Module structure of classical multidimensional systems appearing in mathematical physics

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***Abstract*— In this paper, within the constructive algebraic analysis approach to linear systems, we study classical linear systems of partial differential (PD) equations in two or three independent variables with constant coefficients appearing in mathematical physics and engineering sciences such as the Stokes and Oseen equations studied in hydrodynamics. We first provide a precise algebraic description of the endomorphism ring of the left D-module associated with a linear PD system. Then, we use it to prove that the endomorphism ring of the Stokes and Oseen equations in $\mathbb{R}^2$ is a cyclic D-module, which allows us to conclude about the decomposition and factorization properties of these linear PD systems.**

I. Introduction

Within the constructive algebraic analysis approach to linear systems theory developed in [2], [4], we study linear systems of partial differential (PD) equations in two independent variables with constant coefficients classically encountered in engineering sciences and mathematical physics (e.g., Stokes/Oseen/Euler/Maxwell equations). Since these linear PD systems can be written as $R\,\eta = 0$, where $R \in D^{q\times p}$ is a $q \times p$ matrix with entries in the commutative polynomial ring $D = k[\partial_x, \partial_y]$ or $D = k[\partial_t, \partial_x, \partial_y]$ of PD operators in $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_t = \frac{\partial}{\partial t}$ with coefficients in a field k, they can be interpreted as linear 2-dimensional systems. As explained in [2], a linear PD system can be studied by means of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$, finitely presented by the matrix R of PD operators, and its dual $\hom_D(M, \mathcal{F})$ ([9]), where $\mathcal{F}$ is a left D-module. Indeed, according to Malgrange's remark ([8]), the linear PD system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$, also called behaviour, satisfies $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$. Hence, the linear PD system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the two D-modules M and $\mathcal{F}$. In this paper, we shall study properties of $\ker_{\mathcal{F}}(R.)$ inherited from the algebraic properties of the D-module M, especially those of the endomorphism ring $\mathrm{end}_D(M) = \hom_D(M, M)$. We focus on the factorization and decomposition problems for these classical linear PD systems and particularly the Stokes and Oseen equations. The Euler and Maxwell equations ([6]) can be studied similarly.

II. Endomorphism ring

A. *Characterization of* $\mathrm{end}_D(M)$

We shall give a precise description of the endomorphism ring $\mathrm{end}_D(M)$ of M in terms of generators, relations and multiplication tables. An algorithm to compute such a representation is given. As shown in [4], the existence of a D-endomorphism $f : M \longrightarrow M$ is equivalent to the existence of two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ such that $R\,P = Q\,R$. Moreover, we have $f(\pi(\lambda)) = \pi(\lambda\,P)$ for all $\lambda \in D^{1\times p}$. Such an endomorphism is defined up to homotopy equivalence. Thus, the ring $\mathrm{end}_D(M)$ can be written as the quotient of two left D-modules $\mathrm{end}_D(M) \cong A/(D^{p\times q}\,R)$, where $A = \{P \in D^{p\times p} \mid \exists\, Q \in D^{q\times q} : R\,P = Q\,R\}$. Let us first compute a family of generators of A. Let $E \in D^{q\times p}$ and $F \in D^{r\times s}$ be two matrices with entries in a ring D. Then, the *Kronecker product* of E and F, denoted by $E \otimes F$, is the matrix defined by:

$$E \otimes F = \begin{pmatrix} E_{11}\,F & \dots & E_{1p}\,F \\ \vdots & & \vdots \\ E_{q1}\,F & \dots & E_{qp}\,F \end{pmatrix} \in D^{(q\,r)\times(p\,s)}.$$

If D is a commutative ring, $E \in D^{r\times q}$, $F \in D^{q\times p}$ and $G \in D^{p\times m}$ and $\mathrm{row}(F) \in D^{1\times q\,p}$ denotes the row vector obtained by concatenating the rows of F, then the product of the three matrices can be obtained by:

$$\mathrm{row}(E\,F\,G) = \mathrm{row}(F)\,(E^T \otimes G).$$

Consequently, we have

$$\begin{cases} \mathrm{row}(R\,P) = \mathrm{row}(R\,P\,I_p) = \mathrm{row}(P)\,(R^T \otimes I_p), \\ \mathrm{row}(Q\,R) = \mathrm{row}(I_q\,Q\,R) = \mathrm{row}(Q)\,(I_q \otimes R), \end{cases}$$

so that

$$R\,P = Q\,R \;\Leftrightarrow\; (\mathrm{row}(P) \quad -\mathrm{row}(Q))\;L = 0,$$

where the matrix L is defined by:

$$L = \begin{pmatrix} R^T \otimes I_p \\ I_q \otimes R \end{pmatrix} \in D^{(p^2+q^2)\times q\,p}.$$

Now, there exists a matrix $L_2 \in D^{s\times(p^2+q^2)}$ such that $\ker_D(.L) = D^{1\times s}\,L_2$. Stacking the rows of L_2, we find a set of matrices $\{P_i\}_{i=1,\ldots,s}$ and $\{Q_i\}_{i=1,\ldots,s}$, where $P_i \in D^{p\times p}$ and $Q_i \in D^{q\times q}$, satisfying the relation $R\,P_i = Q_i\,R$ for $i = 1,\ldots,s$. Moreover, we can easily check that every solution $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ of $R\,P = Q\,R$ has the form

$$\begin{cases} P = \sum_{i=1}^s \alpha_i\,P_i, \\ Q = \sum_{i=1}^s \alpha_i\,Q_i, \end{cases}$$

where the α_i's are arbitrary elements of D, $i = 1,\ldots,s$, i.e., $\{P_i\}_{i=1,\ldots,s}$ is a set of generators of the left D-module A. Therefore, the set $\{\overline{P}_i\}_{i=1,\ldots,s}$ of the residue classes of the matrices P_i's in the D-module $A/(D^{p\times q}\,R) \cong \mathrm{end}_D(M)$ generates $A/(D^{p\times q}\,R)$, i.e., $\mathrm{end}_D(M)$ up to isomorphism. If we consider

$$\forall\, i = 1,\ldots,s, \quad \begin{cases} \overline{P}_i = P_i + Z_i\,R, \\ \overline{Q}_i = Q_i + R\,Z_i, \end{cases}$$

for certain matrices $Z_i \in D^{p\times q}$, then $\overline{P_i}$ and $\overline{Q_i}$ satisfy the relation $R\,\overline{P}_i = \overline{Q}_i\,R$ for $i = 1,\ldots,s$, i.e., they induce $f_i \in \mathrm{end}_D(M)$ defined by:

$$\forall\,\lambda \in D^{1\times p}, \quad f_i(\pi(\lambda)) = \pi(\lambda\,\overline{P}_i), \quad i = 1,\ldots,s.$$

