Robust Regulation of SISO Systems: The Fractional Ideal Approach

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Abstract

We solve the robust regulation problem for single-input single-output plants by using the fractional ideal approach and without assuming the existence of coprime factorizations. In particular, we are able to formulate the famous internal model principle for stabilizable plants which do not necessarily admit coprime factorizations. We are able to give a necessary and sufficient solvability condition for the robust regulation problem, which leads to a design method for a robustly regulating controller. The theory is illustrated by examples.

1 Introduction

Robustness is a fundamentally important feature of controllers since it allows them to work under small uncertainties. Robust regulation of finite-dimensional plants is well understood [6, 7, 24], and the finite-dimensional theory have been generalized to infinite-dimensional plants and signals by several authors. See for example [4, 10, 11, 12, 13, 15, 16, 20, 21, 23, 25]. A basic result of robust regulation is the internal model principle, which states that any robustly regulating controller must contain a reduplicated model of the dynamics to be regulated.

In the frequency domain, the robust regulation problem can be reformulated as an algebraic problem. Vidyasagar formulated and solved it by using coprime factorizations over the ring of stable rational transfer matrices [24]. These results state the internal model principle, give a necessary and sufficient solvability condition of the problem, and parameterize all robustly regulating controllers in a remarkably simple form. The results due to Vidyasagar have been generalized to algebraic frameworks suitable for distributed parameter systems and/or infinite-dimensional reference and disturbance signals [4, 10, 12, 15, 16, 20, 25]. The common feature of the existing frequency domain results of robust regulation is that they require existence of coprime factorizations. This is problematic since there exists algebraic frameworks with stabilizable plants that do not possess coprime factorizations [1, 18], or where the existence of coprime factorizations is not known [15, 19, 22]. Thus, there is a need for robust regulation theory for stabilizable plants which do not necessarily admit coprime factorizations.

In this paper, we develop robust regulation theory of single-input single-output plants based on stability results of [22]. In addition, to presenting new results, our intention is to introduce the reader to the factorization approach (or the fractional representation approach) to regulation. Because of this, we provide simple examples illustrating the theory and relating it to the classical results. The advantage of the theory of [22] is that it uses no coprime factorizations and allows us to develop theory in a simple framework with very few assumptions. We only need to define a commutative ring $\mathbf{R}$ of stable elements with a unit and no zero divisors to start with. The plants are just the elements in the field of fractions of the integral domain $\mathbf{R}$. This makes the theory applicable in several different algebraic frameworks, e.g. in those of [16, 17] that are suitable for infinite-dimensional systems.

The abstract algebraic approach to robust regulation has not received much attention this far. Vidyasagar discussed the generalization of finite-dimensional stabilization and regulation theory developed in his book [24] to infinite-dimensional systems in the last chapter of the book. Unfortunately, the part concerning robust regulation uses coprime factorizations and is not applicable for general rings. The same is true for the theory developed in [20]. In addition, both of the above references use topological notions in the study of robustness. It is possible to do without by defining the robustness in the way that stability must imply regulation as was done in [15, 16]. With this approach, robustness of a controller is determined by its robustness of stability which is well-understood in many physically interesting algebraic structures [5, 8, 24].

By using the fractional ideal approach [22], we are able to generalize the theory of [24] to abstract algebraic framework using no coprime factorizations. The main

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contributions of this article are:

- A reformulation of the internal model principle using no coprime factorizations.
- A necessary and sufficient condition for solvability of the robust regulation problem.

The paper is organized as follows. Notations, preliminary results and the problem formulation are given in Section 2. The internal model principle is studied in Section 3. Section 4 contains solvability considerations and, by using the results of the section, we are able to give a procedure to find robustly regulating controllers. We illustrate the results by examples. The concluding remarks are made in Section 5.

