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ORIGINAL ARTICLE

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On a generalization of the Youla–Kučera parametrization. Part II: the lattice approach to MIMO systems

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Abstract Within the lattice approach to analysis and synthesis problems recently developed in Quadrat (Signal Syst, to appear), we obtain a general parametrization of all stabilizing controllers for internally stabilizable multi input multi output (MIMO) plants which do not necessarily admit doubly coprime factorizations. This parametrization is a linear fractional transformation of free parameters and the set of arbitrary parameters is characterized. This parametrization generalizes for MIMO plants the parametrization obtained in Quadrat (Syst Control Lett 50:135–148, 2003) for single input single output plants. It is named general Q-parametrization of all stabilizing controllers as we show that some ideas developed in this paper can be traced back to the pioneering work of Zames and Francis (IEEE Trans Automat control 28:585-601, 1983) on H_∞-control. Finally, if the plant admits a doubly coprime factorization, we then prove that the general Q-parametrization becomes the well-known Youla-Kučera parametrization of all stabilizing controllers (Desoer et al. IEEE Trans Automat control 25:399–412, 1980; Vidyasagar, Control system synthesis: a factorization approach MIT Press, Cambridge 1985).

Keywords Parametrization of all stabilizing controllers · Youla–Kučera parametrization · Internal stabilizability · (Weakly) Left-/right-/doubly coprime factorization · Lattice approach to analysis and synthesis problems

1 Introduction

The Youla–Kučera parametrization of all stabilizing controllers was developed for transfer matrices which admit doubly coprime factorizations [6,13,33,34]. As

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the Youla–Kučera parametrization is a linear fractional transformation of a matrix of free (arbitrary) parameters, it allows us to transform standard non-linear optimal problems into affine, and thus, convex ones [5,33]. This fact explains why it played an important role in the development of the H_∞-control in the 1990s [5,9,33].

However, it is becoming well-known that an internally stabilizable plant does not necessarily admit doubly coprime factorizations [17,18,21,30,31]. In particular, the existence of a left-/right-/doubly coprime factorization is a sufficient but generally not a necessary condition for internal stabilizability. The equivalence between these two concepts is still open for important classes of plants and specially for infinite-dimensional or multidimensional linear systems [5,15–18,30, 31]. Hence, we may wonder whether or not it is possible to parametrize all stabilizing controllers of an internally stabilizable plant which does not necessarily admit doubly coprime factorizations.

A characterization of all stabilizing controllers of multi input multi output (MIMO) stabilizable plants which do not necessarily admit doubly coprime factorizations was first studied in [30,31] and a generalization of the Youla–Kučera parametrization was obtained for plants defined over unique factorization domains (UFDs) [26]. However, in spite of the great novelty of [30,31] this parametrization has the inconvenience of not being explicit in terms of the free parameters. Moreover, we shall prove that the only complex Banach algebra which is a unique factorization domain is $\mathbb{C}$. Hence, we cannot use the parametrization developed in [30,31] for parametrizing all stabilizing controllers of plants defined over non-trivial Banach algebras and, in particular, the ones for which it is not known if an internally stabilizable plant admits doubly coprime factorizations (e.g., the Wiener algebras $\hat{\mathcal{A}}$ and W_+ of bounded input bounded output plants [5,16,33]). Recently, a more explicit parametrization of all stabilizing controllers has been obtained in [17]. However, contrary to the Youla–Kučera parametrization, this parametrization has not the explicit form of a linear fractional transformation and the set of free parameters is not completely characterized.

Within the lattice approach to analysis and synthesis problems developed in [21,23], the purpose of this paper is to obtain a general parametrization of all stabilizing controllers for MIMO stabilizable plants which do not necessarily admit coprime factorizations. This new parametrization only uses the knowledge of a stabilizing controller and it is a linear fractional transformation of the free parameters. The set of free parameters is characterized and a (non-minimal) family of generators of this set is exhibited. Two equivalent forms of this general parametrization are given: the first one only uses the transfer matrices of the plant P and of a stabilizing controller C, whereas the second uses stable fractional representations $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$, $C = Y\,X^{-1} = \tilde{X}^{-1}\,\tilde{Y}$ of P and C. These results generalize for MIMO plants the ones recently obtained in [19] for single input single output (SISO) plants within the fractional ideal approach to analysis and synthesis problems.

If the plant admits a doubly coprime factorization, we then prove that the general parametrization of all stabilizing controllers becomes the well-known Youla–Kučera parametrization [6,13,33,34]. Moreover, we also show that it generalizes the so-called Q-parametrization of all stabilizing controllers developed by Zames and Francis [35] in their pioneering work on H_∞-control for general internally stabilizable plants which do not necessarily admit doubly coprime factorizations. Hence, we shall call it *the general Q-parametrization* of all stabilizing controllers.

Finally, we show that the problem of determining the minimal number of free parameters of the general Q-parametrization is related to the well-known problem in commutative algebra consisting in determining the minimal number of generators of modules. Using Heitmann's generalization of Forster–Swan's theorem [4,7, 10], we give an upper bound on the minimal number of free parameters appearing in the general Q-parametrization.

A part of the results of this paper appeared in the congress papers [23,24].

Notation. In what follows, we only consider an integral domain A (namely, A is a ring with an identity which satisfies $a\,b = b\,a$ for any $a, b \in A$ and $a\,b = 0,\ a \neq 0 \Rightarrow b = 0$) [3,26]. Elements of A^m (resp., $A^{1\times m}$) are column (resp., row) vectors of length m with entries in A. Moreover, $A^{m\times m}$ denotes the ring of $m \times m$ matrices with entries in A and I_m the identity matrix of $A^{m\times m}$. If M and M' are two A-modules, then $M \cong M'$ means that M and M' are isomorphic as A-modules and $M \oplus M'$ denotes the direct sum of M and M' as A-modules [3,26]. Finally, if V is a finite-dimensional K-vector space, then the dimension of V over K is denoted by $\dim_K(V)$.

2 A lattice approach to analysis and synthesis problems

2.1 Introduction to the fractional representation approach

The *fractional representation approach to analysis and synthesis problems* was introduced in the 1980s by Desoer, Vidyasagar and others in order to study in a common mathematical framework analysis and synthesis problems for different classes of linear systems (e.g., continuous/discrete finite-/infinite-dimensional or multidimensional systems). For more details, see [6,16,18,21,33] and references therein.

Within the fractional representation approach, the "universal class of systems" is defined by the set of transfer matrices with entries in the quotient field $Q(A) = \{n/d \mid 0 \neq d,\ n \in A\}$ of *an integral domain A of SISO stable plants*. For instance, we have the following examples of such integral domains $A = RH_\infty$, $H_\infty(\mathbb{D})$, $H_\infty(\mathbb{C}_+)$, W_+, $\hat{\mathcal{A}}$, $A(\mathbb{D})$, $R(z_1, \ldots, z_n)_S$. See [5,6,18,19,21,30,31,33] for more details and examples.

We shortly recall a few definitions [6,18,21,30,31,33].

Definition 1 Let A be an integral domain of SISO stable plants and its quotient field $K = Q(A)$.

- We call *fractional representation* of a transfer matrix $P \in K^{q\times r}$ any representation of the form $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$, where:

$$R = (D \quad -N) \in A^{q\times(q+r)}, \quad \tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r)\times r}.$$

- A transfer matrix $P \in K^{q\times r}$ admits a *weakly left-coprime factorization* if there exists a fractional representation $P = D^{-1}\,N$ satisfying that, for all $\lambda \in K^{1\times q}$ such that $\lambda\,(D \quad -N) \in A^{1\times(q+r)}$, we then have $\lambda \in A^{1\times q}$.

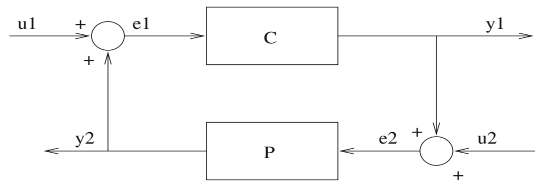

Fig. 1 Closed-loop system

- A transfer matrix $P \in K^{q\times r}$ admits a *weakly right-coprime factorization* if there exists a fractional representation $P = \tilde{N}^{-1}\,\tilde{D}$ satisfying that, for all $\lambda \in K^r$ such that $(\tilde{N}^{\mathrm{T}} \quad \tilde{D}^{\mathrm{T}})^{\mathrm{T}}\,\lambda \in A^{q+r}$, we then have $\lambda \in A^r$.
- A transfer matrix $P \in K^{q\times r}$ admits a *left-coprime factorization* if there exists a fractional representation $P = D^{-1}\,N$ such that the matrix R admits a right-inverse $S = (X^{\mathrm{T}} \quad Y^{\mathrm{T}})^{\mathrm{T}} \in A^{(q+r)\times q}$, namely, we have:

$$R\,S = D\,X - N\,Y = I_q.$$

- A transfer matrix $P \in K^{q\times r}$ admits a *right-coprime factorization* if there exists a fractional representation $P = \tilde{N}^{-1}\,\tilde{D}$ such that the matrix $\tilde{R}$ admits a left-inverse $\tilde{S} = (-\tilde{Y} \quad \tilde{X}) \in A^{(r\times(q+r)}$, namely, we have:

$$\tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r.$$

- A transfer matrix admits a *(weakly) doubly coprime factorization* if it admits a (weakly) left- and a (weakly) right-coprime factorizations.
- A plant $P \in K^{q\times r}$ is said to be *internally stabilizable* iff there exists a controller $C \in K^{r\times q}$ such that all the entries of the following matrix

$$\begin{aligned} H(P,C) = \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} &= \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \\ &= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \end{aligned} \tag{1}$$

  belong to A (see Fig.1). Then, C is called a *stabilizing controller* of P.

We refer to [16,19–21,25,33] for the definition of *strong* (resp., *simultaneous*, *robust*, *optimal*) stabilization.

2.2 Summary of some important results

We have recently developed in [21,23] *a lattice approach to analysis and synthesis problems*. In the next sections, we recall and develop new results which will be useful for the development of a general parametrization of all stabilizing controllers. For more details on the lattice approach, we refer to [19,25] for SISO systems and to [21,23] for MIMO systems.

2.2.1 Characterization in terms of the transfer matrix P

We first recall important definitions.

Definition 2 [3] Let A be an integral domain, $K = Q(A)$ its quotient field, V and W two finite-dimensional K-vector spaces.

1. An A-submodule M of V is called a *lattice* of V if M is contained in a finitely generated A-submodule of V and the K-vector space defined by $K\,M = \{\sum_{i=1}^{n} k_i\, m_i \mid k_i \in K,\ m_i \in M,\ n \in \mathbb{Z}_+\}$ satisfies $K\,M = V$.
2. We denote by $\hom_K(V, W)$ the K-vector space of all K-linear maps from V to W.
3. Let M be a lattice of V and N a lattice of W. Then, we denote by $N : M$ the A-submodule of $\hom_K(V, W)$ formed by all the K-linear maps $f : V \to W$ which satisfy $f(M) \subseteq N$.

We refer to [21,23] for more details. We have the following lemma.

Lemma 1 [3,21] *Let V and W be two finite-dimensional $K = Q(A)$-vector spaces and M (resp., N) a lattice of V (resp., W). Then, the A-module $N : M$ defined in 3 of Definition 2 is a lattice of $hom_K(V, W)$. Moreover, the canonical map $N : M \to hom_A(M, N)$ which maps $f \in N : M$ into the restriction $f_{|M}$ of f to M is an isomorphism, where $hom_A(M, N)$ denotes the A-module of the A-morphisms (i.e., A-linear maps) from M to N.*

The following lattices will play important roles in what follows.

Example 1 [21,23] Let A be an integral domain of SISO stable plants, its quotient field $K = Q(A)$ and $P \in K^{q\times r}$ a transfer matrix. Then, we can easily prove that $\mathcal{L} = (I_q \quad -P)\, A^{q+r}$ is a lattice of $V = K^q$. Hence, we obtain the lattice $A : \mathcal{L} = \{\lambda \in A^{1\times q} \mid \lambda\, P \in A^{1\times r}\}$ of $\hom_K(K^q, K) \cong K^{1\times q}$.

Similarly, we can prove that $\mathcal{M} = A^{1\times(q+r)}\,(P^T \quad I_r^T)^T$ is a lattice of $K^{1\times r}$ and $A : \mathcal{M} = \{\lambda \in A^r \mid P\lambda \in A^q\}$ is a lattice of $\hom_K(K^{1\times r}, K) \cong K^r$.

Example 2 [19,21] If $V = K$, then the lattices of K are just the non-zero *fractional ideals* of A [3,26]. Let us recall that a fractional ideal J of A is an A-submodule of K such that there exists a non-zero $a \in A$ satisfying:

$$(a)\, J \triangleq \{a\, j \mid aj \in J\} \subseteq A.$$

If $p \in K$ is a transfer function, then $\mathcal{L} = A + A\,(-p) = A + A\,p = \mathcal{P}$ is a fractional ideal of A as we have $(d)\,\mathcal{L} = A\,d + A\,n \subseteq A$, where $p = n/d$, $0 \neq d,\ n \in A$, is a fractional representation of p. Then, the ideal defined by

$$A : \mathcal{L} = A : \mathcal{P} = \{k \in K \mid k,\ k\,p \in A\} = \{d \in A \mid d\,p \in A\}$$

is usually called *the ideal of the denominators* of p. Indeed, for all $d \in A : \mathcal{L}$, we have $n = d\,p \in A$ and $p = n/d$ is a fractional representation of p. We point out that $A : \mathcal{L}$ cannot be finitely generated as an A-module [19].

We refer the reader to [21] for more details and examples. We now recall the intrinsic characterizations of the concepts introduced in Definition 1 in terms of the lattices given in Example 1.

Theorem 1 [21] *Let $P \in K^{q\times r}$ be a transfer matrix. Then, we have:*

1. *P admits a weakly left-coprime factorization iff there exists a non-singular matrix $D \in A^{q\times q}$ such that $A : \mathcal{L} = A^{1\times q}\, D$. Then, $P = D^{-1}\, N$ is a weakly left-coprime factorization of P, where $N = D\, P \in A^{q\times r}$.*
2. *P admits a weakly right-coprime factorization iff there exists a non-singular matrix $\tilde{D} \in A^{r\times r}$ such that $A : \mathcal{M} = \tilde{D}\, A^r$. Then, $P = \tilde{N}\, \tilde{D}^{-1}$ is a weakly right-coprime factorization of P, where $\tilde{N} = P\, \tilde{D} \in A^{q\times r}$.*
3. *P admits a left-coprime factorization iff there exists a non-singular matrix $D \in A^{q\times q}$ such that $\mathcal{L} = D^{-1}\, A^q$. Then, $P = D^{-1}\, N$ is a left-coprime factorization of P, where $N = D\, P \in A^{q\times r}$.*
4. *P admits a right-coprime factorization iff there exists a non-singular matrix $\tilde{D} \in A^{r\times r}$ such that $\mathcal{M} = A^{1\times r}\, \tilde{D}^{-1}$. Then, $P = \tilde{N}\, \tilde{D}^{-1}$ is a right-coprime factorization of P, where $\tilde{N} = P\, \tilde{D} \in A^{q\times r}$.*
5. *P is internally stabilizable iff there exists $L = (U^T \quad V^T)^T \in A^{(q+r)\times q}$ which satisfies* $\det U \neq 0$ *and:*
   (a) $L\, P = \begin{pmatrix} U\, P \\ V\, P \end{pmatrix} \in A^{(q+r)\times r}$,
   (b) $(I_q \quad -P)\, L = U - P\, V = I_q$.
   Then, $C = V\, U^{-1}$ is a stabilizing controller of P, $U = (I_q - P\, C)^{-1}$ and $V = C\, (I_q - P\, C)^{-1}$.
6. *P is internally stabilizable if there exists $\tilde{L} = (-\tilde{V} \quad \tilde{U}) \in A^{r\times(q+r)}$ which satisfies* $\det \tilde{U} \neq 0$ *and:*
   (a) $P\, \tilde{L} = (-P\, \tilde{V} \quad P\, \tilde{U}) \in A^{q\times(q+r)}$,
   (b) $\tilde{L} \begin{pmatrix} P \\ I_r \end{pmatrix} = -\tilde{V}\, P + \tilde{U} = I_r$.
   Then, $C = \tilde{U}^{-1}\, \tilde{V}$ is a stabilizing controller of P, $\tilde{U} = (I_r - C\, P)^{-1}$ and $\tilde{V} = (I_r - C\, P)^{-1}\, C$.
7. *P is internally stabilizable iff we have $\mathcal{L} \oplus (A : \mathcal{M}) \cong A^{q+r}$. Then, we have $A : \mathcal{M} = \tilde{L}\, A^{q+r}$, where $\tilde{L} \in A^{r\times(q+r)}$ is a matrix satisfying conditions 6(a), and 6(b), and $\mathcal{M} = A : (A : \mathcal{M}) = \{\lambda \in K^{1\times r} \mid \lambda\, \tilde{L} \in A^{1\times(q+r)}\}$.*
8. *P is internally stabilizable iff we have $\mathcal{M} \oplus (A : \mathcal{L}) \cong A^{1\times(q+r)}$. Then, we have $A : \mathcal{L} = A^{1\times(q+r)}\, L$, where $L \in A^{(q+r)\times q}$ is a matrix satisfying conditions 5(a) and 5(b), and $\mathcal{L} = A : (A : \mathcal{L}) = \{\lambda \in K^q \mid L\, \lambda \in A^{q+r}\}$.*

Remark 1 We note that 1 of Theorem 1 means that $A : \mathcal{L}$ is a *free lattice* of $K^{1\times r}$, i.e., the lattice $A : \mathcal{L}$ of $K^{1\times r}$ is free as an A-module. We recall that a finitely generated A-module M is said to be *free* if it admits a finite basis over A or, equivalently, if M is isomorphic to a finite direct sum A^r of A, i.e., $M \cong A^r$. A similar result holds for the lattices defined in 2–4 of Theorem 1.

