Module structure of classical multidimensional systems appearing in mathematical physics

Thomas Cluzeau
University of Limoges,
CNRS, XLIM UMR 6172,
DMI 123 avenue Albert Thomas,
87060 Limoges cedex, France.
cluzeau@ensil.unilim.fr

Alban Quadrat
INRIA Sophia Antipolis,
2004, Route des Lucioles BP 93,
06902 Sophia Antipolis cedex, France.
Alban.Quadrat@sophia.inria.fr

Abstract—In this paper, within the constructive algebraic analysis approach to linear systems, we study classical linear systems of partial differential (PD) equations in two or three independent variables with constant coefficients appearing in mathematical physics and engineering sciences such as the Stokes and Oseen equations studied in hydrodynamics. We first provide a precise algebraic description of the endomorphism ring of the left $D$-module associated with a linear PD system. Then, we use it to prove that the endomorphism ring of the Stokes and Oseen equations in $\mathbb{R}^2$ is a cyclic $D$-module, which allows us to conclude about the decomposition and factorization properties of these linear PD systems.

I. INTRODUCTION

Within the constructive algebraic analysis approach to linear systems theory developed in [2], [4], we study linear systems of partial differential (PD) equations in two independent variables with constant coefficients classically encountered in engineering sciences and mathematical physics (e.g., Stokes/Oseen/Euler/Maxwell equations). Since these linear PD systems can be written as $R \eta = 0$, where $R \in D^{q \times p}$ is a $q \times p$ matrix with entries in the commutative polynomial ring $D = k[\partial_x, \partial_y]$ or $D = k[\partial_t, \partial_x, \partial_y]$ of PD operators in $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_t = \frac{\partial}{\partial t}$ with coefficients in a field $k$, they can be interpreted as linear 2-dimensional systems. As explained in [2], a linear PD system can be studied by means of the left $D$-module $M = D^{1 \times p}/(D^{1 \times q} R)$, finitely presented by the matrix $R$ of PD operators, and its dual $\text{hom}_D(M, F)$ ([9]), where $F$ is a left $D$-module.

Indeed, according to Malgrange’s remark ([8]), the linear PD system $\ker_{F}(R) = \{ \eta \in F^p \mid R \eta = 0 \}$, also called behaviour, satisfies $\ker_{F}(R) \cong \text{hom}_D(M, F)$. Hence, the linear PD system $\ker_{F}(R)$ can be studied by means of the two $D$-modules $M$ and $F$. In this paper, we shall study properties of $\ker_{F}(R)$ inherited from the algebraic properties of the $D$-module $M$, especially those of the endomorphism ring $\text{end}_D(M) = \text{hom}_D(M, M)$. We focus on the factorization and decomposition problems for these classical linear PD systems and particularly the Stokes and Oseen equations. The Euler and Maxwell equations ([6]) can be studied similarly.

II. EDOMORPHISM RING

A. Characterization of $\text{end}_D(M)$

We shall give a precise description of the endomorphism ring $\text{end}_D(M)$ of $M$ in terms of generators, relations and multiplication tables. An algorithm to compute such a representation is given. As shown in [4], the existence of a $D$-endomorphism $f : M \rightarrow M$ is equivalent to the existence of two matrices $P \in D^{p \times p}$ and $Q \in D^{q \times q}$ such that $RP = QR$. Moreover, we have $f(\pi(\lambda)) = \pi(\lambda P)$ for all $\lambda \in D^{1 \times p}$. Such an endomorphism is defined up to homotopy equivalence. Thus, the ring $\text{end}_D(M)$ can be written as the quotient of two left $D$-modules $\text{end}_D(M) \cong A/(D^{p \times q} R)$, where $A = \{ P \in D^{p \times p} \mid \exists Q \in D^{q \times q} : RP = QR \}$. Let us first compute a family of generators of $A$. Let $E \in D^{q \times p}$ and $F \in D^{r \times s}$ be two matrices with entries in a ring $D$. Then, the Kronecker product of $E$ and $F$, denoted by $E \otimes F$, is the matrix defined by:

$$E \otimes F = \begin{pmatrix} E_{11} F & \ldots & E_{1p} F \\ \vdots & \ddots & \vdots \\ E_{q1} F & \ldots & E_{qp} F \end{pmatrix} \in D^{(qr) \times (ps)}.$$

If $D$ is a commutative ring, $E \in D^{r \times q}$, $F \in D^{q \times p}$ and $G \in D^{p \times m}$ and $\text{row}(F) \in D^{1 \times q}$ denotes the row vector obtained by concatenating the rows of $F$, then the product of the three matrices can be obtained by:

$$\text{row}(EFG) = \text{row}(F) (E^T \otimes G).$$

Consequently, we have

$$\begin{cases} \text{row}(RP) = \text{row}(RP I_p) = \text{row}(P) (R^T \otimes I_p), \\ \text{row}(QR) = \text{row}(I_q Q R) = \text{row}(Q) (I_q \otimes R), \end{cases}$$
so that
\[ R P = Q R \iff (\text{row}(P) - \text{row}(Q)) L = 0, \]
where the matrix \( L \) is defined by:
\[
L = \begin{pmatrix} R^T \otimes I_p \\ I_q \otimes R \end{pmatrix} \in D^{(p^2 + q^2) \times q}. \]

