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Localization and parametrization of linear multidimensional control systems

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Abstract

We study the link existing between the parametrization of differential operators by potential-like arbitrary functions and the localization of differential modules, while applying these results to the parametrization of linear multidimensional control systems. We show that the localization of differential modules is a natural way to generalize some well-known results on transfer matrix, classically obtained by using Laplace transform, to time-varying ordinary differential control systems and to partial differential control systems with variable coefficients. In particular, we show that the parametrizations obtained by localization are simpler than those obtained by formal duality but are worse in the sense of Palamodov–Kashiwara's classification of differential modules.

Keywords: Controllability; Controller form; Transfer matrix; Minimal realization; Parametrization of differential operators; Localization; Formal integrability; Differential module; Homological algebra

1. Introduction

Recently, the language of differential modules has been introduced in control theory to understand intrinsically the structural properties of control systems. Our approach is guided by the wish to have a tool, which formally should look like the Laplace transform, but which could permit a better understanding of the concepts of controllability and observability. For instance, the system $\ddot{y} - \dot{y} - 2y - \dot{u} + 2u = 0$ with $y(0) = \dot{y}(0) = u(0) = 0$ becomes, by the Laplace transform, $\hat{y} = \frac{(s-2)}{(s+1)(s-2)}\hat{u}$, where we denote by $\hat{y}$, $\hat{u}$ the Laplace transform of y, u. The transfer function of the system is $H(s) = \frac{(s-2)}{(s+1)(s-2)} = \frac{1}{s+1}$, after the cancellation of the common factor $(s-2)$. In 1960, Kalman was the first to understand that such a cancellation led to the loss of the system controllability [11]. The use of differential modules prevents any cancellation [1,4–6,19,21]. Indeed, we write the system as $(\frac{\mathrm{d}}{\mathrm{d}t} + 1)(\frac{\mathrm{d}}{\mathrm{d}t} - 2)y - (\frac{\mathrm{d}}{\mathrm{d}t} - 2)u = 0$, i.e. $(\frac{\mathrm{d}}{\mathrm{d}t} - 2)((\frac{\mathrm{d}}{\mathrm{d}t} + 1)y - u) = 0$. The last equation leads to the existence of a *torsion element*, i.e. an element $z = (\frac{\mathrm{d}}{\mathrm{d}t} + 1)y - u$ satisfying the non zero equation $(\frac{\mathrm{d}}{\mathrm{d}t} - 2)z = 0$. Thus, the controllability of control systems can be seen as the lack of any torsion element, that is to say, in the algebraic language, the differential module generated by the control system is a $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$ *torsion-free* module [32]. Hence, the use of modules permits to give an intrinsic definition of controllability, not depending on the choice of inputs and outputs among the control variables. It also provides a clear understanding of the previous

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the D-submodule generated by the linear formal differential components of $\mathscr{D}_1 y$.

Remark 1. We shall use either the language of jet theory for systems of partial differential equations (PDE) or the language of sections for operators [21]. In the first case, we write $d_i y^k_\mu = y^k_{\mu+1_i}$, whereas, in the second case, d_i must be replaced by ∂_i on sections. We shall use the notation $\partial_{ij} = \partial_i \partial_j$.

Remark 2. In this framework, we can study the following types of systems:

1. $D = k[\frac{\mathrm{d}}{\mathrm{d}t}] = k[x]$: ordinary differential systems with constant coefficients.
2. $D = K[\frac{\mathrm{d}}{\mathrm{d}t}]$: ordinary differential systems with variable coefficients (e.g. $K = \mathbb{R}(t)$).
3. $D = k[\frac{\mathrm{d}}{\mathrm{d}t}, \delta_1, \dots, \delta_n]$: delay differential systems.
4. $D = k[\chi_1, \dots, \chi_n]$: n-dimensional systems with constant coefficients.

Example 1. Let us consider the following control system, adapted from a delay differential system presented in [18] but interpreted as a partial differential system ($\partial_1 = \delta, \partial_2 = \frac{\mathrm{d}}{\mathrm{d}t}$):

$$\partial_2 y^1 - \partial_1 y^1 + 2y^1 + 2y^2 - 2\partial_1 y^3 = 0,$$
$$\partial_2 y^1 + \partial_2 y^2 - \partial_1 y^3 = 0.$$

We associate with this system the following differential operator $\mathscr{D}_1 : \eta \to \zeta$ defined by

$$\begin{aligned} &\partial_2 \eta^1 - \partial_1 \eta^1 - 2\partial_1 \eta^3 + 2\eta^1 + 2\eta^2 = \zeta^1, \\ &\partial_2 \eta^1 + \partial_2 \eta^2 - \partial_1 \eta^3 = \zeta^2, \end{aligned} \tag{1}$$

and the D-module $M = Dy/(y^1_2 - y^1_1 - 2y^3_1 + 2y^1 + 2y^2, y^1_2 + y^2_2 - y^3_1)$ where $D = \mathbb{R}[d_1, d_2]$ and $y = (y^1, y^2, y^3)$.

We recall certain properties of a D-module M (see [32] for more details) defined as above and thus finitely generated over D.

Definition 2.
- A D-module M is *free* if M is isomorphic to D^r for a certain $r \in \mathbb{N}$.
- A D-module M is *projective* if there exists a free D-module F and a D-module N such that: $F = M \oplus N$.
- We call *torsion submodule* of M, the D-module defined by $t(M) = \{m \in M \mid \exists 0 \neq a \in D, am = 0\}$. The D-module M is *torsion-free* if $t(M) = 0$ and $M/t(M)$ is a torsion-free D-module.

It is quite easy to show that every free D-module is projective and every projective D-module is torsion-free. This can be summed up by the following module inclusions:

free $\subseteq$ projective $\subseteq \cdots \subseteq$ torsion-free.

We shall see in the next section how to check whether or not a finitely generated left D-module is respectively torsion-free or projective and we shall give a characterization of free modules. We have the following theorem, where part 2 is the famous Quillen–Suslin theorem (see [32,35] for more details).

Theorem 1.
1. *If D is a principal ideal ring (for example $D = K[\frac{\mathrm{d}}{\mathrm{d}t}]$), then every torsion-free D-module is a free D-module.*
2. *Every projective module over a polynomial ring $k[\chi_1, \dots, \chi_n]$, where k is a field, is free. Hence, over $D = k[d_1, \dots, d_n]$, any projective module is free.*

Now, if we consider a differential operator $\mathscr{D}_1 : F_0 \to F_1$ and a section ζ of F_1, then all the necessary conditions for the local existence of a section η of F_0, satisfying the inhomogenous system $\mathscr{D}_1 \eta = \zeta$, are of the form $\mathscr{D}_2 \zeta = 0$. The operator $\mathscr{D}_2$ only depends on the operator $\mathscr{D}_1$ and is called the *compatibility conditions* of $\mathscr{D}_1$. An historical problem was to construct effectively the operator $\mathscr{D}_2$ [2,9,30]. We can construct the operator $\mathscr{D}_2$ by bringing the operator $\mathscr{D}_1$ to *involutiveness* and start anew with $\mathscr{D}_2$. The procedure has to stop after at most $n+1$ steps, where n is the number of partial derivatives (see [3,21,28,29] for more details). We obtain a *formally exact sequence* (that is to say, each $\mathscr{D}_{i+1}$ represents exactly all the compatibility conditions of $\mathscr{D}_i$) of differential operators $\mathscr{D}_i$ on the "right" of $\mathscr{D}_1$ and depending only on $\mathscr{D}_1$:

$$0 \to \Theta \to F_0 \xrightarrow{\mathscr{D}_1} F_1 \xrightarrow{\mathscr{D}_2} \cdots \xrightarrow{\mathscr{D}_{n+1}} F_{n+1} \to 0, \tag{2}$$

where Θ denotes the solutions of $\mathscr{D}_1$. We can reformulate this result into the framework of D-modules [16,17,28]. Corresponding to Eq. (2), we have the following *free resolution* of the D-module M (see e.g. [32]), i.e. the exact sequence of D-modules:

$$0 \leftarrow M \leftarrow D^{r_0} \xleftarrow{.\mathscr{D}_1} D^{r_1} \xleftarrow{.\mathscr{D}_2} \cdots \xleftarrow{.\mathscr{D}_{n+1}} D^{r_{n+1}} \leftarrow 0, \tag{3}$$

where r_i is the fibered dimension of F_i ($r_0 = m, r_1 = l$) and $.\mathscr{D}_1$ means that we let operate a row vector of D^{r_1} on the left of $\mathscr{D}_1$ to obtain a row vector of D^{r_0}. Following Spencer's work [29], it can be shown that the last operator $\mathscr{D}_{n+1}$ determines a projective D-module.

For more details and applications, see the forthcoming publication [27]. We recall that a complete knowledge of M is given by the free resolution (3) and that the length of the exact sequence (3) explains why, in general, it is difficult to treat multidimensional systems with two or more independant unknowns (see discussions in [33,34,36]).

3. Parametrization

Dually, if we have a system of partial differential equations $\mathscr{D}_0\xi = 0$, it is useful to know when the solutions of the systems can be parametrized by certain arbitrary functions called "*potentials*" (for example, $\mathrm{curl}\,\eta = 0 \Rightarrow \exists\xi$ such that $\mathrm{grad}\,\xi = \eta$). The new problem is the following: starting with an operator $\mathscr{D}_1 : F_0 \to F_1$, can we find a criterion to decide about the existence of an operator $\mathscr{D}_0 : F_1 = E \to F_0$ such that $\mathscr{D}_1\eta = 0$ represents exactly all the compatibility conditions of the inhomogeneous system $\mathscr{D}_0\xi = \eta$. If such an operator $\mathscr{D}_0$ exists, we say that $\mathscr{D}_0$ *parametrizes* the operator $\mathscr{D}_1$. Starting anew with $\mathscr{D}_0$, we can try to construct a formally exact differential sequence, on the "left" of $\mathscr{D}_1$ and depending only on $\mathscr{D}_1$,

$$F_{-k-1} \xrightarrow{\mathscr{D}_{-k}} F_{-k} \xrightarrow{\mathscr{D}_{-k+1}} \cdots \xrightarrow{\mathscr{D}_{-1}} F_{-1} \xrightarrow{\mathscr{D}_0} F_0 \xrightarrow{\mathscr{D}_1} F_1, \qquad (4)$$

in which each operator exactly describes the compatibility conditions of the previous one. This criterion exists for syzygies. It was first provided in the seventies by Palamodov for linear partial differential operators with constant coefficients [20] and was extended by Kashiwara for analytic ones [12]. In the framework of D-modules, the criterion says that the obstructions to embed the D-module M into the following exact sequence of D-modules

$$0 \to M \to D^{r_{-1}} \xrightarrow{.\mathscr{D}_{-1}} D^{r_{-2}} \xrightarrow{.\mathscr{D}_{-2}} \cdots \xrightarrow{.\mathscr{D}_{-k}} D^{r_{-k-1}}, \qquad (5)$$

are given by

- for $k = 0$, the injectivity of the canonical map $M \xrightarrow{\phi} \hom_D(\hom_D(M,D),D)$ defined by $\forall x \in M$, $\forall f \in \hom_D(M,D)$: $\phi(x)(f) = f(x)$,
- for $k = 1$, the injectivity *and* the surjectivity of ϕ. In this case M is said to be *reflexive* [27]:
- for $k > 1$, the injectivity and the surjectivity of ϕ plus other conditions on the vanishing of certain extension modules [32] ($\forall i = 1,\dots,k-1$: $\mathrm{ext}^i_D(\mathrm{Hom}_D(M,D),D) = 0$).

The exact sequence (5) is the algebraic counterpart of the formally exact sequence of differential operators (4). This is a beautiful but delicate and powerful tool of homological algebra which is, in its general form, out of our scope now. See [27] for applications of (4) to linear multidimensional control systems. We shall only be interested here in the first step of (5), i.e. the embedding of M into a free D-module, and we shall give an example of a D-module embedded into an exact sequence of two free D-modules.

As far as we are concerned, the quite abstract and previous results of Palamodov–Kashiwara seem to be unknown in control theory, applied mathematics or physics. It is not evident, at the first glance, that the differential operator $\mathscr{D}_1$, which defines the control system, is parametrizable by an operator $\mathscr{D}_0$ if and only if the control system is controllable in the sense of controllability of multidimensional systems [13,18,19,21,26,33,36]. See also [5,6,8]. Let us show this result.

3.1. Controllability

We find the following definition of controllability in [21,26] which is even working for nonlinear analytic control systems but is not the one now adopted for distributed parameter systems.

Definition 3. Let $\mathscr{D}_1 : F_0 \to F_1$ be a differential operator and let M be the D-module associated with $\mathscr{D}_1$.

1. We call *observable* of the control system any scalar function of the inputs, outputs and their derivatives up to a certain order.
2. A system is called *controllable* if every observable of the system does not satisfy by itself any PDE.

For example, the nonlinear system $u\dot{y} - \dot{u} = a = \mathrm{cst}$ is controllable in the previous sense if and only if $a \neq 0$. If $a = 0$, it admits the observable $z = y - \log u$ satisfying $\dot{z} = 0$, a reason for dealing with analytic functions (see [21] and Remark 4).

In the linear case, an observable is an element of M and we may give the following corresponding definition which agrees with the one now adopted for multidimensional control systems [13,18,19,33,36].

Definition 4. A control system, defined by an operator $\mathscr{D}_1$, is controllable if the D-module M determined by $\mathscr{D}_1$ is torsion-free.

As a byproduct, the control system defined by the D-module $M/t(M)$ is always controllable. This algebraic definition of controllability is the intrinsic formulation of the concept of "factor primeness in the generalized sense" ("minor primeness") for matrices of (surjective) operators with constant coefficients [33–36]. This fact first appeared for applications to control theory in [19]. For a more general classification of algebraic properties of D-modules (torsion-free, reflexive, ..., projective) and the different concepts of primeness, see [27].

Example 2.
1. Let the OD control system be defined in Kalman form by
$$\begin{aligned}&\dot{\eta}^1 - \eta^2 - \eta^3 = 0,\\ &\dot{\eta}^2 - \eta^1 + \alpha\eta^3 = 0,\end{aligned} \tag{6}$$
where α is a constant real coefficient. For $\alpha = -1$, the element $z = \eta^1 - \eta^2$ satisfies $\dot{z} + z = 0$ and z is a torsion element in the corresponding D-module M. Moreover, if $\alpha = 1$, we have $z = \eta^1 + \eta^2$ satisfying $\dot{z} - z = 0$ and z is a torsion element of M. At least for two values of the parameter α, the control system is not controllable.
2. Let us consider the following control system, adapted from a delay differential system presented in [18] but interpreted as a partial differential system ($\partial_1 = \delta, \partial_2 = \frac{d}{dt}$):
$$\begin{aligned}&\partial_2 y^1 - \partial_1 y^1 + 2y^1 + 2y^2 - 2\partial_1 y^3 = 0,\\ &\partial_2 y^1 + \partial_2 y^2 - \partial_{12} y^3 = 0.\end{aligned} \tag{7}$$
This system is not controllable because the observable $z = y^1$ satisfies $\partial_{22} z - \partial_{12} z = 0$.