Then, $\{f_i\}_{i=1,\ldots,s}$ is a family of generators of $\mathrm{end}_D(M)$. A D-linear relation $\sum_{j=1}^s d_j\,f_j = 0$ between the f_i's is equivalent to the existence of $Z \in D^{p\times q}$ satisfying $\sum_{j=1}^s d_j\,\overline{P}_j = Z\,R$, i.e.:

$$\sum_{j=1}^s d_j\,\mathrm{row}(\overline{P}_j) - \mathrm{row}(Z)\,(I_p \otimes R) = 0$$

$$\Leftrightarrow \quad (d_1\ \ldots\ d_s\ \ -\mathrm{row}(Z)) \begin{pmatrix} \mathrm{row}(\overline{P}_1) \\ \vdots \\ \mathrm{row}(\overline{P}_s) \\ I_p \otimes R \end{pmatrix} = 0.$$

If we introduce the matrices

$$\begin{cases} U = \left(\mathrm{row}(\overline{P}_1)^T\ \ldots\ \mathrm{row}(\overline{P}_s)^T\right)^T \in D^{s\times p^2}, \\ V = I_p \otimes R \in D^{p\,q\times p^2}, \\ W = (U^T \quad V^T)^T \in D^{(s+p\,q)\times p^2}, \end{cases} \tag{1}$$

then there exist $X \in D^{l\times s}$ and $Y \in D^{l\times p\,q}$ satisfying $\ker_D(.W) = D^{1\times l}\,(X \quad -Y)$. If $Y_{i,j}$ denotes the $i\times j$ entry of the matrix Y and for $i = 1,\ldots,l$,

$$Z_i = \begin{pmatrix} Y_{i,1} & \ldots & Y_{i,q} \\ Y_{i,(q+1)} & \ldots & Y_{i,2\,q} \\ \vdots & & \vdots \\ Y_{i,(p-1)\,q+1} & \ldots & Y_{i,p\,q} \end{pmatrix} \in D^{p\times q},$$

then $\sum_{j=1}^s X_{ij}\,\overline{P}_j = Z_i\,R$, and thus the f_i's satisfy the following D-linear relations:

$$\sum_{j=1}^s X_{ij}\,f_j = 0, \quad i = 1,\ldots,l. \tag{2}$$

Hence, $\mathrm{end}_D(M) \cong D^{1\times s}/(D^{1\times l}\,X)$, i.e., $\mathrm{end}_D(M)$ is finitely presented by $X \in D^{l\times s}$.

Now, we note that

$$A = \{P \in D^{p\times p} \mid \exists\, Q \in D^{q\times q} : R\,P = Q\,R\}$$

is also a ring. Indeed, $0 \in A$, $I_p \in A$ and if $P_1,\,P_2 \in A$, i.e., $R\,P_1 = Q_1\,R$ and $R\,P_2 = Q_2\,R$ for certain matrices $Q_1,\,Q_2 \in D^{q\times q}$, then we have

$$\begin{cases} R\,(P_1 + P_2) = (Q_1 + Q_2)\,R, \\ R\,(P_1\,P_2) = (Q_1\,Q_2)\,R, \end{cases}$$

so that $P_1 + P_2 \in A$ and $P_1\,P_2 \in A$. The other properties of a ring can easily be checked. The ring A is generally a noncommutative ring since $P_1\,P_2$ is generally different from $P_2\,P_1$. Moreover, $I \triangleq D^{p\times q}\,R$ is a two-sided ideal of A. Indeed, if $P_1,\,P_2 \in A$ and $Z_1\,R,\,Z_2\,R \in I$, where $Z_i \in D^{p\times q}$ for $i = 1,\,2$, then:

$$\begin{cases} P_1\,(Z_1\,R) + P_2\,(Z_2\,R) = (P_1\,Z_1 + P_2\,Z_2)\,R, \\ (Z_1\,R)\,P_1 + (Z_2\,R)\,P_2 = (Z_1\,Q_1 + Z_2\,Q_2)\,R. \end{cases}$$

Hence, $B = A/I$ is generally a noncommutative ring and $\kappa = \mathrm{id}_p \otimes \pi : A \longrightarrow B$ is the canonical projection onto B. In particular, the product of B is defined by:

$$\forall\, P_1,\,P_2 \in A, \quad \kappa(P_1)\,\kappa(P_2) = \kappa(P_1\,P_2).$$

We call *opposite ring* of B, denoted by B^{op}, the ring defined by B as an abelian group but equipped with the *opposite multiplication* defined by:

$$\forall\, a,\,b \in A, \quad b \bullet a = a\,b.$$

If $\phi : B \longrightarrow \mathrm{end}_D(M)$ is the abelian group isomorphism defined by

$$\forall\,\lambda \in D^{1\times p}, \quad \phi(\kappa(P))(\pi(\lambda)) = \pi(\lambda\,P),$$

and $f_1 = \phi(\kappa(P_1))$ and $f_2 = \phi(\kappa(P_2))$, then

$$\forall\, \lambda \in D^{1\times p}, \quad (f_2 \circ f_1)(\pi(\lambda)) = \phi(\kappa(P_1)\,\kappa(P_2))(\pi(\lambda)),$$

i.e., using the *opposite ring* $B^{\rm op}$, we obtain:

$$\begin{aligned}\phi(\kappa(P_2) \bullet \kappa(P_1)) &= \phi(\kappa(P_1)\,\kappa(P_2)) \\ &= \phi(\kappa(P_2)) \circ \phi(\kappa(P_1)).\end{aligned}$$

Moreover, $\phi(\kappa(I_p)) = {\rm id}_M$, which proves that ϕ is a ring isomorphism, i.e., ${\rm end}_D(M) \cong B^{\rm op}$.