Now, there exists a matrix \( L_2 \in D^{s \times (p^2 + q^2)} \) such that \( \ker_D(L) = D^{1 \times s} L_2 \). Stacking the rows of \( L_2 \), we find a set of matrices \( \{P_r\}_{1=1, \ldots, s} \) and \( \{Q_s\}_{1=1, \ldots, s} \), where \( P_r \in D^{p \times p} \) and \( Q_s \in D^{q \times q} \), satisfying the relation \( R P_r = Q_s R \) for \( i = 1, \ldots, s \). Moreover, we can easily check that every solution \( P \in D^{p \times p} \) and \( Q \in D^{q \times q} \) of \( R P = Q R \) has the form
\[
\begin{cases}
P = \sum_{i=1}^s \alpha_i P_i, \\
Q = \sum_{i=1}^s \alpha_i Q_i,
\end{cases}
\]
where the \( \alpha_i \)'s are arbitrary elements of \( D \), \( i = 1, \ldots, s \), i.e., \( \{P_r\}_{1=1, \ldots, s} \) is a set of generators of the left \( D \)-module \( A \). Therefore, the set \( \{P_r\}_{1=1, \ldots, s} \) of the residue classes of the matrices \( P_r \)'s in the \( D \)-module \( A/\langle D^{p \times q} \rangle \) generates \( A/\langle D^{p \times q} \rangle \), i.e., \( \text{end}_D(M) \) up to isomorphism. If we consider
\[
\forall i = 1, \ldots, s, \quad \begin{cases} P_i = P_i + Z_i R, \\
Q_i = Q_i + R Z_i,
\end{cases}
\]
for certain matrices \( Z_i \in D^{p \times q} \), then \( P_i \) and \( Q_i \) satisfy the relation \( R P_i = Q_i R \) for \( i = 1, \ldots, s \), i.e., they induce \( f_i \in \text{end}_D(M) \) defined by:
\[
\forall \lambda \in D, \quad f_i(\pi(\lambda)) = \pi(\lambda P_i), \quad i = 1, \ldots, s.
\]
Then, \( \{f_i\}_{1=1, \ldots, s} \) is a family of generators of \( \text{end}_D(M) \). A \( D \)-linear relation \( \sum_{j=1}^s d_j f_j = 0 \) between the \( f_i \)'s is equivalent to the existence of \( Z \in D^{p \times q} \) satisfying \( \sum_{j=1}^s d_j (\text{row}(P_j) - \text{row}(Z)(I_p \otimes R)) = 0 \)
\[
\iff (d_1 \ldots d_s - \text{row}(Z)) \begin{pmatrix}
\text{row}(P_1) \\
\vdots \\
\text{row}(P_s) \\
I_p \otimes R
\end{pmatrix} = 0.
\]
If we introduce the matrices
\[
\begin{align*}
U &= (\text{row}(P_1))^T \ldots (\text{row}(P_s))^T \in D^{p \times p^2}, \\
V &= I_p \otimes R \in D^{p \times q \times p^2}, \\
W &= (U^T V^T)^T \in D^{(s \times p + q) \times p^2},
\end{align*}
\]
then there exist \( X \in D^{1 \times s} \) and \( Y \in D^{1 \times p \times q} \) satisfying \( \ker_D(W) = D^{1 \times l} \). If \( Y_{i,j} \) denotes the \( i \times j \) entry of the matrix \( Y \) and for \( i = 1, \ldots, l \),
\[
Z_i = \begin{pmatrix} Y_{i,1} & \cdots & Y_{i,q} \\
\vdots & \ddots & \vdots \\
Y_{i,(q+1)} & \cdots & Y_{i,2q} \\
\vdots & & \vdots \\
Y_{i,(p-1)q+1} & \cdots & Y_{i,pq} \\
\end{pmatrix} \in D^{p \times q},
\]
then \( \sum_{i=1}^s X_{ij} T_j = Z_i R \), and thus the \( f_i \)'s satisfy the following \( D \)-linear relations:
\[
\sum_{j=1}^s X_{ij} f_j = 0, \quad i = 1, \ldots, l.
\]
Hence, \( \text{end}_D(M) \cong D^{1 \times s}/(D^{1 \times l} X) \), i.e., \( \text{end}_D(M) \) is finitely presented by \( X \in D^{1 \times s} \).

Now, we note that
\[
A = \{ P \in D^{p \times p} \mid \exists Q \in D^{q \times q} : R P = Q R \}
\]
is also a ring. Indeed, \( 0 \in A, I_p \in A \) and if \( P_1, P_2 \in A \), i.e., \( R P_1 = Q_1 R \) and \( R P_2 = Q_2 R \) for certain matrices \( Q_1, Q_2 \in D^{q \times q} \), then we have
\[
\begin{cases}
R(P_1 + P_2) = (Q_1 + Q_2) R, \\
R(P_1 P_2) = (Q_1 Q_2) R,
\end{cases}
\]
so that \( P_1 + P_2 \in A \) and \( P_1 P_2 \in A \). The other properties of a ring can easily be checked. The ring \( A \) is generally a noncommutative ring since \( P_1 \neq P_2 \) is generally different from \( P_2 \). Moreover, \( I \triangleq D^{p \times q} \) is a two-sided ideal of \( A \). Indeed, if \( P_1, P_2 \in A \) and \( Z_1 R, Z_2 R \in I \), where \( Z_i \in D^{p \times q} \) for \( i = 1, 2 \), then:
\[
\begin{cases}
P_1 (Z_1 R) + P_2 (Z_2 R) = (P_1 Z_1 + P_2 Z_2) R, \\
(Z_1 R) P_1 + (Z_2 R) P_2 = (Z_1 Q_1 + Z_2 Q_2) R.
\end{cases}
\]
Hence, \( B = A/I \) is generally a noncommutative ring and \( \kappa = \text{id}_p \otimes \pi : A \to B \) is the canonical projection onto \( B \). In particular, the product of \( B \) is defined by:
\[
\forall P_1, P_2 \in A, \quad \kappa(P_1) \kappa(P_2) = \kappa(P_1 P_2).
\]
We call opposite ring of \( B \), denoted by \( B^{op} \), the ring defined by \( B \) as an abelian group but equipped with the opposite multiplication defined by:
\[
\forall a, b \in A, \quad b \circ a = a b.
\]
If \( \phi : B \to \text{end}_D(M) \) is the abelian group isomorphism defined by
\[
\forall \lambda \in D^{1 \times p}, \quad \phi((\kappa(P))(\pi(\lambda))) = \pi(\lambda P),
\]
and \( f_1 = \phi(\kappa(P_1)) \) and \( f_2 = \phi(\kappa(P_2)) \), then
\[
\forall \lambda \in D^{1 \times p}, \quad (f_2 \circ f_1)(\pi(\lambda)) = \phi(\kappa(P_1) \kappa(P_2)))(\pi(\lambda)),
\]
i.e., using the opposite ring \( B^{op} \), we obtain:
\[
\phi(\kappa(P_2) \kappa(P_1)) = \phi(\kappa(P_1) \kappa(P_2)) = \phi(\kappa(P_2)) \circ \phi(\kappa(P_1)).
\]
Moreover, \( \phi(\kappa(I_\mu)) = id_M \), which proves that \( \phi \) is a ring isomorphism, i.e., \( \text{end}_D(M) \cong B^{op} \).

