A Functorial Approach to the Behaviour of Multidimensional Control Systems

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Abstract

We show how to use the extension and the torsion functors to compute the torsion submodule of a differential module associated with a multidimensional control system. In particular, we show that the concept of weak primeness of matrices corresponds to the torsion-freeness of the corresponding module and we generalize this concept to non full rank matrices.

Keywords: Multidimensional Control Systems, Weak Primeness, Controllability, Algebraic Analysis, Torsion and Extension Functors, Homological Algebra, Non-commutative Algebra.

1 Introduction

It is well known that the controllability of a linear multidimensional control system depends on an algebraic property (namely, the torsion-freeness) of a certain module $M$ associated with the system [4, 5, 6, 7]. The recent survey [15] gives different equivalent formulations of the controllability and, in particular, the equivalence obtained in [5] between the torsion-freeness of $M$ and the definition of the controllability given by Willems [13].

In this paper, we show how to use the powerful tools of homological algebra to compute the torsion submodule $t(M)$ of $M$, i.e. the non-controllable part of a control system. The results obtained show the link between the concept of the weak primeness of a matrix $R$ and the torsion part $t(M)$ of the module $M$ associated with $R$. Moreover, we show how the torsion functor allows to generalize this concept to a non full rank matrix $R$. Finally, we show that the isomorphisms $t(M) \cong \text{tor}_1^A(K/A, M) \cong \text{ext}_1^A(N, A)$, where $N$ is an important module obtained from $M$ and $A$ is any noetherian ring. This result generalizes previous results obtained by the authors for the ring $D$ of differential operators.

2 Torsion functor and weak primeness

In the course of the paper, we shall denote by $A$ a noetherian integral domain which is supposed to be either a commutative ring or a left Ore domain, namely a domain such that, for any couple $(a, b) \in A^2$, there exists a non trivial couple $(u, v) \in A^2$ such that $ua = vb$. Moreover, let $K = Q(A)$ be the quotient field of $A$. If $A$ and $B$ are two integral domains, then we shall denote by $\mathcal{B}M$ a module $M$ with a structure of left $B$-module and of right $A$-module (see [2, 6] for more details in the non-commutative case).

In the literature of robust stabilization, the concept of \textit{weak primeness} is very useful [9, 12].

\textbf{Definition 1.} Let $R$ be an $l \times m$ matrix $(l \leq m)$ with full rank and entries in $A$. $R$ is said to be weakly left-prime if any row vector $z \in K^l$ which satisfies $zR \in A^m$ belongs to $A^l$.

\textbf{Example 1.} The matrix $R = (d_1, d_2, d_3)'$ with entries in $D = \mathbb{R}[d_1, d_2, d_3]$ is not weakly left-prime because there exists $(d_1^{-1} 0 0) \notin D^3$ such that $(d_1^{-1} 0 0)R = 1 \in D$.

Let us interpret the weak primeness in terms of modules. We need a few definitions [10].

\textbf{Definition 2.} \textbullet We call torsion submodule of an $A$-module $M$, the $A$-module defined by:

$$t(M) = \{m \in M \mid \exists 0 \neq a \in A, \text{ am = 0}\}.$$ 

An $A$-module $M$ is said to be torsion-free if $t(M) = 0$ and torsion if $t(M) \neq 0$. We have the following exact sequence:

$$0 \rightarrow t(M) \rightarrow M \rightarrow M/t(M) \rightarrow 0. \quad (1)$$

\textbullet A module $M_A$ (resp. $\mathcal{B}M_A$) is flat if for every exact sequence of left $A$-module $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$, we have the following exact sequence of abelian groups (resp. left $B$-modules):

$$0 \rightarrow M \otimes A N' \rightarrow M \otimes A N \rightarrow M \otimes A N'' \rightarrow 0.$$
$K$ is a flat $A$-module, i.e. the tensor product by $K$ transforms exact sequences of left $A$-modules into exact sequences of left $K$-vector spaces and we have the exact sequence $[10]$

$$0 \rightarrow t(M) \rightarrow M \xrightarrow{\text{id}_K} K \otimes_A M \rightarrow (K/A) \otimes_A M \rightarrow 0.$$

To our knowledge, the following proposition has firstly been obtained in [9] for a general left Ore domain $A$.