2 Problem Formulation

We choose the set of stable elements to be an integral domain $\mathbf{R}$, i.e., a commutative domain that has a unit element. The field of fractions of $\mathbf{R}$, namely

$$Q(\mathbf{R}) := \left\{ \frac{n}{d} \,\middle|\, 0 \neq d,\, n \in \mathbf{R} \right\},$$

is denoted by $\mathbf{F}$. We denote the set of all matrices with entries in $\mathbf{R}$ by $\mathcal{M}(\mathbf{R})$. A matrix $H \in \mathcal{M}(\mathbf{F})$ is *stable* if $H \in \mathcal{M}(\mathbf{R})$ and otherwise, it is *unstable*.

Example 2.1. An important ring of stable transfer functions is the set of proper rational functions with no poles in the closure $\overline{\mathbb{C}_+} := \{s \in \mathbb{C} \,|\, \Re(s) \geq 0\}$ of $\mathbb{C}_+ := \{s \in \mathbb{C} \,|\, \Re(s) > 0\}$. As in the literature, we denote this integral domain by RH_∞ The poles in $\overline{\mathbb{C}_+}$ of an element $f \in RH_\infty$ are called *unstable*.

Definition 1. A controller $C \in \mathbf{F}$ *stabilizes* a plant $P \in \mathbf{F}$ if the closed loop system of Figure 1 from (y_r, d) to (e, u), given by

$$H(P, C) := \begin{pmatrix} (I - P\,C)^{-1} & -(I - P\,C)^{-1}\,P \\ C\,(I - P\,C)^{-1} & (I - P\,C)^{-1} \end{pmatrix},$$

is stable, i.e., if $H(P, C) \in \mathcal{M}(\mathbf{R})$.

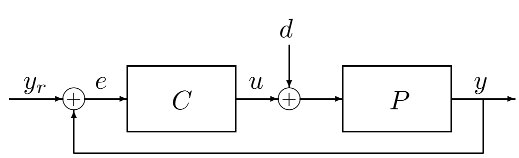

Figure 1: The control configuration.

Definition 2. 1. A fractional representation $\Theta := \frac{\gamma}{\theta}$ of $\Theta \in \mathbf{F}$, where $\gamma,\, \theta \in \mathbf{R}$, is said to be *a coprime factorization* of Θ if γ and θ are coprime, i.e., if there exists $\alpha, \beta \in \mathbf{R}$ such that:

$$\alpha\,\gamma + \beta\,\theta = 1.$$

2. If $f_1, \ldots, f_n \in \mathbf{F}$, then the following $\mathbf{R}$-submodule of $\mathbf{F}$

$$\mathbf{R}\,f_1 + \cdots + \mathbf{R}\,f_n := \left\{ \sum_{i=1}^{n} r_i\,f_i \,\middle|\, r_1, \ldots, r_n \in \mathbf{R} \right\}$$

is simply denoted by $(f_1, \ldots, f_n)$.

3. An $\mathbf{R}$-submodule $\mathbf{I}$ of $\mathbf{F}$ is called *a fractional ideal* of $\mathbf{R}$ if there exists $0 \neq r \in \mathbf{R}$ such that $r\,\mathbf{I} \subseteq \mathbf{R}$.

4. Given two fractional ideals $\mathbf{I}$ and $\mathbf{J}$, we can define the following two fractional ideals:

$$\mathbf{I}\,\mathbf{J} := \left\{ \sum_{i=1}^{n} a_i b_i \,\middle|\, a_i \in \mathbf{I},\, b_i \in \mathbf{J},\, n \geq 0 \right\},$$
$$\mathbf{I} : \mathbf{J} := \{k \in \mathbf{F} \,|\, (k)\,\mathbf{J} \subseteq \mathbf{I}\}.$$

5. A non-zero fractional ideal $\mathbf{I}$ is *invertible* if there exists a fractional ideal $\mathbf{J}$ such that $\mathbf{I}\,\mathbf{J} = \mathbf{R}$. The fractional ideal $\mathbf{J}$ is called the *inverse* of $\mathbf{I}$ and it is denoted by $\mathbf{I}^{-1}$.

6. A fractional ideal $\mathbf{I}$ is called *principal* if there exists $k \in \mathbf{F}$ such that $\mathbf{I} = (k)$.

