Math. Control Signals Systems (2006) 18: 147–186
DOI 10.1007/s00498-005-0159-2

ORIGINAL ARTICLE

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A lattice approach to analysis and synthesis problems

Received: 17 November 2003 / Revised: 16 June 2005 / Published online: December 29, 2005

Abstract Within a lattice approach, the purpose of this paper is to give general necessary and sufficient conditions for internal stabilizability and for the existence of (weakly) left-/right-/doubly coprime factorizations of multi input multi output linear systems. These results extend the ones recently obtained in [24] for single input single output systems. In particular, combining these results with the one obtained in [3,13], we prove that every internally stabilizable multidimensional system admits doubly coprime factorizations, solving Lin's conjecture [17,18].

Keywords Fractional representation approach · Lattice approach · Internal stabilizability · (weakly) left-/right-/doubly coprime factorization · Lin's conjecture · Multidimensional systems · Module theory

1 Introduction

In their pioneering work [5,6,39,40], Desoer and Zames developed the modern concepts of internal stabilizability and parametrization of all stabilizing controllers. They have since played fundamental roles in the study of stabilization problems (e.g., internal/strong/simultaneous/robust/optimal stabilization) and in the successful development of the H_∞-control in the nineties. For more details, see [4,9–11, 19,38] and the references therein.

However, Desoer and Zames did not use the same approaches in order to characterize internal stabilizability and obtain their parametrizations of all stabilizing controllers. Combining the idea of stable factorizations of a plant developed by

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Vidyasagar [38] in the seventies with the concept of coprimeness, Desoer and his co-authors obtained in [6] necessary and sufficient conditions for internal stabilizability in terms of some Bézout equations. Solving these Bézout identities, they obtained the so-called *Youla–Kučera parametrization* of all stabilizing controllers. This approach is nowadays called the *fractional representation approach to analysis and synthesis* and it has been largely developed in the literature for important classes of systems [4,19,38]. In contrast, Zames [39] worked within the *classical approach* directly using the transfer matrices P and C of the plant and the controller [40]. He characterized internal stabilizability in terms of some interpolation problems, and solving them, Zames and Francis [40] were able to obtain a parametrization of all stabilizing controllers for finite-dimensional systems that they called the *Q-parametrization*. In particular, they showed that, after a simplification by a stable factor, the Q-parametrization became the Youla–Kučera one [40].

The fractional representation approach is nowadays a cornerstone in the study of stabilization problems whereas the classical approach and, in particular, Zames' ideas on internal stabilizability and parametrization of all stabilizing controllers, has been mainly forgotten. It can be partially explained by the fact that for finite-dimensional linear systems, every transfer matrix admits doubly coprime factorizations over RH_∞ and different algorithms for their computations are well-known [6,38]. However, this result is generally not true for general linear systems (e.g., infinite-dimensional or multidimensional systems) [20,23,24,33,36,37]. Moreover, for important classes of systems such as infinite-dimensional systems (e.g., differential time-delay systems, systems of partial differential equations, convolutional systems) or multidimensional systems, the question of the equivalence between internal stabilizability and the existence of doubly coprime factorizations is still open [4,18–20,23,24,38]. Hence, we are sometimes led to artificially assume the existence of doubly coprime factorizations for a transfer matrix in order to mimic the results classically obtained for finite-dimensional systems.

Some people have felt that it was important to revisit stabilization problems by using only the concept of internal stabilizability and without assuming that the only interesting plants were the ones admitting doubly coprime factorizations. Such a program was pioneered by the work of Shankar [33] and Sule [36,37] and it has been recently extended in [20,21,23,25,26,28–30]. Within the fractional representation framework, a natural module-theoretic approach to analysis and synthesis problems was developed in [20,23,36,37]. General necessary and sufficient conditions for internal/strong stabilizability and for the existence of (weak) left-/right-/doubly coprime factorizations were then obtained [20,23,33,36,37]. In particular, it was shown that the existence of a left-/right-coprime factorization is a sufficient but generally not a necessary condition for internal stabilizability. Hence, the question of the possibility to parametrize all stabilizing controllers of an internal stabilizable plant which does not necessarily admit doubly coprime factorizations naturally holds [21,24,27,28,33,36,37]. Such a problem has only recently been completely solved in [24] for single input single output (SISO) plants (see [21,33,36,37] for partial answers). However, in order to do that, we had to develop a *fractional ideal approach* to stabilization problems [24,26,30]. The theory of fractional ideals is a framework which allows us to study ideals generated by elements belonging to the quotient field of an integral domain (the classical ideal theory only deals with ideals generated by elements of a ring) [2,32]. Within this mathematical theory, we can simultaneously study the classical approach based on transfer functions and the fractional representation approach based on stable fractional representations of transfer functions. Recently, after the reading of [40], we have realized that the fractional ideal approach gave appropriate tools for investigating and developing Zames' ideas on internal stabilizability and Q-parametrization of all stabilizing controllers for general plants.

A natural challenge was then to find a way to extend the fractional ideal approach to analysis and synthesis problems developed in [24,26,30] for multi input multi output (MIMO) plants. We have recently understood that it could be done by means of the algebraic concept of *lattices of a vector space* [2]. Hence, the purpose of this paper is to develop a lattice approach to analysis and synthesis problems for MIMO plants which generalizes the fractional ideal approach developed in [24] for SISO plants. In this paper, we first explain the concept of lattices of a vector space and show how it generalizes the concept of fractional ideals. Moreover, we also explain elementary operations on the lattices which allow us to develop an algebraic calculus on lattices. Within this framework, we then obtain new necessary and sufficient conditions for internal/strong stabilizability and for the existence of (weakly) left-/right-/doubly coprime factorizations of MIMO plants. We show that the same tools allow us to characterize structural properties of a plant — either defined by a transfer matrix P or by stable fractional representations $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ of P — by means of some lattices intrinsically associated with the matrices $(I \;\; -P)$ and $(P^T \;\; I^T)^T$ or with $(D \;\; -N)$ and $(\tilde{N}^T \;\; \tilde{D}^T)^T$. As in the behavioural and the module-theoretic approaches [22,23,26,36,37], the lattice approach does not separate the inputs and the outputs of plants. We can even prove that a module-theoretic duality exists between the lattice approach developed in this paper and the operator-theoretic one obtained in [4,10,38] using domains and graphs of unbounded linear operators. For the SISO case, see [26] for more details.

We then illustrate this new approach by showing how to combine the previous results with one of [3,13] in order to prove that every internally stabilizable multidimensional system (in the sense of the structural stabilizability [17,18]) admits doubly coprime factorizations. This result positively answers *Lin's conjecture* which was still open in the literature of multidimensional systems (see e.g., [17,18] and the references therein). A part of the results developed here already appeared in the Congress papers [28–30].

Finally, within the lattice approach developed here, a general parametrization of all stabilizing controllers of a MIMO plant, which does not necessarily admit doubly coprime factorizations, will be obtained in [27]. This parametrization generalizes for MIMO plants the parametrization previously obtained in [24] for SISO plants and extends Zames and Francis' ideas on the Q-parametrization of all stabilizing controllers for general linear systems.

Notation In what follows, we shall denote by A an integral domain A (namely, A is a ring with an identity which satisfies $a\,b = b\,a$ for any $a,\, b \in A$ and $a\,b = 0,\; a \neq 0 \Rightarrow b = 0$) [2,32]. Elements of A^m (respectively, $A^{1\times m}$) will be considered as column (respectively, row) vectors of length m with entries in A. We shall denote by $A^{m\times m}$ the ring of $m \times m$ matrices with entries in A and I_m the identity matrix of $A^{m\times m}$. The standard basis of the free A-module A^m is the basis $\{e_i\}_{1\leq i\leq m}$, where e_i is the column vector defined by 1 in the ith component and 0 elsewhere. Finally, if V is a finite-dimensional K-vector space, then we denote its dimension by $\dim_K(V)$.

2 Fractional representation approach

We briefly recall *the fractional representation approach to analysis and synthesis problems*. We refer to [6,19,38] for more details and references.

Let us consider an integral domain A *of (proper) stable SISO plants* [6,19,38]. For instance, we have the following examples of such rings.

Example 1 Let $A = RH_\infty$ be the integral domain of all proper stable real rational functions [38], namely:

$$RH_\infty = \{n/d \mid 0 \neq d,\ n \in \mathbb{R}[s],\ \deg n \leq \deg d,\ d(s_\star) = 0 \Rightarrow \operatorname{Re} s_\star < 0\}.$$

A transfer function p belongs to A iff p is the transfer function of an exponentially stable time-invariant finite-dimensional SISO linear system.

Let $A = H_\infty(\mathbb{C}_+)$ be the integral domain of holomorphic functions in the right half-plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ which are bounded with respect to the norm $\| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)|$.

A transfer function p belongs to A iff we have

$$\| p \|_\infty = \sup_{0 \neq u \in H_2(\mathbb{C}_+)} \frac{\| p\, u \|_2}{\| u \|_2} < +\infty,$$

where $H_2(\mathbb{C}_+)$ denotes the Hilbert space of the holomorphic functions in $\mathbb{C}_+$ which are bounded with respect to the norm:

$$\| f \|_2 = \left(\sup_{\operatorname{Re} x > 0} \int_{-\infty}^{+\infty} |f(x + i\, y)|^2 \, \mathrm{d}y \right)^{1/2}.$$

Therefore, p belongs to A iff p is the transfer function of an $L_2(\mathbb{R}_+)$-stable time-invariant infinite-dimensional SISO linear system [4,10].

Let A be the Wiener algebra [4,6,38] defined by

$$\begin{aligned} \mathcal{A} = \{h(t) = f(t) + \textstyle\sum_{i=0}^{\infty} a_i\, \delta(t - t_i) \mid f \in L_1(\mathbb{R}_+),\ (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+), \\ 0 = t_0 \leq t_1 \leq t_2 \leq \cdots\}, \end{aligned}$$

where h is bounded with respect to the norm:

$$\| h \|_{\mathcal{A}} = \| f \|_{L_1(\mathbb{R}_+)} + \| (a_i)_{i \geq 0} \|_{l_1(\mathbb{Z}_+)} = \int_0^{+\infty} |f(t)| \, \mathrm{d}t + \sum_{i=0}^{+\infty} |a_i|.$$

Then, h belongs to A iff h is the impulse response of an $L_\infty(\mathbb{R}_+)$-stable time-invariant infinite-dimensional SISO linear system (BIBO stability) [4,6]. We also define the integral domain $\hat{\mathcal{A}} = \{\mathcal{L}(f) \mid f \in \mathcal{A}\}$, where $\mathcal{L}(f)$ denotes the Laplace transform.

Let $A = W_+$ be the integral domain of holomorphic functions on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ of $\mathbb{C}$ whose Taylor series converge absolutely, namely,

$$W_+ = \left\{ f(z) = \sum_{n=0}^{+\infty} a_n\, z^n \in H(\mathbb{D}) \mid \sum_{n=0}^{+\infty} |a_n| < +\infty \right\},$$

where $H(\mathbb{D})$ denotes the integral domain of holomorphic functions in $\mathbb{D}$. Then, $p \in W_+$ iff p is the unit-pulse response of a BIBO-stable causal digital filter [38].

Let $A = \mathbb{R}(z_1, \ldots, z_m)_S$ be the integral domain of structural stable multidimensional systems, namely,

$$\mathbb{R}(z_1, \ldots, z_m)_S = \{n/d \mid 0 \neq d,\ n \in \mathbb{R}[z_1, \ldots, z_m],\ \mathrm{d}(z) = 0 \Rightarrow z \notin \overline{\mathbb{D}}^m\},$$

where $\overline{\mathbb{D}}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid |z_i| \leq 1,\ i = 1, \ldots, m\}$ denotes the closed unit polydisc of $\mathbb{C}^m$ [17,18].

If A is an integral domain of (proper) stable SISO systems, then its *quotient field* [2,32], namely, the commutative field defined by

$$Q(A) = \{n/d \mid 0 \neq d,\ n \in A\},$$

corresponds to the universal class of A-stable and A-unstable SISO plants. For instance, the transfer function $p = e^{-s}/(s-1) \notin H_\infty(\mathbb{C}_+)$ because p has an unstable pole in $\mathbb{C}_+$. However, we have $p = n/d$, where

$$n = e^{-s}/(s+1) \in H_\infty(\mathbb{C}_+), \quad d = (s-1)/(s+1) \in H_\infty(\mathbb{C}_+), \tag{1}$$

and thus, $p \in Q(H_\infty(\mathbb{C}_+))$. Similarly, $p \notin \hat{\mathcal{A}}$ but $p = n/d \in Q(\hat{\mathcal{A}})$, where d and n are defined by (1).

More generally, we can consider the class of MIMO plants defined by transfer matrices with entries in $K = Q(A)$. Hence, if $P \in K^{q\times r}$ is a transfer matrix with entries in K, then we can always write P as:

$$P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \quad \text{where} \quad \begin{cases} R = (D \;\; -N) \in A^{q\times(q+r)}, \\ \tilde{R} = (\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{(q+r)\times r}. \end{cases}$$

Indeed, we can take $D = d\,I_q$ and $\tilde{D} = d\,I_r$ where $0 \neq d \in A$ denotes the product of the denominators of all the entries of P, $N = d\,P \in A^{q\times r}$ and $\tilde{N} = P\,d \in A^{q\times r}$. Then, $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$, where $(D \;\; -N) \in A^{q\times(q+r)}$ and $(\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{(q+r)\times r}$, is called *fractional representations* of $P \in K^{q\times r}$.

Example 2 If $A = H_\infty(\mathbb{C}_+)$ and $K = Q(A)$, then

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2 \tag{2}$$

because we have $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$, where:

$$R = \begin{pmatrix} \frac{(s-1)^2}{(s+1)^2} & 0 & -\frac{(s-1)\,e^{-s}}{(s+1)^2} \\ 0 & \frac{(s-1)^2}{(s+1)^2} & -\frac{e^{-s}}{(s+1)^2} \end{pmatrix} \in A^{2\times 3}, \qquad \tilde{R} = \begin{pmatrix} \frac{(s-1)\,e^{-s}}{(s+1)^2} \\ \frac{e^{-s}}{(s+1)^2} \\ \frac{(s-1)^2}{(s+1)^2} \end{pmatrix} \in A^3.$$

The fractional representation approach was developed in the eighties in order to study in a unique mathematical framework several questions on stabilization problems (e.g., existence of internal controllers, parametrization of all stabilizing controllers, strong/simultaneous/robust stabilization, metrics of robustness, H_∞/H_2 controllers). See [6,19,38] for more details.

We recall a few definitions that will constantly be used in what follows.

Definition 1 Let A be an integral domain of stable SISO plants and its quotient field $K = Q(A)$.

- [23] A transfer matrix $P \in K^{q\times r}$ admits a *weakly left-coprime factorization* if there exists a matrix $(D \;\; -N) \in A^{q\times(q+r)}$ such that we have $\det D \neq 0$, $P = D^{-1}\, N$ and:
$$\forall\, \lambda \in K^{1\times q}, \quad \lambda\,(D \;\; -N) \in A^{1\times(q+r)} \;\Rightarrow\; \lambda \in A^{1\times q}.$$
- [23] A transfer matrix $P \in K^{q\times r}$ admits a *weakly right-coprime factorization* if there exists a matrix $(\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{(q+r)\times r}$ such that we have $\det \tilde{D} \neq 0$, $P = \tilde{N}\, \tilde{D}^{-1}$ and:
$$\forall\, \lambda \in K^{r}, \quad \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} \lambda \in A^{q+r} \;\Rightarrow\; \lambda \in A^{r}.$$
- [23] A transfer matrix $P \in K^{q\times r}$ admits a *weakly doubly coprime factorization* iff P admits a weakly left- and a weakly right-coprime factorization.
- [6,19,38] A plant $P \in K^{q\times r}$ is called *internally stabilizable* (IS) if there exists a controller $C \in K^{r\times q}$ such that the transfer matrix defined by
$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P,\, C) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad H(P,\, C) = \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1},$$
is *A-stable* (see Fig. 1), namely, if all the entries of the following matrix
$$H(P,\, C) = \begin{pmatrix} (I_q - P\, C)^{-1} & (I_q - P\, C)^{-1}\, P \\ C\,(I_q - P\, C)^{-1} & I_r + C\,(I_q - P\, C)^{-1}\, P \end{pmatrix} \tag{3}$$
$$= \begin{pmatrix} I_q + P\,(I_r - C\, P)^{-1}\, C & P\,(I_r - C\, P)^{-1} \\ (I_r - C\, P)^{-1}\, C & (I_r - C\, P)^{-1} \end{pmatrix} \tag{4}$$
belongs to A. Then, C is called a *stabilizing controller* of P.
- [19,38] A plant $P \in K^{q\times r}$ is said to be *strongly stabilizable* if P is internally stabilized by a stable controller C, i.e., $C \in A^{r\times q}$.
- [6,19,38] A transfer matrix $P \in K^{q\times r}$ admits a *left-coprime factorization* if there exist $R = (D \;\; -N) \in A^{q\times(q+r)}$ and $S = (X^T \;\; Y^T)^T \in A^{(q+r)\times q}$ such that we have $\det D \neq 0$, $P = D^{-1}\, N$ and:
$$R\, S = D\, X - N\, Y = I_q.$$
- [6,19,38] A transfer matrix $P \in K^{q\times r}$ admits a *right-coprime factorization* if there exist two matrices $\tilde{R} = (\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{(q+r)\times r}$ and $\tilde{S} = (-\tilde{Y} \;\; \tilde{X}) \in A^{r\times(q+r)}$ such that $\det \tilde{D} \neq 0$, $P = \tilde{N}\, \tilde{D}^{-1}$ and:
$$\tilde{S}\, \tilde{R} = -\tilde{Y}\, \tilde{N} + \tilde{X}\, \tilde{D} = I_r.$$

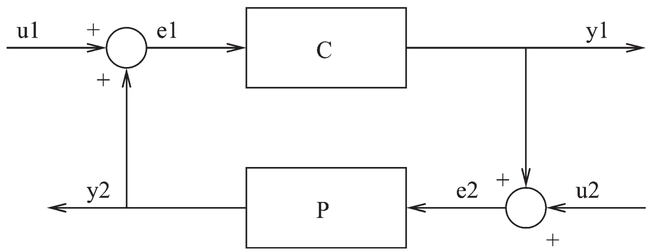

Fig. 1 Closed-loop system

– [6,19,38] A transfer matrix $P \in K^{q \times r}$ admits a *doubly coprime factorization* if P admits a left- and a right-coprime factorizations.

