Second Internation Workshop on
Multidimensional (nD) Systems
27-30 June, 2000, Czocha Castle, Lower Silesia, POLAND

A Functorial Approach to the Behaviour of Multidimensional Control Systems

J.F. Pommaret & A. Quadrat
CERMICS
Ecole Nationale des Ponts et Chaussées
6 et 8 avenue Blaise Pascal,
77455 Marne-La-Vallée Cedex 02, France
e-mail: {pommaret, quadrat}@cermics.enpc.fr

Abstract

We show how to use the extension and the torsion functors to compute the torsion submodule of a differential module associated with a multidimensional control system. In particular, we show that the concept of weak primeness of matrices corresponds to the torsion-freeness of the corresponding module and we generalize this concept to non full rank matrices.

Keywords: Multidimensional Control Systems, Weak Primeness, Controllability, Algebraic Analysis, Torsion and Extension Functors, Homological Algebra, Noncommutative Algebra.

1 Introduction

It is well known that the controllability of a linear multidimensional control system depends on an algebraic property (namely, the *torsion-freeness*) of a certain module M associated with the system [4, 5, 6, 7]. The recent survey [15] gives different equivalent formulations of the controllability and, in particular, the equivalence obtained in [5] between the torsion-freeness of M and the definition of the controllability given by Willems [13].

In this paper, we show how to use the powerful tools of homological algebra to compute the torsion submodule $t(M)$ of M, i.e. the non-controllable part of a control system. The results obtained show the link between the concept of the weak primeness of a matrix R and the torsion part $t(M)$ of the module M associated with R. Moreover, we show how the *tor functor* allows to generalize this concept to a non full rank matrix R. Finally, we show that the isomorphisms $t(M) \cong \mathrm{tor}_1^A(K/A, M) \cong \mathrm{ext}_A^1(N, A)$, where N is an important module obtained from M and A is any noetherian ring. This result generalizes previous results obtained by the authors for the ring D of differential operators.

2 Torsion functor and weak primeness

In the course of the paper, we shall denote by A a noetherian integral domain which is supposed to be either a commutative ring or a *left Ore domain*, namely a domain such that, for any couple $(a, b) \in A^2$, there exists a non trivial couple $(u, v) \in A^2$ such that $u\,a = v\,b$. Moreover, let $K = Q(A)$ be the quotient field of A. If A and B are two integral domains, then we shall denote by ${}_B M_A$ a module M with a structure of left B-module and of right A-module (see [2, 6] for more details in the non-commutative case).

In the literature of robust stabilization, the concept of *weak primeness* is very useful [9, 12].

Definition 1. Let R be an $l \times m$ matrix ($l \leq m$) with full rank and entries in A. R is said to be *weakly left-prime* if any row vector $z \in K^l$ which satisfies $z\,R \in A^m$ belongs to A^l.

Example 1. The matrix $R = (d_1\ d_2\ d_3)'$ with entries in $D = \mathbb{R}[d_1, d_2, d_3]$ is not weakly left-prime because there exists $(d_1^{-1}\ 0\ 0) \notin D^3$ such that $(d_1^{-1}\ 0\ 0)\,R = 1 \in D$.

Let us interpret the weak primeness in terms of modules. We need a few definitions [10].

Definition 2.
- We call *torsion submodule* of an A-module M, the A-module defined by:

  $$t(M) = \{m \in M \mid \exists\, 0 \neq a \in A,\ a\,m = 0\}.$$

  An A-module M is said to be *torsion-free* if $t(M) = 0$ and *torsion* if $t(M) = M$. We have the following exact sequence:

  $$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0. \quad (1)$$

- A module M_A (resp. ${}_B M_A$) is *flat* if for every exact sequence of left A-module $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$, we have the following exact sequence of abelian groups (resp. left B-modules):

  $$0 \longrightarrow M \otimes_A N' \longrightarrow M \otimes_A N \longrightarrow M \otimes_A N'' \longrightarrow 0.$$

K is a flat A-module, i.e. the tensor product by K transforms exact sequences of left A-modules into exact sequences of left K-vector spaces and we have the exact sequence [10]:

$$0 \longrightarrow t(M) \longrightarrow M \xrightarrow{i_K} K \otimes_A M \longrightarrow (K/A) \otimes_A M \longrightarrow 0.$$

To our knowledge, the following proposition has firstly been obtained in [9] for a general left Ore domain A.