If the inverse of a fractional ideal $\mathbf{I}$ exists, then one can prove that $\mathbf{I}^{-1} = \mathbf{R} : \mathbf{I}$. See, e.g., [9]. If $\mathbf{I} = (k)$ is a principal fractional ideal, where $k \in \mathbf{F} \setminus \{0\}$, then we have $\mathbf{I}^{-1} = (k^{-1})$. Finally, if $\mathbf{I} \subseteq \mathbf{J}$, we note that we then have $\mathbf{R} : \mathbf{J} \subseteq \mathbf{R} : \mathbf{I}$.

The theory developed in this article is based on the stability results of [22]. If $P \in \mathbf{F}$, then the fractional ideal $\mathbf{J} := (1, P)$ is called *the fractional ideal associated with P*. We recall the following theorem that is a combination of Theorems 1 and 2 of [22].

Theorem 2.1. *Let $P \in \mathbf{F}$ and $\mathbf{J} := (1, P)$ be the fractional ideal associated with the plant P.*

1. *The following assertions are equivalent:*
   (a) *The plant P is stabilizable.*
   (b) *$\mathbf{J}$ is invertible.*
   (c) *There exist $S, U \in \mathbf{R}$ such that:*

$$\text{(2.1)} \qquad \begin{cases} S - P\,U = 1, \\ P\,S \in \mathbf{R}. \end{cases}$$

Then, the inverse of $\mathbf{J}$ is $\mathbf{J}^{-1} = (S, U)$. If $S \neq 0$, then $C = \frac{U}{S}$ stabilizes P and:

$$S = \frac{1}{1 - P\,C}, \quad U = C\,S = \frac{C}{1 - P\,C}.$$

2. *A controller $C \in \mathbf{F}$ stabilizes P if and only if it is of the form $C = \frac{U}{S}$, where $0 \neq S \in \mathbf{R}$ and $U \in \mathbf{R}$ satisfy* (2.1)*.*

3. *If $0 \neq S \in \mathbf{R}$ and $U \in \mathbf{R}$ satisfy* (2.1)*, then all the stabilizing controllers of P are parametrized by*

$$C(Q_1, Q_2) := \frac{U + Q_1\, S^2 + Q_2\, U^2}{S + Q_1\, P\, S^2 + Q_2\, P\, U^2}, \tag{2.2}$$

where Q_1, $Q_2 \in \mathbf{R}$ are such that:

$$S + Q_1\, P\, S^2 + Q_2\, P\, U^2 \neq 0.$$

The next simple example with rational plant and controller shows how to parameterize all the stabilizing controllers whenever we find one such controller. Although all coprime factorizations can be found easily for the rational plant, one should note that they are not needed at any point. This is especially beneficial if the stabilization is relatively easy when compared to finding coprime factorizations.

Example 2.2. Let us choose $\mathbf{R} := RH_\infty$. Consider the plant $P := \frac{s+1}{s^2+1}$. The plant can be stabilized by using negative feedback. Choosing $C := -5$, we then get:

$$S := (1 - P\, C)^{-1} = \frac{s^2 + 1}{(s+2)\,(s+3)} \in \mathbf{R}.$$

If we consider $U := C\, S = -5\, S$, we have a fractional representation of C satisfying (2.1).

Because of the simplicity of the controller, using (2.2), we can easily find out that all the stabilizing controllers of P are of the form

$$C(Q) := \frac{(-5 + Q\, S)\, S}{(1 + Q\, S\, P)\, S} = \frac{Q\, \frac{s^2+1}{(s+2)\,(s+3)} - 5}{Q\, \frac{s+1}{(s+2)\,(s+3)} + 1}, \tag{2.3}$$

where $Q \in RH_\infty$. The numerator and the denominator on the right-hand side are stable and do not satisfy (2.1).