The ring structure of \( \text{end}_D(M) \) can be characterized by the knowledge of the expansions of the \( f_i \circ f_j \)'s in the family of generators \( \{f_k\}_{k=1,...,s} \) for \( i, j = 1, \ldots, s \):
\[
\forall i, j = 1, \ldots, s, \quad f_i \circ f_j = \sum_{k=1}^s \gamma_{ijk} f_k, \quad \gamma_{ijk} \in D.
\]
(3)
The \( \gamma_{ijk} \)'s look like the structure constants appearing in the theory of finite-dimensional algebras. Hence, if \( F = (f_1 \ldots f_s)^T \), then the matrix \( \Gamma \) formed by the \( \gamma_{ijk} \) satisfies \( F \otimes F = \Gamma F \). \( \Gamma \) is called a multiplication table in group theory. Finally, if \( D(f_1, \ldots f_s) \) is the free associative \( D \)-algebra generated by the \( f_i \)'s and
\[
I = (\sum_{j=1}^s X_{ij} f_j, \quad i = 1, \ldots, l, \quad f_i \circ f_j - \sum_{k=1}^s \gamma_{ijk} f_k, \quad i, j = 1, \ldots, s)
\]
is the two-sided ideal of \( D \) generated by the polynomials corresponding to the identities (2) and (3), then the noncommutative ring \( \text{end}_D(M) \) is defined by
\[
\text{end}_D(M) = D(f_1, \ldots f_s)/I,
\]
which shows that \( \text{end}_D(M) \) can be defined as the quotient of a free associative algebra by a two-sided ideal generated by linear and quadratic relations.

**Algorithm 1:**
- **Input:** A matrix \( R \in D^{q \times p} \) defined over a commutative polynomial ring \( D \) over a computational field \( k \).
- **Output:** A finite family of generators \( \{f_1, \ldots f_s\} \) of the endomorphism ring \( \text{end}_D(M) \) of the \( D \)-module \( M = D^{1 \times p}/(D^{1 \times q} R) \) and a set of \( D \)-linear relations of the \( f_i \)'s generating the \( D \)-module structure of \( \text{end}_D(M) \).

1. Compute \( L = \begin{pmatrix} R^T \otimes I_p \\ I_q \otimes R \end{pmatrix} \).
2. Compute \( L_2 \in D^{s \times (p^2 + q^2)} \) satisfying:
\[
\ker_D(L) = D^{1 \times s} L_2.
\]
3. Construct \( P_i \in D^{p \times p} \) and \( Q_i \in D^{q \times q} \) defined by
\[
P_i(j, k) = L(i, (j-1)p + k), \quad Q_i(l, m) = -L(i, p^2 + (l-1)q + m),
\]
for \( j = 1, \ldots, p, \quad k = 1, \ldots, p, \quad l = 1, \ldots, q \) and \( m = 1, \ldots, q \), where \( L(i, j) \) denotes the \( i \times j \) entry of the matrix \( L \), for \( i = 1, \ldots, s \). Then, we have:
\[
\forall i = 1, \ldots, s, \quad R P_i = Q_i R.
\]
4. Compute a Gröbner basis \( G \) of the rows of \( R \) for a total degree order.
5. For \( i = 1, \ldots, s \), reduce the rows of \( P_i \) with respect to \( G \) by computing their normal forms with respect to \( G \). We obtain the matrices \( \overline{P}_i \) which satisfy \( \overline{P}_i = P_i + Z_i R \) for certain matrices \( Z_i \in D^{p \times q} \) which can be obtained by means of factorizations.
6. For \( i = 1, \ldots, s \), define the following matrices \( \overline{Q}_i = Q_i + R Z_i \). The pair \((\overline{P}_i , \overline{Q}_i)\) then satisfies the relation \( R \overline{P}_i = \overline{Q}_i R \) and the \( D \)-module \( \text{end}_D(M) \) is finitely generated by \( \{f_i\}_{i=1,...,s} \), where \( f_i \in \text{end}_D(M) \) is defined by:
\[
\forall \lambda \in D^{1 \times p}, \quad f_i(\pi(\lambda)) = \pi(\lambda \overline{P}_i).
\]
7. Form the matrices \( U, V \) and \( W \) defined by (1).
8. By means of syzygies computations, compute \( (X - Y) \), where \( X \in D^{1 \times s} \) and \( Y \in D^{1 \times q} \), such that \( \ker_D(W) = D^{1 \times 1} (X - Y) \). Then, the family of generators \( \{f_i\}_{i=1,...,s} \) of the \( D \)-module \( \text{end}_D(M) \) satisfies the \( D \)-linear relations \( X F = 0 \), where \( F = (f_1 \ldots f_s)^T \), i.e.:
\[
\text{end}_D(M) \cong D^{1 \times s}/(D^{1 \times 1} X).
\]
9. For \( i, j = 1, \ldots, s \), compute the normal form of \( \text{row}((P_i, P_j)) \) with respect to a Gröbner basis of the \( D \)-module \( D^{1 \times (s+p+q)} \). With these normal forms, form the matrix \( \Gamma_1, \Gamma_2 \in D^{s \times (s+p+q)} \), where \( \Gamma_1 \in D^{s \times s} \) and \( \Gamma_2 \in D^{s \times p} \). Then, the matrix \( \Gamma_1 \) defines the multiplication table of family of generators \( \{f_i\}_{i=1,...,s} \) of \( \text{end}_D(M) \).

