A generalization of the Youla-Kučera parametrization for stabilizable systems

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Unstable plants

- **Finite-dimensional system:**

$$\dot{x}(t) = x(t)+u(t),\ x(0) = 0 \Rightarrow \widehat{x}(s) = \frac{1}{s-1}\,\widehat{u}(s).$$

- **Delay system:**

$$\begin{cases} \dot{x}(t) = x(t) + u(t),\ x(0) = 0, \\ \\ y(t) = \begin{cases} 0, & 0 \le t \le 1, \\ x(t-1), & t \ge 1, \end{cases} \end{cases}$$

$$\Rightarrow \widehat{y}(s) = \frac{e^{-s}}{s-1}\,\widehat{u}(s).$$

- **System of partial differential equations:**

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ \\ \frac{\partial z}{\partial x}(0,t) = 0,\ \frac{\partial z}{\partial x}(1,t) = u(t), \\ \\ y(t) = \frac{\partial z}{\partial t}(1,t), \end{cases}$$

$$\Rightarrow \widehat{y}(s) = \frac{1+e^{-2\,s}}{1-e^{-2\,s}}\,\widehat{u}(s).$$

- The **poles of the transfer functions** ($1, 1, i\,k\,\pi, k \in \mathbb{Z}$)

$$h_1(s) = \frac{1}{s-1},\ h_2(s) = \frac{e^{-s}}{s-1},\ h_3(s) = \frac{1+e^{-2\,s}}{1-e^{-2\,s}}$$

belong to $\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \ge 0\} \Rightarrow$ **unstability.**

Stabilization by feedback

- $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$,

  $H_\infty(\mathbb{C}_+) = \{$holomorphic functions f in $\mathbb{C}_+ \mid$
  $\| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}$,

  $H_2(\mathbb{C}_+) = \{$holomorphic functions f in $\mathbb{C}_+ \mid$
  $\| f \|_2 = \sup_{x \in \mathbb{R}_+} (\int_{-\infty}^{+\infty} |f(x+iy)|^2 dy)^{1/2} < +\infty\}$
  $= \mathcal{L}(L_2(\mathbb{R}_+))$, $\mathcal{L}(\cdot)$ Laplace transform.

- The transfer functions h_i **do not belong to** $H_\infty(\mathbb{C}_+)$:

$$h_1(s) = \frac{1}{s-1},\ h_2(s) = \frac{e^{-s}}{s-1},\ h_3(s) = \frac{1+e^{-2\,s}}{1-e^{-2\,s}}$$

$\Rightarrow$ we have the **linear unbounded operator**

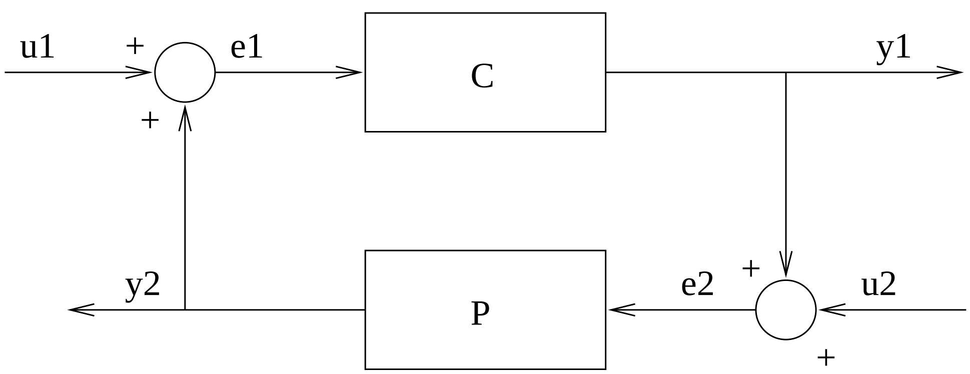
$$\begin{array}{rcl} T_{h_i} : H_2(\mathbb{C}_+) & \longrightarrow & H_2(\mathbb{C}_+), \\ \widehat{u} & \longmapsto & \widehat{y} = h_i\,\widehat{u}, \end{array}$$

$\Rightarrow \mathrm{dom}(T_{h_i}) = \{\widehat{u} \in H_2 \mid \widehat{y} = h_i\,\widehat{u} \in H_2\} \subsetneq H_2$

$\Rightarrow \exists\, \widehat{u} \in H_2(\mathbb{C}_+) :\ \widehat{y} = h_i\,\widehat{u} \notin H_2(\mathbb{C}_+)$.

Robust control

• Is it possible to find a controller C such that the **closed-loop is stable** $\forall\, \widehat{u}_i \in H_2(\mathbb{C}_+)$?



• Is it possible to determine the **set of all stabilizing controllers of** P?

• Is it possible to find **robust/optimal stabilizing controllers** C?

The integral domain RH_∞

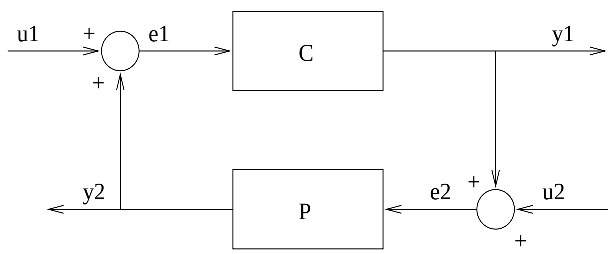
• The ring of **proper stable real rational transfer functions**:

$$RH_\infty = \left\{ \frac{n(s)}{d(s)} \in \mathbb{R}(s) \mid \ \deg n(s) \leq \deg d(s), \ d(s) = 0 \Rightarrow \mathrm{Re}(s) < 0 \right\}$$

• $p \in \mathbb{R}(s)$: **a plant**.

• $c \in \mathbb{R}(s)$: **a controller**.

• The **closed-loop system** is defined by:



u_1, u_2: external inputs, e_1, e_2: internal inputs, y_1, y_2: outputs.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \ \begin{cases} y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{cases}$$

• Definition: c **internally stabilizes** p if we have:

$$\begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \\ \frac{c}{1-p\,c} & \frac{1}{1-p\,c} \end{pmatrix} \in RH_\infty^{2 \times 2}.$$

• **Internal stability** over $RH_\infty \Leftrightarrow$ **exponential stability** (resp. $L_2 - L_2$-stability, $L_\infty - L_\infty$-stability).

Example

- Example: $A = RH_\infty, \quad K = \mathbb{R}(s)$.

$$\begin{cases} p = \frac{s}{s-1}, \\ \\ c = -\frac{(s-1)}{(s+1)}, \end{cases} \Rightarrow \begin{cases} e_1 = \frac{(s+1)}{(2s+1)}\, u_1 + \frac{s(s+1)}{(2s+1)(s-1)}\, u_2, \\ \\ e_2 = \frac{(-s+1)}{(2s+1)}\, u_1 + \frac{(s+1)}{(2s+1)}\, u_2. \end{cases}$$

$\Rightarrow c$ **does not internally stabilize** p because:

$$\frac{s(s+1)}{(2s+1)(s-1)} \notin RH_\infty \ \text{(pole in } 1 \in \mathbb{C}_+).$$

$$u_2 \notin \left(\frac{s-1}{s+1}\right) H_2 \triangleq \left\{\frac{(s-1)}{(s+1)}\, z \mid z \in H_2\right\} \Rightarrow e_1 \notin H_2.$$

(e.g. $u_2 = \frac{1}{s+1}$ i.e. $\mathcal{L}^{-1}(u_2) = e^{-t}\, Y(t)$).

The pole/zero cancellation between p and c leads to an unstability.

- Example: $A = RH_\infty, \quad K = \mathbb{R}(s)$.

$$\begin{cases} p = \frac{s}{s-1}, \\ \\ c = 2, \end{cases} \Rightarrow \begin{cases} e_1 = -\frac{(s-1)}{(s+1)}\, u_1 - \frac{s}{(s+1)}\, u_2, \\ \\ e_2 = -2\,\frac{(s-1)}{(s+1)}\, u_1 - \frac{(s-1)}{(s+1)}\, u_2. \end{cases}$$

$\Rightarrow c$ **internally stabilizes** the plant p.