Condition 7 means that $\mathcal{L}$ is a *projective lattice* of K^q, i.e., the lattice $\mathcal{L}$ of K^q is projective as an A-module. We recall that a finitely generated A-module M is said to be *projective* if there exist an A-module M' and $r \in \mathbb{Z}_+$ such that we have $M \oplus M' \cong A^r$. We note that M' is then also a projective A-module. A similar result holds for the lattice $\mathcal{M}$ defined in 8 of Theorem 1. We refer the reader to Definition 3 and [21] for more details on module theory.

Remark 2 We refer to [28] for the introduction of the concept of lattices in the realization problem. We also refer to [11] for the study of stable factorizations of transfer matrices within the systems over rings approach. The links between the results obtained in Theorem 1 and those of [11] will be studied in a forthcoming publication. Finally, we note that 1–4 of Theorem 1 can be considered over the commutative polynomial ring $A = k[x_1, \ldots, x_n]$, where k is a field (e.g., $k = \mathbb{R}, \mathbb{C}$). Hence, we obtain a lattice approach to multidimensional systems which generalizes for n-D systems some results obtained by Fuhrmann [8] for 1-D systems in.

We have the following corollary of 5 and 6 of Theorem 1 which shows that the existence of a left-/right-coprime factorization is a sufficient condition for internal stabilizability.

Corollary 1 [21] *Let $P \in K^{q \times r}$ be a transfer matrix.*

1. *If $P = D^{-1} N$ is a left-coprime factorization of P, $D X - N Y = I_q$, then the matrix $L = ((X D)^{\mathrm{T}} \quad (Y D)^{\mathrm{T}})^{\mathrm{T}} \in A^{(q+r) \times q}$ satisfies 5(a) and 5(b) of Theorem 1. If* $\det X \neq 0$, *then $C = Y X^{-1}$ internally stabilizes P.*
2. *If $P = \tilde{N} \tilde{D}^{-1}$ is a right-coprime factorization of P, $-\tilde{Y} \tilde{N} + \tilde{X} \tilde{D} = I_r$, then the matrix $\tilde{L} = (-\tilde{D} \tilde{Y} \quad \tilde{D} \tilde{X}) \in A^{r \times (q+r)}$ satisfies 6(a) and 6(b) of Theorem 1. If* $\det \tilde{X} \neq 0$, *then $C = \tilde{X}^{-1} \tilde{Y}$ internally stabilizes P.*

Let us compute a certain lattice which will naturally appear in the general parametrization of all stabilizing controllers developed in this paper.

Example 3 Using the results obtained in Example 1, we have:

$$\begin{aligned}
(A : \mathcal{M}) : \mathcal{L} &= \{Q \in K^{r \times q} \mid Q\,(I_q \quad -P)\,A^{q+r} \subseteq \{\lambda \in A^r \mid P\,\lambda \in A^q\}\} \\
&= \{Q \in K^{r \times q} \mid Q\,A^q,\ Q\,P\,A^r \subseteq \{\lambda \in A^r \mid P\,\lambda \in A^q\}\} \\
&= \{Q \in K^{r \times q} \mid Q\,A^q \subseteq A^r,\ Q\,P\,A^r \subseteq A^r,\ P\,Q\,A^q \subseteq A^q, \\
&\qquad\qquad\qquad\qquad\qquad\qquad P\,Q\,P\,A^r \subseteq A^q\} \\
&= \{Q \in A^{r \times q} \mid Q\,P \in A^{r \times r},\ P\,Q \in A^{q \times q},\ P\,Q\,P \in A^{q \times r}\}.
\end{aligned}$$

Similarly, we can prove that we have:

$$(A : \mathcal{L}) : \mathcal{M} = \{Q \in A^{r \times q} \mid Q\,P \in A^{r \times r},\ P\,Q \in A^{q \times q},\ P\,Q\,P \in A^{q \times r}\}.$$

Let us denote by $\Omega = (A : \mathcal{M}) : \mathcal{L} = (A : \mathcal{L}) : \mathcal{M}$, namely:

$$\Omega = \{Q \in A^{r \times q} \mid Q\,P \in A^{r \times r},\ P\,Q \in A^{q \times q},\ P\,Q\,P \in A^{q \times r}\}. \tag{2}$$

Example 4 Let $p \in Q(A)$ be a transfer function of a SISO plant and the fractional ideal $\mathcal{L} = \mathcal{M} = A + A\,p$ of A generated by 1 and p (see Example 2). By Theorem 1, we find that p admits a weakly coprime factorization iff there exists $0 \neq d \in A$ such that $A : \mathcal{L} = \{l \in A \mid l\,p \in A\} = A\,d$. Then, $p = n/d$ is a weakly coprime factorization of p, where $n = p\,d \in A$. Similarly, we obtain that p admits a coprime factorization iff there exists $0 \neq d \in A$ such that $\mathcal{L} = A\,(1/d)$. Then, $p = n/d$ is a coprime factorization of p, where $n = p\,d \in A$. Moreover, by (5) or (6) of Theorem 1, p is internally stabilizable iff there exist $a,\ b \in A$ such that we have

$$\begin{cases} a - b\,p = 1, \\ a\,p \in A, \end{cases} \tag{3}$$

as $b\,p = a - 1 \in A$. If $a \neq 0$, then $c = b/a$ internally stabilizes p, $a = 1/(1 - p\,c)$ and $b = c/(1 - p\,c)$. If $a = 0$, then $c = 1 - b$ internally stabilizes $p = -1/b$. See [19,21] for more details and explicit examples. Finally, the A-module defined by (2) becomes $\Omega = \{q \in A \mid d\,p,\ d\,p^2 \in A\}$.

The next proposition will be important when finding again the well-known Youla–Kučera parametrization of all stabilizing controllers.

Proposition 1 **We have:**

1. *If $P \in K^{q\times r}$ admits a weakly left-coprime factorization $P = D^{-1}\,N$, then the A-module Ω defined by (2) satisfies:*

$$\Omega = (A^{1\times q} : \mathcal{M})\,D = \{\Lambda \in A^{r\times q} \mid P\,\Lambda \in A^{q\times q}\}\,D.$$

2. *If $P \in K^{q\times r}$ admits a weakly right-coprime factorization $P = \tilde{N}\,\tilde{D}^{-1}$, then the A-module Ω defined by (2) satisfies:*

$$\Omega = \tilde{D}\,(A^r : \mathcal{L}) = \tilde{D}\,\{\Lambda \in A^{r\times q} \mid \Lambda\,P \in A^{r\times r}\}.$$

3. *If $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is a weakly doubly coprime factorization of $P \in K^{q\times r}$, then the A-module Ω defined by (2) satisfies:*

$$\Omega = \tilde{D}\,A^{r\times q}\,D. \tag{4}$$

Proof 1. Using the fact that P admits a weakly left-coprime fatorization $P = D^{-1}\,N$, then, by 1 of Theorem 1, we obtain $A : \mathcal{L} = A^{1\times q}\,D$ (see also Lemmas 1 and 2 of [21]). Therefore, we have:

$$\begin{aligned}\Omega = (A^{1\times q}\,D) : \mathcal{M} &= \left\{Q \in K^{r\times q} \mid A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} Q \subseteq A^{1\times q}\,D\right\} \\ &= \{Q \in K^{r\times q} \mid A^{1\times r}\,Q \subseteq A^{1\times q}\,D,\ A^{1\times q}\,(P\,Q) \subseteq A^{1\times q}\,D\}.\end{aligned}$$

Hence, for $Q \in \Omega$, there exist $\Lambda \in A^{r\times q}$ and $\Theta \in A^{q\times q}$ such that we have $Q = \Lambda\,D$ and $P\,Q = \Theta\,D$. In particular, we obtain $P\,Q = P\,\Lambda\,D = \Theta\,D$, i.e., $(P\,\Lambda - \Theta)\,D = 0$. Now, using the fact that D is non-singular, we obtain $\Theta = P\,\Lambda$, i.e., $\Omega = \{\Lambda \in A^{r\times q} \mid P\,\Lambda \in A^{q\times q}\}\,D$. Finally, we have

$$\begin{aligned}A^{1\times q} : \mathcal{M} &= \{\Lambda \in K^{r\times q} \mid A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} \Lambda \subseteq A^{1\times q}\} \\ &= \{\Lambda \in K^{r\times q} \mid P\,\Lambda \in A^{q\times q},\ \Lambda \in A^{r\times q}\} \\ &= \{\Lambda \in A^{r\times q} \mid P\,\Lambda \in A^{q\times q}\},\end{aligned} \tag{5}$$

which proves 1.

2. Condition 2 can be proved similarly.

3. If $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is a weakly doubly coprime factorization of P, then, by 2, we obtain $\Omega = \{\Lambda \in A^{r\times q} \mid P\,\Lambda \in A^{q\times q}\}\,D$ and, by 2 of Theorem 1, we have $A : \mathcal{M} = \tilde{D}\,A^r$. Hence, every column of $T \in (A^{1\times q} : \mathcal{M})$ belongs to $\{\lambda \in A^r \mid P\,\lambda \in A^q\} = A : \mathcal{M} = \tilde{D}\,A^r$, and thus, $T \in \tilde{D}\,A^{r\times q}$. Finally, we easily check that we have $\tilde{D}\,A^{r\times q} \subseteq \{\Lambda \in A^{r\times q} \mid P\,\Lambda \in A^{q\times q}\}$, which proves $\{\Lambda \in A^{r\times q} \mid P\,\Lambda \in A^{q\times q}\} = \tilde{D}\,A^{r\times q}$, and thus, $\Omega = \tilde{D}\,A^{r\times q}\,D$. □

2.2.2 *Characterizations in terms fractional representations of P*

We now characterize internal stabilizability and the existence of (weakly) left-/right-/doubly coprime factorizations in terms of fractional representations $P = D^{-1} N = \tilde{N}\,\tilde{D}^{-1}$ of P. Let us start with the next important example.

Example 5 Let A be an integral domain of SISO stable plants, $K = Q(A)$ its quotient field, $P \in K^{q\times r}$ and $P = D^{-1} N = \tilde{N}\,\tilde{D}^{-1}$ a fractional representation of P. Then, we can easily prove that $\mathcal{P} = (D \quad -N)\, A^{q+r}$ is a lattice of K^q whereas $\mathcal{Q} = A^{1\times(q+r)}\,(\tilde{N}^T \quad \tilde{D}^T)^T$ is a lattice of $K^{1\times r}$. Hence, we obtain that $A : \mathcal{P} = \{\lambda \in K^{1\times q} \mid \lambda\, R \in A^{1\times(q+r)}\}$ is a lattice of $\hom_K(K^q, K) \cong K^{1\times q}$, whereas $A : \mathcal{Q} = \{\lambda \in K^r \mid \tilde{R}\,\lambda \in A^{q+r}\}$ is a lattice of $\hom_K(K^{1\times r}, K) \cong K^r$. See [21] for more details.

Example 6 Let $p \in K = Q(A)$ be a transfer function of a SISO plant and $p = n/d$, $0 \neq d, n \in A$ a fractional representation of p. By Example 5, we obtain that $\mathcal{P} = \mathcal{Q} = A\,d + A\,n$ is a lattice of K and, more precisely, an ideal of A. Then, $A : \mathcal{P} = \{k \in K \mid k\,d,\ k\,n \in A\}$ is also a lattice of K, namely, a fractional ideal of A (see Example 2).

We have the following important results.

Theorem 2 [21] *Let $P \in K^{q\times r}$ be a transfer matrix and $P = D^{-1} N = \tilde{N}\,\tilde{D}^{-1}$ a fractional representation of P, where $R = (D \quad -N) \in A^{q\times(q+r)}$ and $\tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r)\times r}$. Then, we have:*

1. *P admits a weakly left-coprime factorization iff there exists a non-singular matrix $E \in K^{q\times q}$ such that $A : \mathcal{P} = A^{1\times q}\,E$. Then, if we denote by $D' = E\,D \in A^{q\times q}$ and $N' = E\,N \in A^{q\times r}$, $P = (D')^{-1}\,N'$ is a weakly left-coprime factorization of P.*
2. *P admits a weakly right-coprime factorization iff there exists a non-singular matrix $F \in K^{r\times r}$ such that $A : \mathcal{Q} = F\,A^r$. Then, if we denote by $\tilde{D}' = \tilde{D}\,F \in A^{r\times r}$ and $\tilde{N}' = \tilde{N}\,F \in A^{q\times r}$, $P = \tilde{N}'\,(\tilde{D}')^{-1}$ is a weakly right-coprime factorization of P.*
3. *P admits a left-coprime factorization iff there exists a non-singular matrix $G \in A^{q\times q}$ such that $\mathcal{P} = G\,A^q$. Then, if we denote by $D' = G^{-1}\,D \in A^{q\times q}$ and $N' = G^{-1}\,N \in A^{q\times r}$, $P = (D')^{-1}\,N'$ is a left-coprime factorization of P.*
4. *P admits a right-coprime factorization iff there exists a non-singular matrix $H \in A^{r\times r}$ such that $\mathcal{Q} = A^{1\times r}\,H$. Then, if we denote by $\tilde{D}' = \tilde{D}\,H^{-1}$ and $\tilde{N}' = \tilde{N}\,H^{-1} \in A^{q\times r}$, $P = \tilde{N}'\,(\tilde{D}')^{-1}$ is a right-coprime factorization of P.*
5. *P is internally stabilizable iff there exists $S = (X^T \quad Y^T)^T \in K^{(q+r)\times q}$ which satisfies* $\det X \neq 0$ *and:*
   (a) $S\,R = \begin{pmatrix} X\,D & -X\,N \\ Y\,D & -Y\,N \end{pmatrix} \in A^{(q+r)\times(q+r)}$,
   (b) $R\,S = D\,X - N\,Y = I_q$.
   Then, $C = Y\,X^{-1}$ internally stabilizes the plant P, $X = (D - N\,C)^{-1}$ and $Y = C\,(D - N\,C)^{-1}$.

6. *P is internally stabilizable iff there exists $\tilde{S} = (-\tilde{Y} \quad \tilde{X}) \in K^{r \times (q+r)}$ which satisfies* $\det \tilde{X} \neq 0$ *and:*
   (a) $\tilde{R}\,\tilde{S} = \begin{pmatrix} -\tilde{N}\,\tilde{Y} & \tilde{N}\,\tilde{X} \\ -\tilde{D}\,\tilde{Y} & \tilde{D}\,\tilde{X} \end{pmatrix} \in A^{(q+r)\times(q+r)}$,
   (b) $\tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r$.
   Then, $C = \tilde{X}^{-1}\,\tilde{Y}$ internally stabilizes the plant P, $\tilde{X} = (\tilde{D} - C\,\tilde{N})^{-1}$ and $\tilde{Y} = (\tilde{D} - C\,\tilde{N})^{-1}\,C$.
7. *$P = D^{-1}\,N$ is internally stabilizable iff we have $\mathcal{P} \oplus \ker(R.) \cong A^{q+r}$, where* $\ker(R.) = \{\lambda \in A^{q+r} \mid R\,\lambda = 0\}$. *Then, we have*

$$A : \mathcal{P} = A^{1\times(q+r)}\,S, \quad \mathcal{P} = A : (A : \mathcal{P}) = \{\lambda \in K^q \mid S\,\lambda \in A^{q+r}\},$$

   where $S \in K^{(q+r)\times q}$ is a matrix satisfying conditions 5(a) and 5(b).
8. *$P = \tilde{N}^{-1}\,\tilde{D}$ is internally stabilizable iff we have $\mathcal{Q} \oplus \ker(.\tilde{R}) \cong A^{1\times(q+r)}$, where* $\ker(.\tilde{R}) = \{\lambda \in A^{1\times(q+r)} \mid \lambda\,\tilde{R} = 0\}$. *Then, we have*

$$A : \mathcal{Q} = \tilde{S}\,A^{q+r}, \quad \mathcal{Q} = A : (A : \mathcal{Q}) = \{\lambda \in K^{1\times r} \mid \lambda\,\tilde{S} \in A^{1\times(q+r)}\},$$

   where $\tilde{S} \in K^{r\times(q+r)}$ is a matrix satisfying conditions 6(a) and 6(b).