Theorem 1. *Let M be the left A-module defined by the following free resolution:*

$$\begin{array}{ccccccc} 0 \longrightarrow & A^l & \xrightarrow{.R} & A^m & \xrightarrow{\pi} M \longrightarrow 0. \\ & (z_1 \dots z_l) & \longrightarrow & (z_1 \dots z_l)\, R & \end{array} \quad (2)$$

Then, the following assertions are equivalent:

- *R is weakly left-prime,*
- *$t(M) = 0$, i.e., M is a torsion-free A-module.*

Proof. K is a flat A-module [10], thus by tensoring (2) by K, we obtain the exact sequence

$$0 \longrightarrow K^l \xrightarrow{\mathrm{id}_K \otimes .R} K^m \longrightarrow K \otimes_A M \longrightarrow 0,$$

where $(\mathrm{id}_K \otimes .R)(z) = z\,R,\ \forall\, z \in K^l$. Therefore, we have the commutative exact diagram (3) (see figure 1), where $\kappa : (K/A)^l \longrightarrow (K/A)^m$ is defined by $\kappa(y) = \pi''(z\,R)$, where $\pi'(z) = y$, and $L = \ker(\kappa)$. By the snake lemma [10], we obtain $L \cong t(M)$. But, $0 \neq L \cong t(M) \Leftrightarrow \exists\, 0 \neq y \in (K/A)^l$ such that $\kappa(y) = 0 \Leftrightarrow \exists\, z \in K^l$ such that $\pi'(z) = y \neq 0$ and $\pi''(z\,R) = 0 \Leftrightarrow \exists\, z \notin A^l$ satisfied that $z\,R \in A^m \Leftrightarrow R$ is not weakly left-prime, which proves the proposition. □

Let us notice that $L = \ker(\kappa)$ is, in general, a non-zero A-module because K/A is not a flat A-module. To study the defect of exactness that the tensor product by (K/A) introduces in exact sequences, we shall need the definition of the *torsion functor*.

Definition 3. Let $\dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0$ be a free resolution of the left A-module M and a module N_A (resp. ${}_B N_A$), then the abelian groups (resp. left B-modules) of homology of the complex

$$\dots \xrightarrow{\mathrm{id}_N \otimes d_3} N \otimes_A F_2 \xrightarrow{\mathrm{id}_N \otimes d_2} N \otimes_A F_1 \xrightarrow{\mathrm{id}_N \otimes d_1} N \otimes_A F_0 \longrightarrow 0,$$

do not depend on the choice of the free resolution of M and they are called $\mathrm{tor}_i^A(N, M)$. We have:

$$\begin{cases} \mathrm{tor}_0^A(N, M) = N \otimes_A M, \\ \mathrm{tor}_i^A(N, M) = \ker(\mathrm{id}_N \otimes d_i)/\mathrm{im}(\mathrm{id}_N \otimes d_{i+1}),\ \forall\, i \geq 1. \end{cases}$$

Proposition 1. *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of left A-modules and a module N_A (resp. ${}_B N_A$), then we have the following exact sequence of abelian groups (resp. left B-modules):*

$$\begin{array}{lll} & \dots & \longrightarrow \mathrm{tor}_2^A(N, M'') \longrightarrow \\ \mathrm{tor}_1^A(N, M') & \longrightarrow \mathrm{tor}_1^A(N, M) & \longrightarrow \mathrm{tor}_1^A(N, M'') \longrightarrow \\ N \otimes_A M' & \longrightarrow N \otimes_A M & \longrightarrow N \otimes_A M'' \longrightarrow 0. \end{array}$$

In particular, a module N_A is flat iff for any left A-module M, we have $\mathrm{tor}_i^A(N, M) = 0,\ \forall\, i \geq 1$.