The ring structure of ${\rm end}_D(M)$ can be characterized by the knowledge of the expansions of the $f_i \circ f_j$'s in the family of generators $\{f_k\}_{k=1,\ldots,s}$ for $i, j = 1, \ldots, s$:

$$\forall\, i, j = 1, \ldots, s, \quad f_i \circ f_j = \sum_{k=1}^{s} \gamma_{ijk}\, f_k,\ \gamma_{ijk} \in D. \tag{3}$$

The γ_{ijk}'s look like the *structure constants* appearing in the theory of finite-dimensional algebras. Hence, if $F = (f_1 \ \ldots \ f_s)^T$, then the matrix Γ formed by the γ_{ijk} satisfies $F \otimes F = \Gamma\, F$. Γ is called a *multiplication table* in group theory. Finally, if $D\langle f_1, \ldots f_s\rangle$ is the free associative D-algebra generated by the f_i's and

$$\begin{aligned} I = \langle\ & \textstyle\sum_{j=1}^{s} X_{ij}\, f_j,\ i = 1, \ldots, l, \\ & f_i \circ f_j - \textstyle\sum_{k=1}^{s} \gamma_{ijk}\, f_k,\ i, j = 1, \ldots, s\ \rangle \end{aligned}$$

is the two-sided ideal of D generated by the polynomials corresponding to the identities (2) and (3), then the noncommutative ring ${\rm end}_D(M)$ is defined by

$${\rm end}_D(M) = D\langle f_1, \ldots f_s\rangle / I, \tag{4}$$

which shows that ${\rm end}_D(M)$ can be defined as the quotient of a free associative algebra by a two-sided ideal generated by linear and quadratic relations.

Algorithm 1:
- **Input:** A matrix $R \in D^{q\times p}$ defined over a commutative polynomial ring D over a computational field k.
- **Output:** A finite family of generators $\{f_1, \ldots, f_s\}$ of the endomorphism ring ${\rm end}_D(M)$ of the D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ and a set of D-linear relations of the f_i's generating the D-module structure of ${\rm end}_D(M)$.

1) Compute $L = \begin{pmatrix} R^T \otimes I_p \\ I_q \otimes R \end{pmatrix}$.
2) Compute $L_2 \in D^{s\times(p^2+q^2)}$ satisfying:
$$\ker_D(.L) = D^{1\times s}\, L_2.$$
3) Construct $P_i \in D^{p\times p}$ and $Q_i \in D^{q\times q}$ defined by
$$\begin{aligned} P_i(j,k) &= L(i, (j-1)\,p + k), \\ Q_i(l,m) &= -L(i, p^2 + (l-1)\,q + m), \end{aligned}$$
for $j = 1, \ldots, p$, $\quad k = 1, \ldots, p$, $l = 1, \ldots, q$ and $m = 1, \ldots, q$, where $L(i,j)$ denotes the $i\times j$ entry of the matrix L, for $i = 1, \ldots, s$. Then, we have:
$$\forall\, i = 1, \ldots, s, \quad R\, P_i = Q_i\, R.$$
4) Compute a Gröbner basis G of the rows of R for a total degree order.
5) For $i = 1, \ldots, s$, reduce the rows of P_i with respect to G by computing their normal forms with respect to G. We obtain the matrices $\overline{P}_i$ which satisfy $\overline{P}_i = P_i + Z_i\, R$, for certain matrices $Z_i \in D^{p\times q'}$ which can be obtained by means of factorizations.
6) For $i = 1, \ldots, s$, define the following matrices $\overline{Q}_i = Q_i + R\, Z_i$. The pair $(\overline{P_i}, \overline{Q_i})$ then satisfies the relation $R\,\overline{P}_i = \overline{Q_i}\, R$ and the D-module ${\rm end}_D(M)$ is finitely generated by $\{f_i\}_{i=1,\ldots,s}$, where $f_i \in {\rm end}_D(M)$ is defined by:
$$\forall\, \lambda \in D^{1\times p}, \quad f_i(\pi(\lambda)) = \pi(\lambda\, \overline{P}_i).$$
7) Form the matrices U, V and W defined by (1).
8) By means of syzygies computations, compute $(X \quad -Y)$, where $X \in D^{l\times s}$ and $Y \in D^{l\times p\,q'}$, such that $\ker_D(.W) = D^{1\times l}\,(X \quad -Y)$. Then, the family of generators $\{f_i\}_{i=1,\ldots,s}$ of the D-module ${\rm end}_D(M)$ satisfies the D-linear relations $X\, F = 0$, where $F = (f_1 \ \ldots \ f_s)^T$, i.e.:
$${\rm end}_D(M) \cong D^{1\times s}/(D^{1\times l}\, X).$$
9) For $i, j = 1, \ldots, s$, compute the normal form of ${\rm row}(\overline{P}_i\, \overline{P}_j)$ with respect to a Gröbner basis of the D-module $D^{1\times(s+p\,q)}\, W$. With these normal forms, form the matrix $(\Gamma_1 \quad \Gamma_2) \in D^{s^2\times(s^2+p\,q)}$, where $\Gamma_1 \in D^{s^2\times s}$ and $\Gamma_2 \in D^{s^2\times p\,q}$. Then, the matrix Γ_1 defines the multiplication table of family of generators $\{f_i\}_{i=1,\ldots,s}$ of ${\rm end}_D(M)$.

Algorithm 1 is implemented in the OREMORPHISMS package ([5]) built upon OREMODULES ([3]).

B. *Cyclic D-module ${\rm end}_D(M)$*

We recall that a D-module M is called *cyclic* if there exists $m \in M$ which generates M as a D-module, i.e.:

$$M = D\, m = \{d\, m \mid d \in D\}.$$

Let us consider the Oseen equations in $\mathbb{R}^2$ defined by

$$\begin{cases} \partial_t\, u - \nu\, \Delta\, \vec{u} + (\vec{b}\,.\,\vec{\nabla})\,\vec{u} + \vec{\nabla}\, p = 0, \\ \vec{\nabla}\,.\,\vec{u} = 0, \end{cases} \tag{5}$$

where $\vec{u}$ denotes the velocity, p the pressure, ν the viscosity, $\vec{b} = (b_1 \quad b_2)^T$ a steady velocity, $\vec{\nabla} =$

