g., the ring of differential operators with coefficients in $\mathbb{Q}$ or $\mathbb{Z}$,*
3. *$D = A[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}] \cdots [\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}]$ is a Weyl algebra, where $A = k[x_1, \dots, x_n]$ or $k(x_1, \dots, x_n)$, k is a field of characteristic 0, and moreover*:

$$\begin{cases} \mathrm{rank}_D(\ker_D(.P)) \geqslant 2, \\ \mathrm{rank}_D(\mathrm{im}_D(.P)) \geqslant 2, \end{cases} \qquad \begin{cases} \mathrm{rank}_D(\ker_D(.Q)) \geqslant 2, \\ \mathrm{rank}_D(\mathrm{im}_D(.Q)) \geqslant 2. \end{cases}$$

Then, there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that

$$\overline{R} = V R U^{-1} = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & \overline{R}_2 \end{pmatrix} \in D^{q\times p},$$

where $\overline{R}_1 \in D^{l\times m}$, $\overline{R}_2 \in D^{(p-l)\times(p-m)}$, $m = \mathrm{rank}_D(\ker_D(.P))$ and $l = \mathrm{rank}_D(\ker_D(.Q))$.

Proof. 1. By Lemma 4.5, we know that $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.P)$ and $\mathrm{im}_D(.Q)$ are projective D-modules. By (ii) of Theorem 1.2.9 of [46], D is a left principal ideal domain. Therefore, $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.P)$ and $\mathrm{im}_D(.Q)$ are free left D-modules of rank respectively m, l, $p-m$ and $q-l$ (see [37,46] for more details). Then, the result follows from Theorem 4.2.

2. By Lemma 4.5, we obtain that $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.P)$ and $\mathrm{im}_D(.Q)$ are projective D-modules. As D is a commutative polynomial ring over a field k or a principal ideal domain A, by the famous Quillen-Suslin theorem (see Theorem 4.59 of [66]), we know that they are free D-modules of rank respectively m, l, $p-m$ and $q-l$. Then, the result directly follows from Theorem 4.2.

3. By Lemma 4.5, we obtain that $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.P)$ and $\mathrm{im}_D(.Q)$ are projective left D-modules. A result of J.T. Stafford asserts that projective modules of rank at least 2 over the Weyl algebras $A_n(k)$ and $B_n(k)$, where k is a field of characteristic 0, are free (see Theorem 3.6 of [69]). Then, the result directly follows from Theorem 4.2. □

In order to constructively obtain the unimodular matrices U and V defined in Corollary 4.3, we need to compute bases of the free left D-modules $\ker_D(.P)$ and $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$.

In the first case of Corollary 4.3, we can use Smith or Jacobson forms in order to compute bases of these modules over $D = A[\partial; \sigma, \delta]$ (see [46,54]). In the second case of Corollary 4.3, we can use constructive versions of the famous Quillen-Suslin theorem of Serre's conjecture [43,66]. For more details, see Remarks 3.1 and 4.3. See also [24] for an implementation of the Quillen-Suslin theorem in the package QuillenSuslin. In the last case of Corollary 4.3, we can use the constructive algorithm recently obtained in [29,64] and the implementation of the algorithm developed in [64] in the package Stafford of OreModules [17]. Hence, we get constructive ways to obtain the unimodular matrices U and V defined in Theorem 4.2.

Remark 4.4. Theorem 4.2 supposes that we already know an idempotent f of $\mathrm{end}_D(M)$ defined by idempotents P of $D^{p\times p}$ and Q of $D^{q\times q}$. Algorithm 4.1 gives a way to get an idempotent of $\mathrm{end}_D(M)$ defined by means of a matrix P with a fixed total order in the ∂_i's and a fixed degree in the x_j's for the numerators and denominators. We then need to solve the algebraic Riccati equation (59). Hence, the existence of bounds for total order and degree of P as well as the existence of a solution of (59) need to be studied in great detail in the future for different classes of functional systems appearing in applied mathematics (e.g., control theory, mathematical physics).

All the previous algorithms, implemented in the package Morphisms [21], were recently used to decompose many classical linear functional systems coming from mathematical physics and control theory. Let us illustrate Theorem 4.2 and Corollary 4.3 on different examples.