Applying the previous proposition to the exact sequence (2) and $N = K/A$, we obtain that $\kappa = (\mathrm{id}_{K/A} \otimes .R)$ and $\mathrm{tor}_1^A(K/A, M) = \ker(\mathrm{id}_{K/A} \otimes .R)$. We have seen in the proof of theorem 1 that $t(M) \cong \ker(\kappa)$. Thus, we deduce that $\mathrm{tor}_1^A(K/A, M) \cong t(M)$. In fact, this result is also true if M is defined by a long free resolution. See [10] for a proof of the following proposition.

Proposition 2. *Let M be a left A-module, then:*

$$t(M) \cong \mathrm{tor}_1^A(K/A, M).$$

Hence, the concept of weak left-primeness may be extended to non full rank matrix R as follows: R is a *generalized weak left-prime* matrix if the left A-module M defined by (2) is torsion-free, i.e. $\mathrm{tor}_1^A(K/A, M) = 0 \Leftrightarrow \ker(\mathrm{id}_{K/A} \otimes d_1) = \mathrm{im}(\mathrm{id}_{K/A} \otimes d_2)$. See [9] for more details and applications to stabilization problems.

Remark 1. In [12], Smith shows that if A is a commutative integral domain, then weakly left-primeness implies minor left-primeness and the two concepts are equivalent if A is *Greatest Common Divisor Domain*. See [9, 12] for counter-examples of the fact that minor left-primeness does not imply in general weakly left-primeness. Hence, the concept of weakly left-primeness is the only one which is equivalent for a full rank matrix R, with entries in a left Ore integral domain A, to the torsion-freeness of the corresponding left A-module M. We thank both an anonymous referee and J. Wood to have pointed out to the autors that the concept of generalized weak left-primeness is equivalent to the one of generalized factor left-primeness in the case of a polynomial ring A (see [1, 14] for more details).

Example 2. Let us consider the multidimensional system defined by the gradient in $\mathbb{R}^3$:

$$\begin{cases} d_1\, z = 0, \\ d_2\, z = 0, \\ d_3\, z = 0. \end{cases} \quad (4)$$

We have the following free resolution of the $D = \mathbb{R}[d_1, d_2, d_3]$-module M corresponding to (4):

$$0 \longrightarrow D \xrightarrow{.R_3} D^3 \xrightarrow{.R_2} D^3 \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0,$$

where

$$R_1 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix},$$

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & 0 & & 0 & & t(M) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & & A^l & \xrightarrow{.R} & A^m & \xrightarrow{\pi} & M & \longrightarrow 0 & \\ & & \downarrow & & \downarrow & & \downarrow i_K & & \\ 0 \longrightarrow & & K^l & \xrightarrow{\mathrm{id}_K \otimes .R} & K^m & \longrightarrow & K \otimes_A M & \longrightarrow 0 & \\ & & \downarrow \pi' & & \downarrow \pi'' & & \downarrow & & \\ 0 \longrightarrow & L \longrightarrow & (K/A)^l & \xrightarrow{\kappa} & (K/A)^m & \longrightarrow & (K/A) \otimes_A M & \longrightarrow 0, & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3}$$

