Purity filtration of 2-dimensional linear systems

Alban Quadrat
INRIA Sophia Antipolis,
2004, Route des Lucioles BP 93,
06902 Sophia Antipolis cedex, France.
Alban.Quadrat@sophia.inria.fr

***Abstract*— The purpose of this paper is to show that every linear partial differential (PD) system defined by means of a matrix with entries in the noncommutative polynomial ring $D = A\langle \partial_1, \ldots, \partial_n \rangle$ of PD operators in $\partial_1 = \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n}$ with coefficients in a differential ring A, which satisfies certain regularity conditions, is equivalent to a linear PD system defined by an upper triangular matrix of PD operators formed by three diagonal blocks: the first (resp., second) diagonal block defines a $\dim(D)$-dimensional (resp., $\dim(D)-1$-dimensional) linear PD system and the third one defines a linear PD of dimension less or equal to $\dim(D)-2$. In particular, if $n = 2$, then the equivalent upper triangular matrix corresponds to the purity filtration of the finitely presented left D-module M associated with the linear PD system. Moreover, repeating the same techniques with the linear PD system of dimension less or equal to $\dim(D)-2$, the purity filtration of M can be obtained in the general case (i.e., $n \geq 2$). Finally, this equivalent form of the linear PD system can be used to obtain a Monge parametrization and for closed-form integration of linear PD systems.**

I. ALGEBRAIC ANALYSIS

In this section, we shortly recall a few results on the algebraic analysis approach to linear systems theory ([5]).

Theorem 1 ([4], [9]): Let D be a ring, $R \in D^{q\times p}$ a $q\times p$-matrix with entries in D, $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module finitely presented by R, $\{f_j\}_{j=1,\ldots,p}$ the standard basis of $D^{1\times p}$ (i.e., f_j is defined by 1 at the j^{th} entries and 0 elsewhere), $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M, $y_j = \pi(f_j)$ for $j = 1, \ldots, p$, and $\mathcal{F}$ a left D-module. Then, the abelian group isomorphism

$$\begin{array}{rcl} \chi : \hom_D(M, \mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \\ \phi & \longmapsto & (\phi(y_1) \ \ldots \ \phi(y_p))^T, \end{array} \tag{1}$$

holds, where $\hom_D(M, \mathcal{F})$ is the abelian group of left D-homomorphisms (i.e., left D-linear maps) from M to $\mathcal{F}$.

Theorem 1 shows that there is a one-to-one correspondence between the elements of $\hom_D(M, \mathcal{F})$ and the elements of the linear system (or behaviour) $\ker_{\mathcal{F}}(R.)$. Hence, $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left D-modules M and $\mathcal{F}$. In this paper, we shall study algebraic properties of M and particularly its so-called *purity filtration* ([3]).

Definition 1 ([9]): Let D be a left noetherian domain (namely, a ring without zero-divisors and for which every left ideal is finitely generated as a left D-module) and M a finitely generated left D-module. Then, we have:

1) M is *free* if there exists $r \in \mathbb{N} = \{0, 1, \ldots\}$ such that $M \cong D^{1\times r}$. Then, r is called the *rank* of M.
2) M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module N such that $M \oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules.
3) M is *reflexive* if the canonical left D-homomorphism $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ defined by $\varepsilon(m)(f) = f(m)$ for all $f \in \hom_D(M, D)$ and all $m \in M$, is bijective, i.e., ε is a left D-isomorphism.
4) M is *torsion-free* if the *torsion left D-submodule* of M, namely, $t(M) = \{m \in M \mid \exists\, d \in D\backslash\{0\} : \ d\,m = 0\}$, is trivial, i.e., if $t(M) = 0$. The elements of $t(M)$ are called the *torsion elements* of M. We have the following *short exact sequence* of left D-modules
$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0, \tag{2}$$
i.e., i is injective, ρ is surjective and $\ker \rho = \operatorname{im} i$.
5) M is *torsion* if $t(M) = M$, i.e., if every element of M is a torsion element of M.

If D is a left noetherian ring and M a finitely generated left D-module, then M admits a *finite free resolution*

$$\ldots \xrightarrow{.R_3} D^{1\times r_2} \xrightarrow{.R_2} D^{1\times r_1} \xrightarrow{.R_1} D^{1\times r_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{3}$$

where $R_i \in D^{r_i \times r_{i-1}}$ and $.R_i : D^{1\times r_i} \longrightarrow D^{1\times r_{i-1}}$ is defined by $(.R_i)(\lambda) = \lambda\, R_i$ for all $\lambda \in D^{1\times r_i}$ ([9]).

If $\mathcal{F}$ is a left D-module, then (3) yields the *complex*

$$\ldots \xleftarrow{R_4.} \mathcal{F}^{r_3} \xleftarrow{R_3.} \mathcal{F}^{r_2} \xleftarrow{R_2.} \mathcal{F}^{r_1} \xleftarrow{R_1.} \mathcal{F}^{r_0} \longleftarrow 0, \tag{4}$$

where $R_{i+1} : \mathcal{F}^{r_i} \longrightarrow \mathcal{F}^{r_{i+1}}$ is defined by $(R_i.)(\eta) = R_i\,\eta$ for all $\eta \in \mathcal{F}^{r_i}$ and all $i \in \mathbb{N}$, i.e., $\operatorname{im}_{\mathcal{F}}(R_i.) \subseteq \ker_{\mathcal{F}}(R_{i+1}.)$ for all $i \in \mathbb{N}$. The *defect of exactness* of the complex (4) at $\mathcal{F}^{r_i}$ is the abelian group defined by:

$$\begin{cases} \operatorname{ext}^0_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}), \\ \operatorname{ext}^i_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/\operatorname{im}_{\mathcal{F}}(R_i.), \ i \geq 1. \end{cases}$$

Similarly, if N is a finitely generated right D-module and $\mathcal{G}$ is a right D-module, we can define the $\operatorname{ext}^i_D(N, \mathcal{G})$'s ([9]).

Theorem 2 ([4]): Let D be a noetherian domain with a finite *global dimension* $\operatorname{gld}(D)$ ([9]), $M = D^{1\times p}/(D^{1\times q}\, R)$

This paper is dedicated to Prof. Ulrich Oberst on the occasion of his 70th birthday. His work on mathematical systems theory has alway been a source of inspiration for us and, we are sure, for the next generations.

and the *Auslander transposed* of M, namely, the right D-module $N = D^q/(R\,D^p)$ finitely presented by R.

1) The following left D-isomorphism holds:
$$t(M) \cong \mathrm{ext}^1_D(N, D). \tag{5}$$
2) M is a torsion-free left D-module iff $\mathrm{ext}^1_D(N, D) = 0$.
3) We have the following long exact sequence (6), where ε is defined in 4 of Definition 1.
4) M is reflexive left D-module iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, 2$.
5) M is projective left D-module iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, \ldots, \mathrm{gld}(D)$.

Example 1: $\mathrm{gld}(A\langle\partial_1, \ldots, \partial_n\rangle) = n$, where $A = k$ is a field, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$, $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), and $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$ ([2]).