**Theorem 1.** Let $M$ be the left $A$-module defined by the following free resolution:

$$0 \rightarrow A^1 \xrightarrow{\begin{array}{c} R \\ (z_1 \ldots z_l) \end{array}} A^m \xrightarrow{\pi} M \rightarrow 0,$$

Then, the following assertions are equivalent:

- $R$ is weakly left-prime,
- $t(M) = 0$, i.e., $M$ is a torsion-free $A$-module.

**Proof.** $K$ is a flat $A$-module [10], thus by tensoring (2) by $K$, we obtain the exact sequence

$$0 \rightarrow K \xrightarrow{\text{id}_K \otimes R} K^m \rightarrow K \otimes_A M \rightarrow 0,$$

where $(\text{id}_K \otimes R)(z) = zR$, $\forall z \in K_i$. Therefore, we have the commutative exact diagram (3) (see figure 1), where $\xi : (K/A)^l \rightarrow (K/A)^m$ is defined by $\xi(y) = \pi''(zR)$, where $\pi'(z) = y$, and $L = \ker(\xi)$. By the snake lemma [10], we obtain $L \cong t(M)$. But, $0 \neq L \cong t(M) \lor \exists 0 \neq y \in (K/A)^l$ such that $\xi(y) = 0 \lor \exists z \in K^l$ such that $\pi'(z) = y \neq 0$ and $\pi''(zR) = 0 \lor \exists z \in A^l$ satisfied that $zR \in A^m \Rightarrow R$ is not weakly left-prime, which proves the proposition. \qed

Let us notice that $L = \ker(\xi)$ is, in general, a non-zero $A$-module because $K/A$ is not a flat $A$-module. To study the defect of exactness that the tensor product by $(K/A)$ introduces in exact sequences, we shall need the definition of the torsion functor.

**Definition 3.** Let $\ldots \rightarrow F_2 \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \xrightarrow{\pi} M \rightarrow 0$ be a free resolution of the left $A$-module $M$ and a module $N_A$ (resp. $B$-$N_A$), then the abelian groups (resp. left $B$-modules) of homology of the complex

$$\ldots \rightarrow N \otimes_A F_2 \xrightarrow{\text{id}_N \otimes \text{id}_F} N \otimes_A F_1 \xrightarrow{\text{id}_N \otimes d_0} N \otimes_A F_0 \rightarrow 0,$$

do not depend on the choice of the free resolution of $M$ and they are called $\text{tor}_i^A(N, M)$. We have:

$$\{ \begin{array}{ll}
\text{tor}_0^A(N, M) = N \otimes_A M, \\
\text{tor}_i^A(N, M) = \ker(\text{id}_N \otimes d_i)/\text{im}(\text{id}_N \otimes d_{i+1}), \forall i \geq 1.
\end{array} \$$

**Proposition 1.** If $0 \rightarrow M' \xrightarrow{\pi'} M \xrightarrow{\pi} M'' \rightarrow 0$ is an exact sequence of left $A$-modules and a module $N_A$ (resp. $B$-$N_A$), then we have the following exact sequence of abelian groups (resp. left $B$-modules): 

$$\text{tor}_i^A(N, M') \rightarrow \text{tor}_i^A(N, M) \rightarrow \text{tor}_i^A(N, M'') \rightarrow 0.$$

In particular, a module $N_A$ is flat if for any left $A$-module $M$, we have $\text{tor}_i^A(N, M) = 0$, $\forall i \geq 1$.