Figure 1: commutative exact diagram

and $R_3 = (d_1\ d_2\ d_3)$. The matrix R_1 is not generalized weak left-prime because the following sequence

$$(K/A)^3 \xrightarrow{\mathrm{id}_{K/A} \otimes .R_2} (K/A)^3 \xrightarrow{\mathrm{id}_{K/A} \otimes .R_1} K/A$$

is not exact at $(K/A)^3$, i.e. $\mathrm{tor}_1^A(K/A, M) \neq 0$. Indeed, $y = \pi'((d_1^{-1}\ 0\ 0))$ belongs to $\ker(\mathrm{id}_{K/A} \otimes .R_1)$ but $y \notin \mathrm{im}(\mathrm{id}_{K/A} \otimes .R_2)$ because there is no solution of the system $(t_1\ t_2\ t_3)\, R_2 = (d_1^{-1}\ 0\ 0)$ in K^3. We easily see that y corresponds to the torsion element z of (4) in the isomorphism of proposition 2 (it is just an easy chase in a diagram similar to the one in the proof of theorem 1).

3 Extension functor and behaviours

Definition 4. • Let $\ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0$ be a free resolution of the left A-module M and ${}_AS$ (resp. ${}_AS_B$) a module, then the abelian groups (resp. the right B-modules) of cohomology of the complex

$$\ldots \xleftarrow{d_2^\star} \hom_A(F_1, S) \xleftarrow{d_1^\star} \hom_A(F_0, S) \longleftarrow 0,$$

where $d_i^\star(f) = f \circ d_i$, $\forall\, f \in \hom_A(F_{i-1}, S)$, do not depend on the choice of the free resolution of M and they are called $\mathrm{ext}_A^i(M, S)$. Thus, we have:

$$\begin{cases} \mathrm{ext}_A^0(M, S) = \hom_A(M, S), \\ \mathrm{ext}_A^i(M, S) = \ker d_{i+1}^\star / \mathrm{im}\, d_i^\star, \ \forall\, i \geq 1. \end{cases}$$

• A module ${}_AS$ is called *injective* if for every exact sequence of left A-modules $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$, we have the exact sequence:

$$0 \longleftarrow \hom_A(M', S) \xleftarrow{f^\star} \hom_A(M, S) \xleftarrow{g^\star} \hom_A(M'', S) \longleftarrow 0.$$

Example 3. If M is a left finitely generated A-module, then a free resolution of M can be written on the form

$$\ldots \longrightarrow A^{r_2} \xrightarrow{.R_2} A^{r_1} \xrightarrow{.R_1} A^{r_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where R_i is a $r_i \times r_{i-1}$ matrix with entries in A and $.R_i$ is the A-morphism defined by multiplying a row vector of length r_i on the left of R_i to obtain a row vector of length r_{i-1}. Hence, the extension functor gives the defects of exactness of the following sequence

$$\ldots \longleftarrow S^{r_2} \xleftarrow{R_2.} S^{r_1} \xleftarrow{R_1.} S^{r_0} \longleftarrow 0,$$

where $R_i.$ is the A-morphism defined by multiplying a column vector of length r_{i-1} on the right of R_i to obtain a column vector of length r_i. In particular, $\mathrm{ext}_A^0(M, S) = \hom_A(M, S)$ represents the solution $y \in S^{r_0}$ of the system $R_1\, y = 0$, whereas $\mathrm{ext}_A^1(M, S)$ is the obstruction that a $z \in S^{r_1}$ satisfying $R_2\, z = 0$ to be of the form $z = R_1\, y$ with $y \in S^{r_0}$. When A is the ring D of differential operators with constant coefficients, then $C^\infty, \mathcal{D}'$ and S' are few examples of injective D-modules [3, 4, 11, 15]. In particular, if R_1 is a matrix which defines a multidimensional control system, then $\mathrm{ext}_A^0(M, S) = \hom_A(M, S)$ corresponds to the *behaviour* of the system [13].