Remark 3 As in Remark 1, 1 of Theorem 2 is equivalent to the fact that $A : \mathcal{P}$ is a free lattice of $K^{1\times q}$. A similar result holds for the lattices defined in 2–4 of Theorem 2. Moreover, 5 of Theorem 2 is also equivalent to the fact that $\mathcal{P}$ is a projective lattice of $K^{1\times q}$. A similar result holds for $\mathcal{Q}$.

We have the following trivial corollary of 5 and 6 of Theorem 2 which shows that the existence of a left-/right-coprime factorization is a sufficient condition for internal stabilizability.

Corollary 2 [21] *Let $P \in K^{q\times r}$ be a transfer matrix.*

1. *If $P = D^{-1}\,N$ is a left-coprime factorization of P, $D\,X - N\,Y = I_q$, then the matrix $S = (X^T \quad Y^T)^T \in A^{(q+r)\times q}$ satisfies conditions 5(a) and 5(b) of Theorem 2. If* $\det X \neq 0$, *then $C = Y\,X^{-1}$ internally stabilizes P.*
2. *If $P = \tilde{N}\,\tilde{D}^{-1}$ is a right-coprime factorization of P, $-\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r$, then $\tilde{S} = (-\tilde{Y} \quad \tilde{X}) \in A^{r\times(q+r)}$ satisfies conditions 6(a) and 6(b) of Theorem 2. If* $\det \tilde{X} \neq 0$, *then $C = \tilde{X}^{-1}\,\tilde{Y}$ internally stabilized P.*

The lattice introduced in the next example will naturally appears in the general parametrization of all stabilizing controllers of P.

Example 7 Using the results obtained in Example 5, we have:

$$\begin{aligned}
&(A : \mathcal{P}) : \mathcal{Q} \\
&= \left\{ T \in K^{r\times q} \mid A^{1\times(q+r)} \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T \subseteq \{\lambda \in K^{1\times q} \mid \lambda\,R \in A^{1\times(q+r)}\} \right\} \\
&= \left\{ T \in K^{r\times q} \mid A^{1\times(q+r)} \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T\,(D \quad -N) \subseteq A^{1\times(q+r)} \right\} \\
&= \left\{ T \in K^{r\times q} \mid \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T\,(D \quad -N) \in A^{(q+r)\times(q+r)} \right\}.
\end{aligned}$$

Similarly, we can prove that we have:

$$(A : \mathcal{Q}) : \mathcal{P} = \left\{ T \in K^{r\times q} \mid \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T \,(D \quad -N) \in A^{(q+r)\times(q+r)} \right\}.$$

Let us denote by $\Delta = (A : \mathcal{P}) : \mathcal{Q} = (A : \mathcal{Q}) : \mathcal{L}$, namely:

$$\Delta = \left\{ T \in K^{r\times q} \mid \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T \,(D \quad -N) \in A^{(q+r)\times(q+r)} \right\}. \tag{6}$$

Example 8 Let $p = n/d$ be a fractional representation of p, $0 \neq d,\ n \in A$. By Theorem 2, p admits a weakly coprime factorization iff there exists a non-trivial $e \in K$ such that $A : \mathcal{P} = \{k \in K \mid k\,n,\ k\,d \in A\} = A\,e$. Then, if we denote by $d' = e\,d \in A$ and $n' = e\,n \in A$, $p = n'/d'$ is a weakly coprime factorization of p. Similarly, p admits a coprime factorization iff there exists $0 \neq g \in A$ such that we have $\mathcal{P} = A\,g$. Then, if we define by $d' = d/g \in A$ and $n' = n/g \in A$, $p = n'/d'$ is a coprime factorization of p. Moreover, p is internally stabilizable iff there exist $x, y \in K$ such that we have:

$$\begin{cases} d\,x - n\,y = 1, \\ d\,x,\ d\,y,\ n\,x \in A \quad (n\,y = d\,x - 1 \in A). \end{cases} \tag{7}$$

If $x \neq 0$, then $c = y/x$ internally stabilizes p and we have $x = 1/(d - n\,c)$ and $y = c/(d - n\,c)$. If $x = 0$, then $c = 1 - d\,y$ internally stabilizes $p = -1/(d\,y)$. See also [18,27]. Finally, we easily check that the A-module Δ defined by (6) becomes the fractional ideal $\Delta = \{t \in K \mid t\,d^2,\ t\,d\,n,\ t\,n^2 \in A\}$ of A.

To finish, let us state a new proposition which will play an important role for the Youla–Kučera parametrization.

Proposition 2 We have:

1. *If $P \in K^{q\times r}$ admits a weakly left-coprime factorization $P = D^{-1}\,N$, then the A-module Δ defined by (6) satisfies:*

$$\Delta = A^{1\times q} : \mathcal{Q} = \tilde{D}^{-1}\,(A^{1\times q} : \mathcal{M}) = \tilde{D}^{-1}\,\{\Lambda \in A^{r\times q} \mid P\,\Lambda \in A^{q\times q}\}.$$

2. *If $P \in K^{q\times r}$ admits a weakly right-coprime factorization $P = \tilde{N}\,\tilde{D}^{-1}$, then the A-module Δ defined by (6) satisfies:*

$$\Delta = A^r : \mathcal{P} = (A^r : \mathcal{L})\,D^{-1} = \{\Lambda \in A^{r\times q} \mid \Lambda\,P \in A^{r\times r}\}\,D^{-1}.$$

3. *If $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is a weakly doubly coprime factorization of $P \in K^{q\times r}$, then the A-module Δ defined by (6) satisfies $\Delta = A^{r\times q}$.*

Proof 1. If $P = D^{-1}\,N$ is a weakly left-coprime factorization of P, then, by 1 of Theorem 2 (see also [21]), we have $A : \mathcal{P} = A^{1\times q}$. Therefore, the A-module Δ becomes:

$$\Delta = A^{1\times q} : \mathcal{Q} = \left\{ T \in K^{r\times q} \mid A^{1\times (q+r)} \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T \subseteq A^{1\times q} \right\},$$
$$= \left\{ T \in K^{r\times q} \mid \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T \in A^{(q+r)\times q} \right\}.$$

Hence, $T \in \Delta$ if there exist two matrices $\Lambda_1 \in A^{q\times q}$ and $\Lambda_2 \in A^{r\times q}$ such that we have

$$\begin{cases} \tilde{N}\, T = \Lambda_1, \\ \tilde{D}\, T = \Lambda_2, \end{cases} \Leftrightarrow \begin{cases} T = \tilde{D}^{-1}\, \Lambda_2, \\ P\, \Lambda_2 = \Lambda_1, \end{cases}$$

i.e., we have $\Delta = \tilde{D}^{-1}\, \{\Lambda_2 \in A^{r\times q} \mid P\, \Lambda_2 \in A^{q\times q}\}$ which proves the result once using (5).

2. Condition 2 can be proved similarly.
3. If $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}$ is a doubly weakly coprime factorization of P, then, by 1, we obtain $\Delta = \tilde{D}^{-1}\, \{\Lambda \in A^{r\times q} \mid P\, \Lambda \in A^{q\times q}\} = \tilde{D}^{-1}\; (A^{1\times q} : \mathcal{M})$ and, by 2 of Theorem 1, we have $A : \mathcal{M} = \tilde{D}\, A^r$. Hence, every column of $L \in (A^{1\times q} : \mathcal{M})$ belongs to $\{\lambda \in A^r \mid P\, \lambda \in A^q\} = A : \mathcal{M} = \tilde{D}\, A^r$, and thus, $L \in \tilde{D}\, A^{r\times q}$. Finally, we easily check $\tilde{D}\, A^{r\times q} \subseteq \{\Lambda \in A^{r\times q} \mid P\, \Lambda \in A^{q\times q}\}$, which shows $\{\Lambda \in A^{r\times q} \mid P\, \Lambda \in A^{q\times q}\} = \tilde{D}\, A^{r\times q}$, and thus, $\Delta = A^{r\times q}$.

□

3 A generalization of the Youla–Kučera parametrization

If a plant P admits a doubly coprime factorization, then it is well-known that there exists a parametrization of all its stabilizing controllers called the *Youla–Kučera parametrization* [6,13,33]. However, by Corollaries 1 and 2, we know that the existence of a doubly coprime factorization is a sufficient but not a necessary condition for internal stabilizability. Hence, we may wonder if it is possible to parametrize all stabilizing controllers of an internally stabilizable plant which does not necessarily admit doubly coprime factorizations. The purpose of this section is to show that such a parametrization exists and it can be explicitly computed once a stabilizing controller of the plant is known.

3.1 Characterizations in terms of the transfer matrix P

We start by giving a parametrization of all stabilizing controllers of an internally stabilizable plant without assuming the existence of a doubly coprime factorization for the plant.

Proposition 3 *Let $P \in K^{q\times r}$ be an internally stabilizable plant, $C_\star \in K^{r\times q}$ a stabilizing controller of P and let us denote by:*

$$\begin{cases} U = (I_q - P\, C_\star)^{-1} \in A^{q\times q}, \\ V = C_\star\, (I_q - P\, C_\star)^{-1} \in A^{r\times q}, \\ \tilde{U} = (I_r - C_\star\, P)^{-1} \in A^{r\times r}, \\ \tilde{V} = (I_r - C_\star\, P)^{-1}\, C_\star \in A^{r\times q}. \end{cases}$$

Then, all stabilizing controllers of P are of the form

$$C(Q) = (V + Q)\,(U + P\,Q)^{-1} = (\tilde{U} + Q\,P)^{-1}\,(\tilde{V} + Q), \tag{8}$$

where Q is any matrix which belongs to the A-module Ω defined by (2), namely, $\Omega = \{Q \in A^{r\times q} \mid Q\,P \in A^{r\times r},\ P\,Q \in A^{q\times q},\ P\,Q\,P \in A^{q\times r}\}$, and satisfies the following conditions:

$$\det(U + P\,Q) \neq 0, \quad \det(\tilde{U} + Q\,P) \neq 0. \tag{9}$$

Proof Let us consider two stabilizing controllers C_1 and $C_2 \in K^{r\times q}$ of the plant $P \in K^{q\times r}$ and let us denote by:

$$\begin{cases} U_i = (I_q - P\,C_i)^{-1} \in A^{q\times q}, \\ V_i = C_i\,(I_q - P\,C_i)^{-1} \in A^{r\times q}, \\ \tilde{U}_i = (I_r - C_i\,P)^{-1} \in A^{r\times r}, \\ \tilde{V}_i = (I_r - C_i\,P)^{-1}\,C_i \in A^{r\times q}, \end{cases}$$

Then, we have $C_i = V_i\,U_i^{-1} = \tilde{U}_i^{-1}\,\tilde{V}_i$ for $i = 1, 2$.

We now recall that the following standard identities hold [33]:

$$\begin{cases} (I_q - P\,C_i)^{-1} = P\,(I_r - C_i\,P)^{-1}\,C_i + I_q, \\ (I_q - P\,C_i)^{-1}\,P = P\,(I_r - C_i\,P)^{-1}, \\ C_i\,(I_q - P\,C_i)^{-1} = (I_r - C_i\,P)^{-1}\,C_i, \\ C_i\,(I_q - P\,C_i)^{-1}\,P = (I_r - C_i\,P)^{-1} - I_r. \end{cases} \quad i = 1, 2. \tag{10}$$

By the third relation of (10), we have $C_i\,(I_q - P\,C_i)^{-1} = (I_r - C_i\,P)^{-1}\,C_i$, and thus, we obtain $V_i = \tilde{V}_i$. Secondly, if we denote by

$$L_i = (U_i^{\mathrm{T}} \;\; V_i^{\mathrm{T}})^{\mathrm{T}} \in A^{(q+r)\times q}, \quad \tilde{L}_i = (-\tilde{V}_i \;\; \tilde{U}_i) \in A^{r\times(q+r)}, \quad i = 1, 2,$$

then, using the fact that C_i internally stabilizes P, $i = 1, 2$, we obtain:

$$\begin{cases} (I_q \quad -P)\,L_i = U_i - P\,V_i = I_q, \\ L_i\,(I_q \quad -P) = \begin{pmatrix} U_i & -U_i\,P \\ V_i & -V_i\,P \end{pmatrix} \in A^{(q+r)\times(q+r)}, \\ \tilde{L}_i \begin{pmatrix} P \\ I_r \end{pmatrix} = -\tilde{V}_i\,P + \tilde{U}_i = I_r, \\ \begin{pmatrix} P \\ I_r \end{pmatrix} \tilde{L}_i = \begin{pmatrix} -P\,\tilde{V}_i & P\,\tilde{U}_i \\ -\tilde{V}_i & \tilde{U}_i \end{pmatrix} \in A^{(q+r)\times(q+r)}, \end{cases} \quad i = 1, 2.$$

Moreover, using the two previous equalities, we have:

$$\begin{cases} U_2 - U_1 = P\,V_2 + I_q - P\,V_1 - I_q = P\,(V_2 - V_1), \\ \tilde{U}_2 - \tilde{U}_1 = \tilde{V}_2\,P + I_r - \tilde{V}_1\,P - I_r = (\tilde{V}_2 - \tilde{V}_1)\,P = (V_2 - V_1)\,P. \end{cases}$$

Therefore, using the second identity of (10), we finally obtain

$$\begin{cases} V_2 - V_1 = \tilde{V}_2 - \tilde{V}_1 \in A^{r \times q}, \\ (V_2 - V_1)\, P = \tilde{U}_2 - \tilde{U}_1 \in A^{r \times r}, \\ P\, (V_2 - V_1) = U_2 - U_1 \in A^{q \times q}, \\ P\, (V_2 - V_1)\, P = P\, (\tilde{U}_2 - \tilde{U}_1) = (U_2 - U_1)\, P \in A^{q \times r}, \end{cases}$$

which shows that we have $V_2 - V_1 = \tilde{V}_2 - \tilde{V}_1 \in \Omega$. Therefore, if we denote by $Q = V_2 - V_1 = \tilde{V}_2 - \tilde{V}_1 \in \Omega$, we then have

$$V_2 = V_1 + Q, \quad \tilde{V}_2 = \tilde{V}_1 + Q, \quad \tilde{U}_2 = \tilde{U}_1 + Q\, P, \quad U_2 = U_1 + P\, Q,$$

and, if $\det(U_1 + P\, Q) \neq 0$ and $\det(\tilde{U}_1 + Q\, P) \neq 0$, we finally obtain:

$$\begin{cases} C_2 = V_2\, U_2^{-1} = (V_1 + Q)\, (U_1 + P\, Q)^{-1}, \\ C_2 = \tilde{U}_2^{-1}\, \tilde{V}_2 = (\tilde{U}_1 + Q\, P)^{-1}\, (\tilde{V}_1 + Q). \end{cases}$$

Therefore, if we use the notations $U = U_1$, $V = V_1$, $\tilde{U} = \tilde{U}_1$ and $\tilde{V} = \tilde{V}_1$, then we finally obtain $C_2 = C(Q)$, where $C(Q)$ is defined by (8) for a certain $Q \in \Omega$ which satisfies $\det(U + P\, Q) \neq 0$ and $\det(\tilde{U} + Q\, P) \neq 0$.

Finally, let us prove that the controller $C(Q)$ defined by (8) internally stabilizes P for every $Q \in \Omega$ which satisfies (9). Let us denote by:

$$L(Q) = ((U + P\, Q)^{\mathrm{T}} \;\; (V + Q)^{\mathrm{T}})^{\mathrm{T}}, \quad \tilde{L}(Q) = (-(\tilde{V} + Q) \;\; (\tilde{U} + Q\, P)).$$

Then, using the fact that $Q \in \Omega$, we obtain

$$\begin{cases} V + Q \in A^{r \times q}, U + P\, Q \in A^{q \times q}, \\ L(Q)\, P = \begin{pmatrix} U\, P + P\, Q\, P \\ V\, P + Q\, P \end{pmatrix} \in A^{(q+r) \times r}, \\ (I_q \quad - P)\, L(Q) = U + P\, Q - P\, (V + Q) = U - P\, V = I_q, \end{cases}$$

which shows that $C(Q) = (V + Q)\, (U + P\, Q)^{-1}$ internally stabilizes P by 5 of Theorem 1. Moreover, we have

$$\begin{cases} \tilde{V} + Q \in A^{r \times q}, \tilde{U} + Q\, P \in A^{r \times r}, \\ P\, \tilde{L}(Q) = (-(P\, \tilde{V} + P\, Q) \;\; (P\, \tilde{U} + P\, Q\, P)) \in A^{q \times (q+r)}, \\ \tilde{L}(Q) \begin{pmatrix} P \\ I_r \end{pmatrix} = -\tilde{V}\, P - Q\, P + \tilde{U} + Q\, P = -\tilde{V}\, P + \tilde{U} = I_r, \end{cases}$$

showing that $C(Q) = (\tilde{U} + Q\, P)^{-1}\, (\tilde{V} + Q)$ internally stabilizes P by 6 of Theorem 1. □

Example 9 If the SISO plant p is internally stabilized by a controller $c_\star$, then, by Proposition 3, all stabilizing controllers of p are of the form

$$c(q) = \frac{c_\star/(1 - p\, c_\star) + q}{1/(1 - p\, c_\star) + q\, p} = \frac{c_\star + q\, (1 - p\, c_\star)}{1 + q\, p\, (1 - p\, c_\star)}, \tag{11}$$

where q is any element of the fractional ideal $\Omega = \{l \in A \mid l\, p,\, l\, p^2 \in A\}$ of A satisfying $1/(1 - p\, c_\star) + q\, p \neq 0$. We find the parametrization of all stabilizing controllers of an internally stabilizable SISO plant obtained in [22].