Remark 1 We note that a left-coprime factorization $P = D^{-1} N$ of P, i.e., $R = (D \; - N) \in A^{q \times (q+r)}$, $S = (X^T \quad Y^T) \in A^{(q+r) \times q}$, $R\,S = I_q$, is a weakly left-coprime factorization. Indeed, for all $\lambda \in K^{1 \times q}$ such that $\lambda\, R \in A^{1 \times (q+r)}$, we have $\lambda = \lambda\,(R\,S) = (\lambda\, R)\, S \in A^{1 \times q}$. Similarly, a right-coprime factorization is a weakly right-coprime factorization.

3 Lattices of vector spaces

The purpose of this section is to introduce the concept of a *lattice of a finite-dimensional vector space* [2]. In order to do that, we first recall a few concepts of module theory [2,32].

Definition 2 An A-module M is said to be *free* if M is isomorphic to a direct sum of copies of A.

Equivalently, an A-module M is free iff M has a *basis*, namely, a set $\{e_i\}_{i \in I}$ of elements of M such that every element of M is a unique finite linear combination of the e_is [32]. Then, the *rank* of M is the cardinal of a basis $\{e_i\}_{i \in I}$ of M and it is denoted by $\mathrm{rk}_A(M)$.

Let us now introduce the concept of a lattice of a finite-dimensional vector space [2]. This concept will play a major role in what follows.

Definition 3 [2] Let A be an integral domain, $K = Q(A)$ its quotient field and V a finite-dimensional K-vector space. Then, an A-submodule M of V is called *a lattice of* V if there exist two free A-submodules L_1 and L_2 of V such that $L_1 \subseteq M \subseteq L_2$ and $\mathrm{rk}_A(L_1) = \dim_K(V)$.

To our knowledge, this concept was introduced in control theory by Sontag [35] but it seems that it has not been used since. Let us illustrate the concept of a lattice by means of a few important examples.

Example 3 If $P \in K^{q \times r}$ is a transfer matrix, then the A-module defined by

$$\mathcal{L} = (I_q \;\; - P)\, A^{q+r} = \{\lambda_1 - P\,\lambda_2 \mid \lambda_1 \in A^q,\; \lambda_2 \in A^r\} \tag{5}$$

is a lattice of the finite-dimensional K-vector space K^q. Indeed, we have the following inclusions $L_1 = A^q \subseteq \mathcal{L} = A^q + P\,A^r$ and A^q is a free A-submodule of K^q of rank q. Moreover, if $P = D^{-1}\,N$ is a fractional representation of P, where $(D\ -\ N) \in A^{q\times(q+r)}$, then any element $\lambda \in \mathcal{L}$ has the form $\lambda = \lambda_1 - P\,\lambda_2 = D^{-1}\,(D\,\lambda_1 - N\,\lambda_2)$ for some $\lambda_1 \in A^q$ and $\lambda_2 \in A^r$. Therefore, we have $\mu = D\,\lambda_1 - N\,\lambda_2 \in A^q$, which shows that $\lambda = D^{-1}\,\mu \in D^{-1}\,A^q$. Hence, we have the inclusions $A^q \subseteq \mathcal{L} \subseteq D^{-1}\,A^q$. Finally, $L_2 = D^{-1}\,A^q$ is a free A-submodule of K^q of basis $\{D_1^{-1}, \ldots, D_q^{-1}\}$, where D_i^{-1} is the ith column of D^{-1}.

If $P \in K^{q\times r}$ is a transfer matrix, then we can similarly prove that the A-module defined by

$$\mathcal{M} = A^{1\times(q+r)}\begin{pmatrix} P \\ I_r \end{pmatrix} = \{\lambda_1\,P + \lambda_2 \,|\, \lambda_1 \in A^{1\times q},\ \lambda_2 \in A^{1\times r}\} \tag{6}$$

is a lattice of $K^{1\times r}$.

Example 4 Let $P \in K^{q\times r}$ be a transfer matrix and $P = D^{-1}\,N$ a fractional representation of P (not necessarily a (weakly) left-coprime factorization), where $(D - N) \in A^{q\times(q+r)}$. Then, the A-module defined by

$$\mathcal{P} = (D\ -\ N)\,A^{q+r} = \{D\,\lambda_1 - N\,\lambda_2 \,|\, \lambda_1 \in A^q,\ \lambda_2 \in A^r\} \tag{7}$$

is a lattice of K^q. Indeed, we can easily check that we have the inclusions $D\,A^q \subseteq \mathcal{P} = D\,A^q + N\,A^r \subseteq A^q$. Moreover, $L_1 = D\,A^q$ is a free A-submodule of K^q of basis $\{D_1, \ldots, D_q\}$, where D_i is the ith column of D, and thus, $\mathrm{rk}_A(L_1) = q$. Finally, A^q is trivially a free A-submodule of K^q.

If $P = \tilde{N}\,\tilde{D}^{-1}$ is a fractional representation of a transfer matrix P, where $(\tilde{N}^T\ \ \tilde{D}^T)^T \in A^{(q+r)\times r}$, then we can similarly prove that

$$\mathcal{Q} = A^{1\times(q+r)}\begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} = \{\lambda_1\,\tilde{N} + \lambda_2\,\tilde{D} \,|\, \lambda_1 \in A^{1\times q},\ \lambda_2 \in A^{1\times r}\} \tag{8}$$

is a lattice of $K^{1\times r}$.

We recall that an A-module M is said to be *finitely generated* if there exists a finite set $\{m_i\}_{i\in I}$ of elements of M which generates M, namely, any element $m \in M$ can be written as $m = \sum_{i\in I} a_i\,m_i$ for certain $a_i \in A$ [2,32]. Then, we have the following characterization of a lattice.

Proposition 1 *[2] An A-submodule M of V is a lattice of V iff the K-vector space $K\,M = \{\sum_{i=1}^n k_i\,m_i \,|\, k_i \in K,\ m_i \in M,\ n \in \mathbb{Z}_+\}$ satisfies $K\,M = V$ and M is contained in a finitely generated A-submodule of V.*

Let us again consider Examples 3 and 4 using Proposition 1.

Example 5 Let $P \in K^{q\times r}$ be a transfer matrix and $\mathcal{L} = (I_q\ -\ P)\,A^{q+r}$ the A-submodule of K^q. Then, we have $\mathcal{L} = \sum_{i=1}^q A\,e_i + \sum_{i=1}^r A\,P_i \subseteq K^q$, where $\{e_i\}_{1\le i\le q}$ is the *standard basis* of A^q, namely, e_i is the vector defined by 1 in the ith position and 0 elsewhere, and $P = (P_1\ \cdots\ P_r)$, i.e., $P_i \in A^q$ is the ith column of P. Therefore, $\mathcal{L}$ is finitely generated by $\{e_1, \ldots, e_q, P_1, \ldots, P_r\}$. Finally, we have $K^q \subseteq K\,\mathcal{L} = \sum_{i=1}^q K\,e_i + \sum_{i=1}^r K\,P_i \subseteq K^q$, and thus, $K\,\mathcal{L} = K^q$. Then, by Proposition 1, $\mathcal{L}$ is a lattice of K^q.

Example 6 Let $P \in K^{q \times r}$ be a transfer matrix, $P = D^{-1} N$ a fractional representation of P, $(D \; - N) \in A^{q \times (q+r)}$, and $\mathcal{P} = D\, A^q + N\, A^r$ the A-submodule of K^q. Then, we have $\mathcal{P} \subseteq A^q$, i.e., $\mathcal{P}$ is contained in a finitely generated A-submodule of K^q. Moreover, if we denote by e_i the vector defined by 1 in the ith position and 0 elsewhere and by $D_i^{-1} \in K^q$ the ith column of D^{-1}, then we have $e_i = D\, D_i^{-1} \in K\, \mathcal{P} = D\, K^q + N\, K^r$. Hence, we have $K^q \subseteq K\, \mathcal{P} \subseteq K^q$, i.e., $K\, \mathcal{P} = K^q$, and, by Proposition 1, $\mathcal{P}$ is a lattice of V.

Proposition 2 *[2] Let M be a lattice of a finite-dimensional K-vector space V and L an A-submodule of V.*

- *If there exist non-zero $x, y \in K$ such that $x\, M \subseteq L \subseteq y\, M$, then L is a lattice of V.*
- *If L is a lattice of V, then there exist non-zero $a, b \in A$ such that:*

$$a\, M \subseteq L \subseteq b^{-1}\, M.$$

Example 7 In examples 5 and 6, we proved the following chain of inclusions $D\, A^q \subseteq \mathcal{P} \subseteq A^q \subseteq \mathcal{L} \subseteq D^{-1}\, A^q$. Let us denote by D_c the cofactor matrix of D, namely, the matrix $D_c \in A^{q \times q}$ such that $D_c\, D = D\, D_c = (\det\, D)\, I_q$ (such a matrix always exists [32]). Then, we have:

$$\begin{cases} (\det\, D)\, A^q = D\, (D_c\, A^q) \subseteq D\, A^q, \\ D^{-1}\, A^q = (\det\, D)^{-1}\, (D_c\, A^q) \subseteq (\det\, D)^{-1}\, A^q. \end{cases}$$

We easily check that $(\det\, D)\, A^q$ and $(\det\, D)^{-1}\, A^q$ are two lattices of K^q. Therefore, we have the following long chain of inclusions of lattices of K^q:

$$(\det\, D)\, A^q \subseteq D\, A^q \subseteq \mathcal{P} \subseteq A^q \subseteq \mathcal{L} \subseteq D^{-1}\, A^q \subseteq (\det\, D)^{-1}\, A^q.$$

Then, using the inclusions $(\det\, D)\, A^q \subseteq \mathcal{P} \subseteq A^q$ and $A^q \subseteq \mathcal{L} \subseteq (\det\, D)^{-1}\, A^q$, we finally obtain:

$$\begin{cases} (\det\, D)^2\, \mathcal{L} \subseteq \mathcal{P} \subseteq \mathcal{L}, \\ \mathcal{P} \subseteq \mathcal{L} \subseteq (\det\, D)^{-2}\, \mathcal{P}. \end{cases}$$

The next proposition shows the main interest of using lattices.

Proposition 3 *Let $P = D_1^{-1}\, N_1 = \tilde{N}_1\, \tilde{D}_1^{-1}$ and $P = D_2^{-1}\, N_2 = \tilde{N}_2\, \tilde{D}_2^{-1}$ be two fractional representations of $P \in K^{q \times r}$ and let us define the following lattices of K^q and $K^{1 \times r}$:*

$$\begin{cases} \mathcal{L} = (I_q \quad -P)\, A^{q+r}, \\ \mathcal{P}_1 = (D_1 \quad -N_1)\, A^{q+r}, \\ \mathcal{P}_2 = (D_2 \quad -N_2)\, A^{q+r}, \end{cases} \qquad \begin{cases} \mathcal{M} = A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}, \\ \mathcal{Q}_1 = A^{1\times(q+r)} \begin{pmatrix} \tilde{N}_1 \\ \tilde{D}_1 \end{pmatrix}, \\ \mathcal{Q}_2 = A^{1\times(q+r)} \begin{pmatrix} \tilde{N}_2 \\ \tilde{D}_2 \end{pmatrix}. \end{cases}$$

Then, we have the following isomorphisms $\mathcal{L} \cong \mathcal{P}_1 \cong \mathcal{P}_2$ and $\mathcal{M} \cong \mathcal{Q}_1 \cong \mathcal{Q}_2$.

Proof We easily check that $\mathcal{P}_i = D_i\, \mathcal{L}$ and $\mathcal{Q}_i = \mathcal{M}\, \tilde{D}_i$ for $i = 1, 2$. Hence, every element $p \in \mathcal{P}_1$ has the form $p = D_1\, l$ for a certain $l \in \mathcal{L}$. Such l is uniquely determined by p as we have

$$p = D_1\, l = D_1\, l' \Rightarrow D_1\, (l - l') = 0 \Rightarrow l = l',$$

because D_1 is a non-singular matrix and A is an integral domain. Hence, we obtain $\mathcal{P}_1 \cong \mathcal{L}$ and doing similarly with $\mathcal{P}_2$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, we finally obtain

$$\mathcal{P}_1 \cong \mathcal{L} \cong \mathcal{P}_2, \quad \mathcal{Q}_1 \cong \mathcal{M} \cong \mathcal{Q}_2,$$

which proves the result. □

From Proposition 3, it follows that the lattices introduced in Examples 3 and 4 only depend up to an isomorphism on the plant P. Thus, the structural properties of the plant defined by P can be studied by means of the ones of the lattices $\mathcal{L}$ and $\mathcal{M}$ or the ones of the lattices $\mathcal{P}$ and $\mathcal{Q}$.

The next proposition shows that the set of lattices of a finite-dimensional vector space V is stable by intersections and sums.

Proposition 4 *[2] We have the following two results:*

- *If M_1 and M_2 are two lattices of V, then so are $M_1 \cap M_2$ and $M_1 + M_2$.*
- *If W is a sub-vector space of V and M is a lattice of V, then $M \cap W$ is a lattice of W.*

Example 8 From Examples 3 and 4, we know that $\mathcal{L}$ and $\mathcal{P}$ are two lattices of $V = K^q$. Moreover, using the inclusion $\mathcal{P} \subseteq \mathcal{L}$ obtained in Example 7, we then have $\mathcal{L} \cap \mathcal{P} = \mathcal{P}$ and $\mathcal{L} + \mathcal{P} = \mathcal{L}$, which are obviously two lattices of V.

If V and W are two finite-dimensional K-vector spaces, then we denote by $\hom_K(V, W)$ the set of K-linear maps from V to W.

Definition 4 [2] Let V and W be two finite-dimensional K-vector spaces and M (respectively, N) a lattice of V (respectively, W). Then, we shall denote by $N : M$ the A-submodule of $\hom_K(V, W)$ formed by the K-linear maps $f : V \to W$ which satisfy $f(M) \subseteq N$.

Proposition 5 *[2] Let V and W be two finite-dimensional K-vector spaces and M (respectively, N) a lattice of V (respectively, W). Then, we have:*

1. *$N : M$ is a lattice of* $\hom_K(V, W) = \{f : V \to W \mid f$ *is a K-linear map*$\}$.