Proposition 2.1. ([22]) *A plant $P \in \mathbf{F}$ admits a coprime factorization if and only if the fractional ideal $\mathbf{J} := (1,\, P)$ associated with P is principal. Then, there exists $d \in \mathbf{R} \setminus \{0\}$ such that $\mathbf{J} = (d^{-1})$ and $P = \frac{n}{d}$, where $n := d\, P \in \mathbf{R}$, is a coprime factorization.*

In this article, we assume that all *the reference and disturbance signals* are generated by some fixed *signal generator* $\Theta \in \mathbf{F}$, i.e. every reference signal is of the form $y_r := \Theta\, y_0$ and every disturbance signal is of the form $d := \Theta\, d_0$, where $y_0, d_0 \in \mathbf{R}$.

Definition 3. A controller $C \in \mathbf{F}$ is *regulating a plant $P \in \mathbf{F}$ for a signal generator $\Theta \in \mathbf{F}$* if we have

$$\begin{aligned} e &:= \left((I - P\, C)^{-1} \quad -(I - P\, C)^{-1}\, P \right) \begin{pmatrix} y_r \\ d \end{pmatrix} \\ &= \left((I - P\, C)^{-1} \quad -(I - P\, C)^{-1}\, P \right) \Theta \begin{pmatrix} y_0 \\ d_0 \end{pmatrix} \in \mathbf{R}, \end{aligned}$$

for all y_0 and $d_0 \in \mathbf{R}$.

In what follows, if the context is clear, i.e., if the plant P and the generator Θ are clearly fixed, we say that C is *regulating*. From Definition 3, C is regulating if and only if we have:

$$\left((I - P\, C)^{-1} \quad -(I - P\, C)^{-1}\, P \right) \Theta \in \mathcal{M}\,(\mathbf{R}). \tag{2.4}$$

The next lemma gives a formulation of regulating controllers in terms of fractional ideals.

Lemma 2.1. *Let $P \in \mathbf{F}$ be a plant, $C \in \mathbf{F}$ a controller and $\Theta \in \mathbf{F}$ a signal generator. Let also $S := (1 - P\, C)^{-1}$ and $\mathbf{I} := (S,\, S\, P)$. Then, C is regulating if and only if:*

$$(\Theta) \subseteq \mathbf{R} : \mathbf{I}.$$

Proof. By the definition, we have:

$$\begin{aligned} \mathbf{R} : \mathbf{I} &= \{k \in \mathbf{F} \,|\, (k)\, \mathbf{I} \subseteq \mathbf{R}\} \\ &= \{K \in \mathbf{F} \ \ |\, \textstyle\sum_{i=1}^{n} a_i\, k\, (b_i\, S + c_i\, S\, P) \in \mathbf{R} \\ &\qquad\qquad \forall\, n \geq 0, a_i,\, b_i,\, c_i \in \mathbf{R}\} \\ &= \{k \in \mathbf{F} \,|\, k\, S,\, k\, S\, P \in \mathbf{R}\}\,. \end{aligned}$$

Hence, we obtain:

$$(2.4) \;\Leftrightarrow\; (S \quad S\, P)\, \Theta \in \mathcal{M}\,(\mathbf{R}) \;\Leftrightarrow\; (\Theta) \subseteq \mathbf{R} : \mathbf{I}.$$

□

Definition 4. Given a plant $P \in \mathbf{F}$ and a signal generator $\Theta \in \mathbf{F}$, *the robust regulation problem* aims at finding a controller $C \in \mathbf{F}$ such that:

1. C stabilizes P.

2. C regulates every plant it stabilizes.

A controller C that solves the robust regulation problem is called *robustly regulating.*

3 The Internal Model Principle

We first formulate *the internal model principle* using fractional ideals. The control configuration is as given in Figure 2. The next theorem states that the controller and the signal generator are related as follows

$$\Theta = \alpha + \beta\, C,$$

where $\alpha, \beta \in \mathbf{R}$, which means – roughly speaking – that a robustly regulating controller must contain all unstable modes of the signal generator.