II. Characteristic variety and dimensions

In what follows, we consider the ring $D = A\langle\partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficients in the differential ring A which is either a field k, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$. An element $P \in D$ can be written as $P = \sum_{|\alpha|=0,\ldots,r} a_\alpha\,\partial^\alpha$, where $a_\alpha \in A$ $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$ and $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$. The domain D admits the *order filtration* defined by:
$$\forall\, r \in \mathbb{N}, \quad D_r = \left\{ \sum_{0 \le |\alpha| \le r} a_\alpha\,\partial^\alpha \mid a_\alpha \in A \right\}.$$
The ring D is called a *filtered ring* and an element of D_r is said to have a *degree* less or equal to r. We can easily check that $D_0 = A$ and D_r is a finitely generated A-module.

If $d_1, d_2 \in D$, then $[d_1, d_2] = d_1\,d_2 - d_2\,d_1$. Now, if $d_1 \in D_r$ and $d_2 \in D_s$, then $d_1\,d_2$ and $d_2\,d_1$ belong to D_{r+s} since $D_r\,D_s \subseteq D_{r+s}$ and $D_s\,D_r \subseteq D_{r+s}$. Then, we can check that $[d_1, d_2] \in D_{r+s-1}$, i.e., $[D_r, D_s] \subseteq D_{r+s-1}$.

Let us now introduce the following A-module
$$\mathrm{gr}(D) = \bigoplus_{r \in \mathbb{N}} D_r/D_{r-1},$$
where $D_{-1} = 0$. Let $\pi_r : D_r \longrightarrow D_r/D_{r-1}$ be the canonical projection. Then, $\mathrm{gr}(D)$ inherits a ring structure defined by
$$\begin{cases} \pi_r(d_1) + \pi_s(d_2) \triangleq \pi_t(d_1 + d_2) \in D_t/D_{t-1}, \\ \pi_r(d_1)\,\pi_s(d_2) \triangleq \pi_{r+s}(d_1\,d_2) \in D_{r+s}/D_{r+s-1}, \end{cases}$$
where $t = \max(r, s)$ and for all $d_1 \in D_r$ and all $d_2 \in D_s$. $\mathrm{gr}(D)$ is called the *graded ring* associated with the order filtration of D. If we introduce $\chi_i = \pi_1(\partial_i) \in D_1/D_0$ for $i = 1, \ldots, n$, then $\pi_1([\partial_i, \partial_j]) = 0$ and $\pi_1([\partial_i, a]) = 0$ for all $a \in A$ and all $i, j = 1, \ldots, n$ since $[\partial_i, \partial_j] = 0$ and $[\partial_i, a] \in D_0$, which shows that $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$ is the commutative polynomial ring with coefficients in A.

Definition 2 ([2]): Let M be a finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module.

1) A *filtration* of M is a sequence $\{M_q\}_{q \in \mathbb{N}}$ of A-submodules of M satisfying the following conditions:
   a) For all $q, r \in \mathbb{N}$, $q < r$ implies $M_q \subseteq M_r$.
   b) $M = \bigcup_{q \in \mathbb{N}} M_q$.
   c) For all $q, r \in \mathbb{N}$, we have $D_r\,M_q \subseteq M_{q+r}$.

   The left D-module M is then called a *filtered module*
2) The *graded* $\mathrm{gr}(D)$-module $\mathrm{gr}(M)$ is defined by:
   a) $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q/M_{q-1}$ $(M_{-1} = 0)$.
   b) For every $d \in D_r$ and every $m \in M_q$, we set $\pi_r(d)\,\sigma_q(m) \triangleq \sigma_{q+r}(d\,m) \in M_{q+r}/M_{q+r-1}$, where $\sigma_q : M_q \longrightarrow M_q/M_{q-1}$ is the canonical projection for all $q \in \mathbb{N}$.
3) A filtration $\{M_q\}_{q \in \mathbb{N}}$ is *good* if the $\mathrm{gr}(D)$-module $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q/M_{q-1}$ is finitely generated.

Example 2: Let M be a finitely generated left D-module defined by a family of generators $\{y_1, \ldots, y_p\}$. Then, the filtration $M_q = \sum_{i=1}^p D_q\,y_i$ is a good filtration of M since we then have $\mathrm{gr}(M) = \sum_{i=1}^p \mathrm{gr}(D)\,y_i$, which proves that $\mathrm{gr}(M)$ is a finitely generated left $\mathrm{gr}(D)$-module.

Definition 3: A *proper prime ideal* of a commutative ring A is an ideal $\mathfrak{p} \subsetneq A$ which satisfies that $a\,b \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. The set of all the proper prime ideals of A is denoted by $\mathrm{spec}(A)$ and is a topological space endowed with the *Zariski topology* defined by the Zariski-closed sets $V(I) = \{\mathfrak{p} \in \mathrm{spec}(A) \mid I \subseteq \mathfrak{p}\}$, where I is an ideal of A.

Proposition 1 ([2]): Let M be a finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module and $G = \mathrm{gr}(M)$ the associated graded $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$-module for a good filtration of M. Then, the ideal of $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$ defined by
$$I(M) = \sqrt{\mathrm{ann}(G)} \triangleq \{a \in \mathrm{gr}(D) \mid \exists\, n \in \mathbb{N} :\, a^n\,G = 0\}.$$
does not depend on the good filtration of M. The characteristic variety *of M is then defined by:*
$$\mathrm{char}_D(M) = \{\mathfrak{p} \in \mathrm{spec}(\mathrm{gr}(D)) \mid \sqrt{\mathrm{ann}(G)} \subseteq \mathfrak{p}\}.$$

Definition 4 ([2]): Let M be a finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module. Then, the *dimension* of M is the supremum of the lengths of the chains
$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_d$$
of distinct proper prime ideals in $A[\chi_1, \ldots, \chi_n]/I(M)$.

We shall simply write $\dim(D)$ instead of $\dim_D(D)$.

Example 3 ([2]): We have $\dim(k[x_1, \ldots, x_n]) = n$. If $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then $\dim(A\langle\partial_1, \ldots, \partial_n\rangle) = 2\,n$. If k is a field, then $\dim(k(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_n\rangle) = n$.

Definition 5 ([2], [3]): The *grade* of a non-zero finitely generated left D-module M is defined by:
$$j_D(M) = \min\,\{i \ge 0 \mid \mathrm{ext}^i_D(M, D) \neq 0\}.$$

$$0 \longrightarrow \mathrm{ext}_D^1(N,D) \longrightarrow M \xrightarrow{\varepsilon} \hom_D(\hom_D(M,D),D) \longrightarrow \mathrm{ext}_D^2(N,D) \longrightarrow 0, \tag{6}$$

Theorem 3 ([2], [3]): Let M be a non-zero finitely generated left $D = A\langle\partial_1,\ldots,\partial_n\rangle$-module. Then:

$$j_D(M) = \dim(D) - \dim_D(M). \tag{7}$$

Remark 1: A ring D satisfying (7) for all finitely generated left D-modules M and a dimension function $\dim_D(\,\cdot\,)$ is called a *Cohen-Macaulay ring*. Hence, the previous rings of PD operators are Cohen-Macaulay. Moreover, they are also *Auslander regular rings*, namely, noetherian rings with a finite global dimension which satisfy the *Auslander condition*, namely, for every $i \in \mathbb{N}$, every finitely generated left (resp., right) D-module M and every left (resp., right) D-module $N \subseteq \mathrm{ext}_D^i(M,D)$, then $j_D(N) \geq i$ ([2], [3]).

Theorem 4 ([2], [3]): If M is a non-zero finitely generated left $D = A\langle\partial_1,\ldots,\partial_n\rangle$-module, then:

1) $\dim_D(\mathrm{ext}_D^i(M,D)) \leq \dim(D) - i$.
2) $\dim_D(\mathrm{ext}_D^{j_D(M)}(M,D)) = \dim(D) - j_D(M)$.