If a stabilizing controller $C_\star$ of a plant P is known, then Proposition 3 gives an explicit parametrization of all stabilizing controllers of P. However, we need to characterize the set Ω of free (arbitrary) parameters. By Example 3, we already know that the set Ω of free parameters of (8) is a lattice, and thus, an A-module. The next proposition gives an explicit family of generators of the A-module Ω in terms of a stabilizing controller $C_\star$ of P.

Proposition 4 *Let $P \in K^{q\times r}$ be a stabilizable plant, $C_\star$ a stabilizing controller of P and let us denote by:*

$$\begin{cases} L = \begin{pmatrix} (I_q - P\,C_\star)^{-1} \\ C_\star\,(I_q - P\,C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \tilde{L} = (-(I_r - C_\star\,P)^{-1}\,C_\star \quad (I_r - C_\star\,P)^{-1}) \in A^{r\times(q+r)}. \end{cases}$$

Then, the A-module Ω defined by (2) satisfies

$$\Omega = \tilde{L}\,A^{(q+r)\times(q+r)}\,L, \tag{12}$$

that is, Ω is generated over A by the $(q+r)^2$ matrices $\tilde{L}\,E^i_j\,L$, where E^i_j denotes the matrix defined by 1 in the ith row and jth column and 0 elsewhere, and $i, j = 1, \ldots, q+r$. Equivalently, if we denote by $\tilde{L}_i$ the ith column of $\tilde{L}$ and by L^j the jth row of L, then we have:

$$\Omega = \sum_{i,j=1}^{q+r} A\,(\tilde{L}_i\,L^j). \tag{13}$$

Proof By 7 of Theorem 1, we have $A : \mathcal{M} = \tilde{L}\,A^{q+r}$. Therefore, using Example 3, we obtain:

$$\begin{aligned} \Omega = (\tilde{L}\,A^{q+r}) : \mathcal{L} &= \{Q \in K^{r\times q} \mid Q\,(I_q \quad -P)\,A^{q+r} \subseteq \tilde{L}\,A^{q+r}\} \\ &= \{Q \in K^{r\times q} \mid \exists\ \Lambda \in A^{(q+r)\times(q+r)} :\ Q\,(I_q \quad -P) = \tilde{L}\,\Lambda\}. \end{aligned}$$

Hence, if $Q \in \Omega$, then there exists a matrix $\Lambda \in A^{(q+r)\times(q+r)}$ such that we have $Q\,(I_q \quad -P) = \tilde{L}\,\Lambda$. Now, using the fact that $(I_q \quad -P)\,L = I_q$ (see 5 of Theorem 1), we obtain:

$$Q = Q\,((I_q \quad -P)\,L) = (Q\,(I_q \quad -P))\,L = \tilde{L}\,\Lambda\,L \Rightarrow Q \in \tilde{L}\,A^{(q+r)\times(q+r)}\,L.$$

Conversely, if $Q \in \tilde{L}\,A^{(q+r)\times(q+r)}\,L$, then there exists $\Lambda \in A^{(q+r)\times(q+r)}$ such that $Q = \tilde{L}\,\Lambda\,L$, where L and $\tilde{L}$ satisfy 5 and 6 of Theorem 1. Then, using 5(a) and 6(a) of Theorem 1, we finally obtain

$$\begin{cases} Q\,P = \tilde{L}\,\Lambda\,(L\,P) \in A^{r\times r}, \\ P\,Q = (P\,\tilde{L})\,\Lambda\,L \in A^{q\times q}, \\ P\,Q\,P = (P\,\tilde{L})\,\Lambda\,(L\,P) \in A^{q\times r}, \end{cases}$$

showing that $Q \in \Omega$ and proving (12).

Finally, (13) directly follows from the fact that $A^{(q+r)\times(q+r)}$ is a free A-module of rank $(q+r)^2$ with the standard basis defined by $\{E^i_j\}_{i,j=1,\ldots,q+r}$. Indeed,

$\Lambda \in A^{(q+r)\times(q+r)}$ can be uniquely written as $\Lambda = \sum_{i,j=1}^{q+r} \lambda_i^j \, E_j^i$, with $\lambda_i^j \in A$, and thus, every $Q \in \Omega$ can be written (non-necessarily uniquely) as $Q = \sum_{i,j=1}^{q+r} \lambda_i^j \, (\tilde{L}\, E_j^i \, L)$. Thus, $\{\tilde{L}\, E_j^i \, L\}_{i,j=1,\ldots,q+r}$ is a family of generators of Ω and $\tilde{L}\, E_j^i \, L$ is in the product of the ith column of $\tilde{L}$ by the jth row of L. □

Example 10 Let us consider again Example 9. If $c_\star$ is a stabilizing controller of p and if we denote by $a = 1/(1 - p\, c_\star) \in A$ and $b = c_\star/(1 - p\, c_\star)$, then we have $L = (a \quad b)^T \in A^2$ and $\tilde{L} = (-b \quad a) \in A^{1\times 2}$, and thus, the A-module $\Omega = \tilde{L}\, A^{2\times 2}\, L = A\, a^2 + A\, a\, b + A\, b^2$ is the ideal of A generated by a^2, $a\, b$ and b^2. But, using (3), we obtain $a\, b = b\, a^2 - (a\, p)\, b^2 \in A\, a^2 + A\, b^2$ because b, $a\, p \in A$, showing that $\Omega = A\, a^2 + A\, b^2$. Therefore, an element q of Ω has the form $q = q_1\, a^2 + q_2\, b^2$ where q_1 and $q_2 \in A$ and the parametrization (11) only depends on two free parameters. See [19] for more details and examples.

Combining Propositions 3 and 4, we obtain the first main result.

Theorem 3 *Let $P \in K^{q\times r}$ be a stabilizable plant, $C_\star$ a stabilizing controller of P and let us denote by:*

$$\begin{cases} L = \begin{pmatrix} (I_q - P\, C_\star)^{-1} \\ C_\star\, (I_q - P\, C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \tilde{L} = (-(I_r - C_\star\, P)^{-1}\, C_\star \quad (I_r - C_\star\, P)^{-1}) \in A^{r\times(q+r)}. \end{cases} \tag{14}$$

Let us denote by $\tilde{L}_i$ the ith column of $\tilde{L}$ and by L^j the jth row of L. Then, all stabilizing controllers of P are parametrized by

$$\begin{aligned} C(Q) &= (C_\star\, (I_q - P\, C_\star)^{-1} + Q)\, ((I_q - P\, C_\star)^{-1} + P\, Q)^{-1} \\ &= ((I_r - C_\star\, P)^{-1} + Q\, P)^{-1}\, ((I_r - C_\star\, P)^{-1}\, C_\star + Q), \end{aligned} \tag{15}$$

where Q is any matrix which belongs to $\Omega = \sum_{i,j=1}^{q+r} A\, (\tilde{L}_i\, L^j)$ and satisfies:

$$\det((I_q - P\, C_\star)^{-1} + P\, Q) \neq 0, \quad \det((I_r - C_\star\, P)^{-1} + Q\, P) \neq 0. \tag{16}$$

Remark 4 Parametrization (15) only uses the fact P is internally stabilizable. No assumption on the existence of a doubly coprime factorization for P has been made. Moreover, parametrization (15) only requires the knowledge of a stabilizing controller $C_\star$ and the explicit computation of the three transfer matrices $(I_q - P\, C_\star)^{-1}$, $C_\star\, (I_q - P\, C_\star)^{-1}$ and $C_\star\, (I_q - P\, C_\star)^{-1}\, P$ as we have the following relations (see (10) for more details):

$$\begin{cases} (I_r - C_\star\, P)^{-1}\, C_\star = C_\star\, (I_q - P\, C_\star)^{-1}, \\ (I_r - C_\star\, P)^{-1} = C_\star\, (I_q - P\, C_\star)^{-1}\, P + I_r. \end{cases}$$

For infinite-dimensional systems, it is generally simpler to compute a particular stabilizing controller of a plant (e.g., a finite-dimensional controller, a controller with a special form) rather than to compute a doubly coprime factorization. We shall show in Corollary 4 that if the plant P admits a doubly coprime factorization, then parametrization (15) becomes the Youla–Kučera parametrization. Hence, (15) is a generalization of the Youla–Kučera parametrization for internally stabilizable plants which do not necessarily admit doubly coprime factorization. Finally, as the Youla–Kučera parametrization, (15) is a linear fractional transformation of $Q \in \Omega$ and the set of free parameters Ω is characterized.

Corollary 3 *Let $P \in K^{q \times r}$ be a stabilizable plant and $C_\star$ a stabilizing controller of P. With the notations (14), we then have:*

1. *If P admits a (weakly) left-coprime factorization $P = D^{-1} N$, then the A-module Ω defined by (2) satisfies:*

$$\Omega = \tilde{L}\, A^{(q+r) \times q}\, D.$$

2. *If P admits a (weakly) right-coprime factorization $P = \tilde{N}\, \tilde{D}^{-1}$, then the A-module Ω defined by (2) satisfies:*

$$\Omega = \tilde{D}\, A^{r \times (q+r)}\, L.$$

Proof 1. By 1 of Proposition 1, we know that the A-module Ω defined by (2) satisfies $\Omega = \{\Lambda \in A^{r \times q} \mid P\, \Lambda \in A^{q \times q}\}\, D$. Hence, the result directly follows if we can prove that $\{\Lambda \in A^{r \times q} \mid P\, \Lambda \in A^{q \times q}\} = \tilde{L}\, A^{(q+r) \times q}$.
Let $\Lambda \in A^{r \times q}$ satisfies $P\, \Lambda \in A^{q \times q}$. Then, every column of Λ belongs to the A-module $\{\lambda \in A^r \mid P\, \lambda \in A^q\} = A : \mathcal{M}$. But, by 7 of Theorem 1, we have $A : \mathcal{M} = \{\lambda \in A^r \mid P\, \lambda \in A^q\} = \tilde{L}\, A^{q+r}$. Therefore, there exists $\Theta \in A^{(q+r) \times q}$ such that $\Lambda = \tilde{L}\, \Theta$, i.e., $\Lambda \in \tilde{L}\, A^{(q+r) \times q}$.
Conversely, if $\Lambda \in \tilde{L}\, A^{(q+r) \times q}$, then there exists $\Theta \in A^{(q+r) \times q}$ such that $\Lambda = \tilde{L}\, \Theta$ and, by 6.a of Theorem 1, we obtain $P\, \Lambda = (P\, \tilde{L})\, \Theta \in A^{q \times q}$.
2. 2 can be proved similarly.

□

By Corollary 8 of [21], we know that an internally stabilizable plant admits a weakly left-coprime (resp., right-coprime) factorization iff it admits a left-coprime (resp., weakly right-coprime) factorization. This fact explains why we have added “weakly” in between brackets in Corollary 3.

In the next corollary, we prove that if P admits a doubly coprime factorization, then parametrization (15) becomes the standard *Youla–Kučera parametrization* [6, 13,33,34].

Corollary 4 *If $P \in K^{q \times r}$ admits the doubly coprime factorization*

$$P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}, \quad \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I_{q+r}, \tag{17}$$

then all stabilizing controllers of P are of the form

$$C(\Lambda) = (Y + \tilde{D}\, \Lambda)\, (X + \tilde{N}\, \Lambda)^{-1} = (\tilde{X} + \Lambda\, N)^{-1}\, (\tilde{Y} + \Lambda\, D), \tag{18}$$

where Λ is any matrix of $A^{r \times q}$ satisfying:

$$\det(X + \tilde{N}\, \Lambda) \neq 0, \quad \det(\tilde{X} + \Lambda\, N) \neq 0. \tag{19}$$

Proof If P admits a doubly coprime factorization $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$, then, by 3 of Proposition 1, we have $\Omega = \tilde{D} A^{r \times q} D$. Moreover, by 1 of Corollary 1, we obtain that $C = (Y D)(X D)^{-1} = Y X^{-1}$ is a stabilizing controller of P. Moreover, by 2 of Corollary 1, $C' = (\tilde{D} \tilde{X})^{-1} (\tilde{D} \tilde{Y}) = \tilde{X}^{-1} \tilde{Y}$ is also a stabilizing controller of P. Then, using the Bézout identities (17), we obtain that $-\tilde{Y} X + \tilde{X} Y = 0$, and thus, $C' = C$. Therefore, by Proposition 3 or Theorem 3, we obtain that all stabilizing controllers of P are defined by

$$\begin{aligned} C(\Lambda) &= (Y D + \tilde{D} \Lambda D)(X D + P \tilde{D} \Lambda D)^{-1} \\ &= (Y D + \tilde{D} \Lambda D)(X D + \tilde{N} \Lambda D)^{-1} \\ &= (Y + \tilde{D} \Lambda) D D^{-1} (X + \tilde{N} \Lambda)^{-1} = (Y + \tilde{D} \Lambda)(X + \tilde{N} \Lambda)^{-1} \end{aligned} \tag{20}$$

$$\begin{aligned} C(\Lambda) &= (\tilde{D} \tilde{X} + \tilde{D} \Lambda D P)^{-1} (\tilde{D} \tilde{Y} + \tilde{D} \Lambda D) \\ &= (\tilde{D} \tilde{X} + \tilde{D} \Lambda N)^{-1} (\tilde{D} \tilde{Y} + \tilde{D} \Lambda D) \\ &= (\tilde{X} + \Lambda N)^{-1} \tilde{D}^{-1} \tilde{D} (\tilde{Y} + \Lambda D) = (\tilde{X} + \Lambda N)^{-1} (\tilde{Y} + \Lambda D) \end{aligned} \tag{21}$$

where $\Lambda \in A^{r \times q}$ is any matrix which satisfies (19). □

Remark 5 Using Example 4, we know that a SISO plant $p \in Q(A)$ is internally stabilizable if there exists $a, b \in A$ satisfying (3) or, equivalently, iff there exists $b \in A$ such that we have:

$$\begin{cases} b \in A, \\ p(1 + b p) \in A. \end{cases} \tag{22}$$

Then, $c = b/(1 + b p)$ internally stabilizes p.

Now, if p admits a coprime factorization $p = n/d$, where $d x - n y = 1$, then, by Corollary 2, $c = y/x$ internally stabilizes p, $a = 1/(1 - p c) = d x$ and $b = c/(1 - p c) = d y$. Moreover, by 3 of Proposition 1, we have $\Omega = A d^2$. Therefore, parametrization (9) becomes:

$$c(q) = \frac{d y + q d^2}{d x + q d^2 p} = \frac{y + q d}{x + q n}, \quad \forall q \in A : x + q n \neq 0. \tag{23}$$

These computations correspond to (20) and (21) for SISO plants [19].

We have recently discovered that these computations had already been obtained by Zames and Francis [35] in their pioneering work on H_∞-control over $A = RH_\infty$. They have independently been obtained in [19] as a direct consequence of the general parametrization (9) for internally stabilizable plants which do not necessarily admit coprime factorizations. It is interesting to notice that conditions (22) can be interpreted as an interpolation problem as it is done in [35]. Finally, the left member of (23) was called by Zames and Francis [35] the *Q-parametrization of all stabilizing controllers* of p and it was shown to be equivalent to the Youla–Kučera parametrization, up to a cancellation by a stable factor (see (23)). Hence, we shall call the parametrizations (8) and (15) the *general Q-parametrization* of all stabilizing controllers. Indeed, conditions 5 and 6 of Theorem 1 extend (22) for MIMO plants and solving 5(a), 5(b), 6(a) and 6(b) of Theorem 1 for a general integral domain A, we then obtain the general parametrizations (8) and (15).

In order to copy the standard form of the Youla–Kučera parametrization, in what follows, we shall use (8) instead of (15) for the parametrization of all stabilizing controllers, i.e., we shall not explicitly express the matrices $U,\ V,\ \tilde{U}$ and $\tilde{V}$ in terms of the transfer matrices P and $C_\star$.