$(\partial_x \quad \partial_y)^T$, $\Delta = \partial_x^2 + \partial_y^2$ the Laplacian operator in $\mathbb{R}^2$. The Oseen equations describe the flow of a viscous and incompressible fluid at small Reynolds numbers (linearization of the incompressible Navier-Stokes equations at a steady state) ([7]). Let $D = \mathbb{Q}(\nu, b_1, b_2)[\partial_t, \partial_x, \partial_y]$ be the commutative polynomial ring of PD operators with coefficients in the field $\mathbb{Q}(\nu, b_1, b_2)$,

$$R = \begin{pmatrix} L & 0 & \partial_x \\ 0 & L & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \in D^{3\times 3},$$

where $L = \partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$, $M = D^{1\times 3}/(D^{1\times 3}\,R)$ the D-module finitely presented by R and $\pi : D^{1\times 3} \longrightarrow M$ the canonical projection onto M. Using Algorithm 1, we find that the endomorphism ring $\mathrm{end}_D(M)$ of M is defined by the family of generators $\{f_i\}_{i=1,\ldots,5}$, where $f_i(\pi(\lambda)) = \pi(\lambda\,P_i)$ for all $\lambda \in D^{1\times 3}$ and:

$$P_1 = I_3, \quad P_2 = \begin{pmatrix} 0 & -\partial_y & 0 \\ 0 & \partial_x & 0 \\ 0 & 0 & \partial_x \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & -(\partial_t + b_1\,\partial_x + b_2\,\partial_y) \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & \nu\,\partial_x\,\partial_y & 0 \\ 0 & -(\partial_t + b_1\,\partial_x + b_2\,\partial_y + \nu\,\partial_y^2) & -\partial_y \\ 0 & 0 & \nu\,\partial_y^2 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 0 & \partial_y\,(\partial_t + b_2\,\partial_y - \nu\,\partial_y^2) & \partial_y^2 \\ 0 & -\partial_x\,(\partial_t + b_2\,\partial_y - \nu\,\partial_y^2) & -\partial_x\,\partial_y \\ 0 & 0 & \partial_y^2\,(\nu\,\partial_x - b_1) \end{pmatrix}.$$

The generators f_i's of $\mathrm{end}_D(M)$ satisfy D-linear relations. Using Algorithm 1, we obtain that a generating set of D-linear relations between the generators f_i's of $\mathrm{end}_D(M)$ is defined by $L\,(f_1 \ \ldots \ f_5)^T = 0$, where

$$L = \begin{pmatrix} \partial_x & -1 & 0 & 0 & 0 \\ a & -b_1 & -1 & -1 & 0 \\ 0 & -\nu\,\partial_x & 0 & -1 & 0 \\ 0 & b & -\nu\,\partial_x & -b_1 & -\nu \\ 0 & -\nu^2\,\partial_x\,\partial_y^2 & -\nu\,\partial_y^2 & -(\partial_t + b_2\,\partial_y) & \nu\,\partial_x \\ 0 & 0 & 0 & \nu\,\partial_x - b_1 & -\nu \end{pmatrix},$$

and:

$$\begin{cases} a = -\partial_t - b_2\,\partial_y + \nu\,\partial_y^2, \\ b = -\nu\,(\partial_t + b_1\,\partial_x + b_2\,\partial_y - \nu\,\partial_y^2). \end{cases}$$

Now, following an idea developed for Serre's reduction (see [1]), if we set $\Lambda = (1 \quad 0 \ \ldots \ 0) \in D^{1\times 5}$ and $P = (L^T \quad \Lambda^T)^T \in D^{7\times 5}$, then we can check that P admits a left-inverse S over D, which implies $D^{1\times 5}/(D^{1\times 7}\,P) = 0$. The matrix S can be computed using the package OreModules ([3]). Since $D^{1\times 6}\,L \subseteq D^{1\times 7}\,P$, the classical third isomorphism theorem in module theory (see, e.g., [9]) yields the short exact sequence

$$0 \longrightarrow (D^{1\times 7}\,P)/(D^{1\times 6}\,L) \longrightarrow D^{1\times 5}/(D^{1\times 6}\,L) \\ \longrightarrow D^{1\times 5}/(D^{1\times 7}\,P) = 0,$$

i.e., $D^{1\times 5}/(D^{1\times 6}\,L) = (D^{1\times 7}\,P)/(D^{1\times 6}\,L)$, which proves that the D-module $N = D^{1\times 5}/(D^{1\times 6}\,L)$ of the D-linear relations between the generators $\{f_i\}_{i=1,\ldots,5}$ of $\mathrm{end}_D(M)$ is cyclic and is generated by the residue class of Λ in N. If $S = (S_1 \quad S_2)$, $S_1 \in D^{5\times 6}$, $S_2 = (S_{21} \ \ldots \ S_{25})^T \in D^5$, then $S_1\,L + S_2\,\Lambda = I_5$ and since $L\,f = 0$, where $f = (f_1 \ \ldots \ f_5)^T$, and $\Lambda\,f = f_1 = \mathrm{id}_M$, we obtain

$$f = S_1\,(L\,f) + S_2\,(\Lambda\,f) = S_2\,f_1,$$

$$\Leftrightarrow \quad \begin{cases} f_1 = f_1, \\ f_2 = \partial_x\,f_1, \\ f_3 = -(\partial_t + b_1\,\partial_x + b_2\,\partial_y - \nu\,\Delta)\,f_1, \\ f_4 = -\nu\,\partial_x^2\,f_1, \\ f_5 = -\partial_x^2\,(\nu\,\partial_x - b_1)\,f_1. \end{cases}$$

Therefore, for every $f \in \mathrm{end}_D(M)$, there exist $d_1, \ldots, d_5 \in D$ such that

$$f = \sum_{i=1}^{5} d_i\,f_i = \left(\sum_{i=1}^{5} d_i\,S_{2i}\right)\,f_1,$$

which shows that the endomorphism ring $\mathrm{end}_D(M)$ is generated as a D-module by $f_1 = \mathrm{id}_M$, i.e., $\mathrm{end}_D(M) = D\,f_1$ is a cyclic D-module.

The same technique can be applied to the Stokes, Euler or Maxwell equations in $\mathbb{R}^2$.

Theorem 1: Let D be the appropriate commutative polynomial ring of PD operators with coefficients in a field k. Moreover, let $R \in D^{q\times p}$ be the system matrix of the Stokes, Oseen, Euler or Maxwell equations in $\mathbb{R}^2$ and $M = D^{1\times p}/(D^{1\times q}\,R)$ the D-module finitely presented by R. Then, the endomorphism ring $\mathrm{end}_D(M)$ of M is a cyclic D-module generated by id_M.

III. Decomposition problems

A. Indecomposable D-modules

We recall that a D-module M is called *decomposable* if there exist two proper D-submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$ ([9]). If a D-module M is not decomposable, then M is said to be *indecomposable*. One can prove that the D-module M is decomposable iff the endomorphism ring $E = \mathrm{end}_D(M)$ of M admits a non-trivial *idempotent*, namely, $e \in E \setminus \{0, \mathrm{id}_M\}$ satisfying $e^2 = e$. For more details, see, e.g., [4], [9].