3.2. Torsion-free D-modules

We denote by E a vector bundle over a manifold X, by $T^\star$ the cotangent bundle of X, by $E^\star$ the dual bundle of E and by $\tilde{E} = \bigwedge^n T^\star \otimes E^\star$ its adjoint bundle. The adjoint bundle $\tilde{E}$ is the right generalization of the concept of tensor density in physics [21].

Remark 3. This is the true reason for which, in physics, certain quantities (deformation, electromagnetic field) are tensors whereas the dual quantities (stress, electromagnetic induction) are tensor densities, even if a background metric is not specified.

Definition 5. If $\mathscr{D}_1 : F_0 \to F_1$ is a linear differential operator, its formal adjoint $\tilde{\mathscr{D}}_1 : \tilde{F}_1 \to \tilde{F}_0$ is defined by the following formal rules that are equivalent to an integration by parts:

- the adjoint of a matrix (zero order operator) is the transposed matrix,
- the adjoint of ∂_i is $-\partial_i$;
- for two linear PD operators P, Q that can be composed: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

We can easily check that $\tilde{\tilde{\mathscr{D}}}_1 = \mathscr{D}_1$. It can be proved that, for any section λ of $\tilde{F}_1$, we have the relation
$$\langle \lambda, \mathscr{D}_1\eta\rangle - \langle \tilde{\mathscr{D}}_1\lambda, \eta\rangle = d(\cdot),$$
expressing a difference of two n-forms ($\lambda \in \bigwedge^n T^\star \otimes F_1^\star \Rightarrow \langle \lambda, \mathscr{D}_1\eta\rangle \in \bigwedge^n T^\star$), where d is the standard exterior derivative. We can directly compute the adjoint of an operator by multiplying it by test functions on the left and integrating it by parts.

Example 3.
1. Let us compute the adjoint of the operator (1). We multiply $\mathscr{D}_1\eta$ on the left by $\lambda = (\lambda_1, \lambda_2)$, we integrate the result by parts and we find the operator $\tilde{\mathscr{D}}_1 : \lambda \to \mu$ defined by
$$\begin{aligned}&-\partial_2\lambda_1 + \partial_1\lambda_1 - \partial_2\lambda_2 + 2\lambda_1 = \mu_1,\\ &-\partial_2\lambda_2 + 2\lambda_1 = \mu_2,\\ &2\partial_1\lambda_1 + \partial_1\lambda_2 = \mu_3.\end{aligned} \tag{8}$$
2. In electromagnetism, if A is the 4-potential 1-form and F the electromagnetic field 2-form, the compatibility condition $dF = 0$ describes the first set of Maxwell equations while the adjoint of $F = dA$ describes the second set of Maxwell equations.

As we have noticed in the introduction, it is sometimes useful to parametrize a system of PDE by certain "potentials" considered as arbitrary functions. It leads to the following definition.

Definition 6. We say that an operator $\mathscr{D}_0 : E \to F_0$ (formally) parametrizes the operator $\mathscr{D}_1 : F_0 \to F_1$ if $\mathscr{D}_1\eta = 0$ generates exactly all the compatibility conditions of $\mathscr{D}_0\xi = \eta$.

Let us describe the formal test for checking whether a D-module M is torsion-free or not [21,22,26]. If M is torsion-free then we give a way to compute a parametrization, i.e. a way to embed M into a free D-module (first step in Palamodov-Kashiwara's classification).

Torsion-free test & parametrization:

1. Start with $\mathscr{D}_1$.
2. Construct its adjoint $\tilde{\mathscr{D}}_1$.

3. Find the compatibility conditions of $\tilde{\mathscr{D}}_1\lambda = \mu$ and denote this operator by $\tilde{\mathscr{D}}_0$.
4. Construct its adjoint $\mathscr{D}_0$ $(=\tilde{\tilde{\mathscr{D}}}_0)$.
5. Find the compatibility conditions of $\mathscr{D}_0\xi = \eta$ and call this operator $\mathscr{D}_1'$.

This led to two different cases:

- If $\mathscr{D}_1$ is exactly the compatibility conditions $\mathscr{D}_1'$ of $\mathscr{D}_0$, then the system $\mathscr{D}_1$ determines a torsion-free D-module M and $\mathscr{D}_0$ is a parametrization of $\mathscr{D}_1$.
- Otherwise, the operator $\mathscr{D}_1$ is among, but not exactly, the compatibility conditions of $\mathscr{D}_0$ and we shall write $\mathscr{D}_1 < \mathscr{D}_1'$. The torsion elements of M are all the new compatibility conditions modulo the equations $\mathscr{D}_1\eta = 0$.

Proof. The operator $\tilde{\mathscr{D}}_0$ describes exactly the compatibility conditions of the operator $\tilde{\mathscr{D}}_1$ and we have in particular $\tilde{\mathscr{D}}_0 \circ \tilde{\mathscr{D}}_1 = 0 \Rightarrow \mathscr{D}_1 \circ \mathscr{D}_0 = 0$. Thus, $\mathscr{D}_1$ is among the compatibility conditions of $\mathscr{D}_0$, which are described by the operator $\mathscr{D}_1'$. Now, computing the differential rank of the operators $\mathscr{D}_1'$ and $\mathscr{D}_1$, we find that diff rk $\mathscr{D}_1'$=diff rk $\mathscr{D}_1$ (see [21] for more details). If $\mathscr{D}_1$ is strictly among the compatibility conditions of $\mathscr{D}_0$, then any new single compatibility condition ζ' in $\mathscr{D}_1'$ is a differential consequence of $\mathscr{D}_1$ (diff rk $\mathscr{D}_1'$=diff rk $\mathscr{D}_1$), and we can find an operator $q \in D$ such that $q\zeta' = 0$ whenever $\mathscr{D}_1\eta = 0$. Thus, any new single compatibility condition of $\mathscr{D}_0$ (not in $\mathscr{D}_1$) determines a torsion element. If $\mathscr{D}_1$ describes exactly the compatibility conditions of $\mathscr{D}_0$, then the D-module M determined by $\mathscr{D}_1$ is torsion-free because $M \subseteq D\xi$ and $D\xi$ is a free D-module. □

We can represent the test by the following differential sequences where the numbers indicate the different stages:

$$
\begin{array}{lllll}
 & & & \overset{\mathscr{D}_1'}{\longrightarrow} F_1' & 5 \\
4 & E \overset{\mathscr{D}_0}{\longrightarrow} & F_0 & \overset{\mathscr{D}_1}{\longrightarrow} F_1 & 1 \\
3 & \tilde{E} \overset{\tilde{\mathscr{D}}_0}{\longleftarrow} & \tilde{F}_0 & \overset{\tilde{\mathscr{D}}_1}{\longleftarrow} \tilde{F}_1 & 2
\end{array}
$$

Example 4. Let us try to know if the operator given by (1) is controllable or not. We have already computed the adjoint operator (8) of Eq. (1). The operator (8) admits only one compatibility condition of second order which defines the operator $\tilde{\mathscr{D}}_0 : \mu \longrightarrow \nu$:

$$
\partial_{22}\mu_3 - \partial_{12}\mu_3 - \partial_{12}\mu_2 + 2\partial_{12}\mu_1 - \partial_{11}\mu_2 + 2\partial_1\mu_1 - 2\partial_1\mu_2 = \nu.
$$

Taking its adjoint, we finally find the operator $\mathscr{D}_0$: $\xi \rightarrow \eta$ defined by

$$
\begin{aligned}
&2\partial_{12}\xi - 2\partial_1\xi = \eta^1,\\
&-\partial_{12}\xi - \partial_{11}\xi + 2\partial_1\xi = \eta^2, \qquad (9)\\
&\partial_{22}\xi - \partial_{12}\xi = \eta^3.
\end{aligned}
$$

We let the reader check that the operator $\mathscr{D}_1$ exactly generates the compatibility conditions of $\mathscr{D}_0$ and thus $\mathscr{D}_1$ determines a torsion-free D-module. Hence, we have found a parametrization (9) of the operator (1).