Using Theorem 5 of [21], we answer a question asked by Lin in [15].

Corollary 5 *Let $A = \mathbb{R}(z_1, \ldots, z_m)_S$ be the ring of structural stable multidimensional systems [15,21] and $K = \mathbb{R}(z_1, \ldots, z_m)$ its quotient field. Then, all the stabilizing controllers of an internally stabilizable multidimensional linear system, defined by a transfer matrix with entries in K, are parametrized by means of the standard Youla–Kučera parametrization.*

Remark 6 We note that it is still not known whether or not an internally stabilizable plant over the rings $\mathcal{A}$, $\hat{\mathcal{A}}$ and W_+ admits a doubly coprime factorization, and thus, if the parametrization of all its stabilizing controllers has the form (8) or (18). See [21] for more details.

Since the works of Dedekind in the middle of the $19th$ century, it has been well-known in commutative algebra that projective module over a ring are generally not free, i.e., the hypothesis which ensures an internally stabilizable plant to admit doubly coprime factorizations is rarely satisfied. For instance, this is generally the case for the algebras appearing in algebraic geometry (e.g., coordinate rings of non-singular curves) and in number theory (e.g., integral closures of $\mathbb{Z}$ into finite extensions of $\mathbb{Q}$) [18]. We illustrate this fact by taking an example first considered by Vidyasagar in [33].

Example 11 Let us consider the integral domain

$$A = \mathbb{R}[X_1, X_2, X_3]/(X_1^2 + X_2^2 + X_3^2 - 1) = \mathbb{R}[x_1, x_2, x_3 \mid x_1^2 + x_2^2 + x_3^2 = 1]$$

of polynomials over the 2-dimensional real sphere S^2 and the transfer matrix $P = -(x_2/x_1 \quad x_3/x_1) \in K^{1\times 2}$, where $K = Q(A)$. P admits the left-coprime factorization $P = (1/x_1)\,(-x_2 \quad -x_3)$ because $R = (x_1 \quad x_2 \quad x_3) \in A^{1\times 3}$ satisfies $R\,R^T = 1$. Then, by Corollary 1, we obtain that P is internally stabilizable and, by 3 of Theorem 2, we have $\mathcal{P} = R\,A^3 = A$, and thus, $A : \mathcal{P} = A : A = A$. Therefore, if $P = \tilde{N}\,\tilde{D}^{-1}$ is any right fractional representation of P with $\tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{3\times 2}$, then, by Lemma 5 of [21], we obtain the following *short exact sequence* (see Definition 3 given after):

$$0 \longleftarrow \mathcal{Q} = A^{1\times 3}\,\tilde{R} \xleftarrow{.\tilde{R}} A^{1\times 3} \xleftarrow{.R} A \longleftarrow 0.$$

Moreover, by 8 of Theorem 2, we obtain that $\mathcal{Q} = A^{1\times 3}\,\tilde{R}$ is a projective A-module, and thus, the previous exact sequence *splits* (see Definition 3). Therefore, we obtain that $A^{1\times 3} \cong \mathcal{Q} \oplus A$, showing that $\mathcal{Q}$ is a *stably free A-module* [21,26]. However, it is a well-known result in commutative algebra that $\mathcal{Q} \cong A^{1\times 3}/(A\,R)$ is a stably free but not a free A-module (A is not a *Hermite ring* [21]) [14]. Hence, by 4 of Theorem 2 and Remark 3, we deduce that P does not admit a right-coprime factorization. This result was first obtained by Vidyasagar in [33] (Example 73, Sect. 8.1) using the equivalent topological argument saying that "we cannot comb the hair of a coconut" [14] (see also [18] for another proof). Hence, by Corollary 4, we deduce that it is impossible to parametrize all stabilizing controllers of P by means of the Youla–Kučera parametrization. Using Theorem 3, let us compute a parametrization of all stabilizing controllers of P.

First, we easily check that the matrices

$$U = x_1^2, \quad V = \begin{pmatrix} x_1\,x_2 \\ x_1\,x_3 \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} 1 - x_2^2 & -x_2\,x_3 \\ -x_2\,x_3 & 1 - x_3^2 \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} x_1\,x_2 \\ x_1\,x_3 \end{pmatrix},$$

satisfy conditions 5 and 6 of Theorem 1. Moreover, using 1 of Proposition 1, we obtain:

$$\begin{aligned}\Omega &= \{\Lambda = (\Lambda_1 \;\; \Lambda_2)^T \in A^2 \mid (x_2\,\Lambda_1 + x_3\,\Lambda_2)/x_1 \in A\}\,(x_1\,I_2)\\ &= \{(x_1\,\Lambda_1 \;\; x_1\,\Lambda_2)^T \mid \Lambda_1,\ \Lambda_2 \in A :\ (x_2\,\Lambda_1 + x_3\,\Lambda_2)/x_1 \in A\}.\end{aligned}$$

All the stabilizing controllers of P are then parametrized by (8), i.e.,

$$\begin{aligned}C(\Lambda_1, \Lambda_2) &= \begin{pmatrix} x_1\,(x_2 + \Lambda_1) \\ x_1\,(x_3 + \Lambda_2) \end{pmatrix} \frac{1}{(x_1^2 - x_2\,\Lambda_1 - x_3\,\Lambda_2)}\\ &= \begin{pmatrix} 1 - x_2\,(x_2 + \Lambda_1) & -x_3\,(x_2 + \Lambda_1) \\ -x_2\,(x_3 + \Lambda_2) & 1 - x_3\,(x_3 + \Lambda_2) \end{pmatrix}^{-1} \begin{pmatrix} x_1\,(x_2 + \Lambda_1) \\ x_1\,(x_3 + \Lambda_2) \end{pmatrix},\end{aligned}$$

where Λ_1 and Λ_2 are any elements of A such that $(x_1\,\Lambda_1 \;\; x_1\,\Lambda_2)^{\mathrm{T}} \in \Omega$, i.e., satisfying $(x_2\,\Lambda_1 + x_3\,\Lambda_2)/x_1 \in A$. We let the reader check by himself that all the entries of the matrix $H(P,\ C(\Lambda_1, \Lambda_2))$ defined by (1) belong to A and, in particular, the following two entries:

$$\begin{cases} H(P,\ C(\Lambda_1, \Lambda_2))_{1,2} = (x_2\,(x_2\,\Lambda_1 + x_3\,\Lambda_2 - x_1^2))/x_1 \in A,\\ H(P,\ C(\Lambda_1, \Lambda_2))_{1,3} = (x_3\,(x_2\,\Lambda_1 + x_3\,\Lambda_2 - x_1^2))/x_1 \in A.\end{cases}$$

Let us find a family of generators of the A-module Ω. Using Corollary 3, we find that $\Omega = x_1\,\tilde{L}\,A^3$ where $\tilde{L} = (-\tilde{V} \;\; \tilde{U})$, and thus, Ω is generated by the following three vectors:

$$f_1 = \begin{pmatrix} -x_1^2\,x_2 \\ -x_1^2\,x_3 \end{pmatrix}, \quad f_2 = \begin{pmatrix} x_1\,(1 - x_2^2) \\ -x_1\,x_2\,x_3 \end{pmatrix}, \quad f_3 = \begin{pmatrix} -x_1\,x_2\,x_3 \\ x_1\,(1 - x_3^2) \end{pmatrix}.$$

In particular, using 6(b) of Theorem 1, we then obtain that every element $Q = (x_1\,\Lambda_1 \;\; x_1\,\Lambda_2) \in \Omega$ can be written as:

$$Q = -((x_2\,\Lambda_1 + x_3\,\Lambda_2)/x_1)\,f_1 + \Lambda_1\,f_2 + \Lambda_2\,f_3.$$

Finally, we note that $(x_1\,\Lambda_1 \;\; x_1\,\Lambda_2)^{\mathrm{T}} \in \Omega$ iff there exists $\Lambda_0 \in A$ such that

$$x_1\,\Lambda_0 + x_2\,\Lambda_1 + x_3\,\Lambda_2 = 0 \Leftrightarrow \exists\ \Lambda_0 \in A :\ (\Lambda_0 \;\; \Lambda_1 \;\; \Lambda_2)^{\mathrm{T}} \in \ker\,(x_1 \;\; x_2 \;\; x_3).$$

However, $\ker\,(x_1 \;\; x_2 \;\; x_3)$ is known to be a stably free but not a free A-module [18, 33]. But, we can check that $\ker\,(x_1 \;\; x_2 \;\; x_3)$ is generated by the following three vectors of A^3

$$g_1 = (0 \;\; -x_3 \;\; x_2)^{\mathrm{T}}, \quad g_2 = (x_3 \;\; 0 \;\; -x_1)^{\mathrm{T}}, \quad g_3 = (-x_2 \;\; x_1 \;:\; 0)^{\mathrm{T}},$$

which satisfy the relation $x_1\, g_1 + x_2\, g_2 + x_3\, g_3 = 0$, and thus, we finally obtain:

$$\Omega = \left\{ a_1 \begin{pmatrix} -x_1\, x_3 \\ x_1\, x_2 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ -x_1^2 \end{pmatrix} + a_3 \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix} \in A^2 \mid a_i \in A, \ i = 1, 2, 3 \right\}.$$

If we denote by $h_1 = (-x_1\, x_3 \;\; x_1\, x_2)^{\mathrm{T}}$, $h_2 = (0 \;\; -x_1^2)^{\mathrm{T}}$ and $h_3 = (x_1^2 \;\; 0)^{\mathrm{T}}$, then we check that we have $h_i \in \Omega$ and:

$$f_1 = x_3\, h_2 - x_2\, h_3, \quad f_2 = -x_3\, h_1 + x_1\, h_3, \quad f_3 = x_2\, h_1 - x_1\, h_2,$$

which proves that $\Omega = \sum_{i=1}^{3} A\, f_i = \sum_{i=1}^{3} A\, h_i$. Then, all the stabilizing controllers of P has the form

$$C(a_1, a_2, a_3) = \begin{pmatrix} \frac{a_3\, x_1 + x_2 - a_1\, x_3}{x_1 - a_3\, x_2 + a_2\, x_3} \\ \frac{-a_2\, x_1 + a_1\, x_2 + x_3}{x_1 - a_3\, x_2 + a_2\, x_3} \end{pmatrix}, \tag{24}$$

where the $a_i \in A$ are such that $x_1 - a_3\, x_2 + a_2\, x_3 \neq 0$. Finally, we note that (24) depends on three free parameters whereas, if P would have admitted a doubly coprime factorization, then the Youla–Kučera parametrization would have only depended on two free parameters.

Example 12 In the literature of differential time-delay systems, it is well-known that the unstable plant $p = e^{-s}/(s-1)$ is internally stabilized by the controller $c = -2\, e\, (s-1)/(s+1-2\, e^{-(s-1)})$ involving a distributed delay. This result can be directly checked by computing:

$$\begin{cases} a = \dfrac{1}{(1 - p\, c)} = \dfrac{(s+1-2\, e^{-(s-1)})}{(s+1)} \in H_\infty(\mathbb{C}_+), \\ b = \dfrac{c}{1 - p\, c} = -\dfrac{2\, e\, (s-1)}{(s+1)} \in H_\infty(\mathbb{C}_+), \\ a\, p = \dfrac{p}{(1 - p\, c)} = \dfrac{e^{-s}}{(s+1)}\, \dfrac{(s+1-2\, e^{-(s-1)})}{(s-1)} \in H_\infty(\mathbb{C}_+). \end{cases}$$

Then, all stabilizing controllers of p are parametrized by (11) where, by Example 10, a free parameter $q \in \Omega$ has the form $q = q_1\, a^2 + q_2\, b^2$ with $q_1, q_2 \in H_\infty(\mathbb{C}_+)$. After a few simple computations, we obtain that all stabilizing controllers of p have the form

$$\begin{cases} c(l) = \dfrac{-2\, e + l\, \frac{(s-1)}{(s+1)}}{1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right) + l\, \frac{e^{-s}}{(s+1)}}, \\ l = \left(1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right)\right)^2 q_1 + 4\, e^2\, q_2, \end{cases} \tag{25}$$

for all $q_1, q_2 \in H_\infty(\mathbb{C}_+)$. The previous parametrization is nothing else than the Youla–Kučera parametrization obtained from the following coprime factorization $p = n/d$ (see [19]):

$$n = \frac{e^{-s}}{(s+1)}, \quad d = \frac{(s-1)}{(s+1)}, \quad (-2\, e)\, n - \left(1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right)\right) d = 1.$$

We insist on the fact that we never used any coprime factorization for computing parametrization (25). Hence, parametrization (8) can also be used in order to compute coprime factorizations of a plant when a stabilizing controller is known and coprime factorizations exist for the plant (e.g., plants defined over $H_\infty(\mathbb{C}_+)$). For instance, such a method has recently been used in [2] in order to obtain coprime factorizations for neutral differential time-delay systems which are stabilized by PI controllers.

Let $P \in K^{q\times r}$ be a plant which is internally stabilized by $C_\star \in K^{r\times q}$. Then, with the notations (14), we can define the following two matrices:

$$\begin{aligned} \Pi_1(Q, C_\star) &= L\,(I_q \quad -P) = \begin{pmatrix} (I_q - P\,C_\star)^{-1} & -(I_q - P\,C_\star)^{-1}\,P \\ C_\star\,(I_q - P\,C_\star)^{-1} & -C_\star\,(I_q - P\,C_\star)^{-1}\,P \end{pmatrix}, \\ \Pi_2(Q, C_\star) &= (P^{\mathrm{T}} \quad I_r^{\mathrm{T}})^{\mathrm{T}}\,\tilde{L} = \begin{pmatrix} -P\,(I_r - C_\star\,P)^{-1}\,C_\star & P\,(I_r - C_\star\,P)^{-1} \\ -(I_r - C_\star\,P)^{-1}\,C_\star & (I_r - C_\star\,P)^{-1} \end{pmatrix}. \end{aligned} \tag{26}$$

It is proved in [10,21] that Π_1 and P_2 are two *idempotents* of $A^{(q+r)\times(q+r)}$, namely, $\Pi_i^2 = \Pi_i \in A^{(q+r)\times(q+r)}$, $i = 1, 2$, satisfying $\Pi_1 + \Pi_2 = I_{q+r}$.

Using (1), (12) and (15), we easily obtain the following result.

Corollary 6 *Let us consider an internally stabilizable plant $P \in K^{q\times r}$ and a stabilizing controller $C_\star \in K^{r\times q}$ of P. Then, we have:*

1. *All stable transfer matrices of the closed-loop system (1) are of the form*

$$\begin{aligned} H(P, C(Q)) &= \begin{pmatrix} (I_q - P\,C_\star)^{-1} + P\,Q & (I_q - P\,C_\star)^{-1}\,P + P\,Q\,P \\ C_\star\,(I_q - P\,C_\star)^{-1} + Q & I_r + C_*\,(I_q - P\,C_\star)^{-1}\,P + Q\,P \end{pmatrix} \\ &= \begin{pmatrix} I_q + P\,(I_r - C_\star\,P)^{-1}\,C_\star + P\,Q & P\,(I_r - C_\star\,P)^{-1} + P\,Q\,P \\ (I_r - C_\star\,P)^{-1}\,C_* + Q & (I_r - C_\star\,P)^{-1} + Q\,P \end{pmatrix}, \end{aligned}$$

   where Q is any matrix which belongs to the A-module Ω defined by (12) and satisfies (16).

2. *All the stable idempotents (26) are of the form*

$$\begin{cases} \Pi_1(Q) = \begin{pmatrix} (I_q - P\,C_\star)^{-1} + P\,Q & -(I_q - P\,C_\star)^{-1}\,P - P\,Q\,P \\ C_\star\,(I_q - P\,C_\star)^{-1} + Q & -C_\star\,(I_q - P\,C_\star)^{-1}\,P - Q\,P \end{pmatrix}, \\ \Pi_2(Q) = \begin{pmatrix} -P\,(I_r - C_\star\,P)^{-1}\,C_* - P\,Q & P\,(I_r - C_\star\,P)^{-1} + P\,Q\,P \\ -(I_r - C_\star\,P)^{-1}\,C_* - Q & (I_r - C_\star\,P)^{-1} + Q\,P \end{pmatrix}, \end{cases}$$

   where Q is any matrix which belongs to the A-module Ω defined by (12) and satisfies (16).