The goal of this section is to use Theorem 1 to conclude about the (in)decomposability of the finitely presented D-module M associated to the Stokes, Oseen, Euler or Maxwell equations, and thus of the corresponding linear PD system $\ker_{\mathcal{F}}(R.)$.

We continue with the Oseen equations studied in Section II-B. Let us determine the *annihilator* of the generator $f_1 = \mathrm{id}_M$ of $E = \mathrm{end}_D(M)$, i.e., $\mathrm{ann}_D(f_1) = \{d \in D \mid d\, f_1 = 0\}$. First, using Gröbner basis techniques, we compute $\ker_D(.P)$ and we obtain $\ker_D(.P) = D^{1\times 2}\,(T_1 \quad T_2)$, where $T_1 \in D^{2\times 6}$ and:

$$T_2 = (0 \quad -\nu^2\,\Delta\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta))^T \in D^2.$$

Moreover, we have $L = (I_6 \quad 0)\,P$, which yields ([4])

$$\begin{aligned} D^{1\times 5}/(D^{1\times 6}\,L) &= (D^{1\times 7}\,P)/(D^{1\times 6}\,L) \\ &\cong D^{1\times 7}/\left(D^{1\times 8}\begin{pmatrix} T_1 & T_2 \\ I_6 & 0 \end{pmatrix}\right) \\ &\cong D/(D^{1\times 2}\,T_2) \\ &= D/(D\,(\Delta\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta))), \end{aligned}$$

i.e., $\mathrm{ann}_D(f_1) = D\,(\Delta\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta))$, and thus:

$$E = D\,f_1 \cong D/\mathrm{ann}_D(f_1).$$

Let us study the idempotents of $E = D\,\mathrm{id}_M$. If $\alpha \in D$ then $e = \alpha\,\mathrm{id}_M \in E$ is an idempotent of E iff $e^2 - e = (\alpha^2 - \alpha)\,\mathrm{id}_M = 0$, i.e., iff there exists $\beta \in D$ such that:

$$\alpha\,(\alpha - 1) = \beta\,\Delta\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta). \qquad (6)$$

We first study two simple solutions of (6) leading to the trivial idempotents 0 and id_M of E.

- If $\Delta\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ divides α, i.e., if there exists $\gamma \in D$ such that $\alpha = \gamma\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$, then:
$$e = \alpha\,\mathrm{id}_M = 0.$$
- If $\Delta\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ divides $\alpha - 1$, i.e., $\alpha = 1 + \gamma\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ for a certain $\gamma \in D$, then:
$$e = \alpha\,\mathrm{id}_M = \mathrm{id}_M.$$

We can check that Δ and $\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$ are two irreducible polynomials over the field $k = \mathbb{Q}(\nu, b_1, b_2)$ and their greatest common divisor is 1. Hence, the only two reminding possibilities of (6) are:

- Δ divides α and $\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$ divides $\alpha - 1$, i.e., $\alpha = \gamma\,\Delta$ and $\alpha = 1 + \gamma'\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$, for certain γ, $\gamma' \in D$, which yields
$$\gamma\,\Delta - \gamma'\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) = 1,$$
which is impossible since:
$$(\Delta, \partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) = (\Delta, \partial_t + \vec{b}\,.\,\vec{\nabla}) \subsetneq D.$$
- Δ divides $\alpha - 1$ and $\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$ divides α, i.e., $\alpha = 1 + \gamma\,\Delta$ and $\alpha = \gamma'\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$, for certain γ, $\gamma' \in D$, which yields
$$\gamma'\,(\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) - \gamma\,\Delta = 1,$$
which is impossible.

The above results can be also understood as follows: if $I = (\Delta)$ and $J = (\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$, then $I\,J = I \cap J$ since $\gcd(\Delta, \partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) = 1$, I and J are two prime ideals (i.e., $I\,J = I \cap J$ is a prime decomposition) and the Chinese remainder theorem then shows that $D/(I \cap J) = D/(I\,J) \cong D/I \oplus D/J$ iff $I + J = D$ but $I + J = (\Delta, \partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) \subsetneq D$.

Over the new ring $D' = \mathbb{Q}(\nu, b_1, b_2, i)[\partial_t, \partial_x, \partial_y]$, where i denotes the complex number, we have

$$\Delta = (\partial_x + i\,\partial_y)\,(\partial_x - i\,\partial_y)$$

and $\partial_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$ is irreducible (over any algebraic closure $\overline{k}$ of $k = \mathbb{Q}(\nu, b_1, b_2)$). Then, we can similarly prove that the corresponding module is indecomposable.

Finally, within implicit schemes of the time dependent Navier-Stokes, the term $\partial_t\,u$ in (5) is replaced by $c\,u$, where the constant c corresponds to the inverse of the time step. We can redo the previous computations in this case and prove that the endomorphism ring $\mathrm{end}_E(N)$ of the corresponding $E = \mathbb{Q}(\nu, b_1, b_2, c)[\partial_x, \partial_y]$-module N is a cyclic E-module generated by id_N, which is isomorphic to the E-module $E/(\Delta\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c))$. In particular, this result is also true when $b_1 = 0$ or $b_2 = 0$. We note that the case $b_1 = b_2 = 0$ corresponds to an implicit scheme of the time dependent Stokes equations. Then, $e = \alpha\,\mathrm{id}_N \in \mathrm{end}_E(N)$, where $\alpha \in D$, is an idempotent of $\mathrm{end}_E(N)$ iff:

$$\alpha\,(\alpha - 1) = \beta\,\Delta\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c). \qquad (7)$$

Since Δ and $\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c$ are irreducible over $\mathbb{Q}(\nu, b_1, b_2, c)$, non-trivial solutions of (7) are then:

- Δ divides α and $\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c$ divides $\alpha - 1$, i.e., $\alpha = \gamma\,\Delta$ and $\alpha = 1 + \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c)$, for certain γ, $\gamma' \in D$, which yields:
$$\gamma\,\Delta = 1 + \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c).$$
In particular, we must have $\deg\gamma = \deg\gamma'$ and $(\gamma - \nu\,\gamma')\,\Delta + \gamma'\,\vec{b}\,.\,\vec{\nabla} + \gamma'\,c - 1 = 0$. Moreover, γ' must be a constant as if $\deg\gamma' > 0$, then the constant 1 cannot be cancelled. Then, we obtain $\gamma = \nu\,\gamma'$, $\gamma'\,b_1 = 0$, $\gamma'\,b_2 = 0$ and $\gamma'\,c = 1$, i.e.,

$\gamma' = 1/c$ which yields $\gamma' b_i = b_i/c' = 0$, i.e., $b_i = 0$, for $i = 1, 2$. Hence, if $b_1 = b_2 = 0$, then $\alpha = (\nu/c)\,\Delta$ is a non-trivial solution of (7) and $e = (\nu/c)\,\Delta\,\mathrm{id}_M$ is a non-trivial idempotent of E and M is a decomposable D-module.