2. *The canonical map* $N : M \to \hom_A(M, N)$*, which maps* $f \in N : M$ *into the restriction* $f_{|M}$ *of* f *to* M*, is an isomorphism, where* $\hom_A(M, N)$ *denotes the* A*-module of* A*-linear maps from* M *to* N*.*

Example 9 If $P \in K^{q\times r}$, then according to Example 5, $\mathcal{L} = (I_q \quad -P)\, A^{q+r}$ was a lattice of the K-vector space K^q. Therefore, $A : \mathcal{L}$ is a lattice of $\hom_K(K^q, K) \cong K^{1\times q}$ defined by:

$$\begin{aligned} A : \mathcal{L} = A : ((I_q \quad -P)\, A^{q+r}) &= \{f : K^q \to K \mid f((I_q \quad -P)\, A^{q+r}) \subseteq A\} \\ &= \{\lambda \in K^{1\times q} \mid \lambda\, (I_q \quad -P)\, A^{q+r} \subseteq A\} \\ &= \{\lambda \in K^{1\times q} \mid \lambda \in A^{1\times q},\ \lambda\, P \in A^{1\times r}\} \\ &= \{\lambda \in A^{1\times q} \mid \lambda\, P \in A^{1\times r}\}. \end{aligned}$$

Similarly, we can prove that:

$$A : \mathcal{M} = A : \left(A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}\right) = \{\lambda \in A^r \mid P\, \lambda \in A^q\}.$$

Example 10 Let $P \in K^{q\times r}$ be a transfer matrix, $P = D^{-1}\, N$ a fractional representation of P and $R = (D \quad -N) \in A^{q\times(q+r)}$. Then, from Example 6, $\mathcal{P} = R\, A^{q+r}$ is a lattice of the K-vector space K^q. Therefore, $A : \mathcal{P}$ is a lattice of $\hom_K(K^q, K) \cong K^{1\times q}$ defined by:

$$\begin{aligned} A : \mathcal{P} = A : (R\, A^{q+r}) &= \{\lambda \in K^{1\times q} \mid \lambda\, (R\, A^{q+r}) \subseteq A\} \\ &= \{\lambda \in K^{1\times q} \mid \lambda\, R \in A^{1\times(q+r)}\}. \end{aligned}$$

Similarly, if we denote by $\tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r)\times r}$, then we can prove that we have:

$$A : \mathcal{Q} = A : \left(A^{1\times(q+r)}\, \tilde{R}\right) = \left\{\lambda \in K^r \mid \tilde{R}\, \lambda \in A^{q+r}\right\}.$$

Finally, let us introduce the concept of a *fractional ideal* of A.

Example 11 A *fractional ideal* J of A is an A-submodule of K such that there exists a non-zero $a \in A$ satisfying:

$$a\, J \triangleq \{a\, j \mid j \in J\} \subseteq A.$$

See [2,32] for more details. Hence, if J is a non-zero fractional ideal of A, then J is a lattice of K as we have $b\, A \subseteq J \subseteq (1/a)\, A$, where b is any non-zero element of J, and the non-zero fractional ideals of A are lattices of K.

Conversely, let us suppose that J is a lattice of K. Then, by Definition 3, there exist two free A-submodules L_1 and L_2 of K such that $L_1 \subseteq J \subseteq L_2$ and $\mathrm{rk}_A(L_1) = 1$. Hence, we have $L_1 \neq 0$, and thus, $J \neq 0$. Moreover, we have $K\, L_1 = K \subseteq K\, J \subseteq K\, L_2 \subseteq K$, and thus, $K\, L_2 = K$, which shows that L_2 is a free A-submodule of K of rank 1. Therefore, L_i is isomorphic to A, i.e., there exists $\phi_i \in \hom_A(A, L_i)$ such that $\phi_i(A) = L_i$ and ϕ_i is injective. But, ϕ_i is totally determined by the knowledge of $\phi_i(1) = l_i \in L_i$, and thus, we obtain that $L_i = A\, l_i$ is a non-zero fractional ideal of K generated by $l_i \in K$. Finally, writing

$l_2 = n_2/d_2$, where $0 \neq d_2,\ n_2 \in A$, we obtain that $d_2\, J \subseteq d_2\, L_2 = (n_2) \subseteq A$, which shows that the lattices of K are exactly the non-zero fractional ideals of A. See [2] for more details.

Now, if $p \in K$, then $\mathcal{L} = A + A\,(-p) = A + A\,p = \mathcal{P}$ is a fractional ideal of A as we have $d\,\mathcal{L} = A\,d + A\,n \subseteq A$, where $p = n/d$, $0 \neq d,\ n \in A$, is a fractional representation of p. Then, the ideal defined by

$$A : \mathcal{L} = A : \mathcal{P} = \{k \in K \mid k,\ k\,p \in A\} = \{d \in A \mid d\,p \in A\}$$

is called *the ideal of the denominators* of p [24]. Indeed, for all $0 \neq d \in A : \mathcal{L}$, we have $n = d\,p \in A$, and thus, $p = n/d$ is a fractional representation of p.

Finally, if $p = n/d$ is a fractional representation of p, where $0 \neq d,\ n \in A$, then the lattice $\mathcal{M} = A\,d + A\,(-n) = A\,d + A\,n = \mathcal{Q}$ of K is the integral ideal of A generated by d and n. Then, $A : \mathcal{M} = A : \mathcal{Q} = \{k \in K \mid k\,d,\ k\,n \in A\}$ is also a fractional ideal of A. Note that $A : \mathcal{M}$ cannot be finitely generated as an A-module. See [24] for more details and for an example.

Within the theory of fractional ideals, we have recently obtained in [24] necessary and sufficient conditions for the existence of (weakly) coprime factorizations and internal/strong/bistably stabilizability of SISO systems. The purpose of this paper is to generalize these results for MIMO systems using the lattice approach.

4 Coprime factorizations

Within the lattice approach developed in Sect. 3, the purpose of this section is to give general necessary and sufficient conditions for a transfer matrix to admit (weakly) left-/right-coprime factorizations.

4.1 Weakly left-/weakly right-coprime factorizations

Let us start by the following two lemmas.

Lemma 1 *1. $P = D^{-1}\,N$, $(D\ \ -N) \in A^{q\times(q+r)}$, is a weakly left-coprime factorization of the transfer matrix P iff we have:*

$$A : \mathcal{P} = A^{1\times q}. \tag{9}$$

2. $P = \tilde{N}\,\tilde{D}^{-1}$, $(\tilde{N}^T\ \ \tilde{D}^T)^T \in A^{(q+r)\times r}$, is a weakly right-coprime factorization of the transfer matrix P iff we have:

$$A : \mathcal{Q} = A^{r}. \tag{10}$$

Proof 1. From Example 10, we know that the lattice $\mathcal{P} = (D\ \ -N)\,A^{q+r}$ of K^q is such that $A : \mathcal{P} = \{\lambda \in K^{1\times q} | \lambda(D\ \ -N) \in A^{1\times(q+r)}\}$. By Definition 1, $P = D^{-1}\,N$ is a weakly coprime factorization of P if we have

$$\{\lambda \in K^{1\times q} \mid \lambda\,(D\ \ -N) \in A^{1\times(q+r)}\} = A^{1\times q},$$

which proves the result.

2 can be proved similarly. □

With the notations of Example 3, we easily check that we have $\mathcal{P} = D\,\mathcal{L}$. Let us study the relation existing between the residuals $A : \mathcal{P}$ and $A : \mathcal{L}$.

Lemma 2 *1. If $P = D^{-1}\,N$ is a fractional representation of the transfer matrix P where $(D \quad -N) \in A^{q\times(q+r)}$, then we have:*

$$A : \mathcal{L} = (A : \mathcal{P})\,D. \tag{11}$$

2. If $P = \tilde{N}\,\tilde{D}^{-1}$ is a fractional representation of the transfer matrix P where $(\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r)\times r}$, then we have:

$$A : \mathcal{M} = \tilde{D}\,(A : \mathcal{Q}).$$

Proof 1. From Example 9, we know that $\lambda \in A : \mathcal{L}$ iff $\lambda \in A^{1\times q}$ and $\lambda\,P \in A^{1\times r}$, i.e., iff we have

$$\lambda\,(I_q \quad -P) \in A^{1\times(q+r)} \Leftrightarrow (\lambda\,D^{-1})\,(D \quad -N) \in A^{1\times(q+r)}$$

$$\Leftrightarrow \exists\,\mu \in K^{1\times q} : \begin{cases} \lambda = \mu\,D \\ \mu\,(D \quad -N) \in A^{1\times(q+r)} \end{cases} \Leftrightarrow \lambda \in (A : \mathcal{P})\,D.$$

2 can be proved similarly. □

The next theorem gives necessary and sufficient conditions for the existence of a weakly left-/right-/doubly coprime factorization of a transfer matrix P.

Theorem 1 *1. The following assertions are equivalent:*

(a) $P \in K^{q\times r}$ admits a weakly left-coprime factorization.

(b) There exists a non-singular matrix $D \in A^{q\times q}$ such that:

$$A : \mathcal{L} = A^{1\times q}\,D. \tag{12}$$

If we denote by $N = D\,P \in A^{q\times r}$, $P = D^{-1}\,N$ is then a weakly left-coprime factorization.

(c) The A-module $A : \mathcal{L}$ is a free lattice of $K^{1\times q}$.

2. The following assertions are equivalent:

(a) $P \in K^{q\times r}$ admits a weakly right-coprime factorization.

(b) There exists a non-singular matrix $\tilde{D} \in A^{r\times r}$ such that:

$$A : \mathcal{M} = \tilde{D}\,A^r. \tag{13}$$

If we denote by $\tilde{N} = P\,\tilde{D} \in A^{q\times r}$, $P = \tilde{N}\,\tilde{D}^{-1}$ is then a weakly right-coprime factorization.

(c) The A-module $A : \mathcal{M}$ is a free lattice of K^r.

Proof 1.a $\Rightarrow$ 1.b. If $P = D^{-1}\,N$ is a weakly left-coprime factorization of P, then, by Lemma 1, we have $A : \mathcal{P} = A^{1\times q}$, and thus, by 1 of Lemma 2, we obtain $A : \mathcal{L} = A^{1\times q}\,D$, where $D \in A^{q\times q}$ is a non-singular matrix.

1.b $\Rightarrow$ 1.c. If there exists a non-singular matrix $D \in A^{q\times q}$ such that (12) holds, then the A-module $A : \mathcal{L}$ is generated by the family $\{D^1, \ldots, D^q\}$, where $D^i \in A^{1\times q}$ is the ith row of D. Moreover, the family $\{D^1, \ldots, D^q\}$ is independent as $\lambda\,D = 0$ implies $\lambda = 0$ because D is non-singular. Therefore, $\{D^1, \ldots, D^q\}$ is a basis of $A : \mathcal{L}$.

1.c $\Rightarrow$ 1.a. Let us suppose that $A : \mathcal{L}$ is a free lattice of $K^{1\times q}$. Then, $A : \mathcal{L}$ is a free A-submodule of $K^{1\times q}$ of rank q, and thus, $A : \mathcal{L}$ admits a basis $\{D^1, \ldots, D^q\}$. The fact that $D^i \in A : \mathcal{L}$ implies $D^i \in A^{1\times q}$. Hence, if we denote by $D = ((D^1)^T \; \cdots \; (D^q)^T)^T$, then $D \in A^{q\times q}$ is a non-singular matrix which satisfies (12). Therefore, if we denote by $N = D\,P \in A^{q\times r}$, then $P = D^{-1}\,N$ is a fractional representation of P, using Lemma 2, we obtain that $(A : \mathcal{P})\,D = A^{1\times q}\,D$. Hence, if we take λ in $A : \mathcal{P}$, then there exists $\mu \in A^{1\times q}$ such that $\lambda\,D = \mu\,D$ and, using the fact that D is non-singular, we obtain $\lambda = \mu$, proving that $A : \mathcal{P} \subseteq A^{1\times q}$. Finally, using the trivial fact that $A^{1\times q} \subseteq A : \mathcal{P}$, we obtain $A : \mathcal{P} = A^{1\times q}$, which proves that $P = D^{-1}\,N$ is a weakly coprime factorization by Lemma 1.

2 can be proved similarly. □

In Theorem 1, we characterized the existence of a weakly coprime factorization of a plant P using only the transfer matrix P. In the next corollary, we also characterize the existence of a weakly coprime factorization of P in terms of a fractional representation of P.

Corollary 1 *1. Let $P = D^{-1}\,N$ be a fractional representation of the transfer matrix P and $(D \; -N) \in A^{q\times(q+r)}$. Then, the following assertions are equivalent:*

(a) P admits a weakly left-coprime factorization.

(b) There exists a non-singular matrix $E \in K^{q\times q}$ such that:

$$A : \mathcal{P} = A^{1\times q}\,E. \tag{14}$$

If we denote by $D' = E\,D \in A^{q\times q}$ and $N' = E\,N \in A^{q\times r}$, then $P = (D')^{-1}\,N'$ is a weakly left-coprime factorization of P.

(c) The A-module $A : \mathcal{P}$ is a free lattice of $K^{1\times q}$.

2. Let $P = \tilde{N}\,\tilde{D}^{-1}$ be a fractional representation of the transfer matrix P and $(\tilde{N}^T \; \tilde{D}^T)^T \in A^{(q+r)\times r}$. Then, the following assertions are equivalent:

(a) P admits a weakly right-coprime factorization.

(b) There exists a non-singular matrix $F \in K^{r\times r}$ such that:

$$A : \mathcal{Q} = F\,A^r. \tag{15}$$

If we denote by $\tilde{D}' = \tilde{D}\,F \in A^{r\times r}$ and $\tilde{N}' = \tilde{N}\,F \in A^{q\times r}$, then $P = \tilde{N}'\,(\tilde{D}')^{-1}$ is a weakly right-coprime factorization of P.

(c) The A-module $A : \mathcal{Q}$ is a free lattice of K^r.

Proof 1.a $\Rightarrow$ 1.b. Let $P = (D')^{-1}\,N'$ be a weakly coprime factorization of P. Then, by 1.b of Theorem 1, we have $A : \mathcal{L} = A^{1\times q}\,D'$. Moreover, using 1 of Lemma 2, we obtain that $(A : \mathcal{P})\,D = A^{1\times q}\,D'$, and thus, $A : \mathcal{P} = A^{1\times q}\,E$ where $E = (D'\,D^{-1}) \in K^{q\times q}$ is a non-singular matrix.

1.b $\Rightarrow$ 1.c. We suppose that there exists a non-singular matrix $E \in K^{q\times q}$ such that $A : \mathcal{P} = A^{1\times q}\,E$. Then, $A : \mathcal{P}$ is a free lattice of $K^{1\times q}$ of basis $\{E^1, \ldots, E^q\}$, where E^i denotes the ith row of E.

1.c $\Rightarrow$ 1.a. Let us suppose that $A : \mathcal{P}$ is a free lattice of $K^{1\times q}$. Then, $A : \mathcal{P}$ is a free A-submodule of $K^{1\times q}$ of rank q. Therefore, there exists a basis $\{E^1, \ldots, E^q\}$, where $E^i \in K^{1\times q}$. If we denote by $E = ((E^1)^T \; \cdots \; (E^q)^T)^T$, then E is a non-singular matrix which satisfies (14). In particular, we have $E\,(D \quad -N) \in A^{q\times(q+r)}$ and, if we denote by

$$D' = E\,D \in A^{q\times q}, \quad N' = E\,N \in A^{q\times r},$$

then we have $\det D' \neq 0$ and $P = (D')^{-1}\,N'$. Finally, let us take $\lambda \in K^{1\times r}$ such that $\lambda\,(D' \quad -N') \in A^{1\times(q+r)}$. Then, we have $(\lambda\,E)\,(D \quad -N) \in A^{1\times(q+r)}$, and thus, $\lambda\,E \in A : \mathcal{P} = A^{1\times q}\,E$, i.e., there exists $\mu \in A^{1\times q}$ such that $\lambda\,E = \mu\,E$. Hence, using the fact that E is non-singular, we obtain $\lambda = \mu \in A^{1\times q}$, proving that $P = (D')^{-1}\,N'$ is a weak coprime factorization.

2 can be proved similarly. □

Example 12 If $p \in K$ is a transfer function of a SISO plant, then

$$\mathcal{L} = (1 \quad -p)\,A^2 = A^{1\times 2}\begin{pmatrix} p \\ 1 \end{pmatrix} = \mathcal{M} = \{a + b\,p \mid a,\, b \in A\}$$

is a fractional ideal of A (see Example 11). By 1 or 2 of Theorem 1, p admits a weakly coprime factorization iff $A : \mathcal{L} = \{d \in A \mid d\,p \in A\}$ is a free lattice of K, i.e., a principal ideal of A. Hence, we obtain that p admits a weakly coprime factorization if there exists $0 \neq d' \in A$ such that $A : \mathcal{L} = A\,d'$. Hence, Theorem 1 generalizes 2 of Theorem 1 of [24] for MIMO systems.

If $p = n/d$ is a fractional representation of p, $0 \neq d,\, n \in A$, then, by 1 or 2 of Corollary 1, p admits a weakly coprime factorization if the lattice $\mathcal{P} = (d \quad -n)\,A^2 = A^{1\times 2}(n \quad d)^T = \mathcal{Q}$ of K, i.e., the ideal $\mathcal{P} = A\,d + A\,n$ of A, is such that $A : \mathcal{P} = \{k \in K \mid k\,d,\; k\,n \in A\}$ is a free lattice of K, i.e., a principal fractional ideal of A. Hence, p admits a weakly coprime factorization iff there exists $0 \neq k \in K$ such that $A : \mathcal{P} = A\,k$. Hence, Corollary 1 generalizes 2 of Theorem 3 of [24] for MIMO systems.