In particular, the transfer matrices H, Π_1 and Π_2 are affine, and thus, convex in the free parameter Q, namely, for all $\lambda \in A$, Q_1, $Q_2 \in \Omega$, we have:

$$\begin{cases} H(P, C(\lambda\,Q_1 + (1-\lambda)\,Q_2)) = \lambda\,H(P, C(Q_1)) + (1-\lambda)\,H(P, C(Q_2)), \\ \Pi_i(\lambda\,Q_1 + (1-\lambda)\,Q_2) = \lambda\,\Pi_i(Q_1) + (1-\lambda)\,\Pi_i(Q_2), \quad i = 1, 2. \end{cases}$$

The fact that the parametrization of all stabilizing controllers (8) is a linear fractional transformation of the free parameter Q allows us to rewrite many standard optimization problems as convex ones. For instance, if A is a Banach algebra with respect to the norm $\| \cdot \|_A$ (e.g., $A = H_\infty(\mathbb{C}_+),\ \hat{\mathcal{A}},\ W_+$), then the sensitivity minimization problem becomes

$$\begin{aligned}
&\inf_{C \in \mathrm{Stab}(P)} \| W_1\,(I_q - P\,C)^{-1}\, W_2 \|_A \\
&\quad = \inf_{Q \in \Omega} \| W_1\,((I_q - P\,C_\star)^{-1} + P\,Q)\, W_2 \|_A, \\
&\quad = \inf_{\Lambda \in A^{(q+r)\times(q+r)}} \| W_1\,((I_q - P\,C_\star)^{-1} + (P\,\tilde{L})\,\Lambda\,L)\, W_2 \|_A,
\end{aligned}$$

where $\mathrm{Stab}(P)$ denotes the set of all stabilizing controllers of P, $C_\star$ is a particular stabilizing controller of P, L and $\tilde{L}$ are the matrices defined by (14) and $W_1,\ W_2 \in A^{q\times q}$ are two weighted matrices.

Copying the case $A = H_\infty(\mathbb{C}_+)$ developed in [9], we can also define:

$$b_{\mathrm{opt}}(P) = \sup_{C \in \mathrm{Stab}(P)} \| \Pi_1(P, C) \|_A^{-1}.$$

Therefore, we obtain:

$$\begin{aligned}
b_{\mathrm{opt}}(P)^{-1} &= \inf_{C \in \mathrm{Stab}(P)} \| \Pi_1(P, C) \|_A \\
&= \inf_{Q \in \Omega} \| \Pi_1(P, C(Q)) \|_A \\
&= \inf_{Q \in \Omega} \left\| \begin{pmatrix} (I_q - P\,C_\star)^{-1} + P\,Q & -(I_q - P\,C_\star)^{-1}\,P + P\,Q\,P \\ C_\star\,(I_q - P\,C_\star)^{-1} + Q & -C_\star\,(I_q - P\,C_\star)^{-1}\,P + Q\,P \end{pmatrix} \right\|_A, \\
&= \inf_{\Lambda \in A^{(q+r)\times(q+r)}} \left\| \begin{pmatrix} (I_q - P\,C_\star)^{-1} + (P\,\tilde{L})\,\Lambda\,L & -(I_q - P\,C_\star)^{-1}\,P + (P\,\tilde{L})\,\Lambda\,(L\,P) \\ C_\star\,(I_q - P\,C_\star)^{-1} + \tilde{L}\,\Lambda\,L & -C_\star\,(I_q - P\,C_\star)^{-1}\,P + \tilde{L}\,\Lambda\,(L\,P) \end{pmatrix} \right\|_A.
\end{aligned}$$

However, we need to study whether or not $b_{\mathrm{opt}}(P)$ is still relevant for robust stabilization problems over A as well as when we have the following equality:

$$b_{\mathrm{opt}}(P) = \sup_{C \in \mathrm{Stab}(P)} \| \Pi_2(P, C) \|_A^{-1}.$$

We shall study these questions and the applications of the new parametrization (8) in optimal and robust problems in a forthcoming publication.

3.2 Characterizations in terms of fractional representations of P

The purpose of this section is to give a parametrization of all stabilizing controllers of an internally stabilizable plant which does not necessarily admit doubly coprime factorizations using fractional representations of P and $C_\star$.

Proposition 5 *Let $P \in K^{q\times r}$ be an internally stabilizable plant with a fractional representation $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$, $C_\star \in K^{r\times q}$ a stabilizing controller of P and let us denote by:*

$$\begin{cases} X = (D - N\,C_\star)^{-1} \in K^{q\times q}, \\ Y = C_\star\,(D - N\,C_\star)^{-1} \in K^{r\times q}, \\ \tilde{X} = (\tilde{D} - C_\star\,\tilde{N})^{-1} \in K^{r\times r}, \\ \tilde{Y} = (\tilde{D} - C_\star\,\tilde{N})^{-1}\,C_\star \in K^{r\times q}. \end{cases}$$

Then, all stabilizing controllers of P are of the form

$$C(Q) = (Y + \tilde{D}\, Q)\, (X + \tilde{N}\, Q)^{-1} = (\tilde{X} + Q\, N)^{-1}\, (\tilde{Y} + Q\, D), \tag{27}$$

where Q is any matrix which belongs to the A-module Δ defined by (6), i.e.,

$$\Delta = \left\{ T \in K^{r \times q} \mid \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T\, (D \quad -N) \in A^{(q+r) \times (q+r)} \right\},$$

and satisfies:

$$\det(X + \tilde{N}\, Q) \neq 0, \quad \det(\tilde{X} + Q\, N) \neq 0. \tag{28}$$

Proof Let us consider two stabilizing controllers C_1 and $C_2 \in K^{r \times q}$ of the plant $P \in K^{q \times r}$. Then, we have $C_i = Y_i\, X_i^{-1} = \tilde{X}_i^{-1}\, \tilde{Y}_i$, with the notations

$$\begin{cases} X_i = (D - N\, C_i)^{-1} \in K^{q \times q}, \\ Y_i = C_i\, (D - N\, C_i)^{-1} \in K^{r \times q}, \\ \tilde{X}_i = (\tilde{D} - C_i\, \tilde{N})^{-1} \in K^{r \times r}, \\ \tilde{Y}_i = (\tilde{D} - C_i\, \tilde{N})^{-1}\, C_i \in K^{r \times q}, \end{cases} \quad i = 1, 2,$$

and the matrix $S_i = (X_i^T \quad Y_i^T)^T$ (resp., $\tilde{S}_i = (-\tilde{Y}_i \quad \tilde{X}_i)$) satisfies conditions 5(a) and 5(b) (resp., 6(a) and 6(b)) of Theorem 2, i.e., we have:

$$\begin{cases} D\, X_i - N\, Y_i = I_q, \\ \begin{pmatrix} X_i \\ Y_i \end{pmatrix} (D \quad -N) \in A^{(q+r) \times (q+r)}, \end{cases} \quad \begin{cases} -\tilde{Y}_i\, \tilde{N} + \tilde{X}_i\, \tilde{D} = I_r, \\ \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} (-\tilde{Y}_i \quad \tilde{X}_i) \in A^{(q+r) \times (q+r)}. \end{cases} \tag{29}$$

In particular, from (29), we obtain the following identities:

$$\begin{cases} D\, (X_2 - X_1) = N\, (Y_2 - Y_1), \\ (\tilde{Y}_2 - \tilde{Y}_1)\, \tilde{N} = (\tilde{X}_2 - \tilde{X}_1)\, \tilde{D}. \end{cases} \tag{30}$$

Moreover, using the identities (10), we also have

$$\begin{aligned} Y_i\, D = C_i\, (I_q - P\, C_i)^{-1} &= (I_r - C_i\, P)^{-1}\, C_i = \tilde{D}\, \tilde{Y}_i, \quad i = 1, 2, \\ &\Rightarrow (Y_2 - Y_1)\, D = \tilde{D}\, (\tilde{Y}_2 - \tilde{Y}_1). \end{aligned} \tag{31}$$

Hence, if we denote by $Q = \tilde{D}^{-1}\, (Y_2 - Y_1) = (\tilde{Y}_2 - \tilde{Y}_1)\, D^{-1}$, using (30), we then obtain

$$\begin{cases} Y_2 = Y_1 + \tilde{D}\, Q, \\ X_2 = X_1 + \tilde{N}\, Q, \\ \tilde{Y}_2 = \tilde{Y}_1 + Q\, D, \\ \tilde{X}_2 = \tilde{X}_1 + Q\, N, \end{cases}$$

which gives

$$C_2 = (Y_1 + \tilde{D}\, Q)\, (X_1 + \tilde{N}\, Q)^{-1} = (\tilde{X}_1 + Q\, N)^{-1}\, (\tilde{Y}_1 + Q\, D),$$

when $\det(X_1 + \tilde{N}\, Q) \neq 0$ and $\det(\tilde{X}_1 + Q\, N) \neq 0$.

Finally, using (29), (30) and (31), we obtain

$$\begin{aligned}\begin{pmatrix}\tilde{N}\\ \tilde{D}\end{pmatrix} Q\,(D \quad -N) &= \begin{pmatrix}\tilde{N}\,Q\,D & -\tilde{N}\,Q\,N\\ \tilde{D}\,Q\,D & -\tilde{D}\,Q\,N\end{pmatrix}\\ &= \begin{pmatrix}\tilde{N}\,(\tilde{Y}_2-\tilde{Y}_1) & -\tilde{N}\,(\tilde{Y}_2-\tilde{Y}_1)\,P\\ (Y_2-Y_1)\,D & -(Y_2-Y_1)\,N\end{pmatrix}\\ &= \begin{pmatrix}\tilde{N}\,(\tilde{Y}_2-\tilde{Y}_1) & -\tilde{N}\,(\tilde{X}_2-\tilde{X}_1)\\ (Y_2-Y_1)\,D & -(Y_2-Y_1)\,N\end{pmatrix} \in A^{(q+r)\times(q+r)},\end{aligned}$$

which proves that $Q \in \Delta$. Hence, if we use the notations $X_1 = X$, $Y_1 = Y$, $\tilde{X}_1 = \tilde{X}$ and $\tilde{Y}_1 = \tilde{Y}$, then we have $C_2 = C(Q)$, where $C(Q)$ is defined by (27) for a certain $Q \in \Delta$ which satisfies (28).

Finally, let us prove that, for every $Q \in \Delta$ which satisfies (28), the controller $C(Q) = (Y+\tilde{D}\,Q)\,(X+\tilde{N}\,Q)^{-1} = (\tilde{X}+Q\,N)^{-1}\,(\tilde{Y}+Q\,D)$ internally stabilizes P. Using the fact that $Q \in \Delta$, where Δ is defined by (6), we obtain

$$\begin{cases}\begin{pmatrix}X+\tilde{N}\,Q\\ Y+\tilde{D}\,Q\end{pmatrix}(D \quad -N) = \begin{pmatrix}(X+\tilde{N}\,Q)\,D & -(X+\tilde{N}\,Q)\,N\\ (Y+\tilde{D}\,Q)\,D & -(Y+\tilde{D}\,Q)\,N\end{pmatrix}\\ \qquad\qquad = \begin{pmatrix}X\,D & -X\,N\\ Y\,D & -Y\,N\end{pmatrix} + \begin{pmatrix}\tilde{N}\\ \tilde{D}\end{pmatrix} Q(D-N) \in A^{(q+r)\times(q+r)},\\ (D \quad -N)\begin{pmatrix}X+\tilde{N}\,Q\\ Y+\tilde{D}\,Q\end{pmatrix} = D\,(X+\tilde{N}\,Q) - N\,(Y+\tilde{D}\,Q) = I_q,\end{cases}$$

as we have $D\,\tilde{N} = N\,\tilde{D}$, which shows that $C(Q) = (Y+\tilde{D}\,Q)\,(X+\tilde{N}\,Q)^{-1}$ internally stabilizes P by 5 of Theorem 2. Moreover, we have

$$\begin{cases}\begin{pmatrix}\tilde{N}\\ \tilde{D}\end{pmatrix}(-(\tilde{Y}+Q\,D) \quad (\tilde{X}+Q\,N)) = \begin{pmatrix}-\tilde{N}\,(\tilde{Y}+Q\,D) & \tilde{N}\,(\tilde{X}+Q\,N)\\ -\tilde{D}\,(\tilde{Y}+Q\,D) & \tilde{D}\,(\tilde{X}+Q\,N)\end{pmatrix},\\ \qquad\qquad = \begin{pmatrix}-\tilde{N}\,\tilde{Y} & \tilde{N}\,\tilde{X}\\ -\tilde{D}\,\tilde{Y} & \tilde{D}\,\tilde{X}\end{pmatrix} - \begin{pmatrix}\tilde{N}\\ \tilde{D}\end{pmatrix} Q\,(D-N) \in A^{(q+r)\times(q+r)},\\ (-(\tilde{Y}+Q\,D) \quad (\tilde{X}+Q\,N))\begin{pmatrix}\tilde{N}\\ \tilde{D}\end{pmatrix} = -(\tilde{Y}+Q\,D)\,\tilde{N} + (\tilde{X}+Q\,N)\,\tilde{D} = I_r,\end{cases}$$

as we have $D\,\tilde{N} = N\,\tilde{D}$, which shows that $C(Q) = (\tilde{X}+Q\,N)^{-1}\,(\tilde{Y}+Q\,D)$ internally stabilizes P by 6 of Theorem 2. □

Example 13 If $p = n/d$ is internally stabilized by the controller $c = y/x$, where $0 \neq x, y \in K = Q(A)$ satisfy (7), then all stabilizing controllers of p are of the form

$$c(q) = \frac{y+d\,q}{x+n\,q}, \tag{32}$$

where q is any element of $\Delta = \{l \in K \mid l\,d^2,\ l\,n^2,\ l\,d\,n \in A\}$ satisfying the condition $x+n\,q \neq 0$. We find the parametrization of all stabilizing controllers of an internally stabilizable SISO plant obtained in [19].

Let us now give an explicit description of the A-module Δ in terms of the plant P and a stabilizing controller $C_\star$.

Proposition 6 *Let $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ be a fractional representation of an internally stabilizable plant $P \in K^{q\times r}$, $C_\star \in K^{r\times q}$ a stabilizing controller of P and let us denote by:*

$$\begin{cases} S = \begin{pmatrix} (D - N\, C_\star)^{-1} \\ C_\star\, (D - N\, C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \tilde{S} = (-(\tilde{D} - C_\star\, \tilde{N})^{-1}\, C_\star \quad (\tilde{D} - C_\star\, \tilde{N})^{-1}) \in A^{r\times(q+r)}. \end{cases}$$

Then, the A-module Δ defined by (6) satisfies

$$\Delta = \tilde{S}\, A^{(q+r)\times(q+r)}\, S, \tag{33}$$

that is, Δ is generated over A by the $(q+r)^2$ matrices $\tilde{S}\, E^i_j\, S$, where E^i_j denotes the matrix defined by 1 in the ith row and jth column and 0 elsewhere, and i, $j = 1, \ldots, q+r$. Equivalently, if we denote by $\tilde{S}_i$ the ith column of $\tilde{S}$ and S^j the jth row of S, then we have:

$$\Delta = \sum_{i,j=1}^{q+r} A\,(\tilde{S}_i\, S^j). \tag{34}$$

Proof By 8 of Theorem 2, we have $A : \mathcal{Q} = \tilde{S}\, A^{q+r}$. Therefore, using Example 7, we obtain:

$$\begin{aligned} \Delta &= (\tilde{S}\, A^{q+r}) : ((D \quad -N)\, A^{q+r}) \\ &= \{T \in K^{r\times q} \mid T\,(D \quad -N)\, A^{q+r} \subseteq \tilde{S}\, A^{q+r}\} \\ &= \{T \in K^{r\times q} \mid \exists\, \Lambda \in A^{(q+r)\times(q+r)} :\; T\,(D \quad -N) = \tilde{S}\, \Lambda\}. \end{aligned}$$

Hence, if $T \in \Delta$, then there exists $\Lambda \in A^{(q+r)\times(q+r)}$ such that $T\,(D \quad -N) = \tilde{S}\,\Lambda$. Now, using the fact that $(D \quad -N)\, S = I_q$ (see 5(b) of Theorem 2), we obtain:

$$T = T\,((D \quad -N)\, S) = (T\,(D \quad -N))\, S = \tilde{S}\,\Lambda\, S \Rightarrow T \in \tilde{S}\, A^{(q+r)\times(q+r)}\, S.$$

Conversely, if $T \in \tilde{S}\, A^{(q+r)\times(q+r)}\, S$, then there exists $\Lambda \in A^{(q+r)\times(q+r)}$ such that $T = \tilde{S}\,\Lambda\, S$, where S and $\tilde{S}$ satisfy conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 2. In particular, using 5(a) and 6(a) of Theorem 2, we obtain

$$\begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} T\,(D \quad -N) = \left(\begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} \tilde{S}\right) \Lambda\,(S\,(D \quad -N)) \in A^{(q+r)\times(q+r)},$$

showing that $T \in \Delta$ and proving (33).