Theorem 5 ([2], [3]): Let M be a non-zero a finitely generated left $D = A\langle\partial_1,\ldots,\partial_n\rangle$-module.

1) $\mathrm{ext}_D^j(\mathrm{ext}_D^i(M,D),D) = 0$ for $j < i$.
2) If $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M,D),D)$ is non-zero, then $\dim_D(\mathrm{ext}_D^i(\mathrm{ext}_D^i(M,D),D)) = \dim(D) - i$.
3) $j_D(\mathrm{ext}_D^{j_D(M)}(M,D)) = j_D(M)$.

Definition 6 ([2], [3]): A finitely generated left D-module M is said to be *$j_D(M)$-pure* if $j_D(N) = j_D(M)$ for all non-zero left D-submodules N of M.

Theorem 6 ([2], [3]): If M is a non-zero finitely generated left D-module and $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M,D),D) \neq 0$, then the left D-module $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M,D),D)$ is i-pure.

Example 4: By Theorem 5, if $M = D^{1\times p}/(D^{1\times q}\,R)$, then the left D-module $\hom_D(\hom_D(M,D),D)$ is 0-pure. Hence, if $N = D^q/(R\,D^p)$, then (5) and (6) yield the inclusion $M/t(M) \subseteq \hom_D(\hom_D(M,D),D)$, which shows that the left D-module $M/t(M)$ is either zero or 0-pure.

Example 5: Let $M = D^{1\times p}/(D^{1\times p}\,R)$ be the left D-module finitely presented by a full row rank square matrix $R \in D^{p\times p} \setminus \mathrm{GL}_p(D)$, i.e., $M \neq 0$. Then, M is a torsion left D-module, i.e., $M = t(M)$. Since $N = D^p/(R\,D^p) \cong \mathrm{ext}_D^1(M,D)$, then using (5), we get $M = t(M) \cong \mathrm{ext}_D^1(\mathrm{ext}_D^1(M,D),D) \neq 0$. According to Theorems 5 and 6, $\dim_D(M) = \dim_D(\mathrm{ext}_D^1(\mathrm{ext}_D^1(M,D),D)) = \dim(D) - 1$ and M is a 1-pure left D-module. This result was conjectured by Janet in 1921 and first proved by Johnson in 1978.

III. General results

In what follows, we shall assume that D is a domain which is an Auslander regular ring (see Remark 1). Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be a left D-module finitely presented by $R \in D^{q\times p}$. Since D is a left noetherian ring, we can consider the beginning of a finite free resolution of M:

$$D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0. \tag{8}$$

Then, the defects of exactness of the following complex

$$D^r \xleftarrow{R_2.} D^q \xleftarrow{R.} D^p \longleftarrow 0$$

are the right D-modules defined by:

$$\begin{cases} \mathrm{ext}_D^0(M,D) = \hom_D(M,D) \cong \ker_D(R.), \\ \mathrm{ext}_D^1(M,D) \cong \ker_D(R_2.)/\mathrm{im}_D(R.). \end{cases}$$

Let $N_2 = D^r/(R_2\,D^q)$ (resp., $N = D^p/(R\,D^q)$) be the Auslander transpose right D-module of the left D-module $M_2 = D^{1\times q}/(D^{1\times r}\,R_2)$ (resp., $M = D^{1\times p}/(D^{1\times q}\,R)$). Since D is a right noetherian ring, we can consider the beginning of a finite free resolution of N_2 (resp., N):

$$\begin{array}{cccccccccc} 0 \longleftarrow & N_2 & \xleftarrow{\kappa_2} & D^r & \xleftarrow{R_2.} & D^q & \xleftarrow{R'.} & D^{p'} & \xleftarrow{Q'.} & D^{m'}, \\ 0 & \longleftarrow & N & \xleftarrow{\kappa} & D^q & \xleftarrow{R.} & D^p & \xleftarrow{Q.} & D^m. & \end{array} \tag{9}$$

Now, $\mathrm{im}_D(R.) = R\,D^p \subseteq \ker_D(R_2.) = R'\,D^{p'}$ implies that the columns of the matrix R belong to $R'\,D^{p'}$, and thus there exists a matrix $R'' \in D^{p'\times p}$ such that $R = R'\,R''$. Using $R\,Q = 0$ and $R = R'\,R''$, we get $R'\,(R''\,Q) = 0$, i.e., $(R''\,Q)\,D^m \subseteq \ker_D(R'.) = Q'\,D^{m'}$, and thus there exists $Q'' \in D^{m'\times m}$ such that $R''\,Q = Q'\,Q''$. If we denote by $N' = D^q/(R'\,D^{p'})$ the Auslander transpose right D-module of left D-module $M' = D^{1\times p'}/(D^{1\times q}\,R')$, then we get the following commutative exact diagram of right D-modules:

$$\begin{array}{ccccccccc} 0 \longleftarrow & N' & \xleftarrow{\kappa'} & D^q & \xleftarrow{R'.} & D^{p'} & \xleftarrow{Q'.} & D^{m'}, \\ & & & \| & & \uparrow R''. & & \uparrow Q''. \\ 0 \longleftarrow & N & \xleftarrow{\kappa} & D^q & \xleftarrow{R.} & D^p & \xleftarrow{Q.} & D^m. \end{array} \tag{10}$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,,D)$ to the previous commutative exact diagram ([9]), we obtain the following commutative diagram of left D-modules:

$$\begin{array}{ccccc} D^{1\times q} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{.Q'} & D^{1\times m'} \\ \| & & \downarrow .R'' & & \downarrow .Q'' \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m}. \end{array} \tag{11}$$

Using (5), we obtain:

$$\begin{cases} t(M') \cong \mathrm{ext}_D^1(N',D) \cong \ker_D(.Q')/\mathrm{im}_D(.R'), \\ t(M) \cong \mathrm{ext}_D^1(N,D) \cong \ker_D(.Q)/\mathrm{im}_D(.R). \end{cases} \tag{12}$$

If $\pi' : D^{1\times p'} \longrightarrow M' = D^{1\times p'}/(D^{1\times q}\,R')$ is the canonical projection onto M', then the commutative diagram (11) yields the following well-defined left D-homomorphism:

$$\begin{array}{rcl} \alpha : \ker_D(.Q')/\mathrm{im}_D(.R') & \longrightarrow & \ker_D(.Q)/\mathrm{im}_D(.R), \\ \pi'(\lambda) & \longmapsto & \pi(\lambda\,R''). \end{array} \tag{13}$$

Indeed, if $\pi'(\lambda)=\pi'(\lambda')$, then there exists $\mu \in D^{1\times q}$ such that $\lambda-\lambda'=\mu\, R'$ and, using $R=R'\,R''$, we obtain:

$$\begin{aligned}\alpha(\pi'(\lambda)) &= \pi(\lambda\, R'')=\pi((\lambda'+\mu\, R')\, R'')\\ &= \pi(\lambda'\, R'')+\pi(\mu\, R)=\pi(\lambda'\, R'')=\alpha(\pi'(\lambda')).\end{aligned}$$

The classical third isomorphism theorem in module theory (see, e.g., [9]) yields the following short exact sequence:

$$0 \longrightarrow (R'\, D^{p'})/(R\, D^{p}) \xrightarrow{i} N \xrightarrow{\rho} N' \longrightarrow 0.$$