- Δ divides $\alpha - 1$ and $\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c$ divides α, i.e., $\alpha = 1 + \gamma\,\Delta$ and $\alpha = \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c)$, for certain $\gamma, \gamma' \in D$, which yields:

$$1 + \gamma\,\Delta = \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c).$$

In particular, we must have $\deg \gamma = \deg \gamma'$ and

$$(\gamma - \nu\,\gamma')\,\Delta + \gamma'\,\vec{b}\,.\,\vec{\nabla} + \gamma'\,c + 1 = 0,$$

and thus $\deg \gamma' = 0$ and $\gamma'\,c = -1$, $\gamma = \nu\,\gamma'$, $\gamma'\,b_1 = 0$ and $\gamma'\,b_2 = 0$, i.e., $\gamma' = -1/c$, which yields $\gamma'\,b_i = -b_i/c = 0$, i.e., $b_1 = b_2 = 0$. Therefore, if $b_1 = b_2 = 0$, then $\alpha = 1 - (\nu/c)\,\Delta$ and $e = \alpha\,\mathrm{id}_M$ is a non-trivial idempotent of M and M is a decomposable D-module.

Similar studies for the Euler or Maxwell equations in $\mathbb{R}^2$ can be performed. We summarize the main result of this section in the following theorem.

Theorem 2: Let D be the appropriate commutative polynomial ring of PD operators with coefficients in a field k. If $R \in D^{q\times p}$ is the system matrix of the Oseen, Euler or Maxwell equations (resp., Stokes) in $\mathbb{R}^2$ and $M = D^{1\times p}/(D^{1\times q}\,R)$ the D-module finitely presented by R, then the D-module M is indecomposable (resp., decomposable).

B. Closed-form solutions of the Stokes equations in $\mathbb{R}^2$

We consider an implicit scheme of the time dependent *Stokes equations*, namely:

$$\begin{cases} -\nu\,\Delta\,\vec{u} + c\,\vec{u} + \vec{\nabla}\,p = 0, \\ \vec{\nabla}\,.\,\vec{u} = 0. \end{cases} \tag{8}$$

Let $D = \mathbb{Q}(\nu, c)[\partial_x, \partial_y]$ be the commutative polynomial ring of PD operators with coefficients in the field $\mathbb{Q}(\nu, c)$, R the presentation matrix of (8) defined by

$$R = \begin{pmatrix} -\nu\,\Delta + c & 0 & \partial_x \\ 0 & -\nu\,\Delta + c & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \in D^{3\times 3}, \tag{9}$$

and $M = D^{1\times 3}/(D^{1\times 3}\,R)$ the D-module finitely presented by R. In Section III-A, we proved that M is decomposable when $c \neq 0$. The matrices

$$P = Q = \left(1 - \frac{\nu}{c}\,\Delta\right) I_3$$

define $f \in \mathrm{end}_D(M)$ by $f(\pi(\lambda)) = \pi(\lambda\,P)$ for all $\lambda \in D^{1\times 3}$, where $\pi : D^{1\times 3} \longrightarrow M$ is the canonical projection onto M. Then, using results obtained in [4], we get $\mathrm{coim}\,f = D^{1\times 3}/(D^{1\times 4}\,S)$, where

$$S = \begin{pmatrix} -c\,\partial_y & c\,\partial_x & 0 \\ \nu\,c\,\partial_x\,\partial_y & c\,(\nu\,\partial_y^2 - c) & -c\,\partial_y \\ -c\,\partial_x & -c\,\partial_y & 0 \\ c\,(\nu\,\Delta - c) & 0 & -c\,\partial_x \end{pmatrix},$$

and $\ker f \cong D^{1\times 4}/(D^{1\times 4}\,(L^T \quad S_2^T)^T)$, where

$$L = -\frac{1}{c}\begin{pmatrix} 0 & 0 & 0 & 1 \\ \nu\,\partial_x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and $S_2 = (\nu\,\partial_y^2 - c \quad -\partial_x \quad 0 \quad \partial_y)$ (see [4]). Using a Gröbner basis computation, we get:

$$S\,\zeta = 0 \quad\Leftrightarrow\quad \begin{cases} \Delta\,\zeta_3 = 0, \\ c\,\zeta_2 + \partial_y\,\zeta_3 = 0, \\ c\,\zeta_1 + \partial_x\,\zeta_3 = 0, \end{cases}$$

$$\Leftrightarrow\quad \begin{cases} \zeta_1 = -\dfrac{1}{c}\,\partial_x\,\zeta_3, \\ \zeta_2 = -\dfrac{1}{c}\,\partial_y\,\zeta_3, \\ \zeta_3 = \zeta_3, \\ \Delta\,\zeta_3 = 0. \end{cases}$$

Moreover, we have:

$$\begin{pmatrix} L \\ S_2 \end{pmatrix} \tau = 0 \quad\Leftrightarrow\quad \begin{cases} \tau_2 = -\nu\,\partial_x\,\tau_1, \\ \tau_3 = 0, \\ \tau_4 = 0, \\ (\nu\,\Delta - c)\,\tau_1 = 0. \end{cases}$$

Finally, we have $U_1\,L + U_2\,S_2 + S\,V = I_4$, where

$$U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_x & \partial_y & -c \\ 0 & 0 & 0 \end{pmatrix}, \; U_2 = \frac{1}{c}\begin{pmatrix} -1 \\ 0 \\ 0 \\ \nu\,\partial_y \end{pmatrix},$$

$$V = -\frac{1}{c^2}\begin{pmatrix} \nu\,\partial_y & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \nu^2\,\partial_x\,\partial_y & \nu\,\partial_y & 0 & \nu\,\partial_x \end{pmatrix},$$

i.e., $M \cong \ker f \oplus \mathrm{coim}\,f$ ([4]). We note that U_1, U_2 and V can be computed using the package OREMODULES ([3]). Then, we get $(u_1 \quad u_2 \quad p)^T = \zeta + V\,\tau$, i.e.,

$$\begin{cases} u_1 = -\dfrac{1}{c}\left(\partial_x\,\zeta_3 + \dfrac{\nu}{c}\,\partial_y\,\tau_1\right), \\ u_2 = \dfrac{1}{c}\left(-\partial_y\,\zeta_3 + \dfrac{\nu}{c}\,\partial_x\,\tau_1\right), \\ p = \zeta_3, \end{cases}$$

where ζ_3 (resp., τ_1) satisfies the PD equation $\Delta\,\zeta_3 = 0$ (resp., $(\nu\,\Delta - c)\,\tau_1 = 0$).