Example 4.6. We consider again Example 4.1. Let $D = A_1(\mathbb{Q})$ be the Weyl algebra, $R \in D^{4\times 4}$ the matrix defined by (44) and the left D-module $M = D^{1\times 4}/(D^{1\times 4}R)$. Using Algorithm 4.1, we obtain that the matrices P and Q defined by (45) define an idempotent $f \in \mathrm{end}_D(M)$, which proves that M is decomposable. Moreover, we easily check that $P^2 = P$, i.e., P is an idempotent of $D^{3\times 3}$. Using the fact that the entries of P belong to the field $\mathbb{Q}$, we can easily compute bases of $\ker_k(.P)$ and $\mathrm{im}_k(.P) = \ker_k(.(I_4 - P))$ and we get the following unimodular matrices:

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Moreover, if we denote by

$$V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix},$$

we can then check that we have

$$\begin{cases} \ker_D(.Q) = D^{1\times 2} V_1, \\ \mathrm{im}_D(.Q) = \ker_D(.(I_4 - Q)) = D^{1\times 2} V_2, \end{cases}$$

and $V = (V_1^{\mathrm{T}} \quad V_2^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{GL}_4(D)$. The matrix R is then equivalent to the block-diagonal matrix:

$$\overline{R} = V R U^{-1} = \begin{pmatrix} -\partial & 1 & 0 & 0 \\ t(\partial - 1) & -(\partial + t) & 0 & 0 \\ 0 & 0 & 0 & -\partial \\ 0 & 0 & \partial & (t+1)\partial - t \end{pmatrix}.$$

Moreover, if we denote by

$$E = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -t & -1 & 0 & 0 \\ 0 & 0 & t+1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \mathrm{GL}_4(D),$$

and $W = EV$, we can easily check that we have

$$\overline{\overline{R}} = W R U^{-1} = \begin{pmatrix} \partial & -1 & 0 & 0 \\ t & \partial & 0 & 0 \\ 0 & 0 & \partial & -t \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

If we denote by $W = (W_1^{\mathrm{T}} \quad W_2^{\mathrm{T}})^{\mathrm{T}}$, where $W_1 \in D^{2\times 4}$ and $W_2 \in D^{2\times 4}$, then we have

$$\ker_{k[t]}(.Q) = k[t]^{1\times 2} W_1, \quad \mathrm{im}_{k[t]}(.Q) = k[t]^{1\times 2} W_2,$$

i.e., W_1 (resp., W_2) defines a basis of $\ker_D(.Q)$ (resp., $\mathrm{im}_D(.Q)$) with coefficients in $k[t]$, whereas V_1 (resp., V_2) defines a basis with coefficients in k.

Finally, the diagonal blocks of the matrix $\overline{\overline{R}}$ are equivalent to the two systems that we had to solve in Example 4.1 in order to integrate the solutions of $R\eta = 0$. In particular, the solution of the first diagonal block is $z_1(t) = C_3\mathrm{Ai}(t) + C_4\mathrm{Bi}(t)$ and $z_2(t) = C_3\partial\mathrm{Ai}(t) + C_4\partial\mathrm{Bi}(t)$, whereas the solution of the second diagonal block is $z_3(t) = \frac{1}{2}C_1t^2 + C_2$ and $z_4(t) = C_1$, where C_1, C_2, C_3 and C_4 are four arbitrary constants. Hence, using the fact that $\eta = U^{-1}(z_1, z_2, z_3, z_4)^{\mathrm{T}}$, we find again that the general solution of $R\eta = 0$ is given by (58).

Example 4.7. If we consider $D = A_1(\mathbb{Q})$ and the idempotent $\overline{P}_3 \in D^{2\times 2}$ defined in Example 4.3, then we have $\mathrm{rank}_D(\ker_D(.\overline{P}_3)) = 1$ and $\mathrm{rank}_D(\mathrm{im}_D(.\overline{P}_3)) = 1$. Hence, we cannot use 1 or 3 of Corollary 4.3 in order to conclude that $R = (\partial^2 \quad -t\partial - 1)$ is equivalent to $\overline{R} = (\alpha \quad 0), \alpha \in D$, by means of unimodular matrices over D. Indeed, we can prove that $\ker_D(.\overline{P}_3)) = D(\partial \quad -t)$, which implies that $\ker_D(.\overline{P}_3))$ is a free left D-module of rank 1. However, we have

$$\mathrm{im}_D(.\overline{P}_3) \cong D^{1\times 2}/(D(\partial \quad -t))$$

and it was proved in [64] that the last left D-module was not free. A similar comment holds for $\overline{P}_4$ as we have $\ker_D(.\overline{P}_4) \cong D^{1\times 2}/(D(\partial \quad -t))$. Of course, if we consider the Weyl algebra $B_1(\mathbb{Q})$ instead of $A_1(\mathbb{Q})$, namely, $B_1(\mathbb{Q}) = \mathbb{Q}(t)[\partial; \mathrm{id}_{\mathbb{Q}(t)}, \frac{\mathrm{d}}{\mathrm{d}t}]$, using a computation of a Jacobson form, we can easily prove that R is equivalent to $\overline{R} = (\partial \quad 0)$ (see 1 of Corollary 4.3). However, we point out that singularities then appear in the matrices U and U^{-1} defined in Theorem 4.2 as, for instance, we have $RU^{-1} = (\partial \quad 0)$, where U is the matrix defined by