Algorithm 1 is implemented in the OreMorphisms package ([5]) built upon OreModules ([3]).

**B. Cyclic D-module \( \text{end}_D(M) \)**

We recall that a \( D \)-module \( M \) is called cyclic if there exists \( m \in M \) which generates \( M \) as a \( D \)-module, i.e.:
\[
M = Dm = \{dm \mid d \in D\}.
\]
Let us consider the Oseen equations in \( \mathbb{R}^2 \) defined by
\[
\begin{cases}
\partial_t u - \nu \Delta \bar{u} + (\bar{b}, \nabla) \bar{u} + \nabla p = 0, \\
\nabla \cdot \bar{u} = 0,
\end{cases}
\]
where \( \bar{u} \) denotes the velocity, \( p \) the pressure, \( \nu \) the viscosity, \( \bar{b} = (b_1 \ b_2)^T \) a steady velocity, \( \nabla = \)
The Oseen equations describe the flow of a viscous and incompressible fluid at small Reynolds numbers (linearization of the incompressible Navier-Stokes equations at a steady state) ([7]). Let \( D = \mathbb{Q}\langle \nu, b_1, b_2 \rangle \) be the commutative polynomial ring of PD operators with coefficients in the field \( \mathbb{Q} \). Using Algorithm 1, we can find that the endomorphism ring \( \text{end}_D(M) \) of \( M \) is generated as a \( D \)-module finitely presented by

\[
L = \begin{pmatrix}
L & 0 & \partial_y \\
0 & L & \partial_x \\
\partial_x & \partial_y & 0
\end{pmatrix} \in D^{3 \times 3},
\]

where \( L = \partial_t + \mathbf{u} \cdot \nabla - \nu \Delta, \) \( M = D^{1 \times 3}/(D^{1 \times 3} R) \) the \( D \)-module finitely presented by \( R \) and \( \pi : D^{1 \times 3} \to M \) the canonical projection onto \( M \). Using Algorithm 1, we find that the endomorphism ring \( \text{end}_D(M) \) of \( M \) is defined by the family of generators \( \{ f_i \}_{i=1}^{25} \), where

\[
f_i(\pi(\lambda)) = \pi(\lambda P_i) \quad \text{for all } \lambda \in D^{1 \times 3} \text{ and:}
\]

\[
P_1 = I_3, \quad P_2 = \begin{pmatrix}
0 & -\partial_y & 0 \\
0 & \partial_x & 0 \\
0 & 0 & \partial_x
\end{pmatrix},
\]

\[
P_3 = \begin{pmatrix}
0 & 0 & \partial_x \\
0 & 0 & \partial_y \\
0 & -\partial_x + b_1 \partial_x + b_2 \partial_y & 0
\end{pmatrix},
\]

\[
P_4 = \begin{pmatrix}
0 & \nu \partial_x \partial_y & 0 \\
0 & 0 & \nu \partial_y^2 \\
0 & \partial_y (\partial_t + b_1 \partial_x + b_2 \partial_y + \nu \partial_y^2) & -\partial_y
\end{pmatrix},
\]

\[
P_5 = \begin{pmatrix}
0 & -\partial_x (\partial_t + b_2 \partial_y - \nu \partial_y^2) & \partial_y^2 \\
0 & -\partial_x (\partial_t + b_2 \partial_y - \nu \partial_y^2) & -\partial_x \partial_y \\
0 & \nu \partial_x \partial_y & \nu \partial_x
\end{pmatrix}.
\]

The generators \( f_i \)’s of \( \text{end}_D(M) \) satisfy \( D \)-linear relations. Using Algorithm 1, we obtain that a generating set of \( \text{end}_D(M) \) relationships between the generators \( \{ f_i \}_{i=1}^{25} \) of \( \text{end}_D(M) \) is cyclic and is generated by the residue class of \( \Lambda \) in \( N \). If \( S = \langle S_1, S_2, S_3 \rangle \subseteq D^{1 \times 3}, \) \( S_2 = (S_{21} \cdots S_{25})^T \in D^5, \) then \( S_1 L + S_2 \Lambda = I_5 \) and since \( L f = 0, \) where \( f = (f_1 \cdots f_5)^T, \) and \( \Lambda f = f_1 = \text{id}_M, \) we obtain

\[
f = S_1 (L f) + S_2 (\Lambda f) = S_2 f_1,
\]

\[
\Leftarrow \quad \begin{cases}
f_1 = f_1, \\
f_2 = \partial_x f_1,
\end{cases}
\]

\[
\Leftarrow \quad \begin{cases}
f_3 = -(\partial_t + b_1 \partial_x + b_2 \partial_y - \nu \Delta) f_1, \\
f_4 = -\nu \partial_y^2 f_1,
\end{cases}
\]

\[
\Leftarrow \quad \begin{cases}
f_5 = -\partial_x^2 (\nu \partial_x - b_1) f_1.
\end{cases}
\]

Therefore, for every \( f \in \text{end}_D(M), \) there exist \( d_1, \ldots, d_5 \in D \) such that

\[
f = \sum_{i=1}^{5} d_i f_i = \left( \sum_{i=1}^{5} d_i S_{2i} \right) f_1,
\]

which shows that the endomorphism ring \( \text{end}_D(M) \) is generated as a \( D \)-module by \( f_1 = \text{id}_M, \) i.e., \( \text{end}_D(M) = D f_1 \) is a cyclic \( D \)-module.