C. Decomposition of the Stokes system

We now apply results developed in [4] to obtain a decomposition of the matrix R defined by (9) and defining the Stokes equations (8). More precisely, if

$$\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\; U\,V = V\,U = I_p\},$$

then we compute two matrices $U \in \mathrm{GL}_3(D)$ and $V \in \mathrm{GL}_3(D)$ such that $\overline{R} = V\,R\,U^{-1}$ is a block-diagonal matrix. All the calculations can be performed by means of the OREMORPHISMS package ([5]) built on OREMODULES ([3]).

The isomorphism $M \cong \ker f \oplus \operatorname{coim} f$ was proved in Section III-B. The matrices $P = Q = (1 - \frac{\nu}{c}\,\Delta)\,I_3$ define an idempotent $e \in \mathrm{end}_D(M)$. Moreover, we have $P^2 = P + Z\,R$, where $Z \in D^{3\times 3}$ is defined by:

$$Z = \frac{\nu}{c^2}\begin{pmatrix} -\partial_y^2 & \partial_x\,\partial_y & \partial_x\,(\nu\,\Delta - c) \\ \partial_x\,\partial_y & -\partial_x^2 & \partial_y\,(\nu\,\Delta - c) \\ \partial_x\,(\nu\,\Delta - c) & \partial_y\,(\nu\,\Delta - c) & (\nu\,\Delta - c)^2 \end{pmatrix}.$$

Searching for solutions Λ of the *algebraic Riccati equation* $\Lambda\,R\,\Lambda + (P - I_3)\,\Lambda + \Lambda\,Q + Z = 0$ ([4]) of order 2, we obtain the following solution

$$\Lambda = \frac{1}{c}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \nu\,\partial_x & \nu\,\partial_y & \nu\,(\nu\,\Delta - c) \end{pmatrix},$$

which yields the following idempotent matrices

$$\overline{P} = P + \Lambda\,R = -\frac{1}{c}\begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & -c \end{pmatrix},$$

$$\overline{Q} = Q + R\,\Lambda = \frac{\nu}{c}\begin{pmatrix} \partial_x^2 & \partial_x\,\partial_y & \partial_x\,(\nu\,\Delta - c) \\ \partial_x\,\partial_y & \partial_y^2 & \partial_y\,(\nu\,\Delta - c) \\ -\partial_x & -\partial_y & -\nu\,\Delta + c \end{pmatrix},$$

i.e., $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$ and $R\,\overline{P} = \overline{Q}\,R$, which finally shows that the idempotent e can be defined by means of the idempotent matrices $\overline{P}$ and $\overline{Q}$. For more details, see [4]. Since $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$, the $D = \mathbb{Q}(\nu, c)[\partial_x, \partial_y]$-modules $\ker_D(.\overline{P})$, $\mathrm{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\mathrm{im}_D(.\overline{Q})$ are projective (see [4]), and thus free by the Quillen-Suslin theorem ([9]). Syzygy module computations yield $\ker_D(.\overline{P}) = D^{1\times 3}\,X$, $\ker_D(.\overline{Q}) = D^{1\times 3}\,Y$, where

$$X = \begin{pmatrix} c & 0 & \partial_x \\ -\partial_y & \partial_x & 0 \\ 0 & c & \partial_y \end{pmatrix},\; Y = \begin{pmatrix} 1 & 0 & \nu\,\partial_x \\ -\partial_y & \partial_x & 0 \\ 0 & 1 & \nu\,\partial_y \end{pmatrix},$$

$\mathrm{im}_D(.\overline{P}) = \ker_D(.(I_3 - \overline{P})) = D\,(0 \quad 0 \quad 1)$ and $\mathrm{im}_D(.\overline{Q}) = \ker_D(.(I_3 - \overline{Q})) = D\,(\partial_x \quad \partial_y \quad \nu\,\Delta - c)$. The matrix X does not define a basis of $\ker_D(.\overline{P})$ since $\mathrm{rank}_D(\ker_D(.\overline{P})) \leq 2$ and X has three rows. A similar comment holds for Y and $\ker_D(.\overline{Q})$. Hence, the rows of X and Y are not D-linearly independent, i.e.:

$$\begin{cases} \ker_D(.X) = D\,(-\partial_y \quad -c \quad \partial_x), \\ \ker_D(.Y) = D\,(-\partial_y \quad -1 \quad \partial_x). \end{cases}$$

Hence, if $X_{i\bullet}$ denotes the i^{th} row of X, then we have

$$\begin{cases} c\,X_{2\bullet} = -\partial_y\,X_{1\bullet} + \partial_x\,X_{3\bullet}, \\ Y_{2\bullet} = -\partial_y\,Y_{1\bullet} + \partial_x\,Y_{3\bullet}. \end{cases}$$

Consequently, a basis of $\ker_D(.\overline{P})$ (resp., $\ker_D(.\overline{Q})$) is defined by the first and third rows of X (resp., Y), i.e., $\ker_D(.\overline{P}) = D^{1\times 2}\,U_1$, $\ker_D(.\overline{Q}) = D^{1\times 2}\,V_1$, where:

$$\begin{cases} U_1 = \begin{pmatrix} c & 0 & \partial_x \\ 0 & c & \partial_y \end{pmatrix}, \\ V_1 = \begin{pmatrix} 1 & 0 & \nu\,\partial_x \\ 0 & 1 & \nu\,\partial_y \end{pmatrix}, \end{cases}$$

$$\Rightarrow \begin{cases} U = \begin{pmatrix} c & 0 & \partial_x \\ 0 & c & \partial_y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \\ V = \begin{pmatrix} 1 & 0 & \nu\,\partial_x \\ 0 & 1 & \nu\,\partial_y \\ \partial_x & \partial_y & \nu\,\Delta - c \end{pmatrix} \in \mathrm{GL}_3(D). \end{cases}$$

Thus, using [4], the matrix R is equivalent to:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} -\dfrac{\nu\,\partial_y^2 - c}{c} & \dfrac{\nu\,\partial_x\,\partial_y}{c} & 0 \\ \dfrac{\nu\,\partial_x\,\partial_y}{c} & -\dfrac{\nu\,\partial_x^2 - c}{c} & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$