Proposition 3. *If $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is an exact sequence of left A-modules and ${}_AS$ (resp. ${}_AS_B$) a module, then we have the following exact sequence of abelian groups (resp. right B-modules):*

$$\begin{aligned} 0 &\longrightarrow \hom_A(M'', S) \xrightarrow{g^\star} \hom_A(M, S) \xrightarrow{f^\star} \hom_A(M', S) \\ &\longrightarrow \mathrm{ext}_A^1(M'', S) \longrightarrow \mathrm{ext}_A^1(M, S) \longrightarrow \mathrm{ext}_A^1(M', S) \\ &\longrightarrow \mathrm{ext}_A^2(M'', S) \longrightarrow \ldots \end{aligned}$$

In particular, a module ${}_AS$ is injective iff for any left A-module M, we have $\mathrm{ext}_A^i(M, S) = 0$, $i \geq 1$.

If M is a left A-module defined by a finite presentation $F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0$, then we can define the right

A-module N by:

$$0 \longleftarrow N \longleftarrow F_1^\star \overset{d_1^\star}{\longleftarrow} F_0^\star \longleftarrow M^\star \longleftarrow 0. \qquad (5)$$

Two different free resolutions of M give two different right A-modules N and N', and thus, the right A-module N is not uniquely defined by M. But, it is shown in [8, 9] that N is uniquely defined up to a *projective equivalence* [10], a fact which implies that $\mathrm{ext}_A^i(N,S) \cong \mathrm{ext}_A^i(N',S)$ and $\mathrm{tor}_i^A(N,S) \cong \mathrm{tor}_i^A(N',S)$ for $i \geq 1$ and any left A-module S. The module N plays a crucial role in the study of the algebraic properties of the left A-module M and their correspondences with the different types of primeness [4, 7, 8]. An interesting application of this result is the following. Let us take a free resolution of N of the form

$$0 \longleftarrow N \longleftarrow A^{r_1} \overset{R_1.}{\longleftarrow} A^{r_0} \overset{R_0.}{\longleftarrow} A^{r_{-1}} \longleftarrow \ldots,$$

and deleting N and taking the tensor product by a left A-module S, we obtain the following sequence:

$$0 \longleftarrow S^{r_1} \overset{R_1.}{\longleftarrow} S^{r_0} \overset{R_0.}{\longleftarrow} S^{r_{-1}} \longleftarrow \ldots$$

The defects of exactness are given by $\mathrm{tor}_i^A(N,S)$. Thus, if S is a flat left A-module, we have parametrized the solution $y \in S^{r_0}$ of the system $R_1\, y = 0$ by $y = R_0\, z,\ z \in S^{r_{-1}}$ and so on. For example, when A is the ring D of differential operators with constant coefficients, $\mathcal{S}, \mathcal{D}$ and $\mathcal{E}'$ are flat D-modules [3, 11].

4 Duality between Extension and Torsion functors

Theorem 2. *[9] Let M be a left A-module defined by the finite presentation $F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0$ and an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ of right A-modules. Then, we have the commutative exact diagram (6) (see figure 2), where we have denoted by N the right A-module defined by $0 \longleftarrow N \longleftarrow F_1^\star \overset{d_1^\star}{\longleftarrow} F_0^\star \longleftarrow M^\star \longleftarrow 0$. There exist two connecting maps $\delta : \mathrm{hom}_A(N,Z) \longrightarrow X \otimes_A M$ and $\kappa : \mathrm{tor}_1^A(Z,M) \longrightarrow \mathrm{ext}_A^1(N,X)$ such that the two following sequences are exact:*

$$0 \longrightarrow \mathrm{hom}_A(N,X) \longrightarrow \mathrm{hom}_A(N,Y) \longrightarrow \mathrm{hom}_A(N,Z)$$
$$\xrightarrow{\delta} X \otimes_A M \longrightarrow Y \otimes_A M \longrightarrow Z \otimes_A M \longrightarrow 0,$$

$$\begin{array}{lll} & \ldots & \longrightarrow \mathrm{tor}_2^A(Z,M) \\ \longrightarrow \mathrm{tor}_1^A(X,M) \longrightarrow & \mathrm{tor}_1^A(Y,M) & \longrightarrow \mathrm{tor}_1^A(Z,M) \\ \xrightarrow{\kappa} \mathrm{ext}_A^1(N,X) \longrightarrow & \mathrm{ext}_A^1(N,Y) & \longrightarrow \mathrm{ext}_A^1(N,Z) \\ \longrightarrow \mathrm{ext}_A^2(N,X) \longrightarrow & \ldots & \end{array}$$