The same technique can be applied to the Stokes, Euler or Maxwell equations in \( \mathbb{R}^2 \).

**Theorem 1:** Let \( D \) be the appropriate commutative polynomial ring of PD operators with coefficients in a field \( k. \) Moreover, let \( R \in D^{q \times p} \) be the system matrix of the Stokes, Oseen, Euler or Maxwell equations in \( \mathbb{R}^2 \) and \( M = D^{1 \times p}/(D^{1 \times q} R) \) the \( D \)-module finitely presented by \( R. \) Then, the endomorphism ring \( \text{end}_D(M) \) of \( M \) is a cyclic \( D \)-module generated by \( \text{id}_M. \)

**III. Decomposition Problems**

**A. Indecomposable D-modules**

We recall that a \( D \)-module \( M \) is called **decomposable** if there exist two proper \( D \)-submodules \( M_1 \) and \( M_2 \) of \( M \) such that \( M = M_1 \oplus M_2 \) ([9]). If a \( D \)-module \( M \) is not decomposable, then \( M \) is said to be **indecomposable**. One can prove that the \( D \)-module \( M \) is decomposable
iff the endomorphism ring $E = \text{end}_D(M)$ of $M$ admits a non-trivial idempotent, namely, $e \in E \setminus \{0, \text{id}_M\}$ satisfying $e^2 = e$. For more details, see, e.g., [4], [9].

The goal of this section is to use Theorem 1 to conclude about the (in)decomposability of the finitely presented $D$-module $M$ associated to the Stokes, Oseen, Euler, Maxwell equations, and thus of the corresponding linear PD system $\ker_{x}(R)$. We continue with the Oseen equations studied in Section II-B. Let us determine the annihilator and their greatest common divisor is 1. Hence, the only $\ker$ing linear PD system $\ker D$ presented $e$.

We first study two simple solutions of (6) leading to We continue with the Oseen equations studied in Section II-B. Let us determine the annihilator and their greatest common divisor is 1. Hence, the only $\ker$ing linear PD system $\ker D$ presented $e$.

We first study two simple solutions of (6) leading to

which is impossible since:

- $\Delta$ divides $\alpha - 1$ and $\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta$ divides $\alpha$, i.e., $\alpha = 1 + \gamma \Delta$ and $\alpha = \gamma' (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$, for certain $\gamma, \gamma' \in D$, which yields

$$\gamma' (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta) - \gamma \Delta = 1,$$

which is impossible.

The above results can be also understood as follows: if $I = (\Delta)$ and $J = (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$, then $IJ = I \cap J$ since $\gcd(\Delta, \partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta) = 1$, $I$ and $J$ are two prime ideals (i.e., $IJ = I \cap J$ is a prime decomposition) and the Chinese remainder theorem then shows that $D/(I \cap J) = D/I \oplus D/J$ if $I + J = D$ but $I + J = (\Delta, \partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta) \subseteq D$.

Over the new ring $D' = \mathbb{Q}(\nu, b_1, b_2, i)[\partial_t, \partial_x, \partial_y]$, where $i$ denotes the complex number, we have

$$\Delta = (\partial_x + i \partial_y) (\partial_x - i \partial_y)$$

and $\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta$ is irreducible (over any algebraic closure $k$ of $k = \mathbb{Q}(\nu, b_1, b_2)$). Then, we can similarly prove that the corresponding module is indecomposable.

Finally, within implicit schemes of the time dependent Navier-Stokes, the term $\partial_t u$ in (5) is replaced by $c u$, where the constant $c$ corresponds to the inverse of the time step. We can redo the previous computations in this case and prove that the endomorphism ring $\text{end}_E(N)$ of the corresponding $E = \mathbb{Q}(\nu, b_1, b_2, c)[\partial_x, \partial_y]$-module $N$ is a cyclic $E$-module generated by $i \text{id}_N$, which is isomorphic to the $E$-module $E/(\Delta (\nu \Delta - \vec{b} \cdot \vec{\nabla} - c))$. In particular, this result is also true when $b_1 = 0$ or $b_2 = 0$. We note that the case $b_1 = b_2 = 0$ corresponds to an implicit scheme of the time dependent Stokes equations. Then, $e = \alpha \text{id}_N \in \text{end}_E(N)$, where $\alpha \in D$, is an idempotent of $\text{end}_E(N)$ iff:

$$\alpha (\alpha - 1) = \beta \Delta (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta).$$

(6)

We first study two simple solutions of (6) leading to the trivial idempotents 0 and $i \text{id}_E$.

- If $\Delta (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$ divides $\alpha$, i.e., if there exists $\gamma \in D$ such that $\alpha = \gamma (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$, then:

$$e = \alpha \text{id}_M = 0.$$

- If $\Delta (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$ divides $\alpha - 1$, i.e., $\alpha = 1 + \gamma (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$ for a certain $\gamma \in D$, then:

$$e = \alpha \text{id}_M = i \text{id}_M.$$

We can check that $\Delta$ and $\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta$ are two irreducible polynomials over the field $k = \mathbb{Q}(\nu, b_1, b_2)$ and their greatest common divisor is 1. Hence, the only two remaining possibilities of (6) are:

- $\Delta$ divides $\alpha$ and $\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta$ divides $\alpha - 1$, i.e., $\alpha = \gamma \Delta$ and $\alpha = 1 + \gamma' (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$, for certain $\gamma, \gamma' \in D$, which yields

$$\gamma \Delta - \gamma' (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta) = 1,$$

and

$$\Delta (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta) = (\Delta, \partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta) \subseteq D.$$

- $\gamma'$ divides $\alpha - 1$ and $\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta$ divides $\alpha$, i.e., $\alpha = 1 + \gamma \Delta$ and $\alpha = \gamma' (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta)$, for certain $\gamma, \gamma' \in D$, which yields

$$\gamma' (\partial_t + \vec{b} \cdot \vec{\nabla} - \nu \Delta) - \gamma \Delta = 1,$$

which is impossible.