Remark 2 An algorithm computing weakly doubly coprime factorizations is obtained in [23]. Moreover, we proved in [23] that every transfer matrix admits a weakly doubly coprime factorization iff A is a *coherent Sylvester domain*, namely, a ring A satisfying that, for every $n \in \mathbb{Z}_+$ and every row vector $R \in A^{1\times n}$, the A-module $\ker R = \{\mu \in A^n \mid R\,\mu = 0\}$ is free [7]. For instance, RH_∞, $H_\infty\,(\mathbb{C}_+)$, $\mathbb{R}[s,\, z]$ are coherent Sylvester domains [23].

4.2 Left-/right-coprime factorizations

Let us start with the following simple lemma.

Lemma 3 *1. $P = D^{-1}\,N$, $(D \; -N) \in A^{q\times(q+r)}$, is a left-coprime of the transfer matrix P iff we have $\mathcal{P} = A^q$.*

2. $P = \tilde{N}\,\tilde{D}^{-1}$, $(\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r)\times r}$, is a right-coprime factorization of the transfer matrix P iff we have $\mathcal{Q} = A^{1\times r}$.

Proof The fact that $\mathcal{P} = A^q$ is equivalent to the existence of a matrix $(X^T \quad Y^T)^T \in A^{q\times(q+r)}$ satisfying $(D \quad -N)\,(X^T \quad Y^T)^T = I_q$, and thus, it is equivalent to the fact that $P = D^{-1}\,N$ is a left-coprime factorization.

2 can be proved similarly. □

The following theorem gives necessary and sufficient conditions for the existence of a left-/right-coprime factorization of a transfer matrix P.

Theorem 2 *1. The following assertions are equivalent:*
(a) $P \in K^{q\times r}$ admits a left-coprime factorization.
(b) There exists a non-singular matrix $D \in A^{q\times q}$ such that:

$$\mathcal{L} = D^{-1}\, A^q. \tag{16}$$

If we denote by $N = D\,P \in A^{q\times r}$, $P = D^{-1}\,N$ is then a left-coprime factorization of P.
(c) The A-module $\mathcal{L}$ is a free lattice of K^q.

2. The following assertions are equivalent:
(a) $P \in K^{q\times r}$ admits a right-coprime factorization.
(b) There exists a non-singular matrix $\tilde{D} \in A^{r\times r}$ such that:

$$\mathcal{M} = A^{1\times r}\, \tilde{D}^{-1}. \tag{17}$$

If we denote by $\tilde{N} = P\,\tilde{D} \in A^{q\times r}$, $P = \tilde{N}\,\tilde{D}^{-1}$ is then a right-coprime factorization of P.
(c) $\mathcal{M}$ is a free lattice of $K^{1\times r}$.

Proof 1.a $\Rightarrow$ 1.b. Let $P = D^{-1}\,N$ be a left-coprime factorization of P and $D\,X - N\,Y = I_q$. Then, we have:

$$\mathcal{L} = (I_q \quad -P)\, A^{q+r} = (D^{-1}\,(D \quad -N))\, A^{q+r} = D^{-1}\,\mathcal{P}.$$

Using 1 of Lemma 3, we finally obtain $\mathcal{P} = A^q$, and thus, $\mathcal{L} = D^{-1}\,A^q$.

1.b $\Rightarrow$ 1.c. If there exists a non-singular matrix $D \in A^{q\times q}$ such that we have (16), then the A-module $\mathcal{L}$ is trivially free and a basis of $\mathcal{L}$ is defined by $\{D_1^{-1}, \ldots, D_q^{-1}\}$, where D_i^{-1} denotes the ith column of D^{-1}. Therefore, $\mathcal{L}$ is a free lattice of K^q.

1.c $\Rightarrow$ 1.a. If $\mathcal{L}$ is a free lattice of K^q, then $\mathcal{L}$ is in particular a free A-submodule of K^q of rank q. Hence, if we denote by $\{E_1, \ldots, E_q\}$ a basis of $\mathcal{L}$ and form the non-singular matrix $E = (E_1 \ \cdots \ E_q) \in K^{q\times q}$, then we have $\mathcal{L} = E\,A^q$. Hence, there exists two matrices $(D \quad -N) \in A^{q\times(q+r)}$ and $(X^T \quad Y^T)^T \in A^{(q+r)\times q}$ such that we have

$$\left\{\begin{array}{l} (I_q \quad -P) = E\,(D \quad -N),\\ E = (I_q \quad -P)\begin{pmatrix} X\\ Y\end{pmatrix},\end{array}\right. \Rightarrow \left\{\begin{array}{l} E\,D = I_q,\\ P = E\,N,\\ X - P\,Y = E,\end{array}\right. \Rightarrow \left\{\begin{array}{l} P = D^{-1}\,N,\\ D\,X - N\,Y = I_q,\end{array}\right.$$

which shows that P admits the left-coprime factorization $P = D^{-1}\,N$.

2 can be proved similarly. □

In Theorem 2, we characterized the existence of a coprime factorization of a plant P using only the transfer matrix P. In the next corollary, we characterize the existence of a coprime factorization of P in terms of a fractional representation of P.

Corollary 2 *1. Let $P = D^{-1} N$ be a fractional representation of the transfer matrix P and $(D \; -N) \in A^{q \times (q+r)}$. Then, the following assertions are equivalent:*

(a) P admits a left-coprime factorization.

(b) There exists a non-singular matrix $G \in A^{q \times q}$ such that:

$$\mathcal{P} = G\, A^q. \tag{18}$$

If we denote by $D' = G^{-1} D \in A^{q \times q}$ and $N' = G^{-1} N \in A^{q \times r}$, then $P = (D')^{-1} N'$ is a left-coprime factorization of P.

(c) The A-module $\mathcal{P}$ is a free lattice of K^q.

2. Let $P = \tilde{N}\, \tilde{D}^{-1}$ be a fractional representation of the transfer matrix P and $(\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{(q+r) \times r}$. Then, the following assertions are equivalent:

(a) P admits a right-coprime factorization.

(b) There exists a non-singular matrix $H \in A^{r \times r}$ such that:

$$\mathcal{Q} = A^{1 \times r}\, H. \tag{19}$$

If we denote by $\tilde{D}' = \tilde{D}\, H^{-1} \in A^{r \times r}$ and $\tilde{N}' = \tilde{N}\, H^{-1} \in A^{q \times r}$, then $P = \tilde{N}'\, (\tilde{D}')^{-1}$ is a right-coprime factorization of P.

(c) The A-module $\mathcal{Q}$ is a free lattice of $K^{1 \times r}$.

Proof 1.a $\Rightarrow$ 1.b. If $P = (D')^{-1} N'$ is a coprime factorization of P, then, by 1 of Theorem 2, we have $\mathcal{L} = (D')^{-1} A^q$. Therefore, we obtain:

$$\begin{aligned} \mathcal{L} &= (I_q \;\; -P)\, A^{q+r} = D^{-1}\, ((D \;\; -N)\, A^{q+r}) = D^{-1}\, \mathcal{P} \\ &\Rightarrow D^{-1}\, \mathcal{P} = (D')^{-1} A^q \Rightarrow \mathcal{P} = (D\, (D')^{-1})\, A^q. \end{aligned}$$

Thus, the non-singular matrix $G = D\, (D')^{-1}$ satisfies $\mathcal{P} = G\, A^q$ and, using the fact that $\mathcal{P} \subseteq A^q$, we obtain $G \in A^{q \times q}$.

1.b $\Rightarrow$ 1.c. If there exists a non-singular matrix $G \in A^{q \times q}$ such that we have (18), then $\mathcal{P}$ is a free A-submodule of K^q with basis $\{G_1, \ldots, G_q\}$, where G_i denotes the ith column of G. Thus, $\mathcal{P}$ is a free lattice of K^q.

1.c $\Rightarrow$ 1.a. If $\mathcal{P}$ is a free lattice of K^q, then $\mathcal{P}$ is a free A-submodule of K^q of rank q. Hence, if we denote by $\{G_1, \ldots, G_q\}$ a basis of $\mathcal{P}$ and form the non-singular matrix $G = (G_1 \;\; \cdots \;\; G_q)$, we then have $\mathcal{P} = G\, A^q$. Thus, there exists $(D' \;\; -N') \in A^{q \times (q+r)}$ and $(X^T \;\; Y^T)^T \in A^{(q+r) \times q}$ satisfying

$$\begin{cases} (D \;\; -N) = G\, (D' \;\; -N'), \\ G = (D \;\; -N) \begin{pmatrix} X \\ Y \end{pmatrix}, \end{cases} \Rightarrow \begin{cases} D = G\, D', \\ N = G\, N', \\ D\, X - N\, Y = G, \end{cases} \Rightarrow \begin{cases} P = (D')^{-1} N', \\ D'\, X - N'\, Y = I_q, \end{cases}$$

showing that P admits the left-coprime factorization $P = (D')^{-1} N'$.

2 can be proved similarly. □

We refer to [23,33,36,37] for related results. We thank Prof. Sontag (Rutgers University) for pointing out to us Reference [14] where conditions for the existence of stable factorizations are also obtained within the systems over the rings approach. The links with these results will be studied in the future.

Example 13 If $p \in K$ is a transfer function, then, by 1 or 2 of Theorem 2, p admits a coprime factorization iff the lattice $\mathcal{L} = (1 \quad -p)\, A^2 = \mathcal{M}$ of K is free, i.e., iff $\mathcal{L}$ is a principal fractional ideal of A. Hence, Theorem 2 generalizes 4 of Theorem 1 of [24] for MIMO systems.

If $p = n/d$ is a fractional representation of p, with $0 \neq d,\ n \in A$, then, by 1 or 2 of Corollary 2, p admits a coprime factorization iff the lattice $\mathcal{P} = (d - n)\, A^2 = \mathcal{Q}$ of K is free, i.e., iff the ideal $\mathcal{P} = A\, d + A\, n$ is a principal ideal of A [38]. Hence, Corollary 2 generalizes 4 of Theorem 3 of [24] for MIMO systems.

Remark 3 Vidyasagar proved in [38] that every transfer matrix admits a doubly coprime factorization iff A is a *Bézout domain* [32], namely, an integral domain such that every finitely generated ideal of A is *principal*, namely, generated by an element of A (see also [23] for another proof).

Finally, let us illustrate Theorems 1 and 2 by obtaining Remark 1.

Corollary 3 *If $P = D^{-1}\, N$ (respectively, $P = \tilde{N}\, \tilde{D}^{-1}$) is a left-coprime (respectively, right-coprime) factorization of the transfer matrix P, then $P = D^{-1}\, N$ (respectively, $P = \tilde{N}\, \tilde{D}^{-1}$) is a weakly left-coprime (respectively, weakly right-coprime) factorization of P.*

Proof If $P \in K^{q\times r}$ admits a left-coprime factorization $P = D^{-1}\, N$, then, by 1 of Theorem 2, we have $\mathcal{L} = D^{-1}\, A^q$. Therefore, we obtain:

$$\begin{aligned} A : \mathcal{L} &= \{\lambda \in K^{1\times q} \mid \lambda\, ((I_q \quad -P)\, A^{q+r}) = \lambda\, (D^{-1}\, A^q) \subseteq A\} \\ &= \{\lambda \in K^{1\times q} \mid \lambda\, D^{-1} \in A^{1\times q}\} = A^{1\times q}\, D. \end{aligned}$$

By 1 of Theorem 1, $P = D^{-1}\, N$ is a weakly left-coprime factorization of P.

Similarly, we can prove that a right-coprime factorization is a weakly right-coprime factorization. □

Finally, we want to point out that the lattice approach developed in this paper can also be used for the study of multidimensional systems defined over the polynomial ring $k[x_1, \ldots, x_n]$, where k is a field ($k = \mathbb{R}, \mathbb{C}$). We then obtain a generalization of some results obtained by Fuhrmann [8] for 1-D systems.

5 Elementary characterizations of internal stabilizability

5.1 Characterizations in terms of the transfer matrix P

Let us start by giving elementary characterizations of internal stabilizability.

Proposition 6 *A plant $P \in K^{q\times r}$ is internally stabilizable (IS) iff one of the following equivalent assertions is satisfied:*

1. *There exists $L = (U^T \quad V^T)^T \in A^{(q+r)\times q}$ which satisfies* $\det U \neq 0$ *and:*
   (a) $L\, P = \begin{pmatrix} U\, P \\ V\, P \end{pmatrix} \in A^{(q+r)\times r}$,
   (b) $(I_q \quad -P)\, L = U - P\, V = I_q$.

Then, the controller defined by $C = V\, U^{-1}$ *internally stabilizes* P *and:*

$$\begin{cases} U = (I_q - P\, C)^{-1}, \\ V = C\, (I_q - P\, C)^{-1}. \end{cases}$$

2. *There exists* $\tilde{L} = (-\tilde{V} \quad \tilde{U}) \in A^{r \times (q+r)}$ *which satisfies* $\det \tilde{U} \neq 0$ *and:*
   (a) $P\, \tilde{L} = (-P\, \tilde{V} \quad P\, \tilde{U}) \in A^{q \times (q+r)}$,

   (b) $\tilde{L} \begin{pmatrix} P \\ I_r \end{pmatrix} = -\tilde{V}\, P + \tilde{U} = I_r$.

   Then, the controller defined by $C = \tilde{U}^{-1}\, \tilde{V}$ *internally stabilizes* P *and:*

$$\begin{cases} \tilde{U} = (I_r - C\, P)^{-1}, \\ \tilde{V} = (I_r - C\, P)^{-1}\, C. \end{cases}$$

3. *There exist* $L = (U^T \quad V^T)^T \in A^{(q+r) \times q}$ *and* $\tilde{L} = (-\tilde{V} \quad \tilde{U}) \in A^{r \times (q+r)}$ *which satisfy* $\det U \neq 0$, $\det \tilde{U} \neq 0$ *and:*

$$\begin{pmatrix} I_q & -P \\ -\tilde{V} & \tilde{U} \end{pmatrix} \begin{pmatrix} U & P \\ V & I_r \end{pmatrix} = \begin{pmatrix} U & P \\ V & I_r \end{pmatrix} \begin{pmatrix} I_q & -P \\ -\tilde{V} & \tilde{U} \end{pmatrix} = I_{q+r}, \tag{20}$$

$$\Pi_1 = \begin{pmatrix} U \\ V \end{pmatrix} (I_q \quad - P) \in A^{(q+r) \times (q+r)}, \tag{21}$$

$$\Pi_2 = \begin{pmatrix} P \\ I_r \end{pmatrix} (-\tilde{V} \quad \tilde{U}) \in A^{(q+r) \times (q+r)}. \tag{22}$$

   Then, the controller defined by $C = V\, U^{-1} = \tilde{U}^{-1}\, \tilde{V}$ *internally stabilizes* P *and we have:*

$$\begin{cases} U = (I_q - P\, C)^{-1}, \\ V = C\, (I_q - P\, C)^{-1} = (I_r - C\, P)^{-1}\, C = \tilde{V}, \\ \tilde{U} = (I_r - C\, P)^{-1}. \end{cases}$$

Proof IS $\Rightarrow$ 1. Let us suppose that $P \in K^{q \times r}$ is internally stabilizable, i.e., there exists a controller $C \in K^{r \times q}$ such that we have:

$$\begin{cases} A_1 = (I_q - P\, C)^{-1} \in A^{q \times q}, \\ A_2 = (I_q - P\, C)^{-1}\, P \in A^{q \times r}, \\ A_3 = C\, (I_q - P\, C)^{-1} \in A^{r \times q}, \\ A_4 = I_r + C\, (I_q - P\, C)^{-1}\, P \in A^{r \times r}. \end{cases} \tag{23}$$

From (23), we easily obtain $C = A_3\, A_1^{-1}$. Now, if we define the matrix $L = (A_1^T \quad A_3^T)^T \in A^{(q+r) \times q}$, then we have:

- $L\, (I_q \quad - P) = \begin{pmatrix} A_1 & -A_1\, P \\ A_3 & -A_3\, P \end{pmatrix} = \begin{pmatrix} A_1 & -A_2 \\ A_3 & I_r - A_4 \end{pmatrix} \in A^{(q+r) \times (q+r)}$,
- $(I_q \quad - P)\, L = A_1 - P\, A_3 = (I_q - P\, C)^{-1} - P\, C\, (I_q - P\, C)^{-1} = I_q$.

IS $\Leftarrow$ 1. Let us suppose that there exists $L = (U^T \quad V^T)^T \in A^{(q+r)\times q}$ satisfying $\det U \neq 0$, and 1.a and 1.b. If we define $C = V\,U^{-1} \in K^{r\times q}$, then, using 1.b, we obtain:

$$I_q - P\,C = (U - P\,V)\,U^{-1} = U^{-1} \Rightarrow (I_q - P\,C)^{-1} = U \in A^{q\times q}.$$

Hence, using 1.a and (3), we obtain

$$H(P,\,C) = \begin{pmatrix} U & U\,P \\ V & I_r + V\,P \end{pmatrix} \in A^{(q+r)\times(q+r)},$$

i.e., $C = V\,U^{-1}$ internally stabilizes P and $V = C\,U = C\,(I_q - P\,C)^{-1}$.