Since $\mathrm{ext}^1_D(M,D)\cong (R'\, D^{p'})/(R\, D^p)$, the previous short exact sequence yields the following short exact sequence:

$$0 \longrightarrow \mathrm{ext}^1_D(M,D) \xrightarrow{i'} N \xrightarrow{\rho} N' \longrightarrow 0.$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,,D)$ to the previous short exact sequence, we obtain the long exact sequence of left D-modules defined by (14) (see, e.g., [9]). Since D is an Auslander regular ring (see Remark 1), $\mathrm{ext}^0_D(\mathrm{ext}^1_D(M,D),D)=0$ and using (12), we obtain the following exact sequence of left D-modules

$$0 \longrightarrow \mathrm{ext}^1_D(N',D) \xrightarrow{\alpha} t(M) \xrightarrow{\beta} \mathrm{ext}^1_D(\mathrm{ext}^1_D(M,D),D), \tag{15}$$

which yields the following short exact sequence:

$$0 \longrightarrow \mathrm{ext}^1_D(N',D) \xrightarrow{\alpha} t(M) \longrightarrow \mathrm{coker}\,\alpha \longrightarrow 0. \tag{16}$$

Now, since $\mathrm{ext}^i_D(\,\cdot\,,D^r)=0$ for all $i\geq 1$ ([9]), the following short exact sequence

$$0 \longrightarrow N' \longrightarrow D^r \longrightarrow N_2 \longrightarrow 0,$$

implies the following isomorphisms ([9]):

$$\forall\, i\geq 1, \quad \mathrm{ext}^i_D(N',D)\cong \mathrm{ext}^{i+1}_D(N_2,D). \tag{17}$$

The long exact sequence (15) then yields the following one:

$$0 \longrightarrow \mathrm{ext}^2_D(N_2,D) \xrightarrow{\alpha'} t(M) \xrightarrow{\beta} \mathrm{ext}^1_D(\mathrm{ext}^1_D(M,D),D). \tag{18}$$

If we consider the beginning of a finite free resolution of $L'=D^{1\times m'}/(D^{1\times p'}\, Q')$ (resp., $L=D^{1\times m}/(D^{1\times p}\, Q)$), then, repeating what we have just done for the commutative exact diagram (10), the identity $Q'\, Q''=R''\, Q$ yields the following commutative exact diagram of left D-modules:

$$\begin{array}{ccccccc} D^{1\times u'} & \xrightarrow{.T'} & D^{1\times t'} & \xrightarrow{.S'} & D^{1\times p'} & \xrightarrow{.Q'} & D^{1\times m'} \\ \downarrow .T'' & & \downarrow .S'' & & \downarrow .R'' & & \downarrow .Q'' \\ D^{1\times u} & \xrightarrow{.T} & D^{1\times t} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m}. \end{array} \tag{19}$$

Then, (12) becomes:

$$\begin{cases} \mathrm{ext}^1_D(N',D)\cong (D^{1\times t'}\, S')/(D^{1\times q}\, R'), \\ \mathrm{ext}^1_D(N,D)\cong (D^{1\times t}\, S)/(D^{1\times q}\, R). \end{cases}$$

Now, since $D^{1\times q}\, R'\subseteq D^{1\times t'}\, S'$ and $D^{1\times q}\, R\subseteq D^{1\times t}\, S$, there exist $F'\in D^{q\times t'}$ and $F\in D^{q\times t}$ such that:

$$\begin{cases} R'=F'\, S', \\ R=F\, S. \end{cases} \tag{20}$$

Proposition 2 (Lemma 3.1 of [5]): Let D be a left noetherian ring, $R\in D^{q\times p}$ and $R'\in D^{q'\times p}$ two matrices such that $D^{1\times q}\, R\subseteq D^{1\times q'}\, R'$, i.e., satisfying $R=R''\, R'$ for a certain matrix $R''\in D^{q\times q'}$. Let $R'_2\in D^{r'\times q'}$ be a matrix such that $\ker_D(.R')=D^{1\times r'}\, R'_2$ and let us respectively denote by π and π' the following canonical projections:

$$\begin{aligned} \pi : D^{1\times q'}\, R' &\longrightarrow P=(D^{1\times q'}\, R')/(D^{1\times q}\, R), \\ \pi' : D^{1\times q'} &\longrightarrow P'=D^{1\times q'}/(D^{1\times q}\, R''+D^{1\times r'}\, R'_2). \end{aligned}$$

Then, we have the following left D-isomorphism χ:

$$\begin{array}{llll} \chi : P' & \longrightarrow & P & \\ \pi'(\lambda) & \longmapsto & \pi(\lambda\, R'), \end{array} \qquad \begin{array}{lll} \chi^{-1} : P & \longrightarrow & P' \\ \pi(\lambda\, R') & \longmapsto & \pi'(\lambda). \end{array} \tag{21}$$

Using Proposition 2 and (12), we obtain

$$\begin{array}{rcl} \chi' : L'=D^{1\times t'}/(D^{1\times q}\, F'+D^{1\times u'}\, T') & \longrightarrow & t(M'), \\ \gamma'(\mu) & \longmapsto & \pi'(\mu\, S'), \end{array} \tag{22}$$

$$\begin{array}{rcl} \chi : L=D^{1\times t}/(D^{1\times q}\, F+D^{1\times u}\, T) & \longrightarrow & t(M) \\ \gamma(\nu) & \longmapsto & \pi(\nu\, S), \end{array} \tag{23}$$

where $\gamma' : D^{1\times t'} \longrightarrow L'$ (resp., $\gamma : D^{1\times t} \longrightarrow L$) is the canonical projection onto L (resp., L').

Using (22) and $S'\, R''=S''\, S$, α defined by (13) yields the left D-homomorphism $\overline{\alpha}=\chi^{-1}\circ\alpha\circ\chi' : L' \longrightarrow L$:

$$\begin{aligned} \overline{\alpha}(\gamma'(\mu)) &= (\chi^{-1}\circ\alpha)(\pi'(\mu\, S'))=\chi^{-1}(\pi(\mu\, S'\, R'')) \\ &= \chi^{-1}(\pi((\mu\, S'')\, S))=\gamma(\mu\, S''). \end{aligned} \tag{24}$$

Using the identities $R=R'\, R''$, $S'\, R''=S''\, S$ and (20), we get $F\, S=R=R'\, R''=F'\, S'\, R''=F'\, S''\, S$, and thus $(F-F'\, S'')\, S=0$, i.e., $D^{1\times q}\,(F-F'\, S'')\subseteq \ker_D(.S)=D^{1\times u}\, T$, i.e., there exists $X\in D^{q\times u}$ such that:

$$F=F'\, S''+X\, T. \tag{25}$$

Moreover, using (25) and $T'\, S''=T''\, T$, we have:

$$\left(\begin{array}{c} F' \\ T' \end{array}\right) S''=\left(\begin{array}{c} F-X\, T \\ T''\, T \end{array}\right)=\left(\begin{array}{cc} I_q & -X \\ 0 & T'' \end{array}\right)\left(\begin{array}{c} F \\ T' \end{array}\right).$$