Since $\Delta$ and $\nu \Delta - \vec{b} \cdot \vec{\nabla} - c$ are irreducible over $\mathbb{Q}(\nu, b_1, b_2, c)$, non-trivial solutions of (7) are then:

- $\Delta$ divides $\alpha$ and $\nu \Delta - \vec{b} \cdot \vec{\nabla} - c$ divides $\alpha - 1$, i.e., $\alpha = \gamma \Delta$ and $\alpha = 1 + \gamma' (\nu \Delta - \vec{b} \cdot \vec{\nabla} - c)$, for certain $\gamma, \gamma' \in D$, which yields:

$$\gamma \Delta + \gamma' (\nu \Delta - \vec{b} \cdot \vec{\nabla} - c) = 1.$$
\( \gamma' = 1/c \) which yields \( \gamma' b_i = b_i / c = 0 \), i.e., \( b_1 = 0 \), for \( i = 1, 2 \). Hence, if \( b_1 = b_2 = 0 \), then \( \alpha = (\nu/c) \Delta \) is a non-trivial solution of (7) and \( e = (\nu/c) \Delta \text{id}_M \) is a non-trivial idempotent of \( E \) and \( M \) is a decomposable \( D \)-module.

- \( \Delta \) divides \( \alpha - 1 \) and \( \nu \Delta - b \cdot \nabla - c \) divides \( \alpha \), i.e., \( \alpha = 1 + \gamma \Delta \) and \( \alpha = \gamma' (\nu \Delta - b \cdot \nabla - c) \), for certain \( \gamma, \gamma' \in D \), which yields:

\[
1 + \gamma \Delta = \gamma' (\nu \Delta - b \cdot \nabla - c).
\]

In particular, we must have \( \deg \gamma = \deg \gamma' \) and

\[
(\gamma - \nu \gamma') \Delta + \gamma' \nabla + \gamma' c + 1 = 0,
\]

and thus \( \deg \gamma' = 0 \) and \( \gamma' c = -1, \gamma = (\nu \gamma)' \), \( \gamma' b_1 = 0 \) and \( \gamma' b_2 = 0 \), i.e., \( \gamma' = -1/c \), which yields \( \gamma' b_i = -b_i / c = 0 \), i.e., \( b_1 = b_2 = 0 \). Therefore, if \( b_1 = b_2 = 0 \), then \( \alpha = 1 - (\nu/c) \Delta \) and \( e = \alpha \text{id}_M \) is a non-trivial idempotent of \( M \) and \( M \) is a decomposable \( D \)-module.

Similar studies for the Euler or Maxwell equations in \( \mathbb{R}^2 \) can be performed. We summarize the main result of this section in the following theorem.

**Theorem 2**: Let \( D \) be the appropriate commutative polynomial ring of PD operators with coefficients in a field \( k \). If \( R \in D^{n \times p} \) is the system matrix of the Oseen, Euler or Maxwell equations (resp., Stokes) in \( \mathbb{R}^2 \) and \( M = D^{1 \times p}/(D^{1 \times q} R) \) the \( D \)-module finitely presented by \( R \), then the \( D \)-module \( M \) is indecomposable (resp., decomposable).

**B. Closed-form solutions of the Stokes equations in \( \mathbb{R}^2 \)**

We consider an implicit scheme of the time dependent Stokes equations, namely:

\[
\begin{align*}
-\nu \Delta \vec{u} + c \vec{u} + \nabla p &= 0, \\
\nabla \cdot \vec{u} &= 0.
\end{align*}
\]

Let \( D = \mathbb{Q}[\nu, c][\partial_x, \partial_y] \) be the commutative polynomial ring of PD operators with coefficients in the field \( \mathbb{Q}[\nu, c] \), \( R \) the presentation matrix of (8) defined by

\[
R = \begin{pmatrix}
-\nu \Delta + c & 0 & \partial_x \\
0 & -\nu \Delta + c & \partial_y \\
\partial_x & \partial_y & 0
\end{pmatrix} \in D^{3 \times 3},
\]

and \( M = D^{1 \times 3}/(D^{1 \times 3} R) \) the \( D \)-module finitely presented by \( R \). In Section III-A, we proved that \( M \) is decomposable when \( c \neq 0 \). The matrices

\[
P = Q = \left(1 - \frac{\nu}{c} \Delta \right) I_3
\]

define \( f \in \text{end}_D(M) \) by \( f(\pi(\lambda)) = \pi(\lambda P) \) for all \( \lambda \in D^{1 \times 3}, \) where \( \pi : D^{1 \times 3} \longrightarrow M \) is the canonical projection onto \( M \). Then, using results obtained in [4], we get

\[
f(\text{coim} f) = D^{1 \times 3}/(D^{1 \times 4} S), \quad \text{where}
\]

\[
S = \begin{pmatrix}
-c \partial_y & c \partial_x & 0 \\
\nu c \partial_x \partial_y & c (\nu \partial_y - c) & -c \partial_y \\
-c \partial_x & -c \partial_y & 0 \\
(c (\nu \Delta - c) & 0 & -c \partial_x)
\end{pmatrix},
\]

and \( \ker f \cong D^{1 \times 4}/(D^{1 \times 4} (L^T S_2^T) \tau) \), where

\[
L = \frac{1}{c} \begin{pmatrix}
\nu \partial_x & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

and \( S_2 = (\nu \partial_y^2 - c - \partial_x \nu \partial_y) \) (see [4]). Using a Gröbner basis computation, we get:

\[
S \zeta = 0 \iff \begin{cases}
\Delta \zeta_3 = 0, \\
\nu \zeta_2 + \partial_y \zeta_3 = 0,
\end{cases}
\]