IS $\Rightarrow$ 2. Let us suppose that $P \in K^{q\times r}$ is internally stabilizable, i.e., there exists a controller $C \in K^{r\times q}$ such that we have:

$$\begin{cases} B_1 = I_q + P\,(I_r - C\,P)^{-1}\,C \in A^{q\times q}, \\ B_2 = P\,(I_r - C\,P)^{-1} \in A^{q\times r}, \\ B_3 = (I_r - C\,P)^{-1}\,C \in A^{r\times q}, \\ B_4 = (I_r - C\,P)^{-1} \in A^{r\times r}. \end{cases} \tag{24}$$

From (24), we easily obtain $C = B_4^{-1}\,B_3$. Now, if we define the matrix $\tilde{L} = (-B_3 \quad B_4) \in A^{r\times(q+r)}$, then we have:

- $\begin{pmatrix} P \\ I_r \end{pmatrix} \tilde{L} = \begin{pmatrix} -P\,B_3 & P\,B_4 \\ -B_3 & B_4 \end{pmatrix} = \begin{pmatrix} I_q - B_1 & B_2 \\ -B_3 & B_4 \end{pmatrix} \in A^{(q+r)\times(q+r)}$,
- $\tilde{L} \begin{pmatrix} P \\ I_r \end{pmatrix} = -B_3\,P + B_4 = -(I_r - C\,P)^{-1}\,C\,P + (I_q - C\,P)^{-1} = I_r.$

IS $\Leftarrow$ 2. Let us suppose that there exists $\tilde{L} = (-\tilde{V} \quad \tilde{U}) \in A^{r\times(q+r)}$ satisfying $\det \tilde{U} \neq 0$, 2.a and 2.b. If we define $C = \tilde{U}^{-1}\,\tilde{V} \in K^{r\times q}$, then, using 2.b, we obtain

$$I_r - C\,P = I_r - (\tilde{U}^{-1}\,\tilde{V})\,P = \tilde{U}^{-1}\,(\tilde{U} - \tilde{V}\,P) = \tilde{U}^{-1},$$

and thus, $(I_r - C\,P)^{-1} = \tilde{U} \in A^{r\times r}$. Using point 2.a and (4), we obtain

$$H(P,\,C) = \begin{pmatrix} I_q + P\,\tilde{V} & P\,\tilde{U} \\ \tilde{V} & \tilde{U} \end{pmatrix} \in A^{(q+r)\times(q+r)},$$

i.e., $C = \tilde{U}^{-1}\,\tilde{V}$ internally stabilizes P and $\tilde{V} = \tilde{U}\,C = (I_r - C\,P)^{-1}\,C$.

IS $\Rightarrow$ 3. Let us suppose that $P \in K^{q\times r}$ is internally stabilized by the controller $C \in K^{r\times q}$. Following the proofs of "1 $\Rightarrow$" and "2 $\Rightarrow$", we obtain that $L = (A_1^T \quad A_3^T)^T$ (respectively, $\tilde{L} = (-B_3 \quad B_4)$ satisfies (21) (respectively, (22) and

$$\begin{pmatrix} I_q & -P \\ -B_3 & B_4 \end{pmatrix} \begin{pmatrix} A_1 & P \\ A_3 & I_r \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ -B_3\,A_1 + B_4\,A_3 & I_r \end{pmatrix} = I_{q+r}, \tag{25}$$

because we have:

$$\begin{aligned} B_3\, A_1 &= ((I_r - C\,P)^{-1}\, C)\,(I_q - P\,C)^{-1} \\ &= (I_r - C\,P)^{-1}\,(C\,(I_q - P\,C)^{-1}) = B_4\, A_3. \end{aligned}$$

Therefore, from (25), we obtain

$$\begin{pmatrix} I_q & -P \\ -B_3 & B_4 \end{pmatrix}^{-1} = \begin{pmatrix} A_1 & P \\ A_3 & I_r \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} A_1 & P \\ A_3 & I_r \end{pmatrix} \begin{pmatrix} I_q & -P \\ -B_3 & B_4 \end{pmatrix} = I_{q+r},$$

which proves (20). To finish, let us note that this last equality can be directly checked by using the following well-known identities [38]:

$$\begin{cases} (I_q - P\,C)^{-1} = P\,(I_r - C\,P)^{-1}\, C + I_q, \\ (I_q - P\,C)^{-1}\, P = P\,(I_r - C\,P)^{-1}, \\ C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\, C, \\ C\,(I_q - P\,C)^{-1}\, P = (I_r - C\,P)^{-1} - I_r. \end{cases} \tag{26}$$

IS $\Leftarrow$ 3. Let us suppose that there exist $L = (U^T \quad V^T)^T \in K^{(q+r)\times q}$ and $\tilde{L} = (-\tilde{V} \quad \tilde{U}) \in K^{r\times(q+r)}$ which satisfy (20), (21) and (22). In particular, L (respectively, $\tilde{L}$) satisfies 1.a and 1.b (respectively, 2.a and 2.b), and thus, by point 1 (respectively, point 2), $C_1 = V\,U^{-1}$ (respectively, $C_2 = \tilde{U}^{-1}\,\tilde{V}$) is a stabilizing controller of P. From (20), we deduce that $\tilde{V}\,U = \tilde{U}\,V$, and thus, $C_1 = V\,U^{-1} = \tilde{U}^{-1}\,\tilde{V} = C_2$ internally stabilizes P. □

Example 14 Let us consider the transfer matrix P defined by (2). We can check that the matrix $L = (U^T \quad V^T)^T \in A^{3\times 2}$ defined by

$$L = \begin{pmatrix} \frac{2}{s+1} + b\left(\frac{s-1}{s+1}\right)^3 & 2\,b\left(\frac{s-1}{s+1}\right)^3 - 2\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\,b\,\frac{(s-1)}{(s+1)^3} \\ -a\,\frac{(s-1)^2}{(s+1)^3} & -2\,a\,\frac{(s-1)^2}{(s+1)^3} \end{pmatrix},$$

where a and b are given by

$$\begin{cases} a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \\ b = \frac{(s+25)}{(s+1)} + \frac{4\,(5\,s-3)}{(s+1)}\,\frac{(2-s-e^{-(s-1)})}{(s-1)^2} = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A, \end{cases} \tag{27}$$

satisfies:

$$\begin{cases} L\,(I_2 \quad -P) \in A^{3\times 3}, \\ (I_2 \quad -P)\,L = U - P\,V = I_2. \end{cases}$$

From 1 of Proposition 6, we deduce that P is internally stabilized by the controller $C = V\,U^{-1}$, namely:

$$C = \begin{pmatrix} -a\,\frac{(s-1)^2}{(s+1)^3} & -2\,a\,\frac{(s-1)^2}{(s+1)^3} \end{pmatrix} \begin{pmatrix} \frac{2}{s+1} + b\left(\frac{s-1}{s+1}\right)^3 & 2\,b\left(\frac{s-1}{s+1}\right)^3 - 2\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\,b\,\frac{(s-1)}{(s+1)^3} \end{pmatrix}^{-1},$$

$$= -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3-4\,(5\,s-3)\,e^{-(s-1)})}\,(1 \quad 2).$$

We note that the controller C involves a distributed delay.

Example 15 By 1 or 2 of Proposition 6, a SISO plant defined by a transfer function $p \in K$ is internally stabilizable iff there exists $a,\ b \in A$ such that

$$\begin{cases} a - b\,p = 1, \\ a\,p \in A, \end{cases} \tag{28}$$

as we have $b\,p = a - 1 \in A$. Equivalently, p is internally stabilizable iff there exists $b \in A$ such that $b\,p \in A$ and $(1 + b\,p)\,p \in A$. If $a \neq 0$, then $c = b/a$ is a stabilizing controller of p whereas, if $a = 0$, then $p = (-1)/b$ is internally stabilized by $c = 1 - b$. Hence, Proposition 6 generalizes 3 of Theorem 1 of [24] for MIMO systems.

Remark 4 The remark made in Example 15 concerning SISO plants can be similarly extended to MIMO plants as follows: the plant P is internally stabilizable iff there exists $V \in A^{r\times q}$ such that:

$$V\,P \in A^{r\times r}, \quad P\,V \in A^{q\times q}, \quad (P\,V + I_q)\,P = P\,(V\,P + I_r) \in A^{q\times r}. \tag{29}$$

Then, $C = V\,(P\,V + I_q)^{-1} = (V\,P + I_r)^{-1}\,V$ is a stabilizing controller of P and we have $V = C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C$. We have recently found that such a condition for internal stabilizability was first obtained by Zames and Francis [40] for SISO systems. We shall show in Section 6 that the matrices L and $\tilde{L}$ defined in proposition 6 in fact play a more intrinsic role than V (see also [24] for SISO plants).

We give conditions on the controller C so that it internally stabilizes P.

Corollary 4 *A plant $P \in K^{q\times r}$ is internally stabilized by the controller $C \in K^{r\times q}$ iff one of the following conditions is satisfied:*

1. *The matrix defined by*

$$\Pi_1 = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \tag{30}$$

is an idempotent *of $A^{(q+r)\times(q+r)}$, namely, Π_1 satisfies the condition:*

$$\Pi_1^2 = \Pi_1 \in A^{(q+r)\times(q+r)}.$$

2. *The matrix defined by*

$$\Pi_2 = \begin{pmatrix} -P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \tag{31}$$

is an idempotent *of* $A^{(q+r)\times(q+r)}$, *namely,* Π_2 *satisfies the condition:*

$$\Pi_2^2 = \Pi_2 \in A^{(q+r)\times(q+r)}.$$

Then, Π_1 *and* Π_2 *satisfy the identity* $\Pi_1 + \Pi_2 = I_{q+r}$ *and we have*

$$\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = \Pi_1 \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}, \quad \begin{pmatrix} y_2 \\ e_2 \end{pmatrix} = \Pi_2 \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}, \tag{32}$$

where $e_1,\ e_2,\ u_1,\ u_2,\ y_1$ *and* y_2 *are defined in Fig. 1.*

Proof IS $\Rightarrow$ 1. Let us suppose that $C \in K^{r\times q}$ internally stabilizes $P \in K^{q\times r}$. Then, by 1 of Proposition 6, there exists a matrix $L = (U^T \quad V^T)^T \in A^{(q+r)\times r}$ satisfying 1.a and 1.b of Proposition 6. Let us denote by:

$$\Pi_1 = L\,(I_q \quad -P) = \begin{pmatrix} U & -U\,P \\ V & -V\,P \end{pmatrix} = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}.$$

By 1.a of Proposition 6, we obtain that $\Pi_1 \in A^{(q+r)\times(q+r)}$ and, by 1.b of Proposition 6, we have:

$$\begin{aligned} \Pi_1^2 &= L\,(I_q \quad -P)\,L\,(I_q \quad -P) = L\,((I_q \quad -P)\,L)\,(I_q \quad -P) \\ &= L\,(I_q \quad -P) = \Pi_1, \end{aligned}$$

i.e., Π_1 is an idempotent of $A^{(q+r)\times(q+r)}$.

IS $\Leftarrow$ 1. First of all, we have $\Pi_1 = \begin{pmatrix} (I_q - P\,C)^{-1} \\ C\,(I_q - P\,C)^{-1} \end{pmatrix} (I_q \quad -P)$ which gives

$$\Pi_1^2 = \begin{pmatrix} (I_q - P\,C)^{-1} \\ C\,(I_q - P\,C)^{-1} \end{pmatrix} ((I_q - P\,C)^{-1} - P\,C\,(I_q - P\,C)^{-1})\,(I_q \quad -P) = \Pi_1,$$

i.e., Π_1 is an idempotent of $K^{(q+r)\times(q+r)}$. Now, if Π_1 is an idempotent of $A^{(q+r)\times(q+r)}$, then we have

$$(I_q - P\,C)^{-1} \in A^{q\times q},\ (I_q - P\,C)^{-1}\,P \in A^{q\times r},\ C\,(I_q - P\,C)^{-1} \in A^{r\times q},$$

and $C\,(I_q - P\,C)^{-1}\,P \in A^{r\times r}$, which implies that $H(P, C) \in A^{(q+r)\times(q+r)}$, where $H(P, C)$ is defined (3), and thus, C internally stabilizes P. 2 can be proved similarly. Finally, using (26), we easily check that $\Pi_1 + \Pi_2 = I_{q+r}$ and, using (3), $y_1 = C\,e_1$ and $y_2 = P\,e_2$, we finally obtain (32). $\square$

Remark 5 The fact that Π_1 and Π_2 are two idempotents satisfying the condition $\Pi_1 + \Pi_2 = I_{q+r}$ was already proved in [9] for $A = H_\infty(\mathbb{C}_+)$. Over $A = H_\infty(\mathbb{C}_+)$, every internally stabilizable plant admits doubly coprime factorizations [12,34] (see also Remark 2 and [23]). We point out that such an assumption is not used in Corollary 4.

The two idempotents Π_1 and Π_2 play important roles in robust control and *loop-shaping procedure* [11]. For instance, for $A = H_\infty(\mathbb{C}_+)$, the *robustness radius* is defined by [9]:

$$b_{P,C} = \| \Pi_1 \|_\infty^{-1} = \| \Pi_2 \|_\infty^{-1}. \tag{33}$$

To finish, we prove that the existence of a left-/right-coprime factorization for the transfer matrix P implies internal stabilizability, finding again a well-known result [38].

Corollary 5 *1. If $P \in K^{q \times r}$ admits a left-coprime factorization*

$$P = D^{-1} N, \quad D X - N Y = I_q, \quad \det X \neq 0,$$

with $(X^T \quad Y^T)^T \in A^{(q+r)\times q}$, then $L = ((X D)^T \quad (Y D)^T)^T \in A^{(q+r)\times q}$ satisfies 1.a and 1.b of Proposition 6, and thus, $C = Y X^{-1}$ is a stabilizing controller of P.

2. If $P \in K^{q \times r}$ admits a right-coprime factorization

$$P = \tilde{N} \tilde{D}^{-1}, \quad -\tilde{Y} \tilde{N} + \tilde{X} \tilde{D} = I_r, \quad \det \tilde{X} \neq 0,$$

with $(-\tilde{Y} \quad \tilde{X}) \in A^{r \times (q+r)}$, then $\tilde{L} = (-\tilde{D} \tilde{Y} \quad \tilde{D} \tilde{X}) \in A^{r \times (q+r)}$ satisfies 2.a and 2.b of Proposition 6, and thus, $C = \tilde{X}^{-1} \tilde{Y}$ is a stabilizing controller of P.

Proof If $P = D^{-1} N$, $D X - N Y = I_q$, is a left-coprime factorization of P, then we have:

$$\begin{cases} D X - N Y = I_q \Rightarrow X - D^{-1} N Y = D^{-1} \Rightarrow X - P Y = D^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow (X D) - P (Y D) = I_q, \\ (X D) P = X N \in A^{q \times r}, \quad (Y D) P = Y N \in A^{r \times r}. \end{cases}$$

Therefore, $L = ((X D)^T \quad (Y D)^T)^T \in A^{(q+r)\times q}$ satisfies 1.a and 1.b of Proposition 6, and thus, $C = (Y D)(X D)^{-1} = Y X^{-1}$ internally stabilizes P.

2 can be proved similarly. □

5.2 Characterization in terms of fractional representations of P

In Proposition 6, we characterized internal stabilizability using only transfer matrices. In the next proposition, we characterize internal stabilizability of a plant $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ by means of a fractional representation of P.