Well-known results

• Theorem: (Morse, Vidyasagar) Every transfer function $p \in \mathbb{R}(s)$ admits a **coprime factorization** over RH_∞, i.e. $\exists\, 0 \neq d,\, n,\, x,\, y \in RH_\infty$ such that:

$$\boxed{p = n/d, \quad d\,x - n\,y = 1.}$$

• Theorem: (Youla, Kučera, Desoer and al) **All stabilizing controllers of $p \in \mathbb{R}(s)$ have the form:**

$$\boxed{c(q) = \frac{y + q\,d}{x + q\,n}, \quad \forall\, q \in RH_\infty : \, x + q\,n \neq 0.}$$

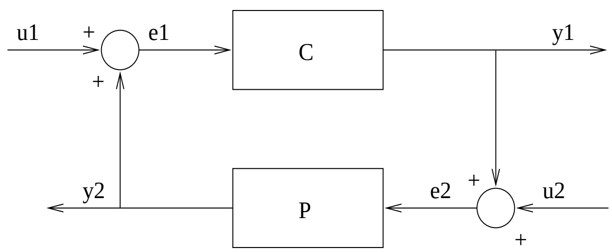
• Interest: Find all the controllers $c \in \mathbb{R}(s)$ such that

$$\inf_{c \in \mathrm{Stab}(p)} \| \, w\,(1 - p\,c)^{-1} \, \|_\infty, \quad w \in RH_\infty,$$

$$\mathrm{Stab}(p) = \{c \in \mathbb{R}(s) \,|\, \tfrac{1}{1-p\,c},\ \tfrac{c}{1-p\,c},\ \tfrac{p}{1-p\,c} \in RH_\infty\}.$$

This **non-linear problem** becomes the **convex** one:

$$\inf_{q \in RH_\infty} \| \, w\,d\,(x + q\,n) \, \|_\infty \, .$$



Extension to other classes of systems

“ The foregoing results about **rational functions** are so elegant that one can hardly resist the temptation **to try to generalize them to non-rational functions**.

But to what class of functions?

Much attention has been devoted in the engineering literature to the **identification of a class that is wide enough to encompass all the functions of physical interest and yet enjoys the structural properties that allow analysis of the robust stabilisation problem**”,

N. Young.

(“Some function-theoretic issues in feedback stabilization”, in *Holomorphy Spaces*, MSRI Publications 33, 1998, 337-349.)

The fractional representation approach

• (Zames) **The set of transfer functions of SISO systems has the structure of an algebra** (parallel +, serie $\circ$, proportional feedback . by scalar in $\mathbb{R}$).

• (Vidyasagar) **Let A be an algebra of transfer functions of SISO stable systems** with a structure of an intregral domain ($a\,b = 0, a \neq 0 \Rightarrow b = 0$) and its **field of fractions**:

$$K = Q(A) = \{n/d \mid 0 \neq d,\, n \in A\}\,.$$

K represents the class of systems

$\Rightarrow$ Any unstable plant is defined by $P \in (K \backslash A)^{q \times r}$.

• (Zames) A **needs to be a normed algebra** $\| \cdot \|_A$ in order to take into account the errors in the modelization & approximation of the real plant by a model.

• (Zames) It is suitable that A is a **complete** k-vector space, i.e. k**-Banach space**

$\Rightarrow$ A is a **Banach algebra**

$$(\| \, a\, b \, \|_A \leq \| \, a \, \|_A \, \| \, b \, \|_A, \quad \| \, 1 \, \|_A = 1).$$

Examples of stable algebras A of SISO systems

1. $RH_\infty = \left\{ \frac{n(s)}{d(s)} \in \mathbb{R}(s) \mid \deg n(s) \leq \deg d(s), \; d(s) = 0 \Rightarrow \mathrm{Re}(s) < 0 \right\}$

- $h_1(s) = \frac{1}{s-1} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{1}{s+1}, \; \frac{s-1}{s+1} \in RH_\infty$

$$\Rightarrow h_1 \in Q(RH_\infty) = \mathbb{R}(s).$$

2. $\mathcal{A} = \{ f(t) + \sum_{i=0}^{+\infty} a_i \, \delta_{t-t_i} \mid f \in L_1(\mathbb{R}_+), \; (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+), \; 0 = t_0 \leq t_1 \leq t_2 ... \}$,

and $\widehat{\mathcal{A}} = \{ \widehat{g} \mid g \in \mathcal{A} \}$, the **Wiener algebras**.

- $h_2(s) = \frac{e^{-s}}{s-1} = \frac{\left(\frac{e^{-s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{e^{-s}}{s+1}, \; \frac{s-1}{s+1} \in \widehat{\mathcal{A}}.$

$$\Rightarrow h_2 \in Q(\widehat{\mathcal{A}}).$$

3. $\mathbb{C}_+ = \{ s \in \mathbb{C} \mid \mathrm{Re}\, s > 0 \}$. The **Hardy algebra**

$H_\infty(\mathbb{C}_+) = \{$ holomorphic functions f in $\mathbb{C}_+ \mid \; \| f \|_\infty = \sup_{s \in \mathbb{C}_+} | f(s) | < +\infty \}$.

- $h_3(s) = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})}, 1+e^{-2\,s}, 1-e^{-2\,s} \in H_\infty(\mathbb{C}_+)$

$$\Rightarrow h_3 \in Q(H_\infty(\mathbb{C}_+)).$$

Fractional representation approach

- Let A be an integral domain and K its quotient field $Q(A) = \{p = n/d \mid 0 \neq d,\, n \in A\}$.

- Definition: $p \in K$ admits a **coprime factorization over** A if $\exists\, 0 \neq d,\, n,\, x,\, y \in A$ such that:

$$p = n/d, \quad d\,x - n\,y = 1.$$

- Definition: $p \in K$ is A**-internally stabilizable** if $\exists\, c \in K = Q(A)$ such that:

$$\begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \\ \frac{c}{1-p\,c} & \frac{1}{1-p\,c} \end{pmatrix} \in A^{2\times 2}.$$

- existence of coprime factorizations over A $\Rightarrow$ A-internal stabilizability.

$\Rightarrow \forall\, p \in \mathbb{R}(s)$ **is** RH_∞**-internally stabilizable**.

- Theorem: (Inouye, Smith) If $A = H_\infty$, then:

H_∞-internal stabilizability $\Leftrightarrow$ existence of coprime factorizations

$\Rightarrow$ **existence of a Youla-Kučera parametrization for every internally stabilizable** $p \in Q(H_\infty)$**.**

Open questions

• **Does A-internal stabilizability imply the existence of coprime factorizations over**:

$$A = \hat{\mathcal{A}} = \{\mathcal{L}(f)(s) + \textstyle\sum_{i=0}^{+\infty} a_i \, e^{-t_i \, s} \mid f \in L_1(\mathbb{R}_+) \\ (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+), \; 0 = t_0 \leq t_1 \leq t_2 \leq \ldots\},$$

(ring of BIBO-stable time-invariant systems)

$$A = W_+ = \left\{\textstyle\sum_{i=0}^{\infty} a_i \, z^i \mid \textstyle\sum_{i=0}^{+\infty} |a_i| < +\infty\right\},$$

(ring of BIBO-stable causal digital filters)

$$A = M_{\mathbb{D}^n} = \{r/s \mid 0 \neq s, \, r \in \mathbb{R}[x_1, \ldots, x_n], \\ s(\underline{x}) = 0 \Rightarrow \underline{x} \notin \mathbb{D}^n\}$$

(ring of nD systems with structural stability) . . . ?

• If it is not the case:

Is it possible to parametrize all stabilizing controllers of a stabilizable plant which does not admit coprime factorizations?

• In this talk, we shall solve the last question.

Theory of fractional ideals

- Let A and $K = Q(A) = \{n/d \,|\, 0 \neq d,\, n \in A\}$.