$$U = \begin{pmatrix} \partial & -t \\ -t\partial + 1 & t^2 \end{pmatrix}.$$

U does belong to $\mathrm{GL}_3(B_1(\mathbb{Q}))$ but not to $\mathrm{GL}_3(A_1(\mathbb{Q}))$ as the matrix U^{-1} is singular at $t = 0$:

$$U^{-1} = \begin{pmatrix} t & 1 \\ \partial & \frac{1}{t}\partial \end{pmatrix}.$$

Finally, even if the hypotheses of Theorem 4.2 are not fulfilled as $\mathrm{im}_D(.\overline{P}_3)$ is not a free left D-module, we can use Theorem 4.1 to compute the solutions of $R\eta = 0$. Indeed, we can check that the matrices defined by $\overline{S}_3 = (\partial - t)$, $\overline{X}_3 = (t + a \quad 0)^{\mathrm{T}}$ and $\overline{L}_3 = \partial$ satisfy $\overline{P}_3 = I_2 - \overline{X}_3\overline{S}_3$,

$R = \overline{L}_3 \overline{S}_3$ and $\ker_D(.\overline{S}_3) = 0$. Hence, we need to solve $\partial \tau = 0$ which gives $\tau = c$, where c is an arbitrary constant, as well as $\partial \zeta_1 - t \zeta_2 = 0$. If $\mathscr{F}$ is any left D-module (e.g., $\mathscr{F} = C^\infty(\mathbb{R})$), then, using the results developed in [16,56], all the $\mathscr{F}$-solutions of $\partial \zeta_1 - t\zeta_2 = 0$ are of the form:

$$\forall \xi_1, \xi_2 \in \mathscr{F}, \quad \begin{cases} \zeta_1(t) = t^2 \xi_1(t) + t\dot{\xi}_2(t) - \xi_2(t), \\ \zeta_2(t) = t\dot{\xi}(t) + 2\xi_1(t) + \ddot{\xi}_2(t). \end{cases}$$

By Theorem 4.1, all the $\mathscr{F}$-solutions of $R\eta = 0$ are then parametrized by

$$\forall \xi_1, \xi_2 \in \mathscr{F}, \forall a, c \in \mathbb{Q}, \quad \begin{cases} \eta_1(t) = (t+a)c + t^2 \xi_1(t) + t\dot{\xi}_2(t) - \xi_2(t), \\ \eta_2(t) = t\dot{\xi}(t) + 2\xi_1(t) + \ddot{\xi}_2(t). \end{cases}$$

Example 4.8. Let us consider the differential time-delay model of a flexible rod with a torque developed in [47]:

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases} \tag{69}$$

Let us define the Ore algebra $D = \mathbb{Q}[\partial_1; \mathrm{id}_{\mathbb{Q}}, \frac{\mathrm{d}}{\mathrm{d}t}][\partial_2; \sigma_2, 0]$ of differential time-delay operators with rational constant coefficients defined in 4 of Example 2.1 and the matrix of the system (69):

$$R = \begin{pmatrix} \partial_1 & -\partial_1\partial_2 & -1 \\ 2\partial_1\partial_2 & -\partial_1\partial_2^2 - \partial_1 & 0 \end{pmatrix} \in D^{2\times 3}.$$

Let us introduce $M = D^{1\times 3}/(D^{1\times 2}R)$. Using Algorithm 4.1, we obtain that the matrices

$$P = \begin{pmatrix} 1+\partial_2^2 & -\frac{1}{2}\partial_2^2(1+\partial_2) & 0 \\ 2\partial_2 & -\partial_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -\frac{1}{2}\partial_2 \\ 0 & 0 \end{pmatrix},$$

define an idempotent $f \in \mathrm{end}_D(M)$. Moreover, we can check that $P^2 = P$ and $Q^2 = Q$. Then, using 2 of Corollary 4.3, we obtain that R is equivalent to a block-diagonal matrix. Let us compute it. Using the implementation of the Quillen-Suslin theorem developed in [24] or the heuristics given in [16], we obtain the following unimodular matrices:

$$U = \begin{pmatrix} -2\partial_2 & \partial_2^2 + 1 & 0 \\ 2\partial_1(1-\partial_2^2) & \partial_1\partial_2(\partial_2^2 - 1) & -2 \\ -1 & \frac{1}{2}\partial_2 & 0 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\partial_2 \end{pmatrix} \in \mathrm{GL}_2(D).$$