We now describe how to compute the torsion elements of a D-module M determined by an operator $\mathscr{D}_1$.

Computation of torsion elements:

1. Compute $\mathscr{D}_1'$ and check that $\mathscr{D}_1$ is strictly among $\mathscr{D}_1'$.
2. For any new "single" compatibility condition of the form $\mathscr{D}_1'\eta = \zeta'$ of $\mathscr{D}_1'$, compute the compatibility conditions of the following system:

   $\mathscr{D}_1\eta = 0,$
   $\mathscr{D}_1'\eta = \zeta'$ (one equation only).
3. We find that ζ' is a torsion element of M satisfying $a\zeta' = 0$ with $0 \neq a \in D$.

We give an example of the search of torsion elements.

Example 5. Let us consider the operator $\mathscr{D}_1 : \eta \rightarrow \zeta$, defined by the system (7). Its formal adjoint $\tilde{\mathscr{D}}_1 : \lambda \rightarrow \mu$ is defined by

$$
\begin{aligned}
&-\partial_2\lambda_1 + \partial_1\lambda_1 - \partial_2\lambda_2 + 2\lambda_1 = \mu_1,\\
&-\partial_2\lambda_2 + 2\lambda_1 = \mu_2,\\
&2\partial_1\lambda_1 - \partial_{12}\lambda_2 = \mu_3.
\end{aligned}
$$

There is one compatibility condition $-\partial_1\mu_2 + \mu_3 = 0$ and thus the operator $\tilde{\mathscr{D}}_0 : \mu \longrightarrow \nu$ is given by

$$
-\partial_1\mu_2 + \mu_3 = \nu.
$$

We find the operator $\mathscr{D}_0 : \xi \rightarrow \eta$ defined by

$$
\begin{aligned}
&0 = \eta^1,\\
&\partial_1\xi = \eta^2,\\
&\xi = \eta^3,
\end{aligned}
$$

and we find the following operator $\mathscr{D}_1' : \eta \rightarrow \zeta'$

$$
\begin{aligned}
&\eta^1 = \zeta'^1,\\
&\partial_1\eta^3 - \eta^2 = \zeta'^2. \qquad (10)
\end{aligned}
$$

Thus, the D-module determined by $\mathscr{D}_1$ admits torsion elements which can be computed by finding the compatibility conditions of the systems

$$\eta^1 = \zeta'^1,$$
$$\partial_2\eta^1 - \partial_1\eta^1 - 2\partial_1\eta^3 + 2\eta^1 + 2\eta^2 = 0,$$
$$\partial_{12}\eta^3 - \partial_2\eta^1 - \partial_2\eta^2 = 0,$$

and

$$\partial_1\eta^3 - \eta^2 = \zeta'^2,$$
$$\partial_2\eta^1 - \partial_1\eta^1 - 2\partial_1\eta^3 + 2\eta^1 + 2\eta^2 = 0,$$
$$\partial_{12}\eta^3 - \partial_2\eta^1 - \partial_2\eta^2 = 0,$$

and we find the two torsion elements satisfying

$$\zeta'^1 = \eta^1,$$
$$\partial_{22}\zeta'^1 - \partial_{12}\zeta'^1 = 0,$$

and

$$\zeta'^2 = \partial_1\eta^3 - \eta^2,$$
$$\partial_{22}\zeta'^2 - \partial_{12}\zeta'^2 = 0.$$

Example 6. When $\mathscr{D}_1$ describes the $n(n+1)/2$ components of the Einstein tensor multiplied by 2 and linearized

$$\omega^{rs}(\partial_{ij}\Omega_{rs} + \partial_{rs}\Omega_{ij} - \partial_{ri}\Omega_{sj} - \partial_{sj}\Omega_{ri}) - \omega_{ij}(\omega^{rs}\omega^{uv}\partial_{rs}\Omega_{uv} - \omega^{ru}\omega^{sv}\partial_{rs}\Omega_{uv}) = 0,$$

where Ω is a perturbation of the Minkowski metric $\omega = \mathrm{diag}(1,1,1,-1)$ produced by the gravitational field, one easily checks the self-dual relation: $\mathscr{D}_1 = \tilde{\mathscr{D}}_1$. It follows that $\tilde{\mathscr{D}}_0$ is the well known covariant divergence and thus $\mathscr{D}_0$ is the Killing operator. As the corresponding linearized curvature $\mathscr{D}'_1$ is generated by $n^2(n^2-1)/12$ equations and $n^2(n^2-1)/12 > n(n+1)/2$ whenever $n \geqslant 4$, it follows that $\mathscr{D}_1 < \mathscr{D}'_1$ *strictly* whenever $n \geqslant 4$ and therefore Einstein equations cannot be parametrized by a potential whenever $n \geqslant 4$ [23].

Let us recall the concept of minimal realization for multidimensional control system [13,19] and let us generalize it for the variable coefficients case.

Definition 7. Let the operator $\mathscr{D}_1 : F_0 \to F_1$ determine a D-module M. Then a minimal realization of the system, corresponding to M, is an operator $\mathscr{D}'_1 : F_0 \to F'_1$ which determines the D-module $M/t(M)$ as a specialization, i.e. $M/t(M) = Dy/(\mathscr{D}'_1 y)$.

Theorem 2. *If $\mathscr{D}_1$ is the operator determining the D-module M, then a minimal realization of the system, or of M, is the torsion-free module $M' = M/t(M)$ determined by the operator $\mathscr{D}'_1 : F_0 \to F'_1$ computed in step 5 of the formal torsion-free test.*

Example 7. A minimal realization of the control system (7) is given by the system corresponding to the operator (10).

3.3. *Projective and free D-modules*

We only give here a characterization of a projective D-module determined by a surjective operator $\mathscr{D}_1$, i.e. for an operator $\mathscr{D}_1$ with no compatibility conditions. We refer the reader to [24,26] for the general case and for their applications to the generalized Bezout identity [10]. The algebraic definition of a projective module is the instrinsic formulation of the concept of *zero left-coprimeness* for matrices of surjective operators with constant coefficients [19,27]. See also [7,18,34–36].