Therefore, if $V\in D^{(q+s')\times(q+s)}$ is the first matrix in the right hand-side of the above equality, then we obtain the following commutative exact diagram of left D-modules:

$$\begin{array}{ccccccc} D^{1\times(q+u')} & \xrightarrow{.(F'^T \quad T'^T)^T} & D^{1\times t'} & \xrightarrow{\gamma'} & L' & \longrightarrow & 0 \\ \downarrow .V & & \downarrow .S'' & & \downarrow \overline{\alpha} & & \\ D^{1\times(q+u)} & \xrightarrow{.(F^T \quad T^T)^T} & D^{1\times t} & \xrightarrow{\gamma} & L & \longrightarrow & 0. \end{array}$$

Then, $\mathrm{coker}\,\overline{\alpha}\cong D^{1\times t}/(D^{1\times t'}\, S''+D^{1\times q}\, F+D^{1\times u}\, T)$ (see [5]). (25) implies $D^{1\times t'}\, S''+D^{1\times q}\, F+D^{1\times u}\, T=D^{1\times r'}\, S''+D^{1\times u}\, T$, and thus:

$$\mathrm{coker}\,\overline{\alpha}=D^{1\times t}/(D^{1\times t'}\, S''+D^{1\times u}\, T). \tag{26}$$

If $L''=\mathrm{coker}\,\overline{\alpha}$ and $\overline{\beta} : L \longrightarrow L''$ is the canonical projection onto L'', then, up to isomorphism, (16) corresponds to the following short exact sequence:

$$0 \longrightarrow L' \xrightarrow{\overline{\alpha}} L \xrightarrow{\overline{\beta}} L'' \longrightarrow 0.$$

$$
\begin{array}{llllll}
0 & \longrightarrow & \mathrm{ext}_D^0(N', D) & \longrightarrow & \mathrm{ext}_D^0(N, D) & \longrightarrow & \mathrm{ext}_D^0(\mathrm{ext}_D^1(M, D), D) \\
 & \xrightarrow{\delta^1} & \mathrm{ext}_D^1(N', D) & \longrightarrow & \mathrm{ext}_D^1(N, D) & \longrightarrow & \mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D) \\
 & \xrightarrow{\delta^2} & \mathrm{ext}_D^2(N', D) & \longrightarrow & \quad \dots & &
\end{array}
\tag{14}
$$

If $\gamma'' : D^{1\times t} \longrightarrow L''$ is the canonical projection and

$$W = \left(\begin{array}{cc} F' & X \\ 0 & I_u \end{array} \right) \in D^{(q+u)\times(t'+u)},$$

then we have the following commutative exact diagram

$$
\begin{array}{ccccccc}
D^{1\times(q+u)} & \xrightarrow{.(F^T \quad T^T)^T} & D^{1\times t} & \xrightarrow{\gamma} & L & \longrightarrow 0 \\
\downarrow .W & & \| & & \downarrow \overline{\beta} & \\
D^{1\times(t'+u)} & \xrightarrow{.(S''^T \quad T^T)^T} & D^{1\times t} & \xrightarrow{\gamma''} & L'' & \longrightarrow 0,
\end{array}
\tag{27}
$$

i.e., $\gamma'' = \overline{\beta} \circ \gamma$ and the left D-homomorphism $\overline{\beta}$ is:

$$
\begin{array}{rcl}
\overline{\beta} : L & \longrightarrow & L'' = \mathrm{coker}\, \overline{\alpha} \\
\gamma(\nu) & \longmapsto & \gamma''(\nu).
\end{array}
\tag{28}
$$

Proposition 3: Let D be a noetherian domain, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q} R)$ the left D-module finitely presented R. Let the matrices $R_2' \in D^{r\times q}$, $R' \in D^{q\times p'}$, $Q' \in D^{p'\times m'}$, $Q \in D^{p\times m}$, $S \in D^{t\times p}$, $S' \in D^{t'\times p'}$, $R'' \in D^{p'\times p}$, $S'' \in D^{t'\times t}$, $T \in D^{u\times t}$, $T' \in D^{u'\times t'}$, $F \in D^{q\times t}$ and $F' \in D^{q\times t'}$ be respectively defined by:

$$
\ker_D(.R) = D^{1\times r} R_2, \quad \left\{ \begin{array}{l} \ker_D(R_2.) = R' D^{p'}, \\ \ker_D(R'.) = Q' D^{m'}, \\ \ker_D(R.) = Q D^m, \end{array} \right.
$$

$$
\left\{ \begin{array}{l} \ker_D(.Q) = D^{1\times t} S, \\ \ker_D(.Q') = D^{1\times t'} S', \end{array} \right. \quad \left\{ \begin{array}{l} R = R' R'', \\ S'' S = S' R'', \end{array} \right.
$$

$$
\left\{ \begin{array}{l} R = F S, \\ R' = F' S', \end{array} \right. \quad \left\{ \begin{array}{l} \ker_D(.S) = D^{1\times u} T, \\ \ker_D(.S') = D^{1\times u'} T'. \end{array} \right.
$$

Then, we have the following results:

1) If we set $N = D^q/(R D^p)$, $N' = D^q/(R' D^{p'})$ and $N_2 = D^r/(R_2 D^q)$, then we have:

$$
\begin{array}{rcl}
t(M) & = & (D^{1\times t} S)/(D^{1\times q} R) \\
 & \cong & \mathrm{ext}_D^1(N, D) \\
 & \cong & L = D^{1\times t}/(D^{1\times q} F + D^{1\times u} T), \\
M/t(M) & = & D^{1\times p}/(D^{1\times t} S), \\
\mathrm{ext}_D^2(N_2, D) & \cong & \mathrm{ext}_D^1(N', D) \\
 & \cong & (D^{1\times t'} S')/(D^{1\times q} R') \\
 & \cong & L' = D^{1\times t'}/(D^{1\times q} F' + D^{1\times u'} T'). \\
L'' & = & D^{1\times t}/(D^{1\times t'} S'' + D^{1\times u} T).
\end{array}
\tag{29}
$$

2) The exact diagram (30) holds.
3) We have the following short exact sequence

$$0 \longrightarrow L' \xrightarrow{\overline{\alpha}} L \xrightarrow{\overline{\beta}} L'' \longrightarrow 0, \tag{31}$$

where the left D-homomorphisms $\overline{\alpha}$ and $\overline{\beta}$ are respectively defined by $\overline{\alpha}(\gamma'(\mu)) = \gamma(\mu\, S'')$ for all $\mu \in D^{1\times t'}$, and $\overline{\beta}(\gamma(\nu))) = \gamma''(\nu)$ for all $\nu \in D^{1\times t}$, with the following canonical projections:

$$\gamma : D^{1\times t} \longrightarrow L, \ \gamma' : D^{1\times t'} \longrightarrow L', \ \gamma'' : D^{1\times t} \longrightarrow L''.$$

IV. Purity filtration

Using (15) and (16), we obtain

$$L'' = \mathrm{coker}\, \overline{\alpha} \cong \mathrm{im}\, \beta \subseteq \mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D),$$

which proves that the left D-module L'' is 1-pure, and thus:

$$\mathrm{codim}_D(L'') = 1.$$

If R_2 has full row rank, namely, $\ker_D(.R_2) = 0$, then we have $N_2 \cong \mathrm{ext}_D^2(M, D)$, which then yields:

$$L' \cong \mathrm{ext}_D^2(N_2, D) \cong \mathrm{ext}_D^2(\mathrm{ext}_D^2(M, D), D). \tag{32}$$

Thus, by 1 of Theorem 6, the left D-module L' is 2-pure. Using the left D-isomorphism χ defined by (23), (2) yields the following short exact sequence:

$$0 \longrightarrow L \xrightarrow{i\,\circ\,\chi} M \xrightarrow{\rho} M/t(M) \longrightarrow 0.$$