Proposition 7 *A plant defined by $P = D^{-1} N = \tilde{N} \tilde{D}^{-1} \in K^{q \times r}$, where $R = (D \quad -N) \in A^{q \times (q+r)}$ and $\tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r)\times r}$, is internally stabilizable (IS) if one of the following equivalent assertions is satisfied:*

1. *There exists* $S = (X^T \;\; Y^T)^T \in K^{(q+r)\times q}$ *satisfying* $\det X \neq 0$ *and:*
   (a) $S\,R = \begin{pmatrix} X\,D & -X\,N \\ Y\,D & -Y\,N \end{pmatrix} \in A^{(q+r)\times(q+r)}$,
   (b) $R\,S = D\,X - N\,Y = I_q$.
   Then, the controller $C = Y\,X^{-1}$ *internally stabilizes the plant* P *and:*

$$\begin{cases} X = (D - N\,C)^{-1}, \\ Y = C\,(D - N\,C)^{-1}. \end{cases}$$

2. *There exists* $\tilde{S} = (-\tilde{Y} \;\; \tilde{X}) \in K^{r\times(q+r)}$ *satisfying* $\det \tilde{X} \neq 0$ *and:*
   (a) $\tilde{R}\,\tilde{S} = \begin{pmatrix} -\tilde{N}\,\tilde{Y} & \tilde{N}\,\tilde{X} \\ -\tilde{D}\,\tilde{Y} & \tilde{D}\,\tilde{X} \end{pmatrix} \in A^{(q+r)\times(q+r)}$,
   (b) $\tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r$.
   Then, the controller $C = \tilde{X}^{-1}\,\tilde{Y}$ *internally stabilizes the plant* P *and:*

$$\begin{cases} \tilde{X} = (\tilde{D} - C\,\tilde{N})^{-1}, \\ \tilde{Y} = (\tilde{D} - C\,\tilde{N})^{-1}\,C. \end{cases}$$

Proof IS $\Rightarrow$ 1. Let $C \in K^{r\times q}$ be a stabilizing controller of the plant P. Then, we have:

$$\begin{cases} A_1 = (I_q - P\,C)^{-1} \in A^{q\times q}, \\ A_2 = (I_q - P\,C)^{-1}\,P \in A^{q\times r}, \\ A_3 = C\,(I_q - P\,C)^{-1} \in A^{r\times q}, \\ A_4 = I_r + C\,(I_q - P\,C)^{-1}\,P \in A^{r\times r}. \end{cases}$$

Let us define the following matrices:

$$X = A_1\,D^{-1}, Y = A_3\,D^{-1}, \quad S = (X^T \;\; Y^T)^T \in K^{(q+r)\times q}.$$

Then, we have $C = A_3\,A_1^{-1} = Y\,X^{-1}$. Moreover, we have

$$\begin{aligned} R\,S &= D\,A_1\,D^{-1} - N\,A_3\,D^{-1} = (D\,A_1 - N\,A_3)\,D^{-1} \\ &= (D - N\,C)\,(I_q - P\,C)^{-1}\,D^{-1} = D\,(I_q - P\,C)\;(I_q - P\,C)^{-1}\,D^{-1} = I_q, \end{aligned}$$

and thus, we obtain $R\,S = D\,X - N\,Y = I_q$ which proves 1.b. Using this identity, we obtain

$$\begin{aligned} A_1^{-1} &= I_q - P\,C = I_q - (D^{-1}\,N)\,(Y\,X^{-1}) \\ &= D^{-1}\,(D\,X - N\,Y)\,X^{-1} = (X\,D)^{-1}, \end{aligned}$$

and thus, we have $A_1 = X\,D$, $A_2 = X\,N$, $A_3 = Y\,D$ and $A_4 = I_r + Y\,N$. Finally, we obtain

$$S\,R = \begin{pmatrix} X\,D & -X\,N \\ Y\,D & -Y\,N \end{pmatrix} = \begin{pmatrix} A_1 & -A_2 \\ A_3 & I_r - A_4 \end{pmatrix} \in A^{(q+r)\times(q+r)},$$

which proves 1.a.

IS $\Leftarrow$ 1. Let us suppose that there exists $S = (X^T \quad Y^T)^T \in K^{(q+r)\times q}$ satisfying 1.a, 1.b and $\det X \neq 0$ and let us define by $C = Y\, X^{-1}$. Then, using point 1.b, we obtain

$$I_q - P\,C = I_q - (D^{-1}\,N)\,(Y\,X^{-1}) = D^{-1}\,(D\,X - N\,Y)\,X^{-1} = (X\,D)^{-1},$$

and thus, we have:

$$\begin{cases} (I_q - P\,C)^{-1} = X\,D, \\ (I_q - P\,C)^{-1}\,P = X\,N, \\ C\,(I_q - P\,C)^{-1} = Y\,D, \\ C\,(I_q - P\,C)^{-1}\,P = Y\,N. \end{cases}$$

Hence, using 1.a and (3), we obtain

$$H(P,C) = \begin{pmatrix} X\,D & X\,N \\ Y\,D & I_r + Y\,N \end{pmatrix} \in A^{(q+r)\times(q+r)},$$

and thus, $C = Y\,X^{-1}$ internally stabilizes the plant $P = D^{-1}\,N$.

2 can be proved similarly (see [30] for more details). □

Example 16 Let us consider the transfer matrix P defined by (2). We can easily check that P admits the fractional representation $P = D^{-1}\,N$, where $R = (D \quad -N) \in A^{2\times 3}$ is defined by:

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{pmatrix}.$$

We can check that the matrix $S = (X^T \quad Y^T)^T \in K^{3\times 2}$ defined by

$$S = \begin{pmatrix} b\,\left(\frac{s-1}{s+1}\right)^2 + \frac{2}{s-1} & 2\,(b-1)\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)}{(s+1)^2} - \frac{1}{s-1} & \frac{2\,b}{s+1} + \frac{s+1}{s-1} \\ -a\,\frac{(s-1)}{(s+1)^2} & -\frac{2\,a}{s+1} \end{pmatrix},$$

where a and b are defined by (27), satisfies:

$$\begin{cases} S\,R \in A^{3\times 3}, \\ R\,S = D\,X - N\,Y = I_2. \end{cases}$$

Therefore, by 1 of Proposition 7, the controller defined by

$$\begin{aligned} C = Y\,X^{-1} &= \begin{pmatrix} -a\,\dfrac{(s-1)}{(s+1)^2} & -\dfrac{2\,a}{s+1} \end{pmatrix} \begin{pmatrix} b\,\left(\frac{s-1}{s+1}\right)^2 + \frac{2}{s-1} & 2\,(b-1)\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)}{(s+1)^2} - \frac{1}{s-1} & \frac{2\,b}{s+1} + \frac{s+1}{s-1} \end{pmatrix}^{-1}, \\ &= -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)})}\,(1 \quad 2). \end{aligned}$$

internally stabilizes P. We again find the controller obtained in Example 14.

Example 17 By 1 or 2 of proposition 7, a SISO plant defined by a transfer function $p = n/d \in K$, $0 \neq d$, $n \in A$, is internally stabilizable iff there exist $x, \ y \in K$ such that

$$\begin{cases} d\,x - n\,y = 1, \\ d\,x, \ d\,y, \ n\,x \in A, \end{cases} \tag{34}$$

as we have $n\,y = d\,x - 1 \in A$. If $x \neq 0$, then $c = y/x$ is a stabilizing controller of p whereas, if $x = 0$, then $c = 1 - y\,d$ stabilizes $p = (-1)/(y\,d)$. Hence, Proposition 7 generalizes 3 of Theorem 3 of [24] for MIMO systems.

The next corollary shows that every stabilizing controller C of P can be written in the form $C = Y\,X^{-1}$ (respectively, $C = \tilde{X}^{-1}\,\tilde{Y}$), where the matrix $S = (X^T \quad Y^T)^T$ (respectively, $\tilde{S} = (-\tilde{X} \quad \tilde{Y})$) satisfies 1.a and 1.b (respectively, 2.a and 2.b) of Proposition 7.

Corollary 6 *Let $C \in K^{r\times q}$ be a stabilizing controller of $P \in K^{q\times r}$.*

1. *Let $P = D^{-1}\,N$ be a fractional representation of P and let us define the matrices $X = (D - N\,C)^{-1} \in K^{q\times q}$ and $Y = C\,(D - N\,C)^{-1} \in K^{r\times q}$. Then, the matrix $S = (X^T \quad Y^T)^T$ satisfies 1.a and 1.b of Proposition 7 and we have $C = Y\,X^{-1}$.*
2. *Let $P = \tilde{N}\,\tilde{D}^{-1}$ be a fractional representation of P and let us define the matrices $\tilde{X} = (\tilde{D} - C\,\tilde{N})^{-1} \in K^{r\times r}$ and $\tilde{Y} = (\tilde{D} - C\,\tilde{N})^{-1}\,C \in K^{r\times q}$. Then, the matrix $\tilde{S} = (-\tilde{Y} \quad \tilde{X})$ satisfies 2.a and 2.b of Proposition 7 and we have $C = \tilde{X}^{-1}\,\tilde{Y}$.*

Proof 1. Let C be a stabilizing controller of $P = D^{-1}\,N$. Then, using the same notations as in the proof of "IS $\Rightarrow$ 1" of Proposition 7, let us define

$$\begin{cases} X = A_1\,D^{-1} = (I_q - P\,C)^{-1}\,D^{-1} = (D - N\,C)^{-1}, \\ Y = A_3\,D^{-1} = C\,(I_q - P\,C)^{-1}\,D^{-1} = C\,(D - N\,C)^{-1}, \end{cases}$$

and $S = (X^T \quad Y^T)^T \in K^{(q+r)\times q}$. Then, we have $C = A_3\,A_1^{-1} = Y\,X^{-1}$ and the result directly follows from the end of the proof of "IS $\Rightarrow$ 1" of Proposition 7.

2 can be proved similarly (see [30] for more details). □

If $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is a fractional representation of the transfer matrix P and $C \in K^{r\times q}$ a controller, then, with the notations of Corollary 6, we have

$$\Pi_1 = \begin{pmatrix} X\,D & -X\,N \\ Y\,D & -Y\,N \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} -\tilde{N}\,\tilde{Y} & \tilde{N}\,\tilde{X} \\ -\tilde{D}\,\tilde{Y} & \tilde{D}\,\tilde{X} \end{pmatrix}, \tag{35}$$

where Π_1 and Π_2 are defined in Corollary 4. Therefore, a similar result as Corollary 4 holds for Π_1 and Π_2 defined by (35).

Finally, let us give the following direct consequence of Proposition 7.

Corollary 7 *1. [38] If $P \in K^{q\times r}$ admits a left-coprime factorization*

$$P = D^{-1}\,N, \quad D\,X - N\,Y = I_q, \quad \det\,X \neq 0,$$

where $S = (X^T \quad Y^T)^T \in A^{(q+r)\times q}$, then S satisfies 1.a *and* 1.b *of Proposition 7, and thus, $C = Y\,X^{-1}$ is a stabilizing controller of P.*

2. *[38] If $P \in K^{q \times r}$ admits a right-coprime factorization*

$$P = \tilde{N}\,\tilde{D}^{-1}, \quad -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r, \quad \det\,\tilde{X} \neq 0,$$

where $\tilde{S} = (-\tilde{Y} \;\; \tilde{X}) \in A^{r \times (q+r)}$, then $\tilde{S}$ satisfies 2.a *and* 2.b *of Proposition 7, and thus, $C = \tilde{X}^{-1}\,\tilde{Y}$ is a stabilizing controller of P.*

In particular, we again find that the existence of a left-/right-coprime factorization is a necessary but not a sufficient condition for internal stabilizability (see also Corollary 5).

6 Module-theoretic interpretations of internal stabilizability

We reinterpret the characterizations of internal stabilizability obtained in Propositions 6 and 7 in terms of *projective lattices*. Such intrinsic characterizations of internal stabilizability will play an important role in the proof of Lin's conjecture [17,18].

6.1 Characterizations in terms of the transfer matrix P

Let us introduce a few definitions. See [2,32] for more details.

Definition 5 1. $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is called *a short exact sequence* of A-modules if the *A-morphisms* (namely, A-linear maps) f and g satisfy that f is injective, g is surjective and $\ker\, g = \mathrm{im}\, f$.

2. A short exact sequence is said to be a *split exact sequence* if one of the following equivalent assertions is satisfied:
   - there exists an A-morphism $h : M'' \to M$ which satisfies $g \circ h = id_{M''}$,
   - there exists an A-morphism $k : M \to M'$ which satisfies $k \circ f = id_{M'}$,
   - there exist two A-morphisms $h : M'' \to M$ and $k : M \to M'$ such that the A-morphisms defined by

$$\phi = \begin{pmatrix} g \\ k \end{pmatrix} : M \longrightarrow M'' \oplus M', \quad \psi = (h \;\; f) : M'' \oplus M' \longrightarrow M$$

   satisfy the following relations:

$$\begin{cases} \phi \circ \psi = \begin{pmatrix} g \\ k \end{pmatrix} (h \;\; f) = \begin{pmatrix} id_{M''} & 0 \\ 0 & id_{M'} \end{pmatrix} = id_{M'' \oplus M'}, \\ \psi \circ \phi = (h \;\; f) \begin{pmatrix} g \\ k \end{pmatrix} = h \circ g + f \circ k = id_M. \end{cases}$$

   Then, we say that M is *isomorphic* to $M'' \oplus M'$, which will be denoted by $M \cong M'' \oplus M'$ ($\oplus$ denotes the *direct sum* of A-modules).

   Finally, we denote a split exact sequence by the following diagram:

$$\begin{array}{c} 0 \longleftarrow M'' \overset{g}{\longleftarrow} M \overset{f}{\longleftarrow} M' \longleftarrow 0. \\ \quad\;\; \underset{h}{\longrightarrow} \quad\; \underset{k}{\longrightarrow} \end{array} \tag{36}$$

3. An A-module M is said to be *finitely generated* if M admits a finite family of generators.
4. A finitely generated A-module M is said to be *free* if M admits a finite basis or, equivalently, if M is isomorphic to a finite power of A, i.e., there exists $r \in \mathbb{Z}_+$ such that $M \cong A^r$.
5. A finitely generated A-module M is said to be *projective* if there exists an A-module M' and $r \in \mathbb{Z}_+$ such that we have $M \oplus M' \cong A^r$, i.e., if M is a direct summand of a finitely generated free A-module.
6. The rank of an A-module M, denoted by $\mathrm{rk}_A(M)$, is the dimension of the $K = Q(A)$-vector space $K \otimes_A M$ formed by extending the scalars of M from A to K. In other words, we have $\mathrm{rk}_A(M) = \dim_K(K \otimes_A M)$, where $\otimes_A$ denotes the *tensor product* of A-modules.

Example 18 Let us consider the following A-morphism (A-linear map):

$$\begin{aligned} g : A^{q+r} &\longrightarrow \mathcal{L} = (I_q \quad -P)\, A^{q+r}, \\ \lambda &\longmapsto (I_q \quad -P)\, \lambda. \end{aligned}$$

Then, g is surjective and, using Example 9, its kernel is defined by:

$$\begin{aligned} \ker g &= \{\lambda = (\lambda_1^T \quad \lambda_2^T)^T \in A^{q+r} \mid \lambda_1 = P\,\lambda_2\} \\ &= \{((P\,\lambda_2)^T \quad \lambda_2^T)^T \in A^{q+r} \mid \lambda_2 \in A^r : \ P\lambda_2 \in A^q\} \\ &= \begin{pmatrix} P \\ I_r \end{pmatrix} \{\lambda_2 \in A^r \mid P\lambda_2 \in A^q\} = \begin{pmatrix} P \\ I_r \end{pmatrix} (A : \mathcal{M}). \end{aligned}$$

Therefore, we have the following exact sequence

$$0 \longleftarrow \mathcal{L} \xleftarrow{g} A^{q+r} \xleftarrow{f} A : \mathcal{M} \longleftarrow 0, \tag{37}$$

where f is defined by $f(\lambda) = (P^T \quad I_r^T)^T\, \lambda$.

Similarly, we can prove that we have the following exact sequence

$$0 \longrightarrow A : \mathcal{L} \xrightarrow{\phi} A^{1\times(q+r)} \xrightarrow{\psi} \mathcal{M} \longrightarrow 0, \tag{38}$$

where the A-morphisms ϕ and ψ are respectively defined by:

$$\begin{aligned} \phi :\ A : \mathcal{L} &\longrightarrow A^{1\times(q+r)}, & \psi : A^{1\times(q+r)} &\longrightarrow \mathcal{M}, \\ \lambda &\longmapsto \lambda\,(I_q \quad -P), & \mu &\longmapsto \mu\,(P^T \quad I_r^T)^T. \end{aligned}$$

We have the following standard lemma in module theory.

Lemma 4 *[2,32] Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a short exact sequence and* M'' *a projective* A*-module. Then, the exact sequence splits and we have*

$$M \cong M' \oplus M''.$$

The next theorem intrinsically characterizes internal stabilizability.

Theorem 3 *A plant* $P \in K^{q\times r}$ *is internally stabilizable iff one of the following equivalent assertions is satisfied:*

1. $\mathcal{L}$ *is a projective* A*-submodule of* K^q *of rank* q*, namely,* $\mathcal{L}$ *is a projective lattice of* K^q.
2. $\mathcal{M}$ *is a projective* A*-submodule of* $K^{1\times r}$ *of rank* r*, namely,* $\mathcal{M}$ *is a projective lattice of* $K^{1\times r}$.

Proof IS $\Rightarrow$ 1. If P is internally stabilizable then, by 1 of Proposition 6, there exists $L = (U^T \quad V^T)^T \in A^{(q+r)\times q}$ such that $L\,(I_q \quad -P) \in A^{(q+r)\times(q+r)}$ and $(I_q \quad -P)\,L = I_q$, and thus, there exists an A-morphism

$$\begin{aligned} h : \mathcal{L} &\longrightarrow A^{q+r},\\ \mu &\longmapsto L\,\mu, \end{aligned} \tag{39}$$

satisfying $g \circ h = id_{\mathcal{L}}$. Hence, by 2 of Definition 5, (37) is a split exact sequence. Therefore, we have

$$A^{q+r} \cong \mathcal{L} \oplus (A : \mathcal{M}), \tag{40}$$

which, by 3 of Definition 5, shows that $\mathcal{L}$ is a projective A-module.