Theorem 3. *A surjective operator $\mathscr{D}_1 : F_0 \to F_1$ determines a projective D-module M iff its adjoint $\tilde{\mathscr{D}}_1$ is injective, i.e. iff there exists an operator $\mathscr{P}_1 : F_1 \to F_0$ such that $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{F_1}$, where id_{F_1} is the identity operator of F_1.*

Proof. If the operator $\tilde{\mathscr{D}}_1$ is injective, then differential algebraic consequences of the equations $\tilde{\mathscr{D}}_1\lambda = 0$ are $\lambda = 0$. Hence, we have $\tilde{\mathscr{D}}_1\lambda = \mu \Rightarrow \lambda = \tilde{\mathscr{P}}_1\mu$ and thus $\tilde{\mathscr{P}}_1 \circ \tilde{\mathscr{D}}_1 = id_{\tilde{F}_1} \Rightarrow \mathscr{D}_1 \circ \mathscr{P}_1 = id_{F_1}$. The operator $\mathscr{P}_1 : F_1 \to F_0$ is a right-inverse of $\mathscr{D}_1$ and we have the following *splitting exact* sequence [32]:

$$0 \leftarrow M \leftarrow D^m \underset{\xrightarrow{\mathscr{P}_1}}{\xleftarrow{\mathscr{D}_1}} D^l \leftarrow 0,$$

that is to say, $D^m \cong M \oplus D^l$ and $\mathscr{D}_1$ determines a projective D-module. □

Finally, we have the following theorem.

Theorem 4. *An operator $\mathscr{D}_1$ determines a free D-module iff there exists an injective parametrization $\mathscr{D}_0$, i.e. iff there exists a left-inverse $\mathscr{P}_0$ of the operator $\mathscr{D}_0$.*

Proof. If $\mathscr{D}_1$ has an injective parametrization operator $\mathscr{D}'_0$, then $\mathscr{D}_0$ has a left-inverse $\mathscr{P}_0$ and we have the following exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
0 & M \arrow[l] \arrow[d] & D^m \arrow[l] \arrow[d, equal] & D^l \arrow[l, "\mathcal{D}_1"'] \arrow[d, equal] \\
0 & D^{r-1} \arrow[l] \arrow[d] \arrow[r, shift right=2, "\mathcal{P}_0"'] & D^m \arrow[l, "\mathcal{D}_0"'] & D^l, \arrow[l, "\mathcal{D}_1"'] \\
 & 0 & &
\end{tikzcd}$$

where the two horizontal sequences are exact and thus we have $M \cong D^{r-1}$, which means that M is a free D-module. □

Example 8. Let us consider the following operator $\mathcal{D}_2 : \zeta \longrightarrow \pi$, defined by

$$\partial_2\zeta^2 - x^2\partial_1\zeta^1 + \zeta^1 = \pi. \tag{11}$$

Its formal adjoint $\tilde{\mathcal{D}}_2 : \kappa \longrightarrow \lambda$ is given by

$$x^2\partial_1\kappa + \kappa = \lambda_1,$$
$$-\partial_2\kappa = \lambda_2,$$

and we easily see that $\tilde{\mathcal{D}}_2$ is an injective operator as we have

$$\kappa = -x^2\partial_2\lambda_1 - (x^2)^2\partial_1\lambda_2 - x^2\lambda_2 + \lambda_1. \tag{12}$$

Thus, the operator $\mathcal{D}_2$ generates a projective D-module, and taking the adjoint of Eq. (12), we obtain a right-inverse $\mathcal{P}_2 : \pi \longrightarrow \zeta$ of $\mathcal{D}_2$:

$$x^2\partial_2\pi + 2\pi = \zeta^1,$$
$$(x^2)^2\partial_1\pi - x^2\pi = \zeta^2.$$

We let the reader check that $\mathcal{D}_2 \circ \mathcal{P}_2 = id_{F_2}$. We obtain the operator $\tilde{\mathcal{D}}_1 : \lambda \longrightarrow \mu$, by substituting Eq. (12) in $\tilde{\mathcal{D}}_2$, and we find:

$$\begin{aligned}
&(x^2)^2\partial_{12}\lambda_1 + (x^2)^3\partial_{11}\lambda_2 + 2(x^2)^2\partial_1\lambda_2 - x^2\partial_1\lambda_1 \\
&\quad + x^2\partial_2\lambda_1 + x^2\lambda_2 = \mu_1, \\
&x^2\partial_{22}\lambda_1 + (x^2)^2\partial_{12}\lambda_2 + 2x^2\partial_1\lambda_2 + x^2\partial_2\lambda_2 = \mu_2.
\end{aligned} \tag{13}$$

Dualizing $\tilde{\mathcal{D}}_1$, we obtain a parametrization $\mathcal{D}_1 : \eta \longrightarrow \zeta$ of $\mathcal{D}_2$:

$$\begin{aligned}
&x^2\partial_{22}\eta^2 + (x^2)^2\partial_{12}\eta^1 + 2\partial_2\eta^2 + 3x^2\partial_1\eta^1 \\
&\quad - x^2\partial_2\eta^1 - \eta^1 = \zeta^1, \\
&(x^2)^2\partial_{12}\eta^2 + (x^2)^3\partial_{11}\eta^1 - x^2\partial_2\eta^2 - 2(x^2)^2\partial_1\eta^1 \\
&\quad + x^2\eta^1 - \eta^2 = \zeta^2.
\end{aligned} \tag{14}$$

We see that we can parametrize the operator $\mathcal{D}_2$ by a non trivial parametrization with two arbitrary functions η^1 and η^2. We start anew with the operator $\mathcal{D}_1$ and we try again to parametrize the operator $\mathcal{D}_1$, i.e. to check whether it determines a controllable system. We take the adjoint of $\mathcal{D}_1$ defined by (13) and we find only one compatibility condition between μ_1 and μ_2:

$$x^2\partial_2\mu_1 - (x^2)^2\partial_1\mu_2 - x^2\mu_2 - \mu_1 = 0.$$

We obtain the operator $\tilde{\mathcal{D}}_0 : \mu \longrightarrow \nu$ defined by

$$x^2\partial_2\mu_1 - (x^2)^2\partial_1\mu_2 - x^2\mu_2 - \mu_1 = \nu,$$

and dualizing it, we find the operator $\mathcal{D}_0 : \xi \longrightarrow \eta$:

$$-x^2\partial_2\xi - 2\xi = \eta^1,$$
$$(x^2)^2\partial_1\xi - x^2\xi = \eta^2$$

which is easily seen to parametrize $\mathcal{D}_1$.

The operator $\mathcal{D}_0$ is an injective operator, as we have

$$\xi = \partial_2\eta^2 + x^2\partial_1\eta^1 - \eta^1,$$

and $\mathcal{D}_1$ determines a free D-module, with ξ for basis. Hence, the operator $\mathcal{D}_2$ admits a parametrization $\mathcal{D}_1$ which itself admits a parametrization $\mathcal{D}_0$, i.e the D-module determined by $\mathcal{D}_2$ can be embedded into an exact sequence of two free D-modules and is therefore reflexive. We have the following formally exact sequence:

$$0 \longrightarrow E \xrightarrow{\mathcal{D}_0} F_1 \xrightarrow{\mathcal{D}_1} F_2 \xrightarrow{\mathcal{D}_2} F_2 \longrightarrow 0.$$

Let us give a useful corollary of Theorem 3, which leads to the generalized Bezout identity [10] for OD control systems. See [26] for the existence and effective computations of the generalized Bezout identity for linear multidimensional control systems.