Moreover, we have:

\[
\begin{pmatrix} L \\ S_2 \end{pmatrix} \tau = 0 \iff \begin{cases}
\tau_2 = -\nu \partial_x \tau_1, \\
\tau_1 = 0, \\
\tau_4 = 0, \\
(\nu \Delta - c) \tau_1 = 0.
\end{cases}
\]

Finally, we have

\[
U_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\partial_x & \partial_y & -c \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
U_2 = \frac{1}{c} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
V = -\frac{1}{c^2} \begin{pmatrix}
\nu \partial_y & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\nu \partial_x \partial_y & \nu \partial_y & 0 & 0 \\
\nu \partial_x \partial_y & \nu \partial_y & 0 & 0
\end{pmatrix},
\]

i.e., \( M \cong \ker f \oplus \text{coim} f \) ([4]). We note that \( U_1, U_2 \) and \( V \) can be computed using the package OreModules ([3]). Then, we get

\[
\begin{pmatrix}
u_1 \\ u_2 \end{pmatrix} = \zeta + V \tau, \quad \text{i.e.,}
\]

\[
\begin{pmatrix}
u_1 \\ u_2 \end{pmatrix} = \frac{1}{c} \begin{pmatrix}
\partial_x \zeta_3 + \nu \zeta_1 \\
\partial_y \zeta_3 + \nu \zeta_1
\end{pmatrix},
\]

\[
\begin{pmatrix} p \\ \zeta_3 \end{pmatrix} = \frac{1}{c} \begin{pmatrix}
\partial_x \zeta_3 + \nu \zeta_1 \\
\partial_y \zeta_3 + \nu \zeta_1
\end{pmatrix}.
\]
where $\zeta_3$ (resp., $\tau_1$) satisfies the PD equation $\Delta \zeta_3 = 0$ (resp., $(\nu \Delta - c) \tau_1 = 0$).

C. Decomposition of the Stokes system

We now apply results developed in [4] to obtain a decomposition of the matrix $R$ defined by (9) and defining the Stokes equations (8). More precisely, if

$$GL_p(D) = \{ U \in D^{p \times p} \mid \exists V \in D^{p \times p} : UV = VU = I_p \},$$

then we compute two matrices $U \in GL_3(D)$ and $V \in GL_3(D)$ such that $\overline{R} = VRU^{-1}$ is a block-diagonal matrix. All the calculations can be performed by means of the OREMORPHISMS package ([5]) built on OREMODULES ([3]).

The isomorphism $M \cong \ker f \oplus \text{coim} f$ was proved in Section III-B. The matrices $P = Q = (1 - \frac{c}{\nu} \Delta) I_3$ define an idempotent $e \in \text{end}_D(M)$. Moreover, we have $P^2 = P + Z R$, where $Z \in D^{1\times 3}$ is defined by:

$$Z = \frac{\nu}{c^2} \begin{pmatrix} -\partial_y & \partial_x \partial y & \partial_x (\nu \Delta - c) \\ \partial_x \partial y & -\partial_y \partial_y & \partial_y (\nu \Delta - c) \\ \partial_x (\nu \Delta - c) & \partial_y (\nu \Delta - c) & (\nu \Delta - c)^2 \end{pmatrix}.$$  

Searching for solutions $\Lambda$ of the algebraic Riccati equation $\Lambda R \Lambda + (P - I_3) \lambda + \Lambda Q + Z = 0$ ([4]) of order 2, we obtain the following solution

$$\Lambda = \frac{1}{c} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & (\nu \Delta - c) \end{pmatrix},$$

which yields the following idempotent matrices

$$P = P + \Lambda R = \frac{1}{c} \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & -c \end{pmatrix},$$

$$Q = Q + R \Lambda = \frac{\nu}{c} \begin{pmatrix} \partial_y^2 & \partial_x \partial y & \partial_x (\nu \Delta - c) \\ \partial_x \partial y & \partial_x \partial_y & \partial_y (\nu \Delta - c) \\ -\partial_x & -\partial_y & -\nu \Delta + c \end{pmatrix},$$

i.e., $P^2 = P$, $Q^2 = Q$ and $R P = Q R$, which finally shows that the idempotent $e$ can be defined by means of the idempotent matrices $P$ and $Q$. For more details, see [4]. Since $P^2 = P$ and $Q^2 = Q$, the $D = Q(\nu, c)[\partial_x, \partial_y]$-modules $\ker_D(P)$, $\text{im}_D(P)$, $\ker_D(Q)$ and $\text{im}_D(Q)$ are projective (see [4]), and thus free by the Quillen-Suslin theorem ([9]). Syzygy module computations yield $\ker_D(P) = D^{1\times 3} X$, $\ker_D(Q) = D^{1\times 3} Y$, where

$$X = \begin{pmatrix} c & 0 & \partial_x \\ -\partial_y & \partial_x & 0 \\ 0 & c & \partial_y \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & \nu \partial_x \\ -\partial_y & \partial_x & 0 \\ 0 & 1 & \nu \partial_y \end{pmatrix}.$$  

$$\text{im}_D(P) = \ker_D((I_3 - P)) = D \langle 0, 0, 1 \rangle$$

and $\text{im}_D(Q) = \ker_D((I_3 - Q)) = D \langle \partial_x, \partial_y, \nu \Delta - c \rangle$. The matrix $X$ does not define a basis of $\ker_D(P)$ since $\text{rank}_D(\ker_D(P)) \leq 2$ and $X$ has three rows. A similar comment holds for $Y$ and $\ker_D(Q)$. Hence, the rows of $X$ and $Y$ are not $D$-linearly independent, i.e.:

$$\left\{ \begin{array}{l} \ker_D(X) = D \langle -\partial_y - c, \partial_x \rangle, \\ \ker_D(Y) = D \langle -\partial_y - 1, \partial_x \rangle. \end{array} \right.$$  