Hence, using (31), we get the following chain of inclusions:

$$0 \subseteq (i \circ \chi \circ \overline{\alpha})(L') \subseteq (i \circ \chi)(L) \subseteq M.$$

If $D = A\langle\partial_1, \dots, \partial_n\rangle$, where A is either a field k or $k[x_1, \dots, x_n]$, $k(x_1, \dots, x_n)$, $k[\![x_1, \dots, x_n]\!]$ where k a field of characteristic 0, or $k\{x_1, \dots, x_n\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, then the filtration $\{M_i\}_{i=0,\dots,3}$ of the left D-module M defined by $M_0 = M$, $M_1 = (i \circ \chi)(L)$, $M_2 = (i \circ \chi \circ \overline{\alpha})(L')$ and $M_3 = 0$, is called a *purity filtration* ([3]) since the successive quotients $M_2/M_3 \cong L'$, $M_1/M_2 \cong L''$ and $M_0/M_1 \cong M/t(M)$ are respectively 2-pure, 1-pure and 0-pure, i.e., by 1 of Theorem 6, are respectively of dimension $\dim(D) - 2$, $\dim(D) - 1$ and $\dim(D)$, where, for instance

$$
\begin{array}{ll}
\dim(k\langle\partial_1, \dots \partial_n\rangle) = n, & \dim(B_n(k')) = n, \\
\dim(A_n(k')) = 2\,n, & \dim\left(A\langle\partial_1, \dots, \partial_n\rangle\right) = 2\,n,
\end{array}
$$

and k (resp., k') is field (resp., a field of characteristic 0) and $A = k[\![x_1, \dots, x_2]\!]$, $\mathbb{R}\{x_1, \dots, x_2\}$ or $\mathbb{C}\{x_1, \dots, x_2\}$.

Another simple case is $\mathrm{gld}(D) \leq 2$ such as, e.g., $D = A\langle\partial_1, \partial_2\rangle$, where $A = k$, $k[x_1, x_2]$, $k(x_1, x_2)$ or $k[\![x_1, x_2]\!]$ and k a field of characteristic 0, or $k\{x_1, x_2\}$, where $k = \mathbb{R}$ or $\mathbb{C}$ (see Example 1). Then, (17) yields $\mathrm{ext}_D^2(N', D) \cong \mathrm{ext}_D^3(N_2, D) = 0$, and thus (14) shows that the left D-homomorphism $\beta : t(M) \longrightarrow \mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D)$, defined by (15), is surjective, i.e.,

$$L'' \cong \mathrm{coker}\, \beta = \mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D)$$

is 1-pure and $\mathrm{codim}_D(L'') = 1$. Therefore, we get:

$$
\left\{ \begin{array}{l}
\mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D) \cong D^{1\times t}/(D^{1\times t'} S'' + D^{1\times u} T), \\
\mathrm{ext}_D^2(\mathrm{ext}_D^2(M, D), D) \cong D^{1\times t'}/(D^{1\times q} F' + D^{1\times u'} T').
\end{array} \right.
\tag{33}
$$

$$
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
0 \longrightarrow \mathrm{ext}^2_D(N_2, D) & \xrightarrow{\alpha'} & t(M) & \xrightarrow{\beta} & \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) \longrightarrow \mathrm{coker}\,\beta \longrightarrow 0. & & \\
 & & \downarrow i & & & & \\
 & & M & & & & \\
 & & \downarrow \rho & & & & \\
 & & M/t(M) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
\tag{30}
$$

V. A BLOCK-TRIANGULAR FORM OF LINEAR SYSTEMS

The next theorem will play crucial role in what follows.

Theorem 7 ([8]): Let $M = D^{1\times p}/(D^{1\times q}\, R)$ and $N = D^{1\times s}/(D^{1\times t}\, S)$ be two finitely presented left D-modules and $R_2 \in D^{r\times q}$ satisfying $\ker_D(.R) = D^{1\times r}\, R_2$. Then, the following short exact sequence holds

$$
e:\ 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \tag{34}
$$

where the left D-module $E = D^{1\times (p+s)}/(D^{1\times (q+t)}\, Q)$ and

$$
Q = \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \in D^{(q+t)\times(p+s)}, \tag{35}
$$

and A is an element of the abelian group

$$
\Omega = \{A \in D^{q\times s} \mid \exists\, B \in D^{r\times t} : R_2\, A = B\, S\}
$$

$$
\begin{array}{rclrcl}
\alpha : N & \longrightarrow & E & \beta : E & \longrightarrow & M \\
\delta(\mu) & \longmapsto & \varrho(\mu\,(0 \quad I_s)), & \varrho(\lambda) & \longmapsto & \pi(\lambda\,(I_p \quad 0)^T),
\end{array}
$$

where $\pi : D^{1\times p} \longrightarrow M$, $\delta : D^{1\times s} \longrightarrow N$, $D^{1\times(p+s} \longrightarrow E$ are canonical projections. More precisely, $A \in D^{q\times s}$ is such as the following commutative exact diagram holds

$$
\begin{array}{ccccccc}
 & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\
 & \downarrow \phi & & \downarrow \psi & & \| & \\
0 \longrightarrow & N & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0,
\end{array}
$$

where the left D-homomorphisms ψ and ϕ are defined by

$$
\begin{array}{rcl}
\psi : D^{1\times p} & \longrightarrow & E \\
f_j & \longmapsto & \varrho\left(f_j \begin{pmatrix} I_p \\ 0 \end{pmatrix}\right),
\end{array}
\quad
\begin{array}{rcl}
\phi : D^{1\times q} & \longrightarrow & N \\
e_i & \longmapsto & \delta(A_{i\bullet}).
\end{array}
$$

and $\{e_i\}_{i=1,\ldots,q}$ is the standard basis of $D^{1\times q}$.

Let us apply Theorem 7 to the short exact sequence (31). Using (27), we have $\gamma'' = \overline{\beta}\circ\gamma$, which yields the following commutative exact diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & D^{1\times t'}\, S'' + D^{1\times u}\, T & \longrightarrow & D^{1\times t} & \xrightarrow{\gamma''} & L'' & \longrightarrow 0 \\
 & \downarrow \varphi & & \downarrow \gamma & & \| & \\
0 \longrightarrow & L' & \xrightarrow{\overline{\alpha}} & L & \xrightarrow{\overline{\beta}} & L'' & \longrightarrow 0,
\end{array}
\tag{36}
$$

where the left D-homomorphism φ is defined by:

$$
\begin{array}{rcl}
\varphi : D^{1\times t'}\, S'' + D^{1\times u}\, T & \longrightarrow & L' \\
\mu_1\, S'' + \mu_2\, T & \longmapsto & \gamma'(\mu_1).
\end{array}
$$

If $\{e_i\}_{i=1,\ldots,t'+u}$ is the standard basis of $D^{1\times(t'+u)}$, then

$$
\begin{array}{rcl}
\phi : D^{1\times(t'+u)} & \longrightarrow & L' \\
e_i & \longmapsto & \begin{cases} \gamma'(e_i), & i = 1, \ldots, t', \\ 0, & i = t'+1, \ldots, t'+u, \end{cases}
\end{array}
$$

and Theorem 7 shows that $A = (I_{t'}^T \quad 0^T)^T \in D^{(t'+u)\times t'}$.