Now, applying Algorithm 1 to the D-module $O = D^{1\times 2}/(D^{1\times 2}\,T)$ finitely presented by the first diagonal block T of $\overline{R}$, we obtain that $\mathrm{end}_D(O)$ is finitely generated by $\{g_i\}_{i=1,\ldots,4}$, where the g_i's are defined by

$g_i(\kappa(\lambda)) = \kappa(\lambda\, P_i)$ for all $\lambda \in D^{1\times 2}$ and $i = 1, \ldots, 4$,

$$P_1 = \left(\begin{array}{cc} 0 & \nu\, \partial_x\, \partial_y \\ 0 & \nu\, \partial_y^2 - c \end{array}\right), \quad P_2 = I_2,$$

$$P_3 = \left(\begin{array}{cc} 0 & -\nu\, \partial_y^2 \\ 0 & \nu\, \partial_x\, \partial_y \end{array}\right), \quad P_4 = \left(\begin{array}{cc} 0 & c\, \partial_y \\ 0 & -c\, \partial_x \end{array}\right),$$

and $\kappa : D^{1\times 2} \longrightarrow O$ is the canonical projection. Moreover, the g_i's satisfies the following D-linear relations:

$$\left(\begin{array}{cccc} -1 & \nu\, \partial_y^2 - c & 0 & 0 \\ -c & 0 & 0 & \nu\, \partial_x \\ -\partial_x & 0 & \partial_y & 1 \\ 0 & c\, \partial_x & 0 & 1 \\ 0 & 0 & c & \nu\, \partial_y \end{array}\right) \left(\begin{array}{c} g_1 \\ g_2 \\ g_3 \\ g_4 \end{array}\right) = 0.$$

The previous D-linear relations show that $\mathrm{end}_D(O)$ is a cyclic D-module generated by $g_2 = \mathrm{id}_O$ since we have $g_1 = (\nu\, \partial_y^2 - c)\, g_2$, $g_3 = \nu\, \partial_x\, \partial_y\, g_2$ and $g_4 = -c\, \partial_x\, g_2$, where g_2 satisfies $(\nu\, \Delta - c)\, g_2 = 0$, i.e.:

$$\mathrm{end}_D(O) = D\, g_2 \cong D/(\nu\, \Delta - c).$$

Hence, if $\alpha \in D$, then $e = \alpha\, g_2$ is an idempotent of $\mathrm{end}_D(O)$ iff $e^2 = e$, i.e., iff there exists $\beta \in D$ such that $\alpha\, (\alpha - 1) = \beta\, (\nu\, \Delta - c)$. Since $\nu\, \Delta - c$ is an irreducible polynomial (see Section III-A) and α and $\alpha - 1$ do not admit a common factor since $\alpha + (\alpha - 1) = 1$, then $\nu\, \Delta - c$ either divides α or $\alpha - 1$, i.e., $\alpha = \gamma\, (\nu\, \Delta - c)$ or $\alpha = 1 + \gamma\, (\nu\, \Delta - c)$, for a certain $\gamma \in D$, which shows that we either have $e = \gamma\, (\nu\, \Delta - c)\, g_2 = 0$ or $e = (1 + \gamma\, (\nu\, \Delta - c))\, g_2 = g_2$. Therefore, $\mathrm{end}_D(O)$ only admits the trivial idempotents 0 and id_O, which finally proves that O is an indecomposable D-module and T is not equivalent to a diagonal matrix over D.

IV. Factorization problems

Finally, we investigate the possible factorizations of the form $R = L\, S$ of the linear PD systems considered above. The existence of such a factorization is equivalent to the existence of a non-injective element in $\mathrm{end}_D(M)$ (see [4]). Now, $f = d\, \mathrm{id}_M$ defines a non-injective D-endomorphism iff there exists $0 \neq m \in M$ such that $f(m) = d\, m = 0$, i.e., iff m is a non-trivial torsion element of M and $d \in \mathrm{ann}_D(\mathrm{id}_M)$. Consequently, computing a primary decomposition $\mathrm{ann}_D(\mathrm{id}_M)$, i.e., finding the possible factors of a generator of $\mathrm{ann}_D(\mathrm{id}_M)$, we get all the possible factorizations. For instance, if we consider again the Oseen equations given by (5), then the endomorphisms of M defined by $f_1 = \Delta\, \mathrm{id}_M$ and $f_2 = (\partial_t + \vec{b}\,.\vec{\nabla} - \nu\, \Delta)\, \mathrm{id}_M$ are not injective since $(\partial_t + \vec{b}\,.\vec{\nabla} - \nu\, \Delta)\, f_1 = 0$ and $\Delta\, f_2 = 0$ and thus $f_1((\partial_t + \vec{b}\,.\vec{\nabla} - \nu\, \Delta)\, m) = 0$ and $f_2(\Delta\, m) = 0$ for all $m \in M$. Then, R admits the two non-trivial factorizations $R = L_1\, S_1$ and $R = L_2\, S_2$ where

$$L_1 = \left(\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right),$$

$$S_1 = \left(\begin{array}{ccc} \nu\, \partial_x & \nu\, \partial_y & -1 \\ \partial_x & \partial_y & 0 \\ \partial_t + \vec{b}\,.\vec{\nabla} - \nu\, \Delta & 0 & \partial_x \\ 0 & \partial_t + \vec{b}\,.\vec{\nabla} - \nu\, \Delta & \partial_y \end{array}\right),$$

$$L_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & -1 \\ b_1 - \nu\, \partial_x & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right),$$

$$S_2 = \left(\begin{array}{ccc} -\partial_y & \partial_x & 0 \\ (\nu\, \partial_x - b_1)\, \partial_y & -b_2\, \partial_y - \partial_t + \nu\, {\partial_y}^2 & -\partial_y \\ -\partial_x & -\partial_y & 0 \\ \nu\, \Delta - \partial_t - \vec{b}\,.\vec{\nabla} & 0 & -\partial_x \end{array}\right).$$

In particular, the matrices S_1 and S_2 are such that $\mathrm{coim}\, f_i = D^{1\times 3}/(D^{1\times 3}\, S_i)$ for $i = 1, 2$. They can be computed using the package OreMorphisms ([5]) and then, L_1 and L_2 are easily obtained by means of a factorization which can be computed by the OreModules package ([3]). Similar factorizations for the Stokes, Euler or Maxwell equations in $\mathbb{R}^2$ can be obtained.

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