- Definition: A **fractional ideal** J **of** A is an A-submodule of K such that $\exists\, 0 \neq d \in A$ satisfying:

$$\boxed{(d)\, J \triangleq \{a\, d \,|\, a \in J\} \subseteq A.}$$

J of A is **integral** if $J \subseteq A$ and **principal** if $\exists\, k \in K$:

$$\boxed{J = (k) \triangleq A\, k = \{a\, k \,|\, a \in A\}.}$$

- Proposition: Let $\mathcal{F}(A)$ be the set of non-zero fractional ideals of A and $I,\ J \in \mathcal{F}(A)$. Then:

$$\begin{cases} I\, J = \{\sum_{\text{finite}} a_i\, b_i \mid a_i \in I,\ b_i \in J\} \in \mathcal{F}(A), \\ I : J = \{k \in K = Q(A) \mid (k)\, J \subseteq I\} \in \mathcal{F}(A). \end{cases}$$

- Example: Let $p \in K$. Then, we have:

$$\boxed{J = A + A\, p = \{\lambda + \mu\, p \,|\, \lambda,\, \mu \in A\} \in \mathcal{F}(A)}$$

$$(p = n/d,\ d,\ n \in A \Rightarrow (d)\, J = A\, n + A\, d \subseteq A).$$

- Definition: $J \in \mathcal{F}(A)$ is **invertible** if $\exists\, I \in \mathcal{F}(A)$:

$$\boxed{I\, J = A.}$$

$$\Rightarrow I = J^{-1} = A : J = \{k \in K \,|\, (k)\, J \subseteq A\}.$$

Stabilization problems

- Theorem: Let $p \in K = Q(A)$ and:

$$J \triangleq (1,\, p) = A + A\,p \in \mathcal{F}(A).$$

1. p has a **weakly coprime factorization** $p = n/d$

$(0 \neq d,\, n \in A, \quad \forall\, k \in K : \; k\,n,\; k\,d \in A \Rightarrow k \in A)$

$$\Leftrightarrow \boxed{A : J \triangleq \{a \in A \,|\, a\,p \in A\} = A\,d.}$$

2. p is **internally stabilizable** $\Leftrightarrow$ $\boxed{J \text{ is invertible}}$, i.e.

$$\boxed{\exists\, a,\, b \in A : \; \begin{cases} a - b\,p = 1, \\ a\,p \in A. \end{cases}}$$

Then, $c = b/a$ internally stabilizes p, $\;J^{-1} = (a,\, b)$.

3. $c \in K$ **internally stabilizes** p

$$\Leftrightarrow \boxed{(1,\, p)\,(1,\, c) = (1 - p\,c).}$$

4. p admits a **coprime factorization** $p = n/d$

$(0 \neq d,\, n \in A, \quad \exists\, x,\, y \in A : \; d\,x - n\,y = 1)$

$$\Leftrightarrow \boxed{J = (1/d)}.$$

5. p is **strongly stabilizable**

$$\Leftrightarrow \boxed{\exists\, c \in A : \; J = (1 - p\,c).}$$

Example

- Let A be the ring of **BIBO-stable causal filters**:

$$A = W_+ = \{f(z) = \sum_{i=0}^{+\infty} a_i\, z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty\}.$$

- Let us consider the **transfer function** $p = e^{-\left(\frac{1+z}{1-z}\right)}$:

$$\begin{cases} n = (1-z)^3\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \\ d = (1-z)^3 \in A, \end{cases} \Rightarrow p = n/d \in Q(A).$$

- Let us consider the **fractional ideal** $J = (1,\, p)$ of A.

$$A : J = \{d \in A \mid d\, p \in A\}.$$

- The ideal $A : J$ **is not finitely generated**

(R. Mortini & M. Von Renteln, **Ideals in Wiener algebra**, J. Austral. Math. Soc., 46 (1989), 220-228),

i.e. $\nexists$ **finite family** $\{d_1, \ldots, d_r\}$, $d_i \in A$, such that:

$$\forall\, d \in A : J, \quad \exists\, a_i \in A : \quad d = \sum_{i=1}^{r} a_i\, d_i.$$

$\Rightarrow p$ **has not weakly coprime factorizations**

p **does not admit coprime factorizations**
$\&$ p **is not internally stabilizable**
(similar results hold over $A = A(\mathbb{D})$)

Interpolation problem

• <u>Proposition</u>: $p \in Q(A)$ is **internally stabilizable** iff there exists $b \in A$ such that

$$1 + b\,p \in A : J = \{d \in A \,|\, d\,p \in A\},$$

$$\text{i.e.} \quad \begin{cases} 1 + b\,p \in A, \\ (1 + b\,p)\,p \in A. \end{cases}$$

Then, $c = b/(1 + b\,p)$ is a **stabilizing controller**, $b = c/(1 - p\,c)$ and:

$$H(p, c) = \begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 + b\,p & p + b\,p^2 \\ b & 1 + b\,p \end{pmatrix}.$$

• <u>Corollary</u>: If $p \in Q(A)$ admits a **weakly coprime factorization** $p = n/d$, then p is **internally stabilizable** iff there exists $b \in A$ such that $1 + b\,p \in (d)$.

Therefore, at any pole of p of order m in $\mathrm{Re}\, s \geq 0$, $1 + b\,p$ and b have at least m zeros.

• <u>Remark</u>: The previous results can be tracked back to G. Zames-B.A. Francis (83) for $A = RH_\infty$.

Example

- Let us consider $A = H_\infty(\mathbb{C}_+)$ and:

$$p = \frac{e^{-s}}{(s-1)} = \frac{\frac{e^{-s}}{(s+1)}}{\frac{(s-1)}{(s+1)}} \in K = Q(A).$$

- Let us define the **fractional ideal** $J = (1,\, p)$ of A

$$\Rightarrow A:\, J = \{d \in A \mid d\, p \in A\} = \left(\frac{s-1}{s+1}\right),$$

because A is a **GCDD** and $\gcd\left(\frac{e^{-s}}{s+1},\, \frac{s-1}{s+1}\right) = 1.$

- p is **internally stabilizable** iff $\exists\, a,\, b \in A : J$ s.t.:

$$a - b\, p = 1 \Leftrightarrow \exists\, x,\ y \in A : \quad \begin{cases} a = \frac{(s-1)}{(s+1)}\, x, \\ b = \frac{(s-1)}{(s+1)}\, y, \\ a - b\, p = 1. \end{cases}$$

$$b = \frac{(a-1)}{p} = \left(\frac{s-1}{s+1}\right) \left(\frac{(s-1)\, x - (s+1)}{e^{-s}}\right)$$

$$\Leftrightarrow y = \frac{(s-1)\, x - (s+1)}{e^{-s}}$$

$$\Leftrightarrow x = \frac{(s+1) + e^{-s}\, y}{s-1}$$

$$\Rightarrow ((s+1) + e^{-s}\, y(s))(1) = 0 \Rightarrow y(1) = -2\, e.$$

- Taking $y(s) = y(1) = -2\, e \in A$, then:

$$x = \frac{(s+1) - 2\, e^{-(s-1)}}{s-1} = 1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right) \in A.$$

• Therefore, we have:

$$\begin{cases} a = \left(\frac{s-1}{s+1}\right)\left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) \in A : J, \\ b = -2\,e\left(\frac{s-1}{s+1}\right) \in A :\ J, \\ a - b\,p = 1, \qquad\qquad (\star) \end{cases}$$

$\Rightarrow$ a **stabilizing controller** c of p is defined by:

$$c \;= \frac{b}{a} = -\frac{2\,e\,(s-1)}{(s-1)+2\,(1-e^{-(s-1)})} = -\frac{2\,e\,(s-1)}{s+1-2\,e^{-(s-1)}}.$$

• $J = (1,\, p)$ is **invertible**, $J^{-1} = A : J = \left(\frac{s-1}{s+1}\right)$

$\Rightarrow J = (J^{-1})^{-1} = \left(\frac{s+1}{s-1}\right)$ is **principal**

$\Rightarrow p = \frac{\frac{e^{-s}}{(s+1)}}{\frac{(s-1)}{(s+1)}}$ is a **coprime factorization**:

$$(\star) \quad \Leftrightarrow \left(\frac{s-1}{s+1}\right)\left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(2\,e\left(\frac{s-1}{s+1}\right)\right) p = 1,$$

$$\Leftrightarrow \left(\frac{s-1}{s+1}\right)\left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) - \left(\frac{e^{-s}}{s+1}\right)(-2\,e) = 1,$$

$$\Rightarrow \begin{cases} x = 1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right) \in A, \\ y = -2\,e \in A, \\ d\,x - n\,y = 1. \end{cases}$$

Example

- Let $A = H_\infty(\mathbb{C}_+)$ and $K = Q(A)$.