Corollary 1. *A surjective OD operator $\mathcal{D}_1$ defines a controllable system iff its adjoint is injective, i.e. iff it exists a right-inverse $\mathcal{P}_1$ of $\mathcal{D}_1$. If $\mathcal{D}_1$ is an operator defining a controllable system, we can find an injective parametrization $\mathcal{D}_0$ which is the controller form of $\mathcal{D}_1$.*

Proof. A D-module M, determined by an OD operator, is torsion-free iff it is a projective D-module (see Theorem 1). Thus, applying Theorem 3, M is a free D-module iff the operator $\tilde{\mathcal{D}}_1$ is injective, i.e. $\tilde{\mathcal{D}}_1$ admits a left-inverse $\tilde{\mathcal{P}}_1$. By duality, $\mathcal{D}_1$ is controllable iff $\mathcal{D}_1$ has a right-inverse $\mathcal{P}_1$. Moreover, in this case, M is free and by Theorem 4, there exists an injective parametrization $\mathcal{D}_0$ of $\mathcal{D}_1$. This is the generalization of the controller [10] for non-surjective time-varying OD control systems [26]. □

Example 9. Let us test the controllability of the OD control system (6). The operator $\mathscr{D}_1$ defined by Eq. (6) is surjective and its formal adjoint is given by

$$-\dot\lambda_1 - \lambda_2 = \mu_1,$$
$$-\dot\lambda_2 - \lambda_1 = \mu_2,$$
$$-\lambda_1 + \alpha\lambda_2 = \mu_3.$$

Let us investigate the injectivity of $\tilde{\mathscr{D}}_1$. Differentiating the zero order equation and substituting it in the others, we find the new zero order equation:

$$(1-\alpha)(1+\alpha)\lambda_2 = \dot\mu_3 - \mu_1 + \alpha\mu_2 - \alpha\mu_3.$$

We can easily verify that $\tilde{\mathscr{D}}_1$ is injective and thus $\mathscr{D}_1$ is controllable iff $(1-\alpha)(1+\alpha) \neq 0$ that is $\alpha \neq 1$ and $\alpha \neq -1$. Finally, we obtain the following tree of integrability conditions:

$(1-\alpha)(1+\alpha) = 0$	$(1-\alpha)(1+\alpha) \neq 0$
not controllable	controllable

See [25] for more general trees of integrability conditions for multidimensional control systems. Let us show the link between *torsion elements* and *first integrals of motion*. If $\alpha = 1$, the operator $\tilde{\mathscr{D}}_1$ is not injective and the solution of $\tilde{\mathscr{D}}_1\lambda = 0$ is, *after one integration*,

$$\lambda_1(t) = \lambda_2(t) = e^{-t}\lambda_1(0).$$

Moreover, we have $\langle\lambda, \mathscr{D}_1\eta\rangle = \langle\tilde{\mathscr{D}}_1\lambda, \eta\rangle + \frac{d}{dt}(\lambda_1\eta^1 + \lambda_2\eta^2)$ and if we take (η, λ) satisfying $\mathscr{D}_1\eta = 0$ and $\tilde{\mathscr{D}}_1\lambda = 0$, we obtain $\frac{d}{dt}(\lambda_1\eta^1 + \lambda_2\eta^2) = 0$ and thus we obtain a first integral of motion:

$$Z(t) = e^{-t}\lambda_1(0)(\eta^1(t) + \eta^2(t)),$$
$$\dot Z(t) = 0.$$

We let the reader do the same for $\alpha = -1$.

We have the following theorem for local solutions of sufficiently regular analytic ordinary differential operators.

Theorem 5. *If $\mathscr{D}_1$ is a surjective operator defining a non controllable system, then the following numbers are equal*:

1. *The number of solutions of $\tilde{\mathscr{D}}_1$ that are linearly independent over the constants k of K.*
2. *The number of torsion elements which are linearly independent over K.*
3. *The number of first integrals that are linearly independent over k.*

Corollary 2. *In the case of Kalman type systems like* (6), *the preceding number is equal to the dimension over K of the jet space of order zero of the corresponding adjoint system $\tilde{\mathscr{D}}_1\lambda = 0$.*

Proof. If we multiply $-\dot y + Ay + Bu = 0$ on the left by a row vector of test functions λ and integrate by parts, we find the kernel of $\tilde{\mathscr{D}}_1$:

$$\begin{cases} \dot\lambda + \lambda A = 0, \\ \lambda B = 0, \end{cases} \Rightarrow \lambda AB = 0 \Rightarrow \lambda A^2B = 0, \ldots,$$

a result directly leading to the controllability matrix.

If now A, B are depending on time t, we get $\lambda B = 0 \Rightarrow \dot\lambda B + \lambda\dot B = 0 \Rightarrow \lambda(AB - \dot B) = 0 \Rightarrow \lambda(A^2B - \dot AB - 2A\dot B + \ddot B) = 0\ldots$ The concept of a controllability matrix still holds and its rank over K provides the number of equations of order zero that are linearly independent over K as we just need to use the fact that $\dot\lambda = -\lambda A$. □

Hence, we would like to point out the importance of torsion elements compared to first integrals in view of the following two comments:

1. The search for torsion elements is purely algebraic while the search for first integrals is purely analytic (integration needed).
2. The concept of torsion elements can be extended to the PD case as the concept of first integrals is only restricted to the OD case.

Remark 4. In the nonlinear framework, a similar comment is still valid but is out of the scope of this paper devoted to linear systems. Shortly, we study here the case of an affine OD system $\dot y = a(y) + \sum_{i=1}^{s} b_i(y)u_i$. We have already indicated in [21] that the number r of functionally independent constrained observables, that is, observables satisfying at least one OD equation, is equal to the corank of the strong controllability matrix generated by $b_i, [b_i, b_j], [a, b_i]\ldots$, because each observable must be killed by this distribution. If we denote by $z_1, \ldots, z_r$ such a functionally independent set, we notice that the derivatives $\dot z_1, \ldots, \dot z_r$ are still constrained observables and we have, according to the implicit function theorem, $\dot z_i = \phi_i(z_1, \ldots, z_r)$. Hence, if $Z = f(t, z_1, \ldots, z_r)$ is a first integral, we have $\dot Z = \frac{\partial f}{\partial t} + \sum_{i=1}^{s} \frac{\partial f}{\partial z_i}\phi_i(z_1, \ldots, z_r) = 0$. Hence, we find by integration (as in the linear case indeed), exactly

r functionally independent first integrals $Z_1,\dots,Z_r$ killed by the vector field $\frac{\partial}{\partial t}+\sum_{i=1}^{s}\phi_i(z)\frac{\partial}{\partial z_i}$. Once more, we notice that $z_i=g_i(y)$ only depend on y while $Z_i=h_i(t,y)$ explicitly depend on t in general. Of course, we notice that first integrals are trivially constrained observables. See [22] for more details.

Remark 5. The interest of the previous test is that is may work even if the system is not in Kalman form and with variable coefficients.

Example 10 (*see also* [25]). Let us consider the second order SISO control system

$$\ddot{y}+\alpha(t)\dot{y}+\dot{\alpha}(t)y+\ddot{u}-u=0.$$

The kernel of the adjoint system is defined by

$$\ddot{\lambda}-\lambda=0,$$
$$\ddot{\lambda}-\alpha(t)\dot{\lambda}=0,$$

or equivalently

$$\ddot{\lambda}-\lambda=0,$$
$$\alpha(t)\dot{\lambda}-\lambda=0.$$

1. If $\alpha=0$ then $\lambda=0$ and the system is controllable.
2. If $\alpha\neq 0$ then differentiating again and substituting, we get $(\dot{\alpha}+\alpha^2-1)\lambda=0$ and the system is *again* controllable if and only if $\dot{\alpha}+\alpha^2-1\neq 0$.

Such an example proves that controllability is not always a generic property of a system in the sense that a system may be controllable even for non-generic values of the coefficients (here $\alpha=0$).

4. Localization

It is useful to have the possibility, in the D-module framework, to invert certain elements of the ring D. This can be done by *localization* of the D-module. We show how the localization procedure formally extends the Laplace transform to time-varying ordinary OD systems or to PD systems with variable coefficients and we give an effective way to compute localization of D-modules.