Applying the previous proposition to the exact sequence (2) and $N = K/A$, we obtain that $\kappa = (\text{id}_K \otimes R)$ and $\text{tor}_i^A(K/A, M) = \ker(\text{id}_K \otimes \pi)$. We have seen in the proof of theorem 1 that $t(M) \cong \ker(\kappa)$. Thus, we deduce that $\text{tor}_i^A(K/A, M) \cong t(M)$. In fact, this result is also true if $M$ is defined by a long free resolution. See [10] for a proof of the following proposition.

**Proposition 2.** Let $M$ be a left $A$-module, then

$$t(M) \cong \text{tor}_i^A(K/A, M).$$

Hence, the concept of weak left-primeness may be extended to non full rank matrix $R$ as follows: $R$ is a generalized weak left-prime matrix if the left $A$-module $M$ defined by (2) is torsion-free, i.e. $\text{tor}_i^A(K/A, M) = 0 \equiv \ker(\text{id}_K \otimes \pi) = \text{im}(\text{id}_K \otimes \pi)$. See [9] for more details and applications to stabilization problems.

**Remark 1.** In [12], Smith shows that if $A$ is a commutative integral domain, then weakly left-primeness implies minor left-primeness and the two concepts are equivalent if $A$ is Greatest Common Divisor Domain. See [9, 12] for counter-examples of the fact that minor left-primeness does not imply in general weakly left-primeness. Hence, the concept of weakly left-primeness is the only one which is equivalent for a full rank matrix $R$, with entries in a left Ore integral domain $A$, to the torsion-freeness of the corresponding left $A$-module $M$. We thank both an anonymous referee and J. Wood to have pointed out to the authors that the concept of generalized weak left-primeness is equivalent to the one of generalized factor left-primeness in the case of a polynomial ring $A$ (see [1, 14] for more details).

**Example 2.** Let us consider the multidimensional system defined by the gradient in $\mathbb{R}^3$:

$$\begin{cases}
d_1 z = 0, \\
d_2 z = 0, \\
d_3 z = 0.
\end{cases}$$

We have the following free resolution of the $D = \mathbb{R}[d_1, d_2, d_3]$-module $M$ corresponding to (4):

$$0 \rightarrow D \xrightarrow{R_1} D^3 \xrightarrow{R_2} D^3 \xrightarrow{R_3} D \xrightarrow{\pi} M \rightarrow 0,$$

where

$$R_1 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix}.$$
and \( R_3 = (d_1 \ d_2 \ d_3) \). The matrix \( R_1 \) is not generalized weak left-prime because the following sequence
\[
(K/A)^3 \xrightarrow{\text{id}_{K/A} \otimes R_2} (K/A)^3 \xrightarrow{\text{id}_{K/A} \otimes R_1} K/A
\]
is not exact at \((K/A)^3\), i.e., \( \text{tor}^2(K/A, M) \neq 0 \). Indeed, \( y = \pi'(010) \in \text{ker}(\text{id}_{K/A} \otimes R_2) \) but \( y \notin \text{im}(\text{id}_{K/A} \otimes R_1) \) because there is no solution of the system \((t_1 \ t_2 \ t_3)R_2 = (0 \ 0 \ 0) \) in \( K^3 \). We easily see that \( y \) corresponds to the torsion element \( x \) of (4) in the isomorphism of proposition 2 (it is just an easy chase in a diagram similar to the one in the proof of theorem 1).

3 Extension functor and behavio-urs

Definition 4. \( \bullet \) Let \( \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \xrightarrow{\pi} 0 \) be a free resolution of the left \( A \)-module \( M \) and \( _A S \) (resp. \( _A S_B \)) a module, then the abelian groups (resp. the right \( B \)-modules) of cohomology of the complex
\[
\ldots \xrightarrow{d_2} \text{hom}_A(F_1, S) \xrightarrow{d_1} \text{hom}_A(F_0, S) \xrightarrow{\pi} 0,
\]
where \( d'_i(f) = f \circ d_i, \forall f \in \text{hom}_A(F_{i-1}, S), \) do not depend on the choice of the free resolution of \( M \) and they are called \( \text{ext}^i_A(M, S) \). Thus, we have:
\[
\begin{cases}
\text{ext}^0_A(M, S) = \text{hom}_A(M, S), \\
\text{ext}^i_A(M, S) = \text{ker} \frac{d_{i+1}}{\text{im} d_i}, i \geq 1.
\end{cases}
\]