- Let us consider the plant $p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})} \in K$.

- We have $J = (1,\, p) = \left(\frac{1}{1-e^{-2\,s}}\right)$ because:

$$\begin{cases} 1 = (1 - e^{-2\,s})\, \frac{1}{(1-e^{-2\,s})}, \\ p = \frac{1+e^{-2\,s}}{1-e^{-2\,s}} = (1 + e^{-2\,s})\, \frac{1}{(1-e^{-2\,s})}, \\ \frac{1}{(1-e^{-2\,s})} = \frac{1}{2} + \frac{1}{2}\, \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})}. \end{cases}$$

$\Rightarrow p$ admits the **coprime factorization**:

$$\begin{cases} p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})}, \\ \frac{1}{2}\,(1 - e^{-2\,s}) + \frac{1}{2}\,(1 + e^{-2\,s}) = 1. \end{cases}$$

$\Rightarrow c = -1$ is a **stable stabilizing controller** of p.

- We check that $\;1 - p\,c = 1 + p = \frac{2}{(1-e^{-2\,s})}$

$$\Rightarrow J = (1,\, p) = (1/(1 - e^{-2\,s})) = (1 - p\,c).$$

Some properties

• "IS" stands for **"internally stabilized/-zable"**.

• "CF" stands for **"coprime factorization"**.

• Proposition: Let $\delta \in A,\ p,\, c \in Q(A)$.

▲ If p is IS by c, then p admits a CF $\Leftrightarrow$ c admits a CF.

▲ p is IS and p admits a weakly CF $\Leftrightarrow$ p admits a CF.

▲ p is IS by $c \Leftrightarrow p + \delta$ is IS by $c/(1 + \delta\, c)$.

▲ p is IS by $c \Leftrightarrow p/(1 + \delta\, p)$ is IS by $c + \delta$.

▲ p is IS by $c \Leftrightarrow 1/p$ is IS by $1/c$.

▲ p is IS and p admits a weakly CF $\Leftrightarrow$ p admits a CF.

▲ p is **externally stabilized** by c (i.e. $pc/(1-pc) \in A$) $\Leftrightarrow (1,\, pc) = (1 - pc)$.

▲ $p = \frac{n}{d},\ c = \frac{s}{r}$ CF. p is IS by $c \Leftrightarrow d\,r - n\,s \in \mathrm{U}(A)$.

▲ p is IB by $c \Leftrightarrow \mathrm{div}(1,\, p) + \mathrm{div}(1,\, c) = \mathrm{div}(1 - p\,c)$.

$$(\mathcal{D}(A) \triangleq \mathcal{F}(A)/\sim, \quad "I \sim J \Leftrightarrow A : I = A : J", \quad \mathrm{div} I = I + \sim).$$

▲ 0 is IS by $c \Leftrightarrow c \in A \ldots$

Robust stabilization

- $c \in K = Q(A)$ **internally stabilizes** $p \in K$ iff:

$$\boxed{(1,\, p)\,(1,\, c) = (1 - p\,c).}$$

- Let $\delta \in A$. c internally stabilizes p and $p + \delta$

$$\text{iff} \quad \begin{cases} (1,\, p)\,(1,\, c) = (1 - p\,c), \\ (1,\, p + \delta)\,(1,\, c) = (1 - (p + \delta)\,c). \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,\, p)\,(1,\, c) = (1 - p\,c), \\ (1,\, p)\,(1,\, c) = (1 - (p + \delta)\,c). \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,\, p)\,(1,\, c) = (1 - p\,c), \\ \left(\frac{1-(p+\delta)\,c}{1-p\,c}\right) = \left(1 - \frac{\delta\,c}{1-p\,c}\right) = A, \end{cases}$$

$\Leftrightarrow$ c stabilizes p and $(1 - \delta\,c)/(1 - p\,c) \in \mathrm{U}(A)$.

- If A is a **Banach algebra**, then (small gain thm):

$$\boxed{\|\, 1 - a \,\|_A < 1 \Rightarrow a \in \mathrm{U}(A).}$$

$\Rightarrow$ **a sufficient condition of robust stabilization**:

$$\boxed{\|\, \delta \,\|_A < (\|\, c/(1 - p\,c) \,\|_A)^{-1}}$$

Parametrizations

- Theorem: Let $p \in Q(A)$ be a **stabilizable plant** and $J = (1, p)$. Then, **all stabilizing controllers** of p have the form

$$c(q_1, q_2) = \frac{b + a^2 q_1 + b^2 q_2}{a + a^2 p q_1 + b^2 p q_2}, \quad (\star)$$

$$\forall\, q_1, q_2 \in A: \; a + a^2 p q_1 + b^2 p q_2 \neq 0,$$

where $c_* = b/a$ is a stabilizing controller of p, i.e.

$$a - b\,p = 1, \quad a\,p \in A,$$

and $\quad a = 1/(1 - p\,c_*), \quad b = c_*/(1 - p\,c_*)$.

1. $(\star)$ has **only one free parameter**

   $\Leftrightarrow p^2$ **admits a coprime factorization** $p^2 = s/r$

$$(\star) \Leftrightarrow c(q) = \frac{b + r\,q}{a + r\,p\,q}, \quad q \in A: \; a + r\,p\,q \neq 0.$$

2. If p admits a **coprime factorization** $p = n/d$,

$$0 \neq d,\; n \in A, \quad d\,x - n\,y = 1, \quad \text{then:}$$

$$(\star) \Leftrightarrow c(q) = \frac{y + d\,q}{x + n\,q}, \quad \forall\, q \in A: \; x + n\,q \neq 0.$$

$$(a = d\,x, \quad b = d\,y, \quad r = d^2)$$

Youla-Kučera parametrization

Example

- Let us consider $A = \mathbb{Z}[i\sqrt{5}]$, $K = \mathbb{Q}(i\sqrt{5})$ and:

$$p = (1 + i\sqrt{5})/2 \in K$$

"On stabilization and existence of coprime factorizations", V. Anantharam, IEEE TAC 30 (1985), 1030-1031.

- Let us define the **fractional ideal** $J = (1,\, p)$.

- Using the relation in A

$$2 \times 3 = (1 + i\,\sqrt{5})\,(1 - i\,\sqrt{5}) = 6,$$

$$\Rightarrow p = (1 + i\sqrt{5})/2 = 3/(1 - i\,\sqrt{5}).$$

$\Rightarrow A : J = (2,\, 1 - i\,\sqrt{5})$ is **not a principal ideal**.

$\Rightarrow p$ **does not admit a (weakly) coprime factorization.**
$\Rightarrow \nexists$ **Youla-Kučera parametrization.**

- $J\,(A : J) = (2,\, 1 + i\,\sqrt{5},\, 1 - i\,\sqrt{5},\, 3) = A$:

$$-2 + 3 = -2 - (-1 + i\,\sqrt{5})\,p = 1$$

$\Rightarrow c = (1 - i\,\sqrt{5})/2$ **internally stabilizes** p.

- $J^{-2} = (A : J)^2 = (2,\, 1 - i\,\sqrt{5})^2 = (2)$

$$\Rightarrow c(q) = \frac{1 - i\,\sqrt{5} - 2\,q}{2 - (1 + i\,\sqrt{5})\,q}, \quad \forall\, q \in A.$$

Example

- Let us consider the ring $A = \mathbb{R}[x^2, x^3]$ of **discrete time delay systems without the unit delay**.

- A has been used for high-speed circuits, computer memory devices... (K. Mori).

- Let us consider $p = (1 - x^3)/(1 - x^2) \in Q(A)$.

- Let us consider the **fractional ideal** $J = (1, p)$.

- Using the relation in A

$$(1 - x^3)(1 + x^3) = (1 - x^2)(1 + x^2 + x^4),$$

we have:

$$p = \frac{(1 - x^3)}{(1 - x^2)} = \frac{(1 + x^2 + x^4)}{(1 + x^3)}.$$

$\Rightarrow A : J = (1 - x^2, 1 + x^3)$ **is not principal** because $x + 1 \notin A$.