4.1. Definition-properties

Let us recall a few results about the localization in the general framework of D-modules. See [12] for more details.

Definition 8. Let S be a *multiplicative* subset of D, i.e. a subset of D satisfying the following properties:

1. $1\in S$,
2. $\forall s,\ t\in S\Rightarrow st\in S$
3. $\forall a\in D,\ s\in S\Rightarrow \exists b\in D,\ t\in S$ such that $ta=bs$.

Let M be a D-module, then we define the $S^{-1}D$-module $S^{-1}M$ as the quotient of the sets $(s,x)\in S\times M$ by the equivalence relation defined by

$$(s_1,x_1)\sim(s_2,x_2)\Leftrightarrow \exists s_1',s_2'\in S \text{ such that } s_1's_1=s_2's_2 \text{ and } s_1'x_1=s_2'x_2.$$

We denote by $s^{-1}x$ the equivalence class of the pair (s,x) and such a procedure is called "left localization". We have $S^{-1}M=S^{-1}D\otimes_D M$.

In particular, if we take $S=D\backslash 0$, we obtain $S^{-1}D=Q(D)$ the left field of fraction of D and we have the following exact sequence

$$0\to t(M)\to M\xrightarrow{i_S} S^{-1}M=Q(D)\otimes_D M,$$

where $t(M)$ is the torsion D-submodule of M. So, if the D-module M is a torsion-free D-module, then the homomorphism $i_S: x\to(1,x)$ is injective and M is embedded into the $Q(D)$-vector space $Q(D)\otimes_D M$. The torsion elements vanish in $Q(D)\otimes_D M$, a fact which is similar, in the constant coefficients case, to the cancellation in a transfer function. The following theorem extends the passage from left-coprime to right-coprime used in classical control theory, i.e. for $D=k[s]$.

Theorem 6. *If $S=D\backslash 0$, then we have*

$$S^{-1}D=DS^{-1}.$$

Let us give an effective proof of this theorem, which makes clear the link between localization techniques and the use of duality through the formal test for checking whether a module is torsion-free or not.

Proof. Let $a\in S$ and $b\in D$, we have to show that $\exists p\in D, q\in S$ such that $a^{-1}b=pq^{-1}$. If $b=0$, then the result is obvious. Let us suppose that $b\neq 0$. We denote by $\mathscr{D}_1:\eta\to\zeta$ the operator defined by

$$a\eta^1-b\eta^2=\zeta.$$

The adjoint $\tilde{\mathscr{D}}_1:\lambda\to\mu$ is defined by

$$\tilde{a}\lambda=\mu_1,\qquad \tilde{b}\lambda=\mu_2.$$

Now, using the fact that D is a left Ore algebra, we can find *one* compatibility condition $\tilde{\mathscr{D}}_0' : \mu \to \nu$ defined by

$$\tilde{p}\mu_1 - \tilde{q}\mu_2 = \nu,$$

with $\tilde{q} \neq 0$ and thus $\tilde{p} \neq 0$. Dualizing, we obtain the operator $\mathscr{D}_0' : \xi \to \eta$ given by:

$$p\xi = \eta^1, \qquad q\xi = \eta^2.$$

Hence, $\xi = q^{-1}\eta^2 \Rightarrow \eta^1 = pq^{-1}\eta^2$. Finally, the kernel of the operator $\mathscr{D}_1$ is defined by $a\eta^1 - b\eta^2 = 0$, $a \neq 0 \Rightarrow \eta^1 = a^{-1}b\eta^2$ and thus $a^{-1}b = pq^{-1}$, which concludes the proof. □

We have the following corollary.

Corollary 3. *D is a right Ore algebra, i.e., $\forall (a,b) \in D^2, \exists (p,q) \in (D\backslash 0)^2$ such that $ap = bq$.*

If we start with an operator with constant coefficients, which determines a torsion-free $D = k[d_1,\dots,d_n]$-module, then we easily obtain a parametrization by means of localizations, using the relation:

$$\partial_i^{-1}\partial_j = \partial_j\partial_i^{-1}.$$

Example 11. Let us find a parametrization of the divergence operator in $\mathbb{R}^3$, defined by

$$\partial_1\eta^1 + \partial_2\eta^2 + \partial_3\eta^3 = \zeta,$$

by means of localizations. We have $\eta^3 = -\partial_3^{-1}(\partial_1\eta^1) - \partial_3^{-1}(\partial_2\eta^2) \Rightarrow \eta^3 = -\partial_1(\partial_3^{-1}\eta^1) - \partial_2(\partial_3^{-1}\eta^2)$. Finally, if we set

$$\xi^1 = \partial_3^{-1}\eta^1, \qquad \xi^2 = \partial_3^{-1}\eta^2,$$

we have the following parametrization of the divergence operator:

$$\begin{aligned} &\partial_3\xi^1 = \eta^1, \\ &\partial_3\xi^2 = \eta^2, \\ &-\partial_1\xi^1 - \partial_2\xi^2 = \eta^3. \end{aligned} \tag{15}$$

We notice that we do not find the "usual" parametrization of the divergence operator by the curl. In fact, this new parametrization (15) cannot be parametrized in its turn whereas the curl is parametrized by the gradient, a fact showing that the differential module determined by the divergence operator is reflexive though not projective. We shall see it is a general fact that the parametrizations found by localization are "simpler" than those obtained using the formal test, but they are "worse" in the sense of Palamodov–Kashiwara's classification of differential modules (5).

The situation of operators with variable coefficients is more complicated. But, the proof of Theorem 7 shows how to use the formal duality and formal integrability to find a parametrization of an operator, when it determines a torsion-free D-module.

Example 12. Let us reconsider system (11). We can solve it with respect to ζ^2:

$$\zeta^2 = \partial_2^{-1}(x^2\partial_1 - 1)\zeta^1.$$

We denote $a = \partial_2$ and $b = x^2\partial_1 - 1$ and we search two elements p and $q \in D$ such that $ap = bq$. This is equivalent to search $\tilde{p}, \tilde{q} \in D$ such that $\tilde{p}\tilde{a} = \tilde{q}\tilde{b}$, i.e. to find one compatibility condition of the following operator:

$$\begin{aligned} &-\partial_2\kappa = \lambda_1, \\ &-x^2\partial_1\kappa - \kappa = \lambda_2. \end{aligned}$$

This operator is injective as we have $\kappa = x^2\partial_2\lambda_2 - (x^2)^2\partial_1\lambda_1 - x^2\lambda_1 - \lambda_2$ and we find two different compatibility conditions of this operator: the first is defined by

$$-\partial_{22}\lambda_2 + x^2\partial_{12}\lambda_1 + 2\partial_1\lambda_1 + \partial_2\lambda_1 = 0,$$

whereas the second is given by

$$-x^2\partial_{12}\lambda_2 + (x^2)^2\partial_{11}\lambda_1 + 2x^2\partial_1\lambda_1 + \partial_1\lambda_2 - \partial_2\lambda_2 + \lambda_1 = 0.$$

In the first case, we have $\tilde{p} = x^2\partial_{12} + 2\partial_1 + \partial_2$ and $\tilde{q} = \partial_{22}$, which give $p = x^2\partial_{12} - \partial_1 - \partial_2$ and $q = \partial_{22}$. Finally, we have

$$\partial_2(x^2\partial_{12} - \partial_1 - \partial_2) = (x^2\partial_1 - 1)\partial_{22},$$

and thus

$$\zeta^2 = \partial_2^{-1}(x^2\partial_1 - 1)\zeta^1 = (x^2\partial_{12} - \partial_1 - \partial_2)\partial_{22}^{-1}\zeta^1.$$

Setting $\eta = \partial_{22}^{-1}\zeta^1$, we obtain the parametrization:

$$\begin{aligned} &\partial_{22}\eta = \zeta^1, \\ &x^2\partial_{12}\eta - \partial_1\eta - \partial_2\eta = \zeta^2. \end{aligned} \tag{16}$$

Similarly, with the second compatibility condition, we shall obtain the parametrization:

$$\begin{aligned} &x^2\partial_{12}\eta + 2\partial_1\eta - \partial_2\eta = \zeta^1, \\ &(x^2)^2\partial_{11}\eta - 2x^2\partial_1\eta + \eta = \zeta^2. \end{aligned} \tag{17}$$

We are in the same situation as in Example 11: we have obtained two different parametrizations of (1) which are simpler than (14), in the sense that they only depend on one arbitrary function η whereas (14) depends on two arbitrary functions. But these parametrizations cannot be parametrized in their turn by an operator whereas (14) is parametrized by an injective operator.