\( \bullet \) A module \( _A S \) is called injective if for every exact sequence of left \( A \)-modules \( 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \), we have the exact sequence:
\[
0 \longrightarrow \text{hom}_A(M', S) \longrightarrow \text{hom}_A(M, S) \longrightarrow \text{hom}_A(M'', S) \longrightarrow 0.
\]

Example 3. If \( M \) is a left finitely generated \( A \)-module, then a free resolution of \( M \) can be written on the form
\[
\ldots \longrightarrow A^n \longrightarrow A^{n-1} \longrightarrow A^{n-2} \longrightarrow A^{n-3} \longrightarrow 0,
\]
where \( R_i \) is a \( r_i \times r_{i-1} \) matrix with entries in \( A \) and \( R_i \) is the \( A \)-morphism defined by multiplying a row vector of length \( r_i \) on the left of \( R_i \) to obtain a row vector of length \( r_{i-1} \). Hence, the extension functor gives the defects of exactness of the following sequence:
\[
\ldots \longrightarrow S^{r_n} \longrightarrow S^{r_{n-1}} \longrightarrow S^{r_{n-2}} \longrightarrow S^{r_{n-3}} \longrightarrow 0,
\]
where \( R_i \) is the \( A \)-morphism defined by multiplying a column vector of length \( r_{i-1} \) on the right of \( R_{i-1} \) to obtain a column vector of length \( r_i \). In particular, \( \text{ext}^0_A(M, S) = \text{hom}_A(M, S) \) represents the solution \( y \in S^{r_0} \) of the system \( R_1 y = 0 \), whereas \( \text{ext}^1_A(M, S) \) is the obstruction that a \( z \in S^{r_1} \) satisfying \( R_2 z = 0 \) to be of the form \( z = R_1 y \) with \( y \in S^{r_0} \). When \( A \) is the ring \( D \) of differential operators with constant coefficients, then \( C^\infty, D' \) and \( S' \) are few examples of injective \( D \)-modules [3, 4, 11, 15]. In particular, if \( R_1 \) is a matrix which defines a multidimensional control system, then \( \text{ext}^i_A(M, S) = \text{hom}_A(M, S) \) corresponds to the behaviour of the system [13].

Proposition 3. If \( 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \) is an exact sequence of left \( A \)-modules and \( _A S \) (resp. \( _A S_B \)) a module, then we have the following exact sequence of abelian groups (resp. right \( B \)-modules):
\[
0 \longrightarrow \text{hom}_A(M'', S) \xrightarrow{\delta} \text{hom}_A(M', S) \xrightarrow{\pi'} \text{hom}_A(M, S) \longrightarrow \text{ext}_A^1(M'', S) \longrightarrow \text{ext}_A^1(M', S) \longrightarrow \text{ext}_A^1(M, S) \longrightarrow \ldots
\]
In particular, a module \( _A S \) is injective iff for any left \( A \)-module \( M \), we have \( \text{ext}_A^i(M, S) = 0, i \geq 1 \).

If \( M \) is a left \( A \)-module defined by a finite presentation \( F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0 \), then we can define the right
A-module $N$ by:

$$0 \leftarrow N \leftarrow F_1^* \underbrace{\rightarrow}_{d_1} F_0^* \leftarrow M^* \leftarrow 0. \quad (5)$$