$\Rightarrow p$ **does not admit a weakly coprime factorization.**

$\Rightarrow p$ **does not admit a coprime factorization**

$\Rightarrow$ **we cannot parametrize all stabilizing controllers of** p **by means of the Youla-Kučera parametrization.**

Example

- $J\,(A:J) = (1-x^2,\; 1+x^3,\; 1-x^3,\; 1+x^2+x^4)$

$\Rightarrow (1+x^3)/2 + (1-x^3)/2 = 1 \in J\,(A:J)$

$\Rightarrow p$ is **internally stabilizable** and $J^{-1} = A : J$.

- $(1+x^3)/2 + (1-x^3)/2 = 1 \in J\,(A:J)$

$$\Leftrightarrow$$

$$(1+x^3)/2 + ((1-x^2)/2)\,p = 1$$

$$\Rightarrow \begin{cases} a = (1+x^3)/2 \in J^{-1}, \\ b = -(1-x^2)/2 \in J^{-1}, \end{cases}$$

$$\Rightarrow c = b/a = -(1-x^2)/(1+x^3)$$
internally stabilizes p.

- $J^{-2} = ((1-x^2)^2,\; (1+x^3)^2)$ **is not principal ideal of** A ($x+1 \notin A$).

- **All stabilizing controllers** of p have the form

$$c(q_1,q_2) = \frac{-(1-x^2)+(1-x^2)^2\,q_1+(1+x^3)^2\,q_2}{(1+x^3)+(1-x^2)\,(1-x^3)\,q_1+(1+x^3)\,(1+x^2+x^4)\,q_2}$$

for all $q_1,\, q_2 \in A$ such that the denominator exists.

Smith predictor

- $p = p_0\, e^{-\tau s}$, where $p_0 \in RH_\infty$ and $\tau \geq 0$.

- $p \in H_\infty(\mathbb{C}_+) \Rightarrow p = n/d$, $n = p_0\, e^{-\tau s}$, $d = 1$, is a **coprime factorization**.

- The **Youla-Kučera parametrization** of all stabilizing controllers of p is given by:

$$c(q) = \frac{q}{1 + q\, p_0\, e^{-\tau s}}, \quad \forall\, q \in H_\infty(\mathbb{C}_+).$$

- Let $c_0 \in RH_\infty$ be a stabilizing controller of $p_0 \in RH_\infty$

$$\Rightarrow \tilde{q} \triangleq \frac{c_0}{(1 - p_0\, c_0)} \in RH_\infty \subseteq H_\infty(\mathbb{C}_+).$$

- The stabilizing controller of p

$$c(\tilde{q}) = \frac{c_0}{1 + p_0\, c_0\, (e^{-\tau s} - 1)} = \frac{c_0}{1 - c_0\, (p_0 - p)}$$

is called the **Smith predictor**.

- The **complementary sensitivity transfer function** has the form

$$t(\tilde{q}) = \frac{p\, c(\tilde{q})}{1 - p\, c(\tilde{q})} = \left(\frac{p_0\, c_0}{1 - p_0\, c_0}\right) e^{-\tau s},$$

showing that the Smith predictor **rejects the delay** $e^{-\tau s}$ **outside the closed-loop** formed by p_0 and c_0.

Picard group

• <u>Definition</u>: Let $\mathcal{P}(A)$ be the **group of non-zero principal fractional ideals of** A:

$$\boxed{\mathcal{P}(A) = \{(k) \triangleq A\,k \mid 0 \neq k \in K\}.}$$

Let $\mathcal{I}(A)$ be the **group of non-zero invertible fractional ideals of** A:

$$\boxed{\mathcal{I}(A) = \{J \in \mathcal{F}(A) \mid \exists\, I \in \mathcal{F}(A) : \; I\,J = A\}.}$$

The **Picard group** of A is the defined by:

$$\boxed{\mathcal{C}(A) = \mathcal{I}(A)/\mathcal{P}(A)}$$

• <u>Proposition</u>: If $\mathcal{C}(A) \cong \mathbb{Z}/2\,\mathbb{Z}$, then every stabilizable plant $p \in Q(A)$ has a **parametrization of all its stabilizing controllers with only one free parameter.**

If $\mathcal{C}(A) \cong 1$, then every stabilizable plant $p \in Q(A)$ has a **Youla-Kučera parametrization** (e.g. $H_\infty(\mathbb{C}_+)$, RH_∞, Bézout domains).

Convexity of $H(p,c)$

- Let $p \in Q(A)$ be an internally stabilizable plant and c_* a **particular stabilizing controller** of p.

- **All stabilizing controllers** of p are given by

$$c(q_1,q_2) = \frac{(1-p\,c_*)\,c_*+q_1+q_2\,{c_*}^2}{(1-p\,c_*)+q_1\,p+q_2\,p\,{c_*}^2}$$

$$\forall\, q_1,\, q_2 : \quad (1-p\,c_*)+q_1\,p+q_2\,p\,{c_*}^2 \neq 0.$$

- The **closed-loop system**

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \\ \frac{c}{1-p\,c} & \frac{1}{1-p\,c} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

becomes:

$$H(p,\, c(q_1,\, q_2)) = \begin{pmatrix} \frac{1}{1-p\,c_*}+q_1\,\frac{p}{(1-p\,c_*)^2}+q_2\,\frac{p\,{c_*}^2}{(1-p\,c_*)^2} & \frac{c_*}{1-p\,c_*}+q_1\,\frac{1}{(1-p\,c_*)^2}+q_2\,\frac{{c_*}^2}{(1-p\,c_*)^2} \\ \frac{p}{1-p\,c_*}+q_1\,\frac{p^2}{(1-p\,c_*)^2}+q_2\,\frac{(p\,c_*)^2}{(1-p\,c_*)^2} & \frac{1}{1-p\,c_*}+q_1\,\frac{p}{(1-p\,c_*)^2}+q_2\,\frac{p\,{c_*}^2}{(1-p\,c_*)^2} \end{pmatrix}$$

- $H(p,c(q_1,q_2))$ is **convex in** $q_1,\, q_2 \in A$: $\forall\, \lambda \in A$,

$$\begin{aligned} &H(p,\, c(\lambda\, q_1+(1-\lambda)\,q_1',\, \lambda\, q_2+(1-\lambda)\,q_2')) \\ &= \\ &\lambda\, H(p,\, c(q_1,\, q_2))+(1-\lambda)\,H(p,\, c(q_1',\, q_2')). \end{aligned}$$

Sensitivity minimization

- Let A be a **Banach algebra** (H_∞, $\widehat{\mathcal{A}}$, W_+,...)

- Let $p \in K = Q(A)$ be a **stabilizable plant**, then

$$\inf_{c \in \mathrm{Stab}(p)} \| \frac{w}{1 - p\,c} \|_A$$
$$=$$
$$\inf_{q_1,\, q_2 \in A} \| \, w\,(a + a^2\, p\, q_1 + b^2\, p\, q_2) \, \|_A \quad (\star)$$

(convex problem)

where $a,\, b \in A$ satisfy $\quad a - b\, p = 1, \quad a\, p \in A,$
and $c_\star = b/a$ is **a stabilizing controller of p.**

- 1. If $p = n/d$ is a **coprime factorization** of p

$$d\,x - n\,y = 1, \quad x,\, y \in A,$$

$$\Rightarrow a = d\,x, \quad b = d\,y,$$

$$\Rightarrow a + a^2\, p\, q_1 + b^2\, p\, q_2 = d\,(x + q\,n),$$
$$q = x^2\, q_1 + y^2\, q_2.$$

2. $\forall \in A,\ \exists\, q_1,\, q_2 \in A : q = x^2\, q_1 + y^2\, q_2,$

with $q_1 = d^2\,(1 - 2\,n\,y)\,q,\ q_2 = n^2\,(1 + 2\,d\,x)\,q,$

$$\left[(d^2\,(1 - 2\,n\,y))\,x^2 + (n^2\,(1 + 2\,d\,x))\,y^2 = 1 \right].$$

$$(\star) \Leftrightarrow \inf_{q \in A} \| \, w\,d\,(x + n\,q) \, \|_A\,.$$

Conclusion

I. Summary:

• We **generalized the Youla-Kučera parametrization** for **SISO stabilizable plants**.

• This parametrization **does not assume the existence of coprime factorizations.**

II. General comments:

When does a stabilizable plant admit a coprime factorization?

• We proved that this problem is equivalent to:

When is an invertible fractional ideal principal?