Proof. First of all, let us notice that if F is a finitely generated projective (free) left A-module and X is a right A-module, then we have $\mathrm{hom}_A(F^\star, X) \cong X \otimes_A F$, where $F^\star = \mathrm{hom}_A(F,A)$ [10]. By taking the tensor product of the finite presentation of M by respect to X, we obtain the following exact sequence $X \otimes_A F_1 \xrightarrow{\mathrm{id}_X \otimes_A d_1} X \otimes_A F_0 \xrightarrow{\mathrm{id}_X \otimes_A \pi} X \otimes_A M \longrightarrow 0$ whereas, by taking the A-morphisms of the exact sequence (5) in X, we obtain the following exact sequence:

$$\begin{array}{ccccc} 0 \longrightarrow \mathrm{hom}_A(N,X) \longrightarrow & \mathrm{hom}_A(F_1^\star, X) & \longrightarrow & \mathrm{hom}_A(F_0^\star, X). \\ & \| & & \| \\ & X \otimes_A F_1 & \longrightarrow & X \otimes_A F_0 \end{array}$$

Therefore, we have the following exact sequence:

$$0 \longrightarrow \mathrm{hom}_A(N,X) \longrightarrow X \otimes_A F_1 \xrightarrow{\mathrm{id}_X \otimes_A d_1} X \otimes_A F_0$$
$$\xrightarrow{\mathrm{id}_X \otimes_A \pi} X \otimes_A M \longrightarrow 0.$$

Finally, we obtain (6) if we notice that we have the following exact sequence $0 \longrightarrow X \otimes_A F_i \longrightarrow Y \otimes_A F_i \longrightarrow Z \otimes_A F_i \longrightarrow 0$, because F_i is a free and thus a flat A-module [10], for $i = 0, 1$. Then, two chases in (6) prove the theorem. □

Lemma 1. *If M a left A-module, then $\mathrm{ext}_A^i(M,A)$ is a finitely generated torsion right A-module for $i \geq 1$.*

Proof. The fact that $\mathrm{ext}_A^i(M,A)$ is a finitely generated right A-module for all $i \geq 1$ can be easily proved [10]. Now, let F be a maximal free A-module included in M, then we have the exact sequence $0 \longrightarrow F \longrightarrow M \longrightarrow M/F \longrightarrow 0$, where M/F is a torsion left A-module. Then, we can apply proposition 3 to the previous exact sequence to obtain the following exact sequences:

$$0 = \mathrm{hom}_A(M/F,A) \longrightarrow \mathrm{hom}_A(M,A) \longrightarrow \mathrm{hom}_A(F,A)$$
$$\longrightarrow \mathrm{ext}_A^1(M/F,A) \longrightarrow \mathrm{ext}_A^1(M,A) \longrightarrow \mathrm{ext}_A^1(F,A) = 0,$$

$$0 = \mathrm{ext}_A^{i-1}(F,A) \longrightarrow \mathrm{ext}_A^i(M/F,A) \longrightarrow \mathrm{ext}_A^i(M,A)$$
$$\longrightarrow \mathrm{ext}_A^i(F,A) = 0, \quad \forall\, i \geq 2.$$