Let us explain why the localization technique gives a parametrization simpler than the one of the formal test: the number of arbitrary functions in $\mathscr{D}_0$ is equal to the number of equations of its adjoint $\tilde{\mathscr{D}}_0$ and when we compute parametrization, by means of localizations, we do not need to compute all the compatibility conditions of $\tilde{\mathscr{D}}_1$ but just a differential transcendence basis. The surprising fact is that, if we only take a differential transcendence basis $\tilde{\mathscr{D}}_0'$ of $\tilde{\mathscr{D}}_0$, this still gives a parametrization of $\mathscr{D}_1$. This result is explained by the following nontrivial theorem.

Theorem 7. *Let $\mathscr{D}_1 : F_0 \to F_0$ be an operator determining a torsion-free D-module and let $\mathscr{D}_0 : E \to F_0$ be a parametrization of $\mathscr{D}_1$ with a kernel having a non zero differential transcendence degree. Then, there exists a parametrization $\mathscr{D}_0' : E' \to F_0$ of $\mathscr{D}_1$ with a kernel having zero differential transcendence degree. We call such a parametrization $\mathscr{D}_0'$ a minimal parametrization of $\mathscr{D}_1$ and a way to obtain $\mathscr{D}_0'$ is to take a differential transcendence basis $\tilde{\mathscr{D}}_0'$ of $\tilde{\mathscr{D}}_0$.*

Proof. If we denote by $\tilde{\mathscr{D}}_0'$ a differential transcendence basis of $\tilde{\mathscr{D}}_0$, we have the following sequence (not formally exact in general):

$$0 \leftarrow \tilde{E}' \xleftarrow{\tilde{\mathscr{D}}_0'} \tilde{F}_0 \xleftarrow{\tilde{\mathscr{D}}_1} \tilde{F}_1.$$

Introducing the differential ranks of operators (see [21] for more details), we have

$$\operatorname{diff\,rk}(\tilde{\mathscr{D}}_0') = \dim(\tilde{F}_0) - \operatorname{diff\,rk}(\tilde{\mathscr{D}}_1) = \dim(\tilde{E}').$$

Recalling that $\operatorname{diff\,rk}(\tilde{\mathscr{D}}_0) = \operatorname{diff\,rk}(\mathscr{D}_0)$, we obtain:

$$\operatorname{diff\,rk}(\mathscr{D}_0') = \dim(F_0) - \operatorname{diff\,rk}(\mathscr{D}_1) = \operatorname{diff\,rk}(\mathscr{D}_0).$$

The D-module M determined by $\mathscr{D}_1$ is a torsion-free module and thus M can be seen as the D-module determined by $\operatorname{im}(\mathscr{D}_0)$. If we denote by M' the D-module determined by $\operatorname{im}(\mathscr{D}_0')$, i.e. the D-module determined by the operator generating the compatibility conditions of the operator $\mathscr{D}_0'$, then we have:

$$\operatorname{im}(\mathscr{D}_0') \subseteq \operatorname{im}(\mathscr{D}_0),$$

which leads to the following specialization:

$$M \xrightarrow{\phi} M' \to 0.$$

The differential rank of M is equal to the one of M' and thus the kernel of ϕ is a torsion submodule of M, which implies that $\ker(\phi) = 0$ because M is a torsion-free D-module. Hence, we have: $M' \cong M$. □

Let us notice that, for once, the proof of this theorem in the framework of D-modules is easier than that the one in the framework of differential operators, which has been added in the appendix.

5. Conclusion

We hope to have convinced the reader that the *localization technique* is the only tool which, at the same time, is coherent with the transfer matrix approach in the case of ordinary control systems with constant coefficients and can be extended to the variable coefficients or to the partial differential case along a procedure which constitutes the core of commutative algebra. The only difficulty met is to adapt such a procedure to the non-commutative case in order to use the Ore property of the ring of differential operators. In this framework, we hope to have proved that the corresponding duality technique, based on a systematic use of the adjoint operator, will play a major constructive and effective role in the study of control theory for partial differential operators and n-dimensional systems.

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Appendix

Let us give a proof of Theorem 7 using differential operators: if the kernel of the parametrization $\mathscr{D}_0$ of $\mathscr{D}_1$ has a zero differential transcendence degree, then $\mathscr{D}_0' = \mathscr{D}_0$. Let us suppose that the kernel of $\mathscr{D}_0$ has a non zero differential transcendance degree [14,31]. Let us select a maximal set of differentially independent compatibility conditions among $\tilde{\mathscr{D}}_0$ (the image of this new operator $\tilde{\mathscr{D}}_0'$ must produce a differential transcendence basis of the kernel of $\tilde{\mathscr{D}}_{-1}$). We have the commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
\tilde{E}_{-1} & \leftarrow & \tilde{E}'' & \leftarrow & \Omega \leftarrow 0 & & \\
\| & & \downarrow & & & & \\
\tilde{E}_{-1} & \xleftarrow{\tilde{\mathscr{D}}_{-1}} & \tilde{E} & \xleftarrow{\tilde{\mathscr{D}}_0} & \tilde{F}_0 & \xleftarrow{\tilde{\mathscr{D}}_1} & \tilde{F}_1 \\
\downarrow & & \downarrow & & \| & & \| \\
0 & \leftarrow & \tilde{E}' & \xleftarrow{\tilde{\mathscr{D}}'_0} & \tilde{F}_0 & \xleftarrow{\tilde{\mathscr{D}}_1} & \tilde{F}_1, \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

where by construction, the transcendence degree of the space of solutions Ω is zero (the corresponding differential module is a torsion module). We notice that the lower row may not be formally exact at $\tilde{F}_0$. Taking the adjoint of all these operators, we get the commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & & E' & \xrightarrow{\mathscr{D}'_0} & F_0 & \xrightarrow{\mathscr{D}_1} & F_1 \\
\downarrow & & \downarrow & & \| & & \| \\
E_{-1} & \xrightarrow{\mathscr{D}_{-1}} & E & \xrightarrow{\mathscr{D}_0} & F_0 & \xrightarrow{\mathscr{D}_1} & F_1 \\
\| & & \downarrow & & \downarrow & & \\
E_{-1} & \rightarrow & E'' & \rightarrow & 0. & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

An easy chase proves that the full diagram is formally exact and then the upper row is formally exact at F_0 as desired, that is, $\mathscr{D}'_0$ is a parametrization of $\mathscr{D}_1$ such that its kernel has a zero differential transcendence degree.

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