IS $\Leftarrow$ 1. If $\mathcal{L}$ is a projective A-module, then, by Lemma 4, (37) is a split exact sequence because (37) ends with a projective A-module. Therefore, there exists an A-morphism $h : \mathcal{L} \to A^{q+r}$ such that $g \circ h = id_{\mathcal{L}}$. But, by 2 of Proposition 5, we have

$$h \in \hom_A(\mathcal{L}, A^{q+r}) \cong A^{q+r} : \mathcal{L} = \{\Lambda \in A^{(q+r)\times q} \mid \Lambda\, P \in A^{(q+r)\times r}\},$$

i.e., $h(\lambda) = L\,\lambda$ for a certain matrix $L \in A^{(q+r)\times q}$ satisfying $L\,P \in A^{(q+r)\times r}$. Moreover, $g \circ h = id_{\mathcal{L}}$ implies that we have $\lambda = ((I_q \quad -P)\,L)\,\lambda$ for every $\lambda \in \mathcal{L}$. Using the fact that $A^q \subseteq \mathcal{L}$, we finally obtain that $((I_q \quad -P)\,L)\,e_i = e_i$, for $i = 1, \ldots, q$, where e_i denotes the ith vector of the standard basis of A^q, which proves that $(I_q \quad -P)\,L = I_q$, and thus, the result by 1 of Proposition 6.

Similarly, 2 can be proved using the exact sequence (38) and 2 of Proposition 6: P is internally stabilizable iff there exists $\tilde{L} = (-\tilde{V} \quad \tilde{U}) \in A^{r\times(q+r)}$ satisfying 2.a and 2.b of Proposition 6, i.e., iff there exists an A-morphism φ defined by

$$\begin{aligned} \varphi : \mathcal{M} &\longrightarrow A^{1\times(q+r)},\\ \nu &\longmapsto \nu\,\tilde{L}, \end{aligned} \tag{41}$$

which satisfies $\psi \circ \varphi = id_{\mathcal{M}}$, i.e., iff $\mathcal{M}$ is a projective A-module. □

Example 19 Theorem 3 generalizes the following result obtained in [24]: $p \in K$ is internally stabilizable iff the fractional ideal $\mathcal{L} = \mathcal{M} = \{\alpha - \beta\, p \mid \alpha,\ \beta \in A\}$ of A is *invertible* [2,24,32], namely, we have

$$\mathcal{L}\,(A : \mathcal{L}) = \{a - b\, p \mid a,\ b \in A : \mathcal{L}\} = A,$$

or, equivalently, iff there exist $a,\ b \in A : \mathcal{L}$ such that $a - b\,p = 1$.

Remark 6 We proved in [23] that every transfer matrix is IS iff A is a *Prüfer domain* [2,32], namely, A is an integral domain such that, for every $p \in Q(A)$, $\mathcal{L} = A + A\,p$ is an invertible fractional ideal of A (see [2,23,32] for different characterizations). Moreover, we can show that an integral domain A is a Bézout domain (see Remark 3) iff A is a Prüfer domain and a coherent Sylvester domain (see Remark 2) [23].

Proposition 8 *Let $P \in K^{q\times r}$ be an internally stabilizable plant.*

1. *If $L \in A^{(q+r)\times q}$ is a matrix satisfying 1 of proposition 6, then we have:*

$$A : \mathcal{L} = A^{1\times(q+r)}\, L, \tag{42}$$
$$\mathcal{L} = A : (A : \mathcal{L}) \tag{43}$$
$$= \{\lambda \in K^q \mid L\,\lambda \in A^{q+r}\}. \tag{44}$$

2. *If $\tilde{L} \in A^{r\times(q+r)}$ is a matrix satisfying 2 of proposition 6, then we have:*

$$A : \mathcal{M} = \tilde{L}\, A^{q+r}, \tag{45}$$
$$\mathcal{M} = A : (A : \mathcal{M}) \tag{46}$$
$$= \{\lambda \in K^{1\times r} \mid \lambda\, \tilde{L} \in A^{1\times(q+r)}\}. \tag{47}$$

Proof Let us firstly prove (42). Let us consider $\lambda \in A : \mathcal{L}$, i.e., $\lambda \in A^{1\times q}$ satisfies $\lambda\, P \in A^{1\times r}$, and let us denote by $\mu = \lambda\,(I_q \quad -P)$. Then, we have $\mu \in A^{1\times(q+r)}$. Using 1.b of Proposition 6, we obtain $\lambda = (\lambda\,(I_q \quad -P))\,L = \mu\, L$ showing that $\lambda \in A^{1\times(q+r)}\, L$.

Conversely, if $\lambda \in A^{1\times(q+r)}\, L$, then there exists $\mu \in A^{1\times(q+r)}$ such that $\lambda = \mu\, L$ and $\lambda \in A^{1\times q}$. By 1.a of Proposition 6, we have $L\, P \in A^{(q+r)\times r}$, and thus, $\lambda\, P = \mu\,(L\, P) \in A^{1\times r}$, i.e., $\lambda \in A : \mathcal{L}$.

Let us secondly prove (43) and (44). Using (42), we obtain that:

$$A : (A : \mathcal{L}) = A : (A^{1\times(q+r)}\, L) = \{\lambda \in K^q \mid L\,\lambda \in A^{q+r}\}.$$

Finally, let us prove that $\mathcal{L} = A : (A : \mathcal{L})$. Let us consider $\kappa \in \mathcal{L}$, i.e., $\kappa = (I_q \quad -P)\,\mu$ for a certain $\mu \in A^{q+r}$. Then, using 1.a of Proposition 6, we obtain $L\,\kappa = (L\,(I_q \quad -P))\,\mu \in A^{q+r}$, i.e., $\kappa \in \{\lambda \in K^q \mid L\,\lambda \in A^{q+r}\}$.

Conversely, if $\kappa \in K^q$ is such that $\mu = L\,\kappa \in A^{q+r}$, then, using 1.b of Proposition 6, we obtain $\kappa = ((I_q \quad -P)\,L)\,\kappa = (I_q \quad -P)\,\mu \in \mathcal{L}$.

(45), (46) and (47) can be proved similarly. □

Remark 7 We point out that Proposition 8 could be directly proved without any computations. Indeed, the fact that P is internally stabilizable implies that the exact sequences (37) and (38) split and then they are dual with each other. Moreover, if P is internally stabilizable, then, using the proof of Theorem 3 and Proposition 8, we easily obtain the following results

$$\begin{cases} A^{q+r} = h(\mathcal{L}) \oplus f(A : \mathcal{M}) = h(\mathcal{L}) \oplus (\Pi_2\, A^{q+r}), \\ A^{1\times(q+r)} = \varphi(\mathcal{M}) \oplus \phi(A : \mathcal{L}) = \varphi(\mathcal{M}) \oplus (A^{1\times(q+r)}\,\Pi_1), \end{cases}$$

where h (resp., φ) is defined by (39) (respectively, (41)) and Π_1 (respectively Π_2) is defined (30) (respectively (31)).

Let us characterize strong stabilizability in terms of lattices.

Proposition 9 *A plant $P \in K^{q\times r}$ is strongly stabilizable iff there exists a controller $C \in A^{r\times q}$ such that we have $\mathcal{L} = (I_q - P\,C)\,A^q$ or $\mathcal{M} = A^{1\times r}\,(I_r - C\,P)$. Then, $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is a doubly coprime factorization of P, where:*

$$D = (I_q - P\,C)^{-1} \in A^{q\times q},\ N = (I_q - P\,C)^{-1}\,P \in A^{q\times r},\ \tilde{D} = (I_r - C\,P)^{-1} \in A^{r\times r}$$

and $\tilde{N} = P\,(I_r - C\,P)^{-1} \in A^{r\times q}$.

Proof $\Rightarrow$ Let us suppose that P is strongly stabilizable. Then, there exists a controller $C \in A^{r\times q}$ such that the matrix defined by $L = (U^T \;\; V^T)^T \in A^{(q+r)\times q}$, where $U = C\,(I_q - P\,C)^{-1}$ and $V = (I_q - P\,C)^{-1}$, satisfies 1 of Proposition 6. But, by Proposition 8, we know that $\mathcal{L} = A : (A^{1\times(q+r)}\,L) = \{\lambda \in K^q \mid L\,\lambda \in A^{q+r}\}$. Using the facts that $L = (I_q^T \;\; C^T)^T\,(I_q - P\,C)^{-1}$ and $C \in A^{r\times q}$, we obtain that $\mathcal{L} = \{\lambda \in K^q \mid (I_q - P\,C)^{-1}\,\lambda \in A^q\} = (I_q - P\,C)\,A^q$.

$\Leftarrow$ If there exists $C \in A^{r\times q}$ such that $\mathcal{L} = (I_q - P\,C)\,A^q$, then there exists $D \in A^{q\times q}$ and $N \in A^{q\times r}$ satisfying:

$$\begin{cases} I_q = (I_q - P\,C)\,D, \\ -P = (I_q - P\,C)\,(-N) \end{cases} \Rightarrow \begin{cases} D = (I_q - P\,C)^{-1} \in A^{q\times q}, \\ N = (I_q - P\,C)^{-1}\,P \in A^{q\times r}. \end{cases}$$

Finally, using the fact that $C \in A^{r\times q}$, we obtain $C\,(I_q - P\,C)^{-1} \in A^{r\times q}$ and $C\,(I_q - P\,C)^{-1}\,P \in A^{r\times r}$, which shows that C strongly stabilizes P.

A similar result can be proved with the lattice $\mathcal{M}$ of $K^{1\times r}$. $\square$

Remark 8 If we use the notations of the proof of Proposition 9, from the trivial inclusion $(I_q - P\,C)\,A^q \subseteq \mathcal{L}$, we obtain that $\mathcal{L} = (I_q - P\,C)\,A^q$ iff we have $D \in A^{q\times q}$ and $N \in A^{q\times r}$. But, using the identity $D = I_q + N\,C$, we finally obtain that P is strongly stabilizable iff there exists $C \in A^{r\times q}$ such that $N = (I_q - P\,C)^{-1}\,P = P\,(I_r - C\,P)^{-1} \in A^{q\times r}$. See [25] for more details on the strong stabilization problem.

Now, let us show that the existence of a weak doubly coprime factorization for an internally stabilizable plant P implies the existence of a doubly coprime factorization of P.

Corollary 8 *[23] A plant $P \in K^{q\times r}$ is internally stabilizable and admits a weakly left-coprime (respectively, weakly right-coprime) factorization $P = D^{-1}\,N$ (respectively, $P = \tilde{N}\,\tilde{D}^{-1}$) iff $P = D^{-1}\,N$ (respectively, $P = \tilde{N}\,\tilde{D}^{-1}$) is a left-coprime (respectively, right-coprime) factorization of P.*

Proof Let us suppose that the stabilizable plant P admits a weakly left-coprime factorization $P = D^{-1}\,N$. Then, by 1 of Theorem 1, we obtain $A : \mathcal{L} = A^{1\times q}\,D$. Moreover, there exists a matrix $L \in A^{(q+r)\times q}$ which satisfies 1 of Proposition 6 and, by Proposition 8, we obtain

$$\mathcal{L} = A : (A : \mathcal{L}) = A : (A^{1\times q}\,D) = D^{-1}\,A^q,$$

which, by 1 of Theorem 2, proves that $P = D^{-1}\,N$ is a coprime factorization. The converse holds as a left-coprime factorization is a weakly left-coprime factorization (see Remark 1 or Corollary 3) and the existence of a left-coprime factorization implies internal stabilizability by Corollaries 5 and 7. $\square$

6.2 Characterizations in terms of fractional representations of P

Let us give an intrinsic reformulation of Proposition 7 by finding again a result first obtained by Sule [36,37]. See also [23] for different proofs.

Theorem 4 *[36,37] The plant $P = D^{-1} N = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times r}$ is internally stabilizable iff one of the following equivalent assertions is satisfied:*

1. *$\mathcal{P}$ is a projective A-submodule of K^q of rank q, namely, $\mathcal{P}$ is a projective lattice of K^q.*
2. *$\mathcal{Q}$ is projective A-submodule of $K^{1\times r}$ of rank r, namely, $\mathcal{Q}$ is a projective lattice of $K^{1\times r}$.*

Proof Let $P = D^{-1} N$ be a fractional representation of P and let us denote by $R = (D \; - N) \in A^{q \times (q+r)}$. Then, we have the following exact sequence

$$\begin{array}{c} 0 \longrightarrow \ker(R.) \xrightarrow{i} A^{q+r} \xrightarrow{R.} \mathcal{P} = R\, A^{q+r} \longrightarrow 0 \\ \lambda \longmapsto R\,\lambda, \end{array} \tag{48}$$

i.e., the A-morphism $(R.) : A^{q+r} \to \mathcal{P}$ is surjective and i is the canonical injection.

IS $\Rightarrow$ 1. If P is internally stabilizable then, by Proposition 7, there exists a matrix $S = (X^T \;\; Y^T)^T \in K^{(q+r)\times q}$ which satisfies 1.a and 1.b. Hence, we can define the following A-morphism:

$$\begin{array}{c} \mathcal{P} \xrightarrow{S.} A^{q+r} \\ \mu \longmapsto \begin{pmatrix} X \\ Y \end{pmatrix} \mu. \end{array}$$

Indeed, $(S.)$ is trivially A-linear and it is well-defined as, for all $\mu \in \mathcal{P}$, there exists a certain $\nu \in A^{q+r}$ satisfying $\mu = R\,\nu$, and thus, using the fact that we have $S\,R \in A^{(q+r)\times(q+r)}$, we obtain $S\,\mu = (S\,R)\,\nu \in A^{q+r}$. Moreover, 1.b means that $(R.) \circ (S.) = (R\,S). = (I_q.)$, i.e., the A-morphism $(S.)$ is a right-inverse of the A-morphism $(R.)$. Therefore, the exact sequence (48) splits and we have $\mathcal{P} \oplus \ker(R.) \cong A^{q+r}$. Therefore, $\mathcal{P}$ is a direct summand of the free A-module A^{q+r}, i.e., $\mathcal{P}$ is a projective A-module.

IS $\Leftarrow$ 1. We suppose that $\mathcal{P}$ is a projective A-module. Then, by Lemma 4, the short exact sequence (48) splits, and thus, there exists an A-morphism $s : \mathcal{P} \to A^{q+r}$ such that $(R.) \circ s = id_{\mathcal{P}}$. But, using Proposition 5, we have:

$$s \in \hom_A(\mathcal{P}, A^{q+r}) \cong A^{q+r} : \mathcal{P} = \{T \in K^{(q+r)\times q} \mid T\,R \in A^{(q+r)\times(q+r)}\}.$$

Therefore, there exists a matrix $S \in K^{(q+r)\times q}$ satisfying $S\,R \in A^{(q+r)\times(q+r)}$ and such that $s(\lambda) = S\,\lambda$ for all $\lambda \in (R\,A^{q+r})$. Hence, we have

$$(R.) \circ s = (R.) \circ (S.) = (R\,S). = id_{R\,A^{q+r}},$$

and thus, we obtain $(R\,S)(R\,A^{q+r}) = R\,A^{q+r}$, i.e., $R\,S\,R = R$. Finally, we have $(R\,S - I_q)\,R = 0$ and using the fact that $R = (D - N)$ has full row rank, we obtain $R\,S = I_q$. Then, by 1 of Proposition 6, we obtain that P is internally stabilizable.

A similar proof can be obtained using the A-module $\mathcal{Q}$ and the following short exact sequence:

$$\begin{array}{c} 0 \longrightarrow \ker(.\tilde{R}) \longrightarrow A^{1\times(q+r)} \xrightarrow{.\tilde{R}} \mathcal{Q} = A^{1\times(q+r)}\,\tilde{R} \longrightarrow 0, \\ \lambda \longmapsto \lambda\,\tilde{R}. \end{array} \tag{49}$$

□

Example 20 If $p = n/d$, $0 \neq d, n \in A$, is a fractional representation of a transfer function $p \in K = Q(A)$, then, by Theorem 4, we know that p is internally stabilizable iff the lattice $\mathcal{P} = \mathcal{Q} = A\,d + A\,n$ of K is projective or, equivalently, iff the fractional ideal $\mathcal{P}$ is invertible [2,24], namely, we have

$$\mathcal{P}\,(A : \mathcal{P}) = \{d\,x - n\,y \mid x,\ y \in A : \mathcal{P}\} = A,$$

i.e., iff there exist $x, y \in A : \mathcal{P}$ such that $d\,x - n\,y = 1$. This characterization was proved in [23] to be equivalent to the first general characterization of internal stabilizability for SISO plants obtained in the pioneering work [33].

Let us characterize the A-modules $\ker(R.)$ and $\ker(.\tilde{R})$.