From the second exact sequence, we deduce that $\mathrm{ext}_A^i(M,A) \cong \mathrm{ext}_A^i(M/F,A)$, $\forall\, i \geq 2$. Now, using the fact that K is a flat A-module, then we have $K \otimes_A \mathrm{ext}_A^i(M/F,A) \cong \mathrm{ext}_K^i(K \otimes_A M/F, K) = 0$, $\forall\, i \geq 1$, because M/F is a torsion left A-module [10]. Therefore, we have $K \otimes_A \mathrm{ext}_A^i(M,A) = 0$, $\forall\, i \geq 2$, i.e. $\mathrm{ext}_A^i(M,A)$ is a torsion right A-module for all $i \geq 2$. Finally, if we take the tensor product of the first exact sequence with respect to K, then we obtain that $K \otimes_A \mathrm{ext}_A^1(M,A) = 0$, i.e. $\mathrm{ext}_A^1(M,A)$ is a torsion right A-module. □

Corollary 1. *[9] We have the commutative exact diagram (7) of left A-modules (see figure 3) and the following exact sequence of left A-modules:*

$$0 \longrightarrow \mathrm{hom}_A(N,A) \longrightarrow \mathrm{hom}_A(N,K) \longrightarrow \mathrm{hom}_A(N,K/A)$$
$$\xrightarrow{\delta} M \xrightarrow{i_K} K \otimes_A M \longrightarrow (K/A) \otimes_A M \longrightarrow 0.$$

Moreover, we have the following isomorphisms:

$$t(M) \cong \mathrm{ext}_A^1(N,A) \cong \mathrm{tor}_1^A(K/A, M). \qquad (8)$$

$$
\begin{tikzcd}
 & & & & \vdots \arrow[d] & \\
 & & & & \mathrm{tor}_1^A(X,M) \arrow[d] & \\
 & & & & \mathrm{tor}_1^A(Y,M) \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \mathrm{tor}_1^A(Z,M) \arrow[d] & \\
0 \arrow[r] & \hom_A(N,X) \arrow[r] \arrow[d] & X \otimes_A F_1 \arrow[r] \arrow[d] & X \otimes_A F_0 \arrow[r] \arrow[d] & X \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_A(N,Y) \arrow[r] \arrow[d] & Y \otimes_A F_1 \arrow[r] \arrow[d] & Y \otimes_A F_0 \arrow[r] \arrow[d] & Y \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_A(N,Z) \arrow[r] \arrow[d] & Z \otimes_A F_1 \arrow[r] \arrow[d] & Z \otimes_A F_0 \arrow[r] \arrow[d] & Z \otimes_A M \arrow[r] \arrow[d] & 0 \\
 & \mathrm{ext}_A^1(N,X) \arrow[d] & 0 & 0 & 0 & \\
 & \mathrm{ext}_A^1(N,Y) \arrow[d] & & & & \\
 & \mathrm{ext}_A^1(N,Z) \arrow[d] & & & & \\
 & \vdots & & & &
\end{tikzcd} \tag{6}
$$

Figure 2: commutative exact diagram

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \mathrm{tor}_1^A(K/A,M) \arrow[d] & \\
0 \arrow[r] & \hom_A(N,A) \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_A(N,K) \arrow[r] \arrow[d] & K \otimes_A F_1 \arrow[r] \arrow[d] & K \otimes_A F_0 \arrow[r] \arrow[d] & K \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_A(N,K/A) \arrow[r] \arrow[d] & (K/A) \otimes_A F_1 \arrow[r] \arrow[d] & (K/A) \otimes_A F_0 \arrow[r] \arrow[d] & (K/A) \otimes_A M \arrow[r] \arrow[d] & 0. \\
 & \mathrm{ext}_A^1(N,A) \arrow[d] & 0 & 0 & 0 & \\
 & 0 & & & &
\end{tikzcd} \tag{7}
$$