Hence, if $X_i$ denotes the $i$th row of $X$, then we have

$$\left\{ \begin{array}{l} c X_2 \beta = -\partial_y X_1 \beta + \partial_x X_3 \beta, \\ Y_2 \beta = -\partial_y Y_1 \beta + \partial_x Y_3 \beta. \end{array} \right.$$  

Consequently, a basis of $\ker_D(P)$ (resp., $\ker_D(Q)$) is defined by the first and third rows of $X$ (resp., $Y$), i.e., $\ker_D(P) = D^{1\times 2} U_1$, $\ker_D(Q) = D^{1\times 2} V_1$, where:

$$U_1 = \begin{pmatrix} c & 0 & \partial_x \\ 0 & c & \partial_y \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & 0 & \nu \partial_x \\ 0 & 1 & \nu \partial_y \end{pmatrix}.$$  

Thus, using [4], the matrix $R$ is equivalent to:

$$\overline{R} = V R U^{-1} = \begin{pmatrix} -\frac{\nu \partial_y^2 - c}{c} & \frac{\nu \partial_x \partial_y}{c} & 0 \\ \frac{\nu \partial_x \partial_y}{c} & -\frac{\nu \partial_y^2 - c}{c} & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$  

Now, applying Algorithm 1 to the $D$-module $O = D^{1\times 2}/(D^{1\times 2} T)$ finitely presented by the first diagonal block $T$ of $\overline{R}$, we obtain that $\text{end}_D(O)$ is finitely generated by $\{ g_i \}_{i=1, \ldots, 4}$, where the $g_i$’s are defined by
The previous D-linear relations show that end\(_D(O)\) is a cyclic D-module generated by \(g_2 = \text{id}_O\) since we have 
\[g_1 = (\nu \partial_y^2 - c) g_2,\] 
\[g_3 = \nu \partial_y g_2\] 
and \(g_4 = -c \partial_x g_2\), where \(g_2\) satisfies \((\nu \Delta - c) g_2 = 0\), i.e.:
\[\text{end}_D(O) = D g_2 \cong D/(\nu \Delta - c).\]

Hence, if \(\alpha \in D\), then \(e = \alpha g_2\) is an idempotent of \(\text{end}_D(O)\) iff \(e^2 = e\), i.e., iff there exists \(\beta \in D\) such that \((\alpha - 1) = \beta (\nu \Delta - c)\). Since \(\nu \Delta - c\) is an irreducible polynomial (see Section III-A) and \(\alpha\) and \(\alpha - 1\) do not admit a common factor since \(\alpha + (\alpha - 1) = 1\), then \(\nu \Delta - c\) either divides \(\alpha\) or \(\alpha - 1\), i.e., \(\alpha = \gamma (\nu \Delta - c)\) or \(\alpha = 1 + \gamma (\nu \Delta - c)\), for a certain \(\gamma \in D\), which shows that we either have \(e = \gamma (\nu \Delta - c) g_2 = 0\) or \(e = (1 + \gamma (\nu \Delta - c)) g_2 = g_2\). Therefore, \(\text{end}_D(O)\) only admits the trivial idempotents 0 and \(\text{id}_O\), which finally proves that \(O\) is an indecomposable D-module and \(T\) is not equivalent to a diagonal matrix over \(D\).

IV. Factorization problems

Finally, we investigate the possible factorizations of the form \(R = L S\) of the linear PD systems considered above. The existence of such a factorization is equivalent to the existence of a non-injective element in \(\text{end}_D(M)\) (see [4]). Now, \(f = d \text{id}_M\) defines a non-injective D-endomorphism iff there exists \(0 \neq m \in M\) such that \(f(m) = d m = 0\), i.e., iff \(m\) is a non-trivial torsion element of \(M\) and \(d \in \text{ann}_D(\text{id}_M)\). Consequently, computing a primary decomposition \(\text{ann}_D(\text{id}_M)\), i.e., finding the possible factors of a generator of \(\text{ann}_D(\text{id}_M)\), we get all the possible factorizations. For instance, if we consider again the Oseen equations given by (5), then the endomorphisms of \(M\) defined by \(f_1 = \Delta \text{id}_M\) and \(f_2 = (\partial_t + \bar{b} \cdot \bar{\nabla} - \nu \Delta)\text{id}_M\) are not injective since \((\partial_t + \bar{b} \cdot \bar{\nabla} - \nu \Delta) f_1 = 0\) and \(\Delta f_2 = 0\) and thus \(f_1((\partial_t + \bar{b} \cdot \bar{\nabla} - \nu \Delta) m) = 0\) and \(f_2(\Delta m) = 0\) for all \(m \in M\). Then, \(R\) admits the two non-trivial factorizations \(R = L_1 S_1\) and \(R = L_2 S_2\) where

\[
L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad 
S_1 = \begin{pmatrix} \nu \partial_x & \nu \partial_y & -1 \\ \partial_x & \partial_y & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
L_2 = \begin{pmatrix} b_1 - \nu \partial_x & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad 
S_2 = \begin{pmatrix} \nu \partial_y & \nu \partial_y & \nu \partial_y & 0 \\ -\partial_y & \partial_y & -\partial_y & -\partial_y \\ -\partial_x & -\partial_x & \nu \Delta - \partial_t \cdot \bar{b} \cdot \bar{\nabla} & -\partial_x \end{pmatrix}.
\]

In particular, the matrices \(S_1\) and \(S_2\) are such that \(\text{coim} f_i = D^{1\times 3}/(D^{1\times 3} S_i)\) for \(i = 1, 2\). They can be computed using the package OREMORPHISMS ([5]) and then, \(L_1\) and \(L_2\) are easily obtained by means of a factorization which can be computed by the OREMORPHISMS package ([3]). Similar factorizations for the Stokes, Euler or Maxwell equations in \(\mathbb{R}^2\) can be obtained.

REFERENCES