Theorem 8: We have the following left D-isomorphisms

$$
t(M) \cong L \cong D^{1\times(t+t')}/(D^{1\times(t'+u+q+u')}\, U), \tag{37}
$$

where the matrix $U \in D^{(t'+u+q+u')\times(t+t')}$ is defined by:

$$
U = \begin{pmatrix} S'' & -I_{t'} \\ T & 0 \\ 0 & F' \\ 0 & T' \end{pmatrix}. \tag{38}
$$

Proposition 4: Let $E = D^{1\times(t+t')}/(D^{1\times(t'+u+q+u')}\, U)$ the left D-module finitely presented by the matrix U defined by (38) and $\varrho : D^{1\times(t+t')} \longrightarrow E$ the canonical projection onto E. Then, we have the following left D-isomorphisms

$$
\begin{array}{rclrcl}
\chi : L & \longrightarrow & t(M) & \phi : L & \longrightarrow & E \\
\gamma(\nu) & \longmapsto & \pi(\nu\, S), & \gamma(\nu) & \longmapsto & \varrho(\nu\,(I_t \quad 0)),
\end{array}
\tag{39}
$$

$$
\begin{array}{rcl}
\phi^{-1} : E & \longrightarrow & L \\
\varrho(\mu) & \longmapsto & \gamma\left(\mu \begin{pmatrix} I_t \\ S'' \end{pmatrix}\right).
\end{array}
\tag{40}
$$

Corollary 1: If $\mathcal{F}$ is a left D-module, then we have $\ker_{\mathcal{F}}(V.) \cong \ker_{\mathcal{F}}(U.)$, where $V = (F^T \quad T^T)^T$, i.e.,

$$
\left\{ \begin{array}{l} F\,\theta = 0, \\ T\,\theta = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} S''\,\tau - \upsilon = 0, \\ T\,\tau = 0, \\ F'\,\upsilon = 0, \\ T'\,\upsilon = 0, \end{array} \right. \tag{41}
$$

and the following invertible transformations:

$$
\begin{array}{rcl}
\delta : \ker_{\mathcal{F}}(U.) & \longrightarrow & \ker_{\mathcal{F}}(V.) \\
\begin{pmatrix} \tau \\ \upsilon \end{pmatrix} & \longmapsto & \theta = \tau, \\
\delta^{-1} : \ker_{\mathcal{F}}(V.) & \longrightarrow & \ker_{\mathcal{F}}(U.) \\
\theta & \longmapsto & \begin{pmatrix} \tau \\ \upsilon \end{pmatrix} = \begin{pmatrix} \theta \\ S''\,\theta \end{pmatrix},
\end{array}
\tag{42}
$$

Example 6: Let us consider the $D = \mathbb{Q}[\partial_1, \partial_2]$-module $M = D^{1\times 3}/(D^{1\times 3}\, R)$ finitely presented by:

$$R = \begin{pmatrix} 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 \\ \partial_1 & -\partial_1 & -2\,\partial_1 \end{pmatrix} \in D^{3\times 3}.$$

The D-module M admits the finite free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R} D^{1\times 3} \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_2 = (\partial_1 \quad -\partial_1 \quad \partial_2)$. Then, the defects of exactness of the complex $0 \longleftarrow D \xleftarrow{R_2.} D^3 \xleftarrow{R.} D^3 \longleftarrow 0$ are:

$$\begin{cases} \mathrm{ext}^0_D(M, D) \cong \ker_D(R.) = Q\,D, \\ \mathrm{ext}^1_D(M, D) \cong \ker_D(R_2.)/(R\,D^3) = (R'\,D^2)/(R\,D^3), \\ \mathrm{ext}^2_D(M, D) \cong D/(R_2\,D^3), \end{cases}$$

$$Q = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad R' = \begin{pmatrix} 1 & 0 \\ 1 & -\partial_2 \\ 0 & -\partial_1 \end{pmatrix}.$$

We have $Q' = 0$, $S' = I_2$, $F' = R'$, $T' = 0$,

$$S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad T = 0,$$

$$R'' = \begin{pmatrix} 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 \\ -1 & 1 & 2 \end{pmatrix},$$

$$S'' = \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ -1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ \partial_2 & -\partial_1 \\ \partial_1 & -\partial_1 \end{pmatrix}.$$

Then, (29) and (33) yield:

$$\begin{cases} t(M) \cong (D^{1\times 2}\,S)/(D^{1\times 3}\,R) \cong L = D^{1\times 2}/(D^{1\times 3}\,F), \\ \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) \cong L'' = D^{1\times 2}/(D^{1\times 2}\,S''), \\ \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D) \cong L' = D^{1\times 2}/(D^{1\times 3}\,R'). \end{cases}$$

Using Theorem 8, $t(M) \cong E = D^{1\times 4}/(D^{1\times 5}\,U)$, where:

$$U = \begin{pmatrix} 0 & \partial_2 - \partial_1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\partial_2 \\ 0 & 0 & 0 & -\partial_1 \end{pmatrix}.$$

Using (41), the following equivalences hold:

$$\begin{cases} \partial_2\,\theta_2 - \partial_1\,\theta_2 = 0, \\ \partial_2\,\theta_1 - \partial_1\,\theta_2 = 0, \\ \partial_1\,\theta_1 - \partial_1\,\theta_2 = 0, \end{cases} \Leftrightarrow \begin{cases} \partial_2\,\tau_2 - \partial_1\,\tau_2 - \upsilon_1 = 0, \\ -\tau_1 + \tau_2 - \upsilon_2 = 0, \\ \upsilon_1 = 0, \\ \upsilon_1 - \partial_2\,\upsilon_2 = 0, \\ -\partial_1\,\upsilon_2 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \tau_1 = \tau_2 - \upsilon_2, \\ \partial_2\,\tau_2 - \partial_1\,\tau_2 = 0, \\ \upsilon_1 = 0, \\ \partial_1\,\upsilon_2 = 0, \\ \partial_2\,\upsilon_2 = 0. \end{cases} \tag{43}$$

Then, we obtain $\upsilon_1 = 0$, $\upsilon_2 = c$, $\tau_2 = f(x_1 + x_2)$ and $\tau_1 = f(x_1 + x_2) + c$, where f is an arbitrary smooth function and c an arbitrary constant, and using (42), we obtain the following general solution

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} f(x_1 + x_2) + c \\ f(x_1 + x_2) \end{pmatrix}$$

of the first linear PD system defined in (43).