• This is a **difficult problem** studied in:

– **algebra**: algebraic K-theory (Serre's conjecture (55) $A = k[x_1, \ldots, x_n]$, solved by Quillen-Suslin (76)),

– **number theory**: number fields,

– **algebraic geometry**: function fields,

– **topology**: triviality of vector bundles,

– **operator theory**: topological K-theory ($C^\star$-algebra).

this problem could be difficult for $\widehat{\mathcal{A}}$, W_+...

Well-known results

- Theorem: (Morse, Vidyasagar) Every transfer matrix $P \in \mathbb{R}(s)^{q \times r}$ admits a **doubly coprime factorization** over RH_∞, i.e.:

$$P = D^{-1} N = \tilde{N}\, \tilde{D}^{-1},$$

$$\begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I,$$

where D, N, $\tilde{N}$, $\tilde{N}$, X, Y, $\tilde{X}$, $\tilde{Y} \in M(RH_\infty)$.

- Theorem: (Youla, Kučera, Desoer) **All stabilizing controllers of $P \in \mathbb{R}(s)^{q \times r}$ have the form:**

$$\boxed{C(Q) = (\tilde{X} - Q\, N)^{-1}\, (\tilde{Y} - Q\, D) = (Y + \tilde{D}\, Q)\, (X + \tilde{N}\, Q)^{-1}}$$

for every $Q \in RH_\infty^{r \times q}$ such that:

$$\det(\tilde{Y} - Q\, N) \neq 0, \quad \det(X - \tilde{N}\, Q) \neq 0.$$

- Interest: Find the controllers $C \in \mathbb{R}(s)^{r \times q}$ s.t.:

$$\inf_{C \in \mathrm{Stab}(P)} \| W_1\, (I - P\, C)^{-1}\, W_2 \|_\infty,$$

$$\mathrm{Stab}(P) = \{ C \in \mathbb{R}(s)^{r \times q} \,|\, (I - P\, C)^{-1},\, (I - P\, C)^{-1}\, P, \\ C\, (I - P\, C)^{-1},\, C\, (I - P\, C)^{-1}\, P \in M(RH_\infty) \}$$

This **non-linear problem** becomes the **convex** one:

$$\inf_{Q \in RH_\infty^{r \times q}} \| W_1\, (X + \tilde{N}\, Q)\, D\, W_2 \|_\infty \,.$$

Fractional representation approach

- Let A be an **integral domain** and K its quotient field $Q(A) = \{n/d \mid 0 \neq d,\, n \in A\}$.

- <u>Definition</u>: $P \in M(K)$ has a **doubly coprime factorization** over A if there exist

$$\exists\, D,\, N,\, \tilde{D},\, \tilde{N},\, X,\, Y,\, \tilde{X},\, \tilde{Y} \in M(A) \text{ such that:}$$

$$P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1},$$

$$\begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I.$$

- <u>Definition</u>: $P \in K^{q \times r}$ is A**-internally stabilizable** if $\exists\, C \in K^{r \times q}$ such that:

$$\begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \in M(A).$$

- existence of a doubly coprime factorization over $A \Rightarrow A$-internal stabilizability.

$\Rightarrow P \in M(\mathbb{R}(s))$ **is** RH_∞**-internally stabilizable**.

- <u>Theorem</u>: (Smith) If $A = H_\infty(\mathbb{C}_+)$, then:

$H_\infty(\mathbb{C}_+)$-internal stabilizability

$\Leftrightarrow$

existence of doubly coprime factorizations

$\Rightarrow \exists$ **Youla-Kučera parametrization.**

Open questions

- **Does A-internal stabilizability imply the existence of doubly coprime factorizations over**:

$$A = \hat{A} = \{\mathcal{L}(f)(s) + \sum_{i=0}^{+\infty} a_i\, e^{-t_i\, s} \,|\, f \in L_1(\mathbb{R}_+) \\ (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\ 0 = t_0 \leq t_1 \leq t_2 \leq \ldots\},$$

(ring of BIBO-stable time-invariant systems)

$$A = W_+ = \left\{\sum_{i=0}^{\infty} a_i\, z^i \,|\, \sum_{i=0}^{+\infty} |a_i| < +\infty\right\},$$

(ring of BIBO-stable causal digital filters)

$$A = M_{\mathbb{D}^n} = \{r/s \,|\, 0 \neq s,\, r \in \mathbb{R}[x_1, \ldots, x_n], \\ s(\underline{x}) = 0 \Rightarrow \underline{x} \notin \mathbb{D}^n\}$$

(ring of nD systems with structural stability) . . . ?

- If it is not the case:

Is it possible to parametrize all stabilizing controllers of a stabilizable plant which does not admit doubly coprime factorizations?

- In this talk, we shall solve the last question.

Lattices

- Let V be a **finite-dimensional K-vector space.**

- Definition: An A-submodule M of V is a **lattice of** V if $\exists\, L_1,\, L_2$ two **free A-submodules of** V s.t.:

$$\begin{cases} L_1 \subseteq M \subseteq L_2, \\ \mathrm{rk}_A(L_1) = \dim_K(V). \end{cases}$$

- Example: The **lattices of** $V = K$ are just the non-zero **fractional ideals of** A**.**

- Proposition: An A-submodule M of V is **a lattice of** V iff

$$\begin{cases} K\,M \triangleq \{k\,m \mid k \in K,\ m \in M\} = V, \\ M \subseteq P, \end{cases}$$

where P is a finitely generated A-submodule of V.

- Example: Let $P \in K^{q \times r}$, then the A-module

$$(I_q \,:\, -P)\, A^{q+r}$$

is a **lattice of** the K-vector space K^q.

- Example: Let $P \in K^{q \times r}$, then the A-module

$$A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$$

is a **lattice of** the K-vector space $K^{1 \times r}$.

Lattices

- Let V and W be finite-dimensional K-vector spaces.

- Let M (resp. N) be a lattice of V (resp. W).

- Definition: $N : M$ **is the** A**-submodule of**

$\hom_K(V, W) = \{f : V \to W \mid f$ is a $K-$linear map$\}$

formed by the K**-linear maps** $f : V \to W$ **which satisfy** $f(M) \subseteq N$.

- Proposition: 1. $N : M$ is **a lattice of** $\hom_K(V, W)$.

2. The **map**

$$\begin{array}{rcl} N : M & \to & \hom_A(M, N) = \{f : M \to N \mid \\ & & \qquad f \text{ is a } A - \text{linear map}\}, \\ f & \mapsto & f_{|M}, \end{array}$$

is bijective.

- Example: Let $P \in K^{q \times r}$ and $M = (I_q : -P)\, A^{q+r}$. Then, we have:

$$\begin{aligned} A : M &= \{f : K^q \to K \mid f(M) \subseteq A\} \\ &= \{\lambda \in K^{1 \times q} \mid \lambda\, (I_q : -P)\, A^{q+r} \subseteq A\} \\ &= \{\lambda \in K^{1 \times q} \mid \lambda \in A^{1 \times q},\ \lambda\, P \in A^{1 \times r}\} \\ &= \{\lambda \in A^{1 \times q} \mid \lambda\, P \in A^{1 \times r}\}. \end{aligned}$$

Weakly coprime factorizations

• Definition: $P \in K^{q\times r}$ admits a **weakly left-coprime factorization** if $\exists\ R = (D\ :\ -N) \in A^{q\times(q+r)}$ s.t.:

$$P = D^{-1}\, N,$$

$$\forall\ \lambda \in K^{1\times q}:\ \lambda\, R \in A^{1\times(q+r)} \Rightarrow \lambda \in A^{1\times q}.$$

• Definition: $P \in K^{q\times r}$ admits a **weakly right-coprime factorization** if $\exists\ \tilde{R} = (\tilde{N}^T\ :\ \tilde{D}^T)^T \in A^{(q+r)\times r}$ such that:

$$P = \tilde{N}\, \tilde{D}^{-1},$$

$$\forall\ \lambda \in K^{r}:\ \tilde{R}\, \lambda \in A^{p} \Rightarrow \lambda \in A^{r}.$$

• Proposition: $P \in K^{q\times r}$ **admits a weakly left-coprime factorization** iff $\exists\ D \in A^{q\times q}$ such that

$$\begin{aligned} A : ((I_q\ :\ -P)\, A^{q+r}) &= \{\lambda \in A^{1\times q} \mid \lambda\, P \in A^{1\times r}\} \\ &= A^{1\times q}\, D, \end{aligned}$$

i.e. is a **free lattice of** $K^{1\times q}$**.**

• Proposition: $P \in K^{q\times r}$ **admits a weakly right-coprime factorization** iff $\exists\ \tilde{D} \in A^{r\times r}$ such that

$$\begin{aligned} A : \left(A^{1\times(q+r)} \begin{pmatrix} P \\ I_{p-q} \end{pmatrix}\right) &= \{\lambda \in A^{r} \mid P\, \lambda \in A^{q}\} \\ &= \tilde{D}\, A^{r}, \end{aligned}$$

i.e. is **free lattice of** K^{r}**.**

Coprime factorizations

• Let A be an **integral domain** and $K = Q(A)$.