Lemma 5 *Let $P \in K^{q\times r}$ be a transfer matrix, $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ a fractional representation of P, and:*

$$R = (D \quad -N) \in A^{q\times(q+r)}, \quad \tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r)\times r}.$$

Then, we have

$$\begin{cases} \ker(.\tilde{R}) = (A : \mathcal{P})\,R, \\ \ker(R.) = \tilde{R}\,(A : \mathcal{Q}), \end{cases}$$

and the following two short exact sequences:

$$0 \longleftarrow \mathcal{Q} \xleftarrow{.\tilde{R}} A^{1\times(q+r)} \xleftarrow{.R} A : \mathcal{P} \longleftarrow 0, \tag{50}$$

$$0 \longrightarrow A : \mathcal{Q} \xrightarrow{\tilde{R}.} A^{q+r} \xrightarrow{R.} \mathcal{P} \longrightarrow 0. \tag{51}$$

Proof Let us prove the first equality (the second can be obtained similarly). By Proposition 2.8 of part I of [23], we know that $\ker(.\tilde{R}) = \overline{A^{1\times q}\,R}$, where $\overline{A^{1\times q}\,R}$ denotes the A-*closure* of $A^{1\times q}\,R$ in $A^{1\times(q+r)}$, namely:

$$\overline{A^{1\times q}\,R} = \{\lambda \in A^{1\times(q+r)} \mid \exists\, 0 \neq a \in A,\ a\,\lambda \in A^{1\times q}\,R\}.$$

By Lemma 2.3 of part I of [23], we also know that:

$$\overline{A^{1\times q}\,R} = (K^{1\times q}\,R) \cap A^{1\times(q+r)}.$$

But, from Example 10, we have $A : \mathcal{P} = \{\lambda \in K^{1\times q} \mid \lambda\,R \in A^{1\times(q+r)}\}$ showing that:

$$(K^{1\times q}\,R) \cap A^{1\times(q+r)} = \{\lambda\,R \in A^{1\times(q+r)} \mid \lambda \in K^{1\times q}\} = (A : \mathcal{P})\,R.$$

Then, the exactness of (50) directly follows from the exact sequence (48).

A similar proof can be given for $\ker(R.) = \tilde{R}\,(A : \mathcal{Q})$ and (51). □

Remark 9 If $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is an internally stabilizable plant, then it follows from Lemma 5 and the proof of Theorem 4 that the direct complement to $\mathcal{P}$ (respectively $\mathcal{Q}$) in A^{q+r} is defined by $\tilde{R}\,(A : \mathcal{Q})$ (respectively $(A : \mathcal{P})\,R$).

In the case where the plant P is internally stabilizable, then the A-modules $A : \mathcal{P}$ and $A : \mathcal{Q}$ can be characterized in terms of the matrices S and $\tilde{S}$ defined in Proposition 7.

Proposition 10 *Let P be an internally stabilizable plant, $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}$ a fractional representation of P and:*

$$R = (D \;\; -N) \in A^{q \times (q+r)}, \quad \tilde{R} = (\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{(q+r) \times r}.$$

1. *If S is a matrix satisfying 1 of Proposition 7, then we have:*

$$A : \mathcal{P} = A^{1 \times (q+r)}\, S, \tag{52}$$
$$\mathcal{P} = A : (A : \mathcal{P}) = \{\lambda \in K^q \mid S\, \lambda \in A^{q+r}\}. \tag{53}$$

2. *If $\tilde{S}$ is a matrix satisfying 2 of Proposition 7, then we have:*

$$A : \mathcal{Q} = \tilde{S}\, A^{q+r}, \tag{54}$$
$$\mathcal{Q} = A : (A : \mathcal{Q}) = \{\lambda \in K^{1 \times r} \mid \lambda\, \tilde{S} \in A^{1 \times (q+r)}\}. \tag{55}$$

Hence, we have

$$\begin{cases} \ker(.\tilde{R}) = A^{1 \times (q+r)}\, (S\, R) = A^{1 \times (q+r)}\, \Pi_1, \\ \ker(R.) = (\tilde{R}\, \tilde{S})\, A^{q+r} = \Pi_2\, A^{q+r}, \end{cases}$$

where Π_1 and Π_2 are the idempotents of $A^{(q+r) \times (q+r)}$ defined by (35).

Proof The proof is exactly the same as the one given for Proposition 8. □

Finally, similarly as Proposition 9, using Proposition 10, we can easily prove the following result.

Proposition 11 *A plant $P \in K^{q \times r}$ is strongly stabilizable iff there exists a controller $C \in A^{r \times q}$ such that we have $\mathcal{P} = (D - N\, C)\, A^q$ or $\mathcal{Q} = A^{1 \times r}\, (\tilde{D} - C\, \tilde{N})$. Then, $P = (D')^{-1}\, N' = \tilde{N}'\, (\tilde{D}')^{-1}$ is a doubly coprime factorization of P, with the notations $D' = (D - N\, C)^{-1}\, D \in A^{q \times q}$, $N' = (D - N\, C)^{-1}\, N \in A^{q \times r}$, $\tilde{D}' = \tilde{D}\, (\tilde{D} - C\, \tilde{N})^{-1} \in A^{r \times r}$ and $\tilde{N}' = \tilde{N}\, (\tilde{D} - C\, \tilde{N})^{-1} \in A^{r \times q}$.*

Remark 10 As in Remark 8, we can prove that $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times r}$ is strongly stabilizable iff there exists $C \in A^{r \times q}$ such that

$$(D - N\, C)^{-1}\, N = \tilde{N}\, (\tilde{D} - C\, \tilde{N})^{-1} \in A^{q \times r}.$$

We also refer to [25] for more information on the strong stabilization problem.

6.3 Proof of Lin's conjecture

If we take $M' = 0$ in 5 of Definition 5, we then obtain that a free A-module is a projective A-module. Hence, using Theorems 2 and 3, we again obtain the existence of a left-/right-coprime factorization for the transfer matrix P is a sufficient condition for internal stabilizability (but generally not a necessary one). See Corollaries 5 and 7 for more details.

Corollary 9 *[23] If A is a* projective-free ring *[15], namely, a ring such that any finitely generated projective A-module is free, then every internally stabilizable plant admits a doubly coprime factorization. In particular, this result holds for* $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$ *and* $H_\infty(\mathbb{D})$.

Corollary 9 directly follows from Theorems 2, 3 and 4 and Corollary 2. However, we do not know whether or not the converse of Corollary 9 is true, i.e., whether or not the fact that every internally stabilizable plant admits a doubly coprime factorization implies that A is projective-free (at least, we know that every projective A-module of the form $\mathcal{P} = R\,A^{q+r}$ is free, where $R \in A^{q\times(q+r)}$ has full row rank). Such an important question will be investigated in the future.

Remark 11 We note the fact that every internally stabilizable plant over $H_\infty(\mathbb{C}_+)$ admits a doubly coprime factorization was first proved by Inouye [12] and Smith [34] by investigating the properties of the ring $H_\infty(\mathbb{C}_+)$. In particular, we can prove that $H_\infty(\mathbb{C}_+)$ is a coherent Sylvester domain (see Remark 2 and [23]), and thus, a projective-free ring as a coherent Sylvester domain is a projective-free coherent domain with $\mathrm{w.gl.dim}(A) \leq 2$, where $\mathrm{w.gl.dim}(A)$ denotes the *weak global dimension*. See [7,23] for more details.

In a series of papers (see [17,18] and the references therein), Lin has conjectured that every internally stabilizable multidimensional linear systems admits doubly coprime factorizations. The next corollary gives a positive answer to this conjecture.

Theorem 5 *[29] The ring* $\mathbb{R}(z_1,\ldots,z_m)_S$ *of structural stable linear multidimensional systems is projective-free (Corollary 2.2.4 of [3] and Theorem A.3 of [13]). Therefore, every internally stabilizable plant defined by a transfer matrix* $P \in K^{q\times r}$, *where* $K = \mathbb{R}(z_1,\ldots,z_m)$, *admits a doubly coprime factorization over* $\mathbb{R}(z_1,\ldots,z_m)_S$.

The fact that the ring $\mathbb{R}(z_1,\ldots,z_m)_S$ is projective-free is a non-trivial result explaining why the equivalence between internal stabilizability and the existence of coprime factorizations for multidimensional systems was still open up to now (the central part of the proof in [13] comes from a private communication by A. Tannenbaum to the Fields medalist P. Deligne). However, we do not have a constructive proof of Corollary 2.2.4 of [3] and of theorem A.3 of [13]. Such a problem needs to be investigated in the future.

Corollary 10 *[29] The ring* $\mathbb{R}(z_1,z_2)_S$ *is a coherent Sylvester domain.*

Therefore, $\mathbb{R}(z_1,z_2)_S$ *is a* greatest common divisor domain, *namely, any two elements in* $\mathbb{R}(z_1,z_2)_S$ *have a greatest common divisor, and every transfer matrix with entries in* $\mathbb{R}(z_1,z_2)$ *admits a doubly weakly coprime factorization.*

Proof From Remark 11 (see [7,23] for more details), we need to prove that $\mathbb{R}(z_1, z_2)_S$ is a projective-free coherent domain with $\text{w.gl.dim}(\mathbb{R}(z_1, z_2)_S) \leq 2$. First of all, if we denote by $S = \{d \in \mathbb{R}[z_1, z_2] \mid \forall\, (z_1, z_2) \in \overline{\mathbb{D}}^2,\ d(z_1, z_2) \neq 0\}$, then we have [3,13]:

$$\mathbb{R}(z_1, z_2)_S = S^{-1}\,\mathbb{R}[z_1, z_2] \triangleq \{n/d \mid n \in \mathbb{R}[z_1, z_2],\ d \in S\}.$$

But, $\mathbb{R}[z_1, z_2]$ is a *noetherian ring* [2,23,32] and any localization of a noetherian ring by a multiplicatively closed subset of $\mathbb{R}[z_1, z_2]$ is also noetherian [2,32]. Therefore, $\mathbb{R}(z_1, z_2)_S = S^{-1}\,\mathbb{R}[z_1, z_2]$ is a noetherian ring, and thus, a coherent ring [2,23,32]. Moreover, it is a well known fact in homological algebra that $\text{w.gl.dim}(\mathbb{R}(z_1, z_2)_S) = \text{w.gl.dim}(S^{-1}\,\mathbb{R}[z_1, z_2]) \leq \text{w.gl.dim}(\mathbb{R}[z_1, z_2])$ and $\text{w.gl.dim}(\mathbb{R}[z_1, z_2]) = 2$ [2,32]. Finally, by theorem 5, we know that $\mathbb{R}(z_1, z_2)_S$ is projective-free, proving that $\mathbb{R}(z_1, z_2)_S$ is a coherent Sylvester domain.

Using the fact that a coherent Sylvester domain is the greatest common divisor domain (see [23] for a proof), we deduce that $\mathbb{R}(z_1, z_2)_S$ is the greatest common divisor domain.

Finally, the fact that every transfer matrix with entries in $\mathbb{R}(z_1, z_2)$ admits a doubly weakly coprime factorization directly follows from Remark 2 (see [23] for a proof). □

Let us point out that the previous result is no more true if we consider the ring $\mathbb{R}(z_1, \ldots, z_m)_S$ where $m \geq 3$.

Let us introduce the concept of a stably free module.

Definition 6 [32] A finitely generated A-module M is called *stably free* if there exists $r,\ s \in \mathbb{Z}_+$ such that $M \oplus A^s \cong A^r$.

A free module is then stably free and a stably free module is projective.

Corollary 11 *[38] A is a* Hermite ring *[15], namely, every finitely generated stably free A-module is free, iff every transfer matrix which admits a left-coprime (respectively, right-coprime) factorization admits a doubly coprime factorization.*

In particular, this result holds for the rings $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$, $H_\infty(\mathbb{D})$, W_+, $\mathbb{R}(z_1, \ldots, z_m)_S$ and the disc algebra $A(\mathbb{D})$ of the bounded holomorphic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ which are continuous on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ [30].

Proof $\Rightarrow$. If $P = D^{-1}\,N$ is a left-coprime factorization of P and if we denote by $R = (D\,{-}\,N) \in A^{q\times(q+r)}$, then we have $\mathcal{P} = R\,A^{q+r} = A^q$ by Lemma 3, i.e., $\mathcal{P}$ is free, and thus, a projective A-module. Therefore, by Lemma 4, the exact sequence (51) splits, and thus, there exists $S \in A^{(q+r)\times q}$ such that $S\,R \in A^{(q+r)\times(q+r)}$ and $R\,S = I_q$. Hence, we have $(.R) \circ (.S) = .(S\,R) = .I_q$, and thus, the exact sequence (50) splits and we obtain

$$A^{1\times(q+r)} \cong \mathcal{Q} \oplus (A : A^q) \;\Leftrightarrow\; A^{1\times(q+r)} \cong \mathcal{Q} \oplus A^{1\times q},$$

i.e., $\mathcal{Q}$ is a stably free A-module of rank $q + r - q = r$. Then, using the fact that A is a Hermite ring, $\mathcal{Q}$ is a free A-module of rank r. Then, the result directly follows from Corollary 2.

$\Leftarrow$ Let $R = (d \; - n_1 \cdots - n_{r-1}) \in A^{1\times r}$ be such that there exists $S \in A^r$ satisfying $R\,S = 1$ and let us suppose without loss of generality that $d \neq 0$. Then, the transfer matrix $P = (n_1/d \cdots n_{r-1}/d) \in K^{1\times(r-1)}$ admits the left-coprime factorization $P = d^{-1}\,(n_1 \cdots n_{r-1})$, and thus, a doubly coprime factorization by hypothesis. Therefore, there exists two matrices $\tilde{R} = (\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{r\times(r-1)}$ and $\tilde{S} = (-\tilde{Y} \;\; \tilde{X}) \in A^{(r-1)\times r}$ such that $(R^T \;\; \tilde{S}^T)^T\,(S \;\; \tilde{R}) = I_r$. Therefore, for every $r > 0$ and every $R \in A^{1\times r}$ which admits a right-inverse $S \in A^r$, there exists a matrix $U \in A^{r\times r}$ such that $U^{-1} \in A^{r\times r}$ and $R = (1 \;\; 0\cdots0)\,U$. It is known in commutative algebra that this fact is an equivalent formulation for A to be a Hermite ring [15,32].

A projective-free ring is in particular a Hermite ring. Thus, from Corollary 9 and Theorem 5, we obtain that $\mathbb{R}(z_1,\ldots,z_m)_S$, RH_∞, $H_\infty(\mathbb{C}_+)$ and $H_\infty(\mathbb{D})$ are Hermite rings. Finally, the *stable range* of $A = W_+$ (respectively, $A(\mathbb{D})$) is equal to 1 (see [25,30] for more details), namely, whenever we have $A\,a + A\,b = A$ for $a,\ b \in A$, there exists $c \in A$ such that $(a + c\,b)^{-1} \in A$. Therefore, by Theorem 20.13 of [16], we obtain that any stably free A-module is free, i.e., W_+ and $A(\mathbb{D})$ are Hermite rings (see [30] for a different proof based on strong stabilizability). $\square$

There is a long tradition in commutative algebra of studying rings over which projective/stably free modules are not free. For instance, such a question is investigated in the so-called *algebraic K-theory* [31] and examples of no-Hermite or no projective-free rings are nowadays well-known. However, a K-theory for Banach/operators algebras has only been recently developed in order to study such types of modules over Banach algebras (e.g., over $C^\star$-algebras which naturally appear in functional analysis) [1]. To our knowledge, it seems that the question of whether or not there exists projective/stably free but not free modules over the Banach algebras $\mathcal{A}$, $\hat{\mathcal{A}}$ and W_+ (see Example 1) has not been solved in K-theory yet. Hence, it is not known whether or not an internally stabilizable plant over these rings admits doubly coprime factorizations. As many different infinite-dimensional linear systems (e.g., differential time-delay systems, systems of partial differential equations, convolutional systems) are defined by means of transfer matrices with entries in the quotient fields of these algebras [4,19], this question should draw more attention in the future.

7 Conclusion

In this paper, we have shown how a lattice approach to analysis and synthesis problems allowed us to give new, necessary and sufficient conditions for internal stabilizability and for the existence of (weakly) left-/right-/doubly coprime factorizations. Moreover, combining these results with one of [3,13], we proved that every internally stabilizable multidimensional system admitted doubly coprime factorizations, answering positively to an open question in the literature of multidimensional systems theory [17,18,36,37]. In [27], we showed how to use the previous lattice approach in order to obtain a general parametrization of all stabilizing controllers of internally stabilizable plants which do not necessarily admit doubly coprime factorizations. In particular, this new parametrization of all stabilizing controllers is a generalization of the Youla–Kučera parametrization. Finally, within the lattice approach developed in this paper, the results obtained in [26] for

SISO plants can be generalized for MIMO plants. In particular, they characterize the graphs and the domains of internally stabilizable plants which do not necessarily admit doubly coprime factorizations and give the relations between the graph of a plant and the one of any of its stabilizing controllers.

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