We obtain that R is equivalent to the following block-diagonal matrix:

$$\overline{R} = VRU^{-1} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, we get the following isomorphisms:

$$M \cong D^{1\times 3}/(D^{1\times 2}\overline{R}) = D/(D\partial_1) \oplus D^{1\times 2}/(D(1 \quad 0)) \cong D/(D\partial_1) \oplus D,$$

which show that $t(M) \cong D/(D\partial_1)$, $M/t(M) \cong D$ and M is *extended* from the ring $\mathbb{Q}[\partial_1; \mathrm{id}_{\mathbb{Q}}, \frac{\mathrm{d}}{\mathrm{d}t}]$ (see [66] for more details). This last result shows that, as in Example 4.5 for the tank model, the first scalar diagonal block corresponds to the autonomous subsystem, whereas the second diagonal block defines the flat subsystem (see [47]).

Finally, all smooth solutions of $\overline{R}z = 0$ are defined by $z = (c, 0, z_3)^{\mathrm{T}}$, where $c \in \mathbb{R}$ and z_3 is an arbitrary smooth function. Hence, all smooth solutions of (69) are then parametrized by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} c \\ 0 \\ z_3(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c - z_3(t-2) - z_3(t) \\ c - 2z_3(t-1) \\ \dot{z}_3(t-2) - \dot{z}_3(t) \end{pmatrix},$$

where c is an arbitrary real constant and z_3 an arbitrary smooth function.

We refer the reader to [21,24] for more examples of decomposable modules coming from mathematical physics and control theory such as the Cauchy-Riemann equations, the Beltrami equations, acoustic waves, transmission lines, 2-D rotational isentropic flow, wind tunnel/(stirred) tank/string/network models… Finally, we refer to [21] for a description of the package Morphisms in which the different algorithms described in this paper were implemented.

5. Conclusion

Within a constructive homological algebra approach, we have obtained new and general results on the factorization and decomposition problems of linear functional systems over a certain class of Ore algebras. We point out that no particular assumption on the linear functional systems was required. Hence, the different results of the paper can be applied to determined, over-determined and under-determined linear functional systems. In particular, we have shown how some classical results of the literature of the factorization and decomposition problems such as the ones using the concept of the eigenring [4,11,18,31,61,68,74,75] could be seen as particular cases of Theorems 3.1, 3.2, 4.1 and 4.2. However, we point out that some open questions need to be studied in the future such as the computation of bounds for α, β and γ in Algorithms 2.2 and 4.1 for important classes of linear functional systems, the structure of the set of idempotents of an endomorphism ring, the existence of solutions of the Riccati equation.

We have shown how these results could be applied in mathematical physics (e.g., computation of quadratic first integrals of motion and quadratic conservation laws, testing the equivalence of linear systems of PDEs appearing in mathematical physics, factoring, decomposing and computing Galois transformations of the classical linear systems of PDEs appearing in elasticity theory, electromagnetism, hydrodynamics) and in control theory (factorization, decomposition and computation of Galois transformations of classical linear functional systems appearing in the literature of control theory, parametrizations, decoupling the autonomous and the controllable subsystems). We refer the reader to [20,21] for more examples and applications (e.g., study of the KdV equation by means of the eigenring and Lax pairs).

Moreover, all the algorithms presented in the paper have been implemented in the package Morphisms [21] of OreModules (see [17]). This package is available with a library of examples, including the ones of the paper, which illustrates the main results obtained in this paper and the main functions of the package Morphisms. In the case of a commutative polynomial ring, tools of homological algebra such as the computation of morphisms have also recently been implemented in the package homalg [3].

This work opens interesting questions such as proving whether or not the differential modules associated with the main linear systems of PDEs appearing in mathematical physics (e.g., the Maxwell equations, the Navier equations in elasticity theory, the linearized Einstein equations) are simple or indecomposable. This problem will be studied in the future. Applications of our results to the important issue of characterizing the minimal number of generators of a finitely presented module will soon be developed. For some results going in this direction, see [21]. Computing a minimal set of generators for the classical functional systems appearing in mathematical physics seems to us an important issue of constructive algebraic analysis.

The results obtained in this paper can also be used for studying classical problems in control theory such as the decoupling problem or equivalence problems (e.g., when is a multidimensional system equivalent to its Smith form?). We also need to study how the algebraic properties of the underlying module (e.g., torsion-free, reflexive, projective) can be taken into account to characterize the endormorphism ring and the set of its idempotents (e.g., projective modules). Finally, applications of these results to the behavioural approach to multidimensional systems need to be developed following the lines developed in [28,53].

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