Two different free resolutions of $M$ give two different right $A$-modules $N$ and $N'$, and thus, the right $A$-module $N$ is not uniquely defined by $M$. But, it is shown in [8, 9] that $N$ is uniquely defined up to a projective equivalence [10], a fact which implies that $\text{ext}_A^i(N, S) \cong \text{ext}_A^i(N', S)$ and $\text{tor}_A^i(N, S) \cong \text{tor}_A^i(N', S)$ for $i \geq 1$ and any left $A$-module $S$. The module $N$ plays a crucial role in the study of the algebraic properties of the left $A$-module $M$ and their correspondences with the different types of primeness [4, 7, 8]. An interesting application of this result is the following. Let us take a free resolution of $N$ of the form

$$0 \leftarrow N \leftarrow A^n \underbrace{\rightarrow}_{R_1} A^m \leftarrow A^n \leftarrow \ldots,$$

and deleting $N$ and taking the tensor product by a left $A$-module $S$, we obtain the following sequence:

$$0 \leftarrow S^n \underbrace{\rightarrow}_{R_1} S^m \leftarrow S^n \leftarrow S^n \leftarrow \ldots$$

The defects of exactness are given by $\text{tor}_A^i(N, S)$. Thus, if $S$ is a flat left $A$-module, we have parametrized the solution $y \in S^n$ of the system $R_1 y = 0$ by $y = R_0 z$, $z \in S^n$ and so on. For example, when $A$ is the ring $D$ of differential operators with constant coefficients, $S, D$ and $E'$ are flat $D$-modules [3, 11].

4 Duality between Extension and Torsion functors

Theorem 2. [9] Let $M$ be a left $A$-module defined by the finite presentation $F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ and an exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ of right $A$-modules. Then, we have the commutative exact diagram (6) (see figure 2), where we have denoted by $N$ the right $A$-module defined by $0 \leftarrow N \leftarrow F_1^* \underbrace{\rightarrow}_{d_1} F_0^* \leftarrow M^* \leftarrow 0$. There exist two connecting maps $\delta : \text{hom}_A(N, Z) \rightarrow X \otimes_A M$ and $\kappa : \text{tor}_A^1(Z, M) \rightarrow \text{ext}_A^1(N, X)$ such that the following two sequences are exact:

$$0 \rightarrow \text{hom}_A(N, X) \rightarrow \text{hom}_A(N, Y) \rightarrow \text{hom}_A(N, Z) \rightarrow X \otimes_A M \rightarrow Y \otimes_A M \rightarrow Z \otimes_A M \rightarrow 0,$$

$$\rightarrow \text{tor}_A^1(Z, M) \rightarrow \text{tor}_A^1(Z, M) \rightarrow \text{ext}_A^1(N, X) \rightarrow \text{ext}_A^1(N, Y) \rightarrow \text{ext}_A^1(N, Z) \rightarrow \ldots$$

Proof. First of all, let us notice that if $F$ is a finitely generated projective (free) left $A$-module and $X$ is a right $A$-module, then we have $\text{hom}_A(F^*, X) \cong X \otimes_A F$, where $F^* = \text{hom}_A(F, A)$ [10]. By taking the tensor product of the finite presentation of $M$ with respect to $X$, we obtain the following exact sequence $X \otimes_A F_0 \rightarrow X \otimes_A F_1 \rightarrow X \otimes_A F_2 \rightarrow \ldots$ and $\text{tor}_A^1(X, M) = 0$, whereas by taking the $A$-morphisms of the exact sequence (5) in $X$, we obtain the following exact sequence:

$$0 \rightarrow \text{hom}_A(N, X) \rightarrow \text{hom}_A(F_1^*, X) \rightarrow \text{hom}_A(F_0^*, X).$$

Therefore, we have the following exact sequence:

$$0 \rightarrow \text{hom}_A(N, X) \rightarrow X \otimes_A F_1 \rightarrow X \otimes_A F_0 \rightarrow X \otimes_A F_2 \rightarrow \ldots$$

Finally, we obtain (6) if we notice that we have the following exact sequence $0 \rightarrow X \otimes_A F_1 \rightarrow Y \otimes_A F_1 \rightarrow Z \otimes_A F_1 \rightarrow 0$, because $F_1$ is a free and thus a flat $A$-module [10], for $i = 0, 1$. Then, two chases in (6) prove the theorem.