Finally, (34) follows from the same proof as the one given in Proposition 4. □

Example 14 Let us consider again Example 13. If $c = y/x$ is a stabilizing controller of p where $x, y \in Q(A)$ satisfy (7), then we obtain $S = (x \quad y)^T$ and $\tilde{S} = (-y \quad x)$, and thus, by Proposition 6, the A-module $\Delta = \tilde{S}\, A^{2\times 2}\, S = A\, x^2 + A\, x\, y + A\, y^2$ is the fractional ideal of A generated by x^2, $x\, y$ and y^2. Now, using (7), we obtain $x\, y = (y\, d)\, x^2 - (x\, n)\, y^2 \in A\, x^2 + A\, y^2$ because $y\, d,\ x\, n \in A$. Therefore, we obtain $\Delta = A\, x^2 + A\, y^2$ and the free parameter q of (32) has the form $q = q_1\, x^2 + q_2\, y^2$ where q_1 and q_2 are any elements of A satisfying $x + n\, q \neq 0$. Hence, parametrization (32) only depends on two free parameters. See [19] for more details and examples.

Combining Propositions 5 and 6, we obtain the second main result.

Theorem 4 *Let $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ be a fractional representation of an internally stabilizable plant $P \in K^{q\times r}$, $C_\star \in K^{r\times q}$ a stabilizing controller of P and let us denote by:*

$$\begin{cases} S = \begin{pmatrix} (D - N\,C_\star)^{-1} \\ C_\star\,(D - N\,C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \tilde{S} = (-(\tilde{D} - C_\star\,\tilde{N})^{-1}\,C_\star \quad (\tilde{D} - C_\star\,\tilde{N})^{-1}) \in A^{r\times(q+r)}. \end{cases} \tag{35}$$

Let us denote by $\tilde{S}_i$ the ith column of $\tilde{S}$ and by S^j the jth row of S. Then, all stabilizing controllers of P are parametrized by

$$\begin{aligned} C(Q) &= (C_\star\,(D - N\,C_\star)^{-1} + \tilde{D}\,Q)\,((D - N\,C_\star)^{-1} + \tilde{N}\,Q)^{-1} \\ &= ((\tilde{D} - C_\star\,\tilde{N})^{-1} + Q\,N)^{-1}\,((\tilde{D} - C_\star\,\tilde{N})^{-1}\,C_\star + Q\,D), \end{aligned} \tag{36}$$

where Q is any matrix which belongs to $\Delta = \sum_{i,j=1}^{q+r} A\,(\tilde{S}_i\,S^j)$ and satisfies:

$$\det((D - N\,C_\star)^{-1} + \tilde{N}\,Q) \neq 0, \quad \det((\tilde{D} - C_\star\,\tilde{N})^{-1} + Q\,N) \neq 0.$$

Corollary 7 *Let $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ be a fractional representation of an internally stabilizable plant $P \in K^{q\times r}$ and $C_\star \in K^{r\times q}$ a stabilizing controller of P. Then, with the notations (35), we have:*

1. *If $P = D^{-1}\,N$ is a (weakly) left-coprime factorization of P, then the A-module Δ defined by (6) satisfies $\Delta = \tilde{S}\,A^{(q+r)\times q}$.*
2. *If $P = \tilde{N}\,\tilde{D}^{-1}$ is a (weakly) right-coprime factorization of P, then the A-module Δ defined by (6) satisfies $\Delta = A^{r\times(q+r)}\,S$.*

Proof 1. If we denote by $\tilde{R} = (\tilde{N}^{\mathrm{T}} \quad \tilde{D}^{\mathrm{T}})^{\mathrm{T}} \in A^{(q+r)\times r}$, then, by Proposition 2, we have $\Delta = A^{1\times q} : \mathcal{Q} = \{T \in K^{r\times q} \mid \tilde{R}\,T \in A^{(q+r)\times q}\}$. Using 6(b) of Theorem 2, we easily check that $\tilde{S}\,A^{(q+r)\times q} \subseteq \Delta$, where $\tilde{S}$ is defined by (35). Now, by 8 of Theorem 2, we have $A : \mathcal{Q} = \{\lambda \in K^r \mid \tilde{R}\,\lambda \in A^{q+r}\} = \tilde{S}\,A^{q+r}$. Hence, every column of $T \in \Delta$ belongs to $\tilde{S}\,A^{q+r}$, i.e., $T \in \tilde{S}\,A^{(q+r)\times q}$, which proves the result.
2. Condition 2 can be proved similarly.

□

Finally, if P admits a doubly coprime factorization

$$P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \quad D\,X - N\,Y = I_q, \quad -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r,$$

then, using 3 of Proposition 2, Corollary 2 and Proposition 5 or Theorem 4, we obtain that $C_\star = Y\,X^{-1} = \tilde{X}^{-1}\,\tilde{Y}$ is a stabilizing controller of P and all stabilizing controllers of P are parametrized by (27) where Q is any element of $\Delta = A^{r\times q}$. Therefore, parametrization (27) becomes the well-known Youla–Kučera parametrization (18) and we find again Corollary 4.

To finish, let us explain the relations between the two parametrizations (15) and (36).

Proposition 7 *Let $P = D^{-1} N = \tilde{N}\, \tilde{D}^{-1}$ be a fractional representation of an internally stabilizable plant P, Ω (resp., Δ) the A-module defined by (2) (resp., (6)). Then, we have:*

$$\Omega = \tilde{D}\, \Delta\, D. \tag{37}$$

Moreover, if L and $\tilde{L}$ are two matrices satisfying conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 1, then the matrices $S = L\, D^{-1}$ and $\tilde{S} = \tilde{D}^{-1}\, \tilde{L}$ satisfy conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 2. Conversely, if the matrices S and $\tilde{S}$ satisfy conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 2, then the matrices $L = S\, D$ and $\tilde{L} = \tilde{D}\, \tilde{S}$ satisfy conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 1. Hence, parametrizations (8) and (36) are the same.

Proof Let $T \in \Delta$, i.e., by (6), the matrix $T \in K^{r \times q}$ is such that we have $(\tilde{N}^T \quad \tilde{D}^T)^T\, T\, (D \quad -N) \in A^{(q+r)\times(q+r)}$. Hence, if we denote by $Q = \tilde{D}\, T\, D$, then we have

$$\begin{cases} Q \in A^{r \times q}, \\ P\, Q = \tilde{N}\, T\, D \in A^{q \times q}, \\ Q\, P = \tilde{D}\, T\, N \in A^{r \times r}, \\ P\, Q\, P = \tilde{N}\, T\, N \in A^{q \times r}, \end{cases}$$

which shows that $Q \in \Delta$, and thus, $\tilde{D}\, \Delta\, D \subseteq \Omega$. Conversely, let $Q \in \Omega$ and let us denote by $T = \tilde{D}^{-1}\, Q\, D^{-1} \in K^{r \times q}$. Then, we have

$$\begin{cases} \tilde{D}\, T\, D = Q \in A^{r \times q}, \\ \tilde{N}\, T\, D = P\, Q \in A^{q \times q}, \\ \tilde{D}\, T\, N = Q\, P \in A^{r \times r}, \\ \tilde{N}\, T\, N = P\, Q\, P \in A^{q \times r}, \end{cases}$$

which shows that $T \in \Delta$, and thus, we obtain the equality $\Omega = \tilde{D}\, \Delta\, D$.

Now, if L satisfies conditions 5(a) and 5(b) of Theorem 1, then the matrix defined by $S = L\, D^{-1} \in K^{(q+r)\times q}$ satisfies

$$\begin{cases} S\, (D \quad -N) = L\, (I_q \quad -P) \in A^{(q+r)\times(q+r)}, \\ (D \quad -N)\, S = D\, (I_q \quad -P)\, L\, D^{-1} = D\, D^{-1} = I_q, \end{cases}$$

showing that S satisfies conditions 5(a) and 5(b) of Theorem 2.

Now, if $\tilde{L}$ satisfies conditions 6(a) and 6(b) of Theorem 1, then the matrix defined by $\tilde{S} = \tilde{D}^{-1}\, \tilde{L} \in K^{r \times (q+r)}$ satisfies

$$\begin{cases} \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} \tilde{S} = \begin{pmatrix} P \\ I_r \end{pmatrix} \tilde{L} \in A^{(q+r)\times(q+r)}, \\ \tilde{S} \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} = \tilde{D}^{-1}\, \tilde{L} \begin{pmatrix} P \\ I_r \end{pmatrix} \tilde{D} = \tilde{D}^{-1}\, \tilde{D} = I_r, \end{cases}$$

which shows that $\tilde{S}$ satisfies conditions 6(a) and 6(b) of Theorem 2.

The converse results can be proved similarly.

Finally, let us show that parametrizations (15) and (36) are in fact the same. Let $L = (U^{\mathrm{T}} \quad V^{\mathrm{T}})^{\mathrm{T}}$ and $\tilde{L} = (-\tilde{V} \quad \tilde{U})$ be two matrices satisfying conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 1 and $S = L\, D^{-1}$ and $\tilde{S} = \tilde{D}^{-1}\,\tilde{L}$ the matrices satisfying conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 2. Then, we have:

$$\begin{cases} X = U\, D^{-1}, \\ Y = V\, D^{-1}, \\ \tilde{X} = \tilde{D}^{-1}\,\tilde{U}, \\ \tilde{Y} = \tilde{D}^{-1}\,\tilde{V}, \end{cases} \Leftrightarrow \begin{cases} U = X\, D, \\ V = Y\, D, \\ \tilde{U} = \tilde{D}\,\tilde{X}, \\ \tilde{V} = \tilde{D}\,\tilde{Y}. \end{cases}$$

Now, using the fact that $Q \in \Omega$ has the form $Q = \tilde{D}\, T\, D$ for a certain $T \in \Delta$ and, conversely, $T \in \Delta$ has the form $T = \tilde{D}^{-1}\, Q\, D^{-1}$ for a certain $Q \in \Omega$, we then obtain

$$\begin{aligned} C(Q) &= (V + Q)\,(U + P\,Q)^{-1}, \\ &= (Y\,D + \tilde{D}\,T\,D)\,(X\,D + P\,\tilde{D}\,T\,D)^{-1}, \\ &= (Y + \tilde{D}\,T)\,D\,D^{-1}\,(X + \tilde{N}\,T)^{-1}, \\ &= (Y + \tilde{D}\,T)\,(X + \tilde{N}\,T)^{-1}, \end{aligned}$$

and

$$\begin{aligned} C(Q) &= (\tilde{U} + Q\,P)^{-1}\,(\tilde{V} + Q), \\ &= (\tilde{D}\,\tilde{X} + \tilde{D}\,T\,D\,P)^{-1}\,(\tilde{D}\,\tilde{Y} + \tilde{D}\,\tilde{T}\,D), \\ &= (\tilde{X} + T\,N)^{-1}\,\tilde{D}\,\tilde{D}^{-1}\,(\tilde{Y} + T\,D), \\ &= (\tilde{X} + T\,N)^{-1}\,(\tilde{Y} + T\,D), \end{aligned}$$

which proves the result. □

By Proposition 7, we obtain that parametrizations (15) and (36) or, equivalently, parametrizations (8) and (27) are the same but either expressed in terms of the transfer matrices P and $C_\star$ (classical approach [35]) or in terms of the stable matrices $D, N, \tilde{D}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y}$ (fractional representation approach [6,33]). Moreover, Proposition 7 shows that all the results concerning internal stabilizability stated in terms of fractional representations of P and C can be deduced from the results only using the transfer matrices P and C and conversely. We have deliberately chosen to write these results in a rather independent way. We hope that these redundancies would have some pedagogical virtues by showing different formulations of equivalent results. Moreover, they show that the lattice approach to analysis and synthesis problems developed in [21] and continued in this paper is a mathematical framework in which the classical and the fractional representation approaches can be simultaneously studied by means of similar concepts, methods and results.

To finish, let us point out that, in his pioneering work [30,31], Sule developed a parametrization of all stabilizing controllers for a plant which does not necessarily admit doubly coprime factorizations. Unfortunately, this parametrization is not explicit in terms of free parameters and it requires a large amount of local computations. Moreover, it exists over a *unique factorization domain* (UFD or factorial ring) [26]. We recall that an integral domain A is called a UFD if every non-zero element of A can be written as a unique product of primes, up to multiplication by units and renumbering the prime factors. The next results show that no Banach algebra over the field of complex numbers $\mathbb{C}$ is a UFD.

Theorem 5 We have:

1. *[29] Let A be a complex Banach algebra which is also a UFD. If, for every prime element $a \in A$, the ideal (a) is closed, then A is isomorphic to $\mathbb{C}$.*
2. *[32] If A is a commutative complex Banach algebra, then every prime element $a \in A$ has close range, i.e., (a) is a closed ideal of A.*

We obtain the following direct consequence of the two previous results, answering to an open question asked in [29].

Corollary 8 *A complex Banach algebra which is a UFD is isomorphic to $\mathbb{C}$.*

Hence, we cannot use the parametrization of all stabilizing controllers developed in [30,31] over the non-trivial Banach algebras $\mathcal{A}$, $\hat{\mathcal{A}}$ and W_+. We recall that we do not know whether or not every internally stabilizable plant defined over $\mathcal{A}$, $\hat{\mathcal{A}}$ and W_+ admits doubly coprime factorizations. However, we can use parametrizations (15) and (36) or, equivalently, parametrizations (8) and (27), as no restriction on the integral domain A is required. Let us point out that extensions of the results of this paper are possible over non-integral domains using techniques similar as the ones developed in [12].

Finally, Mori [17] has recently developed a parametrization of all stabilizing controllers for internally stabilizable plants. His parametrization has the advantage to be more explicit than the one obtained by Sule. However, it is less explicit than parametrizations (8) and (27) as, for instance, it has not the explicit form of a linear fractional transformation. Moreover, the set of free parameters is not explicitly characterized contrary to the A-modules Ω and Δ (see (2), (6), Propositions 1, 2, 4 and 6, and Corollaries 3 and 7).

4 Minimal generating systems of Ω and Δ

The first purpose of this last section is to give a homological algebra interpretation of the parametrization of all stabilizing controllers (8) and (27) in terms of *split exact sequences* [3,26]. Then, using this new interpretation, we prove that Ω and Δ are two projective A-modules of rank $r \times q$. Finally, this result is used in order to give an upper bound on the minimal number of free parameters appearing in parametrizations (8) and (27).

In order to do that, we first recall a few definitions [3,21,26].

Definition 3 We have:

1. $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is called a short exact sequence of A-modules if the A-morphisms f and g (i.e., A-linear maps) satisfy that f is injective, g is surjective and $\ker g = \operatorname{im} f$.
2. A short exact sequence is said to be a split exact sequence if one of the following equivalent assertions is satisfied:
   - there exists an A-morphism $h : M'' \to M$ which satisfies $g \circ h = id_{M''}$,

– there exists an A-morphism $k : M \to M'$ which satisfies $k \circ f = id_{M'}$,
– there exist two A-morphisms $h : M'' \to M$ and $k : M \to M'$ such that the A-morphisms defined by

$$\phi = \begin{pmatrix} g \\ k \end{pmatrix} : M \longrightarrow M'' \oplus M', \quad \psi = (h \;\; f) : M'' \oplus M' \longrightarrow M$$

satisfy the following relations:

$$\begin{cases} \phi \circ \psi = \begin{pmatrix} g \\ k \end{pmatrix} (h \;\; f) = \begin{pmatrix} id_{M''} & 0 \\ 0 & id_{M'} \end{pmatrix} = id_{M'' \oplus M'}, \\ \psi \circ \phi = (h \;\; f) \begin{pmatrix} g \\ k \end{pmatrix} = h \circ g + f \circ k = id_M. \end{cases}$$

Then, we have $M \cong M'' \oplus M'$.

Finally, we denote a split exact sequence by the following diagram:

$$0 \longleftarrow M'' \underset{\xrightarrow{h}}{\xleftarrow{g}} M \underset{\xrightarrow{k}}{\xleftarrow{f}} M' \longleftarrow 0. \tag{38}$$

3. An A-module M is said to be finitely generated if M admits a finite family of generators.
4. A finitely generated A-module M is said to be free if M admits a finite basis or, equivalently, if M is isomorphic to a certain power of A, i.e., there exists $r \in \mathbb{Z}_+$ such that $M \cong A^r$.
5. A finitely generated A-module M is said to be projective if there exist an A-module N and $r \in \mathbb{Z}_+$ such that we have $M \oplus N \cong A^r$, i.e., if M is a direct summand of a finitely generated free A-module.
6. The rank of an A-module M, denoted by $\mathrm{rk}_A(M)$, is the dimension of the $K = Q(A)$-vector space $K \otimes_A M$ formed by extending the scalars of M from A to K. In other words, we have $\mathrm{rk}_A(M) = \dim_K(K \otimes_A M)$, where $\otimes_A$ denotes the *tensor product* of A-modules.

Let us first start with the following lemma.