Figure 3: commutative exact diagram

Proof. Using the fact that K is a flat A-module and $\text{ext}^1_A(N, A)$ is a torsion left A-module (see lemma 1), we obtain $\text{ext}^1_A(N, K) \cong K \otimes_A \text{ext}^1_A(N, A) = 0$ and $\text{tor}^A_1(K, M) = 0$. Now, applying theorem 2 to the exact sequence $0 \longrightarrow A \longrightarrow K \longrightarrow K/A \longrightarrow 0$ of both left and right A-modules, we obtain the commutative exact diagram (7). Finally, the snake lemma gives the isomorphisms (8). $\square$

Example 4. Let us reconsider example 2. The D-module N corresponding to M is defined by the following exact sequence $0 \longleftarrow N \longleftarrow D^3 \xleftarrow{R_1 \cdot} D \longleftarrow 0$. Therefore, dualizing this exact sequence, we have the sequence $0 \longrightarrow N^\star \longrightarrow D^3 \xrightarrow{\cdot R_1} D \longrightarrow 0$ and we obtain $\text{ext}^1_D(N, D) = M$. Finally, we have $t(M) = M$ and M is a torsion D-module.

5 Conclusion

We hope to have convinced the reader that homological tools such as extension and torsion functors are very useful and powerful in the study of multidimensional control systems. They have allowed to show the link existing between the concept of weak primeness and the one of torsion module. Moreover, we have given a purely algebraic proof of the isomorphism between $t(M)$ and $\text{ext}^1_A(N, A)$, for any noetherian left Ore integral domain A and any finitely generated A-module M. This result generalizes the ones obtained in [7] for rings of differential operators.

References

[1] Fornasini, E. and Valcher, M. E. (1997). "nD polynomial matrices with applications to multidimensional signal analysis", *Multidimensional Systems and Signal Processing*, vol. 8, n° 4, pp. 387-408.

[2] Kashiwara, M. (1970). *Algebraic Study of Systems of Partial Differential Equations*, Mémoires de la Société Mathématiques de France, n° 63 (1995).

[3] Malgrange, B. (1966). *Ideals of Differential Functions*, Oxford University Press.

[4] Oberst, U. (1990). "Multidimensional constant linear systems", *Acta Appl. Math.*, vol. 20, pp. 1-175.

[5] Pillai, H. K. and Shankar, S. (1999). "A behavioural approach to control of distributed systems", SIAM J. Control Optim., vol. 37, pp. 388-408.

[6] Pommaret, J. F. and Quadrat, A. (1999). "Localization and parametrization of linear multidimensional control systems", Systems & Control Letters, vol. 37, pp. 247-260.

[7] Pommaret, J. F. and Quadrat, A. (1999). "Algebraic analysis of linear multidimensional control systems", IMA J. Control and Optimization, vol. 16, pp. 275-297.

[8] Pommaret, J. F. and Quadrat, A. (2000). "Equivalences of linear control systems", to appear in the MTNS 2000 proceedings, Perpignan, France.

[9] Quadrat, A. (1999). *Analyse algébrique des systèmes de contrôle linéaires multidimensionnels*, PhD. thesis, Ecole des Ponts et Chaussées, France, available soon on //www.cermics.enpc.fr.

[10] Rotman, J. J. (1979). *An Introduction to Homological Algebra*, Academic Press.

[11] Shankar, S. (2000). "The fundamental principle over S'", to appear.

[12] Smith, M. C. (1989). "On stabilization and the existence of coprime factorizations", *IEEE Trans. Automatic Control*, vol. 34, n° 9, pp. 1005-1007.

[13] Willems, J. C. (1991). "Paradigms and puzzles in the theory of dynamical systems", *IEEE Trans. Automatic Control*, vol. 36, pp. 259-294.

[14] Wood, J., Rogers, E. and Owens, D. H. (1998). "A formal theory of matrix primeness", vol. 11, n° 1, pp. 40-78.

[15] Wood, J. (2000). "Modules and behaviours in nD systems theory", to appear in *Multidimensional Dimensional Systems and Signal Processing*.