Lemma 1. If $M$ a left $A$-module, then $\text{ext}_A^1(M, A)$ is a finitely generated torsion right $A$-module for $i \geq 1$.

Proof. The fact that $\text{ext}_A^1(M, A)$ is a finitely generated right $A$-module for all $i \geq 1$ can be easily proved [10]. Now, let $F$ be a maximal free $A$-module included in $M$, then we have the exact sequence $0 \rightarrow F \rightarrow M \rightarrow M/F \rightarrow 0$, where $M/F$ is a torsion left $A$-module. Then, we can apply proposition 3 to the previous exact sequence to obtain the following exact sequences:

$$0 = \text{hom}_A(M/F, A) \rightarrow \text{hom}_A(M, A) \rightarrow \text{hom}_A(F, A) \rightarrow \text{ext}_A^1(M, A) \rightarrow \text{ext}_A^1(M, A) \rightarrow \text{ext}_A^1(F, A) = 0,$$

$$0 = \text{ext}_A^1(F, A) \rightarrow \text{ext}_A^1(M, F/A) \rightarrow \text{ext}_A^1(M, A) \rightarrow \text{ext}_A^1(F, A) = 0, \forall i \geq 2.$$
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{hom}_A(N, X) & \rightarrow & X \otimes_A F_1 & \rightarrow & X \otimes_A F_0 & \rightarrow & X \otimes_A M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \text{hom}_A(N, Y) & \rightarrow & Y \otimes_A F_1 & \rightarrow & Y \otimes_A F_0 & \rightarrow & Y \otimes_A M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \text{hom}_A(N, Z) & \rightarrow & Z \otimes_A F_1 & \rightarrow & Z \otimes_A F_0 & \rightarrow & Z \otimes_A M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\text{ext}^1_A(N, X) & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

(6)

Figure 2: commutative exact diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \text{hom}_A(N, A) & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{hom}_A(N, K) & \rightarrow & K \otimes_A F_1 & \rightarrow & K \otimes_A F_0 & \rightarrow & K \otimes_A M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{hom}_A(N, K/A) & \rightarrow & (K/A) \otimes_A F_1 & \rightarrow & (K/A) \otimes_A F_0 & \rightarrow & (K/A) \otimes_A M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{ext}^1_A(N, A) & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

(7)

Figure 3: commutative exact diagram
Proof. Using the fact that $K$ is a flat $A$-module and $\text{ext}^1_A(N, A)$ is a torsion left $A$-module (see lemma 1), we obtain $\text{ext}^1_A(N, K) \cong K \otimes_A \text{ext}^1_A(N, A) = 0$ and $\text{tor}^1_A(K, M) = 0$. Now, applying theorem 2 to the exact sequence $0 \rightarrow A \rightarrow K \rightarrow K/A \rightarrow 0$ of both left and right $A$-modules, we obtain the commutative exact diagram (7). Finally, the snake lemma gives the isomorphisms (8).

Example 4. Let us reconsider example 2. The $D$-module $N$ corresponding to $M$ is defined by the following exact sequence $0 \leftarrow N \leftarrow D^3 \xrightarrow{R_1} D \leftarrow 0$. Therefore, dualizing this exact sequence, we have the sequence $0 \rightarrow N^* \rightarrow D^3 \xrightarrow{R_1^*} D \rightarrow 0$ and we obtain $\text{ext}^1_B(N, D) = M$. Finally, we have $\tau(M) = M$ and $M$ is a torsion $D$-module.

5 Conclusion

We hope to have convinced the reader that homological tools such as extension and torsion functors are very useful and powerful in the study of multidimensional control systems. They have allowed to show the link existing between the concept of weak primeness and the one of torsion module. Moreover, we have given a purely algebraic proof of the isomorphism between $\tau(M)$ and $\text{ext}^1_A(N, A)$, for any noetherian left Ore integral domain $A$ and any finitely generated $A$-module $M$. This result generalizes the ones obtained in [7] for rings of differential operators.

References