Lemma 2 *Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a split exact sequence. Then, we have the following results:*

1. *All the A-morphisms $h : M'' \longrightarrow M$ satisfying $g \circ h = id_{M''}$ are of the form $h = h_\star + f \circ l$, where $h_\star : M'' \longrightarrow M$ is a particular A-morphism (A-linear map) satisfying $g \circ h_\star = id_{M''}$ and l is any element of $hom_A(M'', M')$, namely any A-morphism from M'' to M'.*
2. *All the A-morphisms $k : M \longrightarrow M'$ satisfying $k \circ f = id_{M'}$ are of the form $k = k_\star + l \circ g$, where $k_\star : M' \longrightarrow M$ is a particular A-morphism satisfying $k_\star \circ f = id_{M'}$ and l is any element of $hom_A(M'', M')$, namely any A-morphism from M'' to M'.*
3. *Moreover, if $k_\star \circ h_\star = 0$, then, for every $l \in hom_A(M'', M')$, we have:*

$$\begin{cases} \begin{pmatrix} g \\ k_\star - l \circ g \end{pmatrix} (h_\star + f \circ l \;\; f) = \begin{pmatrix} id_{M''} & 0 \\ 0 & id_{M'} \end{pmatrix}, \\ (h_\star + f \circ l \;\; f) \begin{pmatrix} g \\ k_\star - l \circ g \end{pmatrix} = id_M. \end{cases}$$

Proof 1. We easily check that the A-morphism $h = h_\star + f \circ l : M'' \to M$, where l is any element of $\hom_A(M'', M')$, is a right-inverse of g as we have $g \circ h = g \circ h_\star + g \circ f \circ l = id_{M''}$ because $g \circ f = 0$ as (38) is an exact sequence.
   If $h_1, h_2 : M'' \to M$ are two A-morphisms satisfying $g \circ h_1 = id_{M''}$ and $g \circ h_2 = id_{M''}$, then we have $g \circ (h_2 - h_1) = 0$, i.e., for every $m'' \in M''$, we have $(h_2 - h_1)(m'') \in \ker g$. Using the fact that (38) is a short exact sequence, for every $m'' \in M''$, there exists a unique element $m' \in M'$ such that we have $(h_2 - h_1)(m'') = f(m')$. Let us denote by $l : M'' \to M'$ the A-morphism which maps an element $m'' \in M''$ to the unique element $m' \in M'$ which satisfies $(h_2 - h_1)(m'') = f(m')$. Then, we obtain $h_2 - h_1 = f \circ l$ where $l \in \hom_A(M'', M')$.
2. Condition 2 can be proved similarly and
3. Condition 3 can be directly checked by computations.

□

Let us recall the following result obtained in [21].

Proposition 8 We have:

1. *[21] Let $P \in K^{q \times r}$ be a transfer matrix, $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ a fractional representation of P and let us denote by:*

$$R = (D \quad -N) \in A^{q \times (q+r)}, \quad \tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{(q+r) \times r}.$$

   Then, we have the following two exact sequences

$$0 \longleftarrow \mathcal{L} \xleftarrow{g} A^{q+r} \xleftarrow{f} A : \mathcal{M} \longleftarrow 0, \tag{39}$$

$$0 \longrightarrow A : \mathcal{L} \xrightarrow{\phi} A^{1 \times (q+r)} \xrightarrow{\psi} \mathcal{M} \longrightarrow 0, \tag{40}$$

   with the notations:

$$\begin{aligned} f : A : \mathcal{M} &\longrightarrow A^{q+r}, & g : A^{q+r} &\longrightarrow \mathcal{L}, \\ \lambda &\longmapsto \begin{pmatrix} P \\ I_r \end{pmatrix} \lambda, & \mu &\longmapsto (I_q \quad -P)\, \mu, \end{aligned}$$

$$\begin{aligned} \phi : A : \mathcal{L} &\longrightarrow A^{1 \times (q+r)}, & \psi : A^{1 \times (q+r)} &\longrightarrow \mathcal{M}, \\ \lambda &\longmapsto \lambda\, (I_q \quad -P), & \mu &\longmapsto \mu \begin{pmatrix} P \\ I_r \end{pmatrix}. \end{aligned}$$

2. *[3,26] If $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is a short exact sequence and M'' is a projective A-module, then the short exact sequence splits.*

   The next lemma directly follows from Lemma 1 and Examples 3 and 7.

Lemma 3 *The A-module Ω defined by (2) satisfies:*

$$\Omega \cong hom_A(\mathcal{L}, \ A : \mathcal{M}) \cong hom_A(\mathcal{M}, A : \mathcal{L}).$$

Similarly, the A-module Δ defined by (6) satisfies:

$$\Delta \cong hom_A(\mathcal{Q}, A : \mathcal{P}) \cong hom_A(\mathcal{P}, A : \mathcal{Q}).$$

We are now in position to interpret parametrizations (8) and (27) in terms of split exact sequences. If P is internally stabilized by the controller C_*, then, by 5 and 6 of Theorem 1, $L = (U^T \;\; V^T)^T \in A^{(q+r)\times q}$ and $\tilde{L} = (-\tilde{V} \;\; \tilde{U}) \in A^{r\times(q+r)}$ satisfy conditions 5(a), 5(b), 6(a) and 6(b) of Theorem 1. Then, the A-morphism $h_\star : \mathcal{L} \to A^{q+r}$ defined by $h_\star(\mu) = L\,\mu$ satisfies $f \circ h_\star = id_{\mathcal{L}}$, and thus, (39) is a split exact sequence

$$0 \longleftarrow \mathcal{L} \overset{g}{\underset{\overrightarrow{h_\star}}{\longleftarrow}} A^{q+r} \overset{f}{\underset{\overrightarrow{k_\star}}{\longleftarrow}} A : \mathcal{M} \longleftarrow 0, \tag{41}$$

where $k_\star : A^{q+r} \to A : \mathcal{M}$ is defined by $k_\star(\lambda) = \tilde{L}\,\lambda$ for all $\lambda \in A^{q+r}$. Then, using Lemma 2, we obtain

$$\begin{cases} \begin{pmatrix} g \\ k_\star - l \circ g \end{pmatrix} (h_\star + f \circ l \;\; f) = id_{\mathcal{L}\oplus(A:\mathcal{M})}, \\ (h_\star + f \circ l \;\; f) \begin{pmatrix} g \\ k_\star - l \circ g \end{pmatrix} = id_{A^{q+r}}, \end{cases}$$

where, by Lemma 3, the arbitrary A-morphism l belongs to:

$$\hom_A(\mathcal{L}, A : \mathcal{M}) \cong (A : \mathcal{M}) : \mathcal{L}) = \Omega. \tag{42}$$

Therefore, every right inverse of g has the form $h_\star + f \circ l$, whereas every left-inverse of f has the form $k_\star - l \circ g$, where l belongs to the A-module defined by (42), and thus, we have

$$\begin{array}{llll} \mathcal{L} \xrightarrow{h_\star + f\circ l} A^{q+r}, & & A^{q+r} \xrightarrow{k_\star - l\circ g} A : \mathcal{M}, & \\ \nu \longmapsto \begin{pmatrix} U + P\,Q \\ V + Q \end{pmatrix} \nu, & & \mu \longmapsto (-(\tilde{V} + Q) \;\; (\tilde{U} + Q\,P))\,\mu, & \end{array}$$

for every $Q \in \Omega$. Then, by 5 and 6 of Theorem 1, we finally obtain that every controller of P has the form (8), where Q is any element of Ω satisfying (9).

Remark 7 Let us point out that parametrization (8) of all stabilizing controllers was firstly obtained in [23] by means of the previous module-theoretic proof. Similar arguments using the split exact sequences defined in Lemma 5 of [21] give another proof of parametrization (27) of all stabilizing controllers.

Finally, for SISO plants, the split exact sequence (41) was proved in [22] to be the cornerstone for the development of a module-theoretical duality between the fractional ideal approach [19,21] and the operator-theoretic approach [5,9,33] to stabilization problems. In particular, the results of [22] give the general forms of the domain and graph of an internally stabilizable SISO plant which does not necessarily admit doubly coprime factorizations, generalizing the different results existing in the literature. Hence, combining (41) with the approach developed in [22], we then can extend the previous results for MIMO plants.

Now, using Lemma 3, we obtain the next important result.

Corollary 9 *If $P \in K^{q\times r}$ is internally stabilizable, then the A-module Ω (resp., Δ) defined by (2) (resp., (6)) is projective of rank $r \times q$.*

Proof The fact that P is internally stabilizable implies that $\mathcal{L}$ is a projective A-module of rank q (see 7 of Theorem 1 and Remark 1), and thus, by 3 of Proposition 8, the exact sequence (41) splits and we obtain:

$$\mathcal{L} \oplus (A : \mathcal{M}) \cong A^{q+r}.$$

See also 7 of Theorem 1. Then, using the well-known fact that

$$\hom_A(\mathcal{L} \oplus \mathcal{M}, P) \cong \hom_A(\mathcal{L}, P) \oplus \hom_A(\mathcal{M}, P),$$

(see [3,26] for more details) and Lemma 3, we obtain

$$\begin{aligned} A^{(q+r)\times(q+r)} &\cong \hom_A(A^{q+r}, A^{q+r}) \\ &\cong \mathrm{end}_A(\mathcal{L}) \oplus \mathrm{end}_A(\mathcal{M}) \ \oplus \ \hom_A(\mathcal{M}, \mathcal{L}) \ \oplus \Omega, \end{aligned}$$

where $\mathrm{end}_A(\mathcal{L}) = \hom_A(\mathcal{L}, \mathcal{L})$ denotes the A-module of endomorphisms of $\mathcal{L}$. Hence, Ω is a summand of the free A-module $A^{(q+r)\times(q+r)}$, i.e., Ω is a finitely generated projective A-module (see 5 of Definition 3). Finally, using the fact that $K = Q(A)$ is a *flat A-module* [3,26], we obtain:

$$\begin{aligned} \mathrm{rank}_A\, \Omega = \dim_K\,(K \otimes_A \Omega) &= \dim_K\,(K \otimes_A \hom_A(\mathcal{L}, \mathcal{M})) \\ &= \dim_K\,(\hom_K(K \otimes_A \mathcal{L}, K \otimes_A \mathcal{M})) = \dim_K\,(\hom_K(K^q, K^r)) \\ &= \dim_K\,(K^{r\times q}) = r \times q. \end{aligned}$$

A similar proof can be obtained for the A-module Δ. □

Let us now study the question of the minimal number of free parameters appearing in the parametrizations (8) and (27). The number of free parameters in these parametrizations is related to the number of elements in the shortest system of generators of the projective A-module Ω (resp., Δ), i.e., is the cardinal $\mu_A(\Omega)$ (resp., $\mu_A(\Delta)$) of the *minimal generating system* of Ω (resp., Δ). Using 5 and 6 of Theorem 1, we note that we have

$$\Omega = \begin{pmatrix} P\,V + I_q \\ V \end{pmatrix} A^{(q+r)\times(q+r)}\,(-V \quad I_r + V\,P) \tag{43}$$

where the matrix $V \in A^{r\times q}$ satisfies the conditions:

$$V\,P \in A^{r\times r}, \quad P\,V \in A^{q\times q}, \quad (P\,V + I_q)\,P = P\,(V\,P + I_r) \in A^{q\times r}.$$

See Remark 4 of [21] for more details. But, contrary to the SISO case (see Example 10 and [19]), from (43), it is not easy to obtain an explicit minimal family of generators of Ω. A similar comment holds for Δ.

Fortunately, the computation of the cardinal of minimal generating systems is an active problem in commutative algebra. In order to state Heitmann's generalization of Forster–Swan's theorem, let us first introduce a few definitions [3,26].

Definition 4 We have:

– *An ideal $\mathfrak{p}$ of A is said to be prime if $a\,b \in \mathfrak{p}$ and $a \notin \mathfrak{p}$ implies $b \in \mathfrak{p}$. We denote by Spec A the set of prime ideals of A endowed with the Zariski topology [3,26].*

- *An ideal* $\mathfrak{m}$ *is called maximal if A is the only ideal strictly containing* $\mathfrak{m}$.
- *The radical Rad(A) of A is the intersection of the maximal ideals of A.*
- *The Krull dimension* $\dim A$ *of a commutative ring A is the supremum of the lengths of chains*

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_d$$

  of distinct proper prime ideals of A.
- *An A-module M is locally generated by d elements if, for all* $\mathfrak{p} \in \operatorname{Spec} A$, *we have* $\mu_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq d$, *where* $A_{\mathfrak{p}} = \{a/b \mid a \in A,\ b \in A \backslash \mathfrak{p}\}$ *and* $M_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A M$ *is the* $A_{\mathfrak{p}}$*-module obtained by extending the scalars of M from A to* $A_{\mathfrak{p}}$.

We are now in position to state Heitmann's generalization of Forster-Swan's result using the Krull dimension and without the noetherianity hypothesis.

Theorem 6 [7,10] *Let A be a commutative ring of Krull dimension m and M a finitely generated A-module which is locally generated by d elements, then M is generated by* $d + m$ *elements, i.e.,* $\mu_A(M) \leq d + m$.
This result also holds if we take the Krull dimension of the ring $A/\mathrm{Rad}\, A$.

Let us note that there exist various refinements of Theorem 6 due to Eisenbud and Evans [7], Heitmann [10] and Coquand et al. [4]. A version of Theorem 6 exists using the concept of the *j-dimension* $j\text{-}\dim A$ instead of the Krull dimension $\dim A$. Such a dimension satisfies $j\text{-}\dim A \leq \dim A$ [10]. Recently, this result has even been improved in [4] using the concept of *H-dimension* $H\text{-}\dim A$ which satisfies $H\text{-}\dim A \leq j\text{-}\dim A$. We shall not enter into the details and we only use here the Krull dimension for a sake of simplicity. We obtain the following corollary of Theorem 6.

Corollary 10 *If A is an integral domain of Krull dimension m, then we have*

$$\begin{cases} \mu_A(\Omega) \leq r \times q + m, \\ \mu_A(\Delta) \leq r \times q + m, \end{cases}$$

i.e., the A-module Ω *(resp.,* Δ*) defined by (2) (resp., (6)) can be generated by* $r \times q + m$ *elements.*

Proof By Corollary 9, Ω is a projective module over an integral domain A of Krull dimension m. In module theory, it is well-known that a projective module of rank l over an integral domain is locally a free module of rank l [3,26]. Therefore, for all $\mathfrak{p} \in \operatorname{Spec}(A)$, $\Omega_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank $r \times q$, i.e., Ω is locally generated by $r \times q$ elements. Then, the result directly follows from Theorem 6. A similar proof can be obtained for the A-module Δ. □

Example 15 We have:

- If the transfer matrix P admits a doubly (weakly) coprime factorization $P = D^{-1} N = \tilde{N}\, \tilde{D}^{-1}$, then, by Propositions 1 and 2, we know that $\Omega = \tilde{D}\, A^{r \times q}\, D$ and $\Delta = A^{r \times q}$, i.e., Ω and Δ are two free A-modules of rank $r \times q$. Therefore, a minimal generating system of Ω is defined by $\{\tilde{D}_i\, D^j\}_{1 \leq i \leq r, 1 \leq j \leq q}$, where $\tilde{D}_i$ (resp., D^j) denotes the ith column (resp., jth row) of $\tilde{D}$ (resp., D). Similarly, a minimal generating system of Δ is defined by $\{E^i_j\}_{1 \leq i \leq r, 1 \leq j \leq q}$, where E^i_j is the matrix defined by 1 in the ith row and jth column and 0 elsewhere.

- The Krull dimension of the ring A defined in Example 11 equals 2. Therefore, by Corollary 10, we have $\mu_A(\Omega) \leq r \times q + 2$ and $\mu_A(\Delta) \leq r \times q + 2$.
- The ring $A = \mathbb{Z}[i\sqrt{5}]$ considered in [1,16,17,19] is a *Dedekind domain*, and thus, the Krull dimension of A equals 1 [3,18,26]. Therefore, by Corollary 10, we have $\mu_A(\Omega) \leq r \times q + 1$ and $\mu_A(\Delta) \leq r \times q + 1$.

The Krull dimensions of the rings $\mathcal{A}$, $\hat{\mathcal{A}}$ and W_+ will be investigated in the future.

5 Conclusion

In this paper, we have obtained a general parametrization of all stabilizing controllers of a MIMO stabilizable plant which did not necessarily admit doubly coprime factorizations. This new parametrization is a linear fractional transformation of the free parameters and the set of free parameters has been characterized. Moreover, if the transfer matrix admitted a doubly coprime factorization, then we have shown that the general parametrization became the standard Youla–Kučera parametrization of all stabilizing controllers. Finally, the study of the minimal number of free parameters appearing in this parametrization was reduced to the knowledge of the minimal generating systems of the two finitely generated projective modules Ω and Δ. Using Heitmann's generalization of Forster-Swan's theorem, we then gave upper bounds on the cardinal of the minimal generating systems of Ω and Δ, i.e., on the minimal number of free parameters appearing in this general parametrization of all stabilizing controllers.

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