• Proposition: $P \in K^{q \times r}$ admits the **left-coprime factorization**

$$P = D^{-1} N, \quad D\,X - N\,Y = I_q,$$

iff $\exists\ D \in A^{q \times q}$ such that

$$\begin{aligned}(I_q\,:\,-P)\,A^{q+r} &\triangleq \{\lambda_1 - P\,\lambda_2 \mid \lambda_1 \in A^q,\ \lambda_2 \in A^r\}\\ &= D^{-1}\,A^q,\end{aligned}$$

i.e. iff $(I_q\,:\,-P)\,A^p$ is a **free lattice of** K^q**.**

• Proposition: If $P \in K^{q \times r}$ admits a **right-coprime factorization**

$$P = \tilde{N}\,\tilde{D}^{-1}, \quad -\tilde{Y}\,X + \tilde{X}\,\tilde{D} = I_r,$$

iff $\exists\ \tilde{D} \in A^{r \times r}$ such that

$$\begin{aligned}A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} &\triangleq \{\lambda_1\,P + \lambda_2 \mid \\ &\qquad (\lambda_1\,:\,\lambda_2) \in A^{1 \times (q+r)}\}\\ &= A^{1 \times (q+r)}\,\tilde{D}^{-1},\end{aligned}$$

i.e. iff $A^{1 \times (q+r)}\,(P^T\,:\,I_r)^T$ is a **free lattice of** $K^{1 \times r}$**.**

Stabilizability

• <u>Theorem</u>: $P \in K^{q \times r}$ is **internally stabilizable** iff one of the following conditions is satisfied:

1. $(I_q \,:\, -P)\, A^{q+r}$ is a **projective lattice of** K^q, namely $\exists$ A-module M such that:

$$(I_q \,:\, -P)\, A^{q+r} \oplus M \cong A^{q+r}.$$

2. $A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$ is a **projective lattice of** $K^{1 \times r}$, namely $\exists$ A-module N such that:

$$A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} \oplus N \cong A^{1 \times (q+r)}.$$

• Let $R = (I_q \,:\, -P), \quad Q = \begin{pmatrix} P \\ I_r \end{pmatrix}, \quad p = q + r,$ then we have the following **split exact sequences**:

$$0 \longleftarrow (I_q \,:\, -P)\, A^p \underset{\xrightarrow{S.}}{\xleftarrow{R.}} A^p \underset{\xrightarrow{T.}}{\xleftarrow{Q.}} A : \left(A^{1 \times p} \begin{pmatrix} P \\ I_r \end{pmatrix}\right) \longleftarrow 0,$$

$$0 \longrightarrow A : ((I_q \,:\, -P)\, A^p) \underset{\xleftarrow{.S}}{\xrightarrow{.R}} A^{1 \times p} \underset{\xleftarrow{.T}}{\xrightarrow{.Q}} A^{1 \times p} \begin{pmatrix} P \\ I_r \end{pmatrix} \longrightarrow 0.$$

$\Rightarrow \Pi_1 = S\, R, \; \Pi_2 = Q\, T$ are **projectors** of $A^{p \times p}$.

Stabilizability

• Theorem: $P \in K^{q\times r}$ is **internally stabilizable** iff one of the following conditions is satisfied:

$\mathbf{C_1}$. $\exists\, S = (U^T : \ V^T)^T \in A^{(q+r)\times q}$ such that:

$$\begin{aligned} S\,P &= \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{(q+r)\times r}, \\ (I_q : \ -P)\,S &= U - P\,V = I_q. \end{aligned}$$

Then, $C = V\,U^{-1}$ is **a stabilizing controller of** P**.**

$\mathbf{C_2}$. $\exists\, T = (-X : \ Y) \in A^{r\times(q+r)}$ such that:

$$\begin{aligned} P\,T &= (P\,X : \ P\,Y) \in A^{q\times(q+r)}, \\ T \begin{pmatrix} P \\ I_r \end{pmatrix} &= -X\,P + Y = I_r. \end{aligned}$$

Then, $C' = Y^{-1}\,X$ is **a stabilizing controller of** P**.**

• Proposition: If P is **internally stabilizable**, then $\exists\, S \in A^{(q+r)\times q}$, $T \in A^{r\times(q+r)}$ satisfying $\mathbf{C_1}$, $\mathbf{C_2}$,

$$T\,S = -X\,U + Y\,V = 0,$$

i.e. $\exists$ **a stabilizing controller of** P **of the form:**

$$\boxed{C = V\,U^{-1} = Y^{-1}\,X.}$$

Example

- Let us consider the transfer matrix ($A = H_\infty(\mathbb{C}_+)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2, \quad K = Q(A).$$

- The matrix $S = (U^T : \, V^T)^T \in A^{3\times 2}$ defined by

$$S = \begin{pmatrix} \frac{2}{s+1} + b\left(\frac{s-1}{s+1}\right)^3 & 2\,b\left(\frac{s-1}{s+1}\right)^3 - 2\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\,b\,\frac{(s-1)}{(s+1)^3} \\ -a\,\frac{(s-1)^2}{(s+1)^3} & -2\,a\,\frac{(s-1)^2}{(s+1)^3} \end{pmatrix}$$

with $$\begin{cases} a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \\ b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A, \end{cases}$$

satisfies

$$\begin{cases} S P \in A^{3\times 1}, \\ (I_2 : \, -P)\,S = U - P\,V = I_2, \end{cases}$$

$\Rightarrow P$ is **internally stabilized** by the controller:

$$\begin{aligned} C &= V\,U^{-1} \\ &= -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)})}\,(1 : \, 2). \end{aligned}$$

Stabilizability

• <u>Corollary</u>: $P \in K^{q \times r}$ is **internally stabilized by the controller** $C \in K^{r \times q}$ iff one of the following conditions is satisfied:

1. The matrix

$$\Pi_1 = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}$$

is a **projector of** $A^{(q+r)\times(q+r)}$, i.e.:

$$\boxed{\Pi_1^2 = \Pi_1 \in A^{(q+r)\times(q+r)}.}$$

2. The matrix

$$\Pi_2 = \begin{pmatrix} -P\,(I_{p-q} - C\,P)^{-1}\,C & P\,(I_{p-q} - C\,P)^{-1} \\ -(I_{p-q} - C\,P)^{-1}\,C & (I_{p-q} - C\,P)^{-1} \end{pmatrix}$$

is a **projector of** $A^{(q+r)\times(q+r)}$, i.e.:

$$\boxed{\Pi_2^2 = \Pi_2 \in A^{(q+r)\times(q+r)}.}$$

Then, we have $\boxed{\Pi_1 + \Pi_2 = I_{q+r}.}$

• <u>Remark</u>: This result was known for $A = H_\infty(\mathbb{C}_+)$. The **robustness radius** is defined by (loop-shaping):

$$\boxed{b_{P,C} \triangleq \| \Pi_1 \|_\infty^{-1} = \| \Pi_2 \|_\infty^{-1}.}$$

Stabilizability

- Fact 1: P admits a **doubly coprime factorization**

$$\Leftrightarrow (I_q \, : \, -P) \, A^{q+r} \quad \& \quad A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$$

are **free A-modules.**

- Fact 2: P is **internally stabilizable**

$$\Leftrightarrow (I_q \, : \, -P) \, A^{q+r} \quad \& \quad A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$$

are **projective A-modules.**

- Fact 3: **A free A-module is projective.**

- Corollary:

If $P \in K^{q\times r}$ admits a **left-coprime factorization**

$$P = D^{-1} \, N, \quad D \, X - N \, Y = I_q,$$

then $S = ((X \, D)^T \, : \, (Y \, D)^T)^T$ satisfies $\mathbf{C_1}$

$$\boxed{\Rightarrow C = (Y \, D) \, (X \, D)^{-1} = Y \, X^{-1} \in \mathsf{Stab}(P).}$$

If $P \in K^{q\times r}$ admits a **right-coprime factorization**

$$P = \tilde{N} \, \tilde{D}^{-1}, \quad -\tilde{Y} \, X + \tilde{X} \, \tilde{D} = I_r,$$

then $T = (-\tilde{D} \, \tilde{Y} \, : \, \tilde{D} \, \tilde{X})$ satisfies $\mathbf{C_2}$

$$\boxed{\Rightarrow C = (\tilde{D} \, \tilde{X})^{-1} \, (\tilde{D} \, \tilde{Y}) = \tilde{X}^{-1} \, \tilde{Y} \in \mathsf{Stab}(P).}$$

Structural stabilizable n-D systems

- $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n \,|\, |z_i| \leq 1, i = 1, \ldots, n\}$ unit polydisc of $\mathbb{C}^n$.

- Let A be the ring of **structural stabilizable n-D systems**:

$$M_{\overline{\mathbb{D}^n}} = \{r/s \,|\, 0 \neq s,\, r \in \mathbb{R}[z_1, \ldots, z_n], \\ s(\underline{z}) = 0 \Rightarrow \underline{z} \notin \overline{\mathbb{D}^n}\}$$

- **Z. Lin's conjecture**:

> "Determine whether or not an internally stabilizable n-D linear system defined by a transfer matrix P with entries in $\mathbb{R}(z_1, \ldots, z_n)$ admits a doubly coprime factorization over A".

- Theorem: (Byrnes-Spong-Tarn, Kamen-Khargonekar-Tannenbaum 84): A **is a projective-free ring.**

- Remark: B-S-T & K-K-T obtain this result in their study of **differential time-delay neutral systems**.

- Remark: This result is not trivial: the **proof was given by P. Deligne** in K-K-T.

- Corollary: **Z. Lin's conjecture is solved**.

- Open problem: **Effective proof.**

Parametrization

• Theorem: Let $P \in K^{q \times r}$ be a **stabilizable plant**. **All stabilizing controllers of P have the form**

$$\begin{aligned} C(Q) &= (V + Q)\,(U + P\,Q)^{-1} \\ &= (Y + Q\,P)^{-1}\,(X + Q), \end{aligned}$$

where C_* is **a particular stabilizing controller of P** and:

$$\begin{cases} U = (I_q - P\,C_*)^{-1}, \\ V = C_*\,(I_q - P\,C_*)^{-1}, \\ X = (I_r - C_*\,P)^{-1}\,C_*, \\ Y = (I_r - C_*\,P)^{-1}, \end{cases}$$

and Q is **every matrix which belongs to**

$$\Omega = \{L \in A^{r \times q} \mid \; L\,P \in A^{r \times r},\, P\,L \in A^{q \times q},\, P\,L\,P \in A^{q \times r}\}$$

such that $\det(U + P\,Q) \neq 0$ and $\det(Y + Q\,P) \neq 0$.

(Ω is a **projective A-module of rank $q \times r$**).

Study of the A-module Ω

• Open question: **Find a family of generators of the projective A-module of rank** $q \times r$

$$\Omega = \{L \in A^{r \times q} \mid \; L\,P \in A^{r \times r},\; P\,L \in A^{q \times q},\; P\,L\,P \in A^{q \times r}\},$$

i.e. a finite family $\{L_i\}_{1 \leq i \leq s}$ such that:

$$\forall\, L \in \Omega,\; \exists\, L = \sum_{i=1}^{s} \lambda_i\, L_i, \quad \lambda_i \in A.$$

• Proposition: If $P \in Q(A)^{q \times r}$ admits a **weakly left-coprime factorization** $P = D^{-1}\, N$, then:

$$\Omega = \{L \in A^{r \times q} \mid \; P\,L \in A^{q \times q}\}\, D.$$

• Proposition: If $P \in Q(A)^{q \times r}$ admits a **weakly right-coprime factorization** $P = \tilde{N}\, \tilde{D}^{-1}$, then:

$$\Omega = \tilde{D}\, \{L \in A^{r \times q} \mid \; L\,P \in A^{r \times r}\}.$$

Youla-Kučera parametrization

• <u>Corollary</u>: Let $P \in Q(A)^{q \times r}$ be a plant which admits a **doubly coprime factorization**:

$$\left\{ \begin{array}{l} P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \\ \left(\begin{array}{cc} D & -N \\ -\tilde{Y} & \tilde{X} \end{array} \right) \left(\begin{array}{cc} X & \tilde{N} \\ Y & \tilde{D} \end{array} \right) = I_{q+r}. \end{array} \right.$$

Then, the A-module

$$\begin{array}{ll} \Omega = \{L \in A^{r \times q} \mid & L\,P \in A^{r \times r},\ P\,L \in A^{q \times q}, \\ & P\,L\,P \in A^{q \times r}\} \end{array}$$

is the **free A-module** defined by:

$$\boxed{\begin{array}{ll} \Omega & = \tilde{D}\,A^{r \times q}\,D \\ & = \{L \in A^{r \times q} \mid L = \tilde{D}\,R\,D,\ \forall\,R \in A^{r \times q}\}. \end{array}}$$

⇒ **All stabilizing controllers of P have the form**

$$\boxed{C(Q) = (Y + \tilde{D}\,Q)\,(X + \tilde{N}\,Q)^{-1} = (\tilde{X} + Q\,N)^{-1}\,(\tilde{Y} + Q\,D),}$$

where $Q \in A^{r \times q}$ is **every matrix such that:**

$$\det(X + \tilde{N}\,Q) \neq 0, \quad \det(\tilde{X} + Q\,N) \neq 0.$$

Sensitivity minimization

- Let A be a **Banach algebra** ($H_\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}$, W_+,...)

- Let $P \in Q(A)^{q\times r}$ be a **stabilizable plant**, then

$$\inf_{C\in \text{Stab}(P)} \| W_1 \, (I_q - P\,C)^{-1} \, W_2 \|_A$$
$$=$$
$$\inf_{Q\in\Omega} \| W_1 \, (U + P\,Q) \, W_2 \|_A \quad (\star),$$

(**convex problem**)

$C_\star = V\,U^{-1}$ is **a stabilizing controller of** P and:

$$U = (I_q - P\,C_*)^{-1}, \quad V = C_* \, (I_q - P\,C_*)^{-1}.$$

- If P admits a **doubly coprime factorization**

$$\begin{cases} P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \\ \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I_{q+r}. \end{cases}$$

$$\Rightarrow \begin{cases} Q \in \Omega & = \tilde{D}\,A^{r\times q}\,D, \\ U + P\,Q & = X\,D + \tilde{N}\,\tilde{D}^{-1}\,(\tilde{D}\,R\,D), \\ & = (X + \tilde{N}\,R)\,D, \end{cases}$$

$$(\star) \Leftrightarrow \inf_{R\in A^{r\times q}} \| W_1 \, (X + \tilde{N}\,R)\,D\,W_2 \|_A \,.$$

Conclusion

I. Summary:

• We **generalized the Youla-Kučera parametrization** for **MIMO stabilizable plants**.

• This parametrization **does not assume the existence of doubly coprime factorizations.**

II. General comments:

When does a stabilizable plant admit a doubly coprime factorization?

• We proved that this problem is equivalent to:

When is a projective A-module free?

• This is a **difficult problem** studied for years in:

– **algebra**: algebraic K-theory (Serre's conjecture (55) $A = k[x_1, \ldots, x_n]$, solved by Quillen-Suslin (76)),

– **number theory**: number fields,

– **algebraic geometry**: function fields,

– **topology**: triviality of vector bundles,

– **operator theory**: topological K-theory ($C^\star$-algebra).

this problem could be difficult for $\widehat{\mathcal{A}}$, W_+...