Using the isomorphisms $\chi : L \longrightarrow t(M)$ defined by (23) and $\phi^{-1} : E \longrightarrow L$ defined by (40), we obtain the following short exact sequence of left D-modules:

$$0 \longrightarrow E \xrightarrow{i\,\circ\,\chi\,\circ\,\phi^{-1}} M \xrightarrow{\rho} M/t(M) \longrightarrow 0. \tag{44}$$

Since $M/t(M) = D^{1\times p}/\ker_D(.Q) = D^{1\times p}/(D^{1\times t}\,S)$, we get the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times t}\,S & \longrightarrow & D^{1\times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \pi & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{i\,\circ\,\chi} & M & \xrightarrow{\rho} & M/t(M) & \longrightarrow & 0, \end{array}$$

where $\varphi : D^{1\times t}\,S \longrightarrow L$ is defined by $\varphi(S_{j\bullet}) = \gamma(g_j)$, where $\{g_j\}_{j=1,\ldots,t}$ is the standard basis of $D^{1\times t}$. Using the left D-homomorphism $\phi : L \longrightarrow E$ defined by (39), we get the following commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times t}\,S & \longrightarrow & D^{1\times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \pi & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{i\,\circ\,\chi} & M & \xrightarrow{\rho} & M/t(M) & \longrightarrow & 0 \\ & & \downarrow \phi & & \| & & \| & & \\ 0 & \longrightarrow & E & \xrightarrow{i\,\circ\,\chi\,\circ\,\iota^{-1}} & M & \xrightarrow{\rho} & M/t(M) & \longrightarrow & 0, \end{array}$$

which induces $\tau : D^{1\times t}\,S \longrightarrow E$ defined by $\tau = \phi \circ \varphi$, and thus the following left D-homomorphism:

$$\begin{array}{rcl} \theta : D^{1\times t} & \longrightarrow & E \\ f_j & \longmapsto & \rho(f_j\,(I_t \quad 0)), \quad j = 1, \ldots, t. \end{array}$$

Finally, applying Theorem 7 to the short exact sequence (44) with θ, we obtain the following main theorem.

Theorem 9: We have the following left D-isomorphism

$$M \cong D^{1\times (p+t+t')}/(D^{1\times (t+t'+u+q+u')}\,P), \tag{45}$$

where $P \in D^{(t+t'+u+q+u')\times(p+t+t')}$ is defined by:

$$P = \begin{pmatrix} S & -I_t & 0 \\ 0 & S'' & -I_{t'} \\ 0 & T & 0 \\ 0 & 0 & F' \\ 0 & 0 & T' \end{pmatrix}. \tag{46}$$

Proposition 5: Let O be the left D-module finitely presented by the matrix P defined by (46), namely, $O = D^{1\times(p+t+t')}/(D^{1\times(t+t'+u+q+u')}\,P)$, and $\vartheta : D^{1\times(p+t+t')} \longrightarrow O$ the canonical projection onto O. Then, the left D-homomorphism defined by

$$\begin{array}{rcl} \varpi : M & \longrightarrow & O \\ \pi(\lambda) & \longmapsto & \vartheta(\lambda\,(I_p \quad 0)), \end{array} \tag{47}$$

is an isomorphism and its inverse ϖ^{-1} is defined by

$$\begin{array}{rcl} \varpi^{-1}: O & \longrightarrow & M \\ \vartheta(\mu) & \longmapsto & \pi\left(\mu\left(\begin{array}{c} I_p \\ S \\ S''\,S \end{array}\right)\right). \end{array} \tag{48}$$

Corollary 2: If $\mathcal{F}$ is a left D-module, then we have

$$R\,\eta = 0 \quad \Leftrightarrow \quad \left\{\begin{array}{l} S\,\eta - \tau = 0, \\ S''\,\tau - \upsilon = 0, \\ T\,\tau = 0, \\ F'\,\upsilon = 0, \\ T'\,\upsilon = 0. \end{array}\right. \tag{49}$$

and the following invertible transformations:

$$\begin{array}{rcl} \chi: \ker_{\mathcal{F}}(P.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \left(\begin{array}{c} \zeta \\ \tau \\ \upsilon \end{array}\right) & \longmapsto & \eta = \zeta, \\ \chi^{-1}: \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(P.) \\ \eta & \longmapsto & \left(\begin{array}{c} \zeta \\ \tau \\ \upsilon \end{array}\right) = \left(\begin{array}{c} \eta \\ S\,\eta \\ S''\,S\,\eta \end{array}\right). \end{array} \tag{50}$$

The linear system $R\,\eta = 0$ can be integrated in cascade:

1) We first integrate the linear system in υ formed by last two equations of (49). This linear system has dimension less or equal to $\dim(D) - 2$ (see 1 of Theorem 4). If R_2 has full row rank, then, using (32), this linear system has exactly $\dim(D) - 2$.
2) We integrate the inhomogeneous linear system in τ formed by the second and third equations of (49). Its homogenous part has dimension $\dim(D) - 1$.
3) We integrate the linear inhomogeneous linear system $S\,\eta = \tau$ using the results developed in [4], [8]. In particular, if $\mathcal{F}$ is an injective left D-module, the linear system $\ker_{\mathcal{F}}(S.)$ admits the parametrization $\ker_{\mathcal{F}}(S.) = Q\,\mathcal{F}^m$. Hence, we only need to find a particular solution $\eta_\star \in \mathcal{F}^p$ of the inhomogeneous linear system $S\,\eta = \tau$ to get the general solution $\eta = \eta_\star + Q\,\xi$ for all $\xi \in \mathcal{F}^m$ of $R\,\eta = 0$.

Example 7: We continue Example 6. Corollary 2 yields

$$\left\{\begin{array}{l} \partial_2\,\eta_2 - \partial_1\,\eta_2 + \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_2\,\eta_1 - \partial_1\,\eta_2 - \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_1\,\eta_1 - \partial_1\,\eta_2 - 2\,\partial_1\,\eta_3 = 0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} \zeta_1 - \zeta_3 - \tau_1 = 0, \\ \zeta_2 + \zeta_3 - \tau_2 = 0, \\ \partial_2\,\tau_2 - \partial_1\,\tau_2 - \upsilon_1 = 0, \\ -\tau_1 + \tau_2 - \upsilon_2 = 0, \\ \upsilon_1 = 0, \\ \upsilon_1 - \partial_2\,\upsilon_2 = 0, \\ -\partial_1\,\upsilon_2 = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \zeta_1 - \zeta_3 + \tau_2 - \upsilon_2 = 0, \\ \zeta_2 + \zeta_3 - \tau_2 = 0, \\ \tau_1 = \tau_2 - \upsilon_2, \\ \partial_2\,\tau_2 - \partial_1\,\tau_2 = 0, \\ \upsilon_1 = 0, \\ \partial_1\,\upsilon_2 = 0, \\ \partial_2\,\upsilon_2 = 0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} \zeta_1 = \zeta_3 + f(x_1 + x_2) - c, \\ \zeta_2 = -\zeta_3 - f(x_1 + x_2), \\ \tau_1 = f(x_1 + x_2) + c, \\ \tau_2 = f(x_1 + x_2), \\ \upsilon_1 = 0, \\ \upsilon_2 = c, \end{array}\right.$$

where ζ_3 is an arbitrary function of $C^\infty(\mathbb{R}^2)$, f an arbitrary function of $C^\infty(\mathbb{R})$ and c an arbitrary constant. Finally, using (50), the general solution of $R\,\eta = 0$ is defined by:

$$\left(\begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \end{array}\right) = \left(\begin{array}{c} \zeta_3(x_1, x_2) - f(x_1 + x_2) + c \\ -\zeta_3(x_1, x_2) - f(x_1 + x_2) \\ \zeta_3(x_1, x_2) \end{array}\right).$$

Finally, if $\ker_D(.R_2) \neq 0$, then the purity filtration of the left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module M can similarly be obtained by studying the left D-module L''. For more details, see [7].

The existence of the purity filtration of the left D-module M is proved by means of *spectral sequences* ([3]). The spectral sequences computing the purity filtration of differential modules have recently been implemented in the GAP4 package `homalg` by Barakat ([1]). In this paper (see also [7]), we have shown how the purity filtration of a left $D = A\langle\partial_1, \partial_2\rangle$-module M can be characterized and computed by generalizing the ideas developed in [4]. The results are implemented in the package PURITYFILTRATION.

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