THEOREM 3.1. *A stabilizing controller C is robustly regulating if and only if we have $(\Theta) \subseteq (1, C)$ or equivalent if and only if we have*

$$(1,\, \Theta) \subseteq (1,\, C), \tag{3.5}$$

i.e., the fractional ideal associated with Θ is contained in the fractional ideal associated with C.

Proof. Let C be a stabilizing controller of the plant P, $S := (1 - P\,C)^{-1}$, $\mathbf{I} := (S,\, S\,P)$ and $\mathbf{J} := (1,\, P)$. Since $\mathbf{I} = (S)\,\mathbf{J}$ and $\mathbf{J}$ is invertible by 1.b of Theorem 2.1, we obtain that $\mathbf{I}$ is also an invertible ideal. We then have:

$$\begin{aligned}\mathbf{R} : \mathbf{I} = \mathbf{I}^{-1} &= ((S)\,(1,\, P))^{-1} = (S)^{-1}\,(1,\, P)^{-1}\\ &= (S)^{-1}\,\mathbf{J} = (S)^{-1}\,(S,\, U) = (1,\, C).\end{aligned}$$

Using Lemma 2.1, we then obtain that the stabilizing controller C regulates P if and only if $(\Theta) \subseteq (1,\, C)$, which equivalent to $(1,\, \Theta) \subseteq (1,\, C)$ since $1 \in (1,\, C)$.

Finally, if C also stabilizes the plant P', then we have $(1 - P'\,C)^{-1}, (1 - P'\,C)^{-1}\,C, (1 - P'\,C)^{-1}\,P' \in \mathbf{R}$. Since we have $(\Theta) \subseteq (1,\, C)$, i.e., $\Theta = \alpha + \beta\,C$ for certain $\alpha,\, \beta \in \mathbf{R}$, we get $(1 - P'\,C)^{-1}\,\Theta$, $(1 - P'\,C)^{-1}\,P'\,\Theta \in \mathbf{R}$, which shows that C is robustly regulating. □

The above proof actually shows the well-known result that, for SISO systems, every stabilizing controller that is regulating is also robustly regulating. This is a consequence of the facts that an n-copy of the exosystem as an internal model is sufficient to make a controller robustly regulating [7], where n is the dimension of the output space, and any regulating controller contains at least one copy of the exosystem [2].

COROLLARY 3.1. *A SISO controller is robustly regulating if and only if it is stabilizing and regulating.*

We have the following corollary of Theorem 3.1.

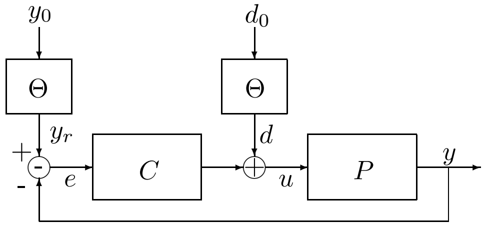

Figure 2: The control configuration for the robust regulation problem.

COROLLARY 3.2. *Let $C \in \mathbf{F}$ be a robustly regulating controller for a signal generator $\Theta \in \mathbf{F}$. Then, we have*

$$\begin{aligned}\mathbf{R} : (1,\, C) &= \{r \in \mathbf{R} \,|\, r\,C \in \mathbf{R}\}\\ &\subseteq \mathbf{R} : (1,\, \Theta) = \{\theta \in \mathbf{R} \,|\, \theta\,\Theta \in \mathbf{R}\}\,,\end{aligned} \tag{3.6}$$

i.e., the ideal formed by all the denominators of the controller C is contained in the ideal formed by all the denominators of signal generator Θ.

Proof. By Theorem 3.1, we have (3.5). Then, we get $\mathbf{R} : (1,\, C) \subseteq \mathbf{R} : (1,\, \Theta)$. Finally, we have

$$\begin{aligned}\mathbf{R} : (1,\, C) &:= \{k \in \mathbf{F} \,|\, (k)\,(1,\, C) \subseteq \mathbf{R}\}\\ &= \{k \in \mathbf{F} \,|\, k, k\,C \in \mathbf{R}\}\\ &= \{r \in \mathbf{R} \,|\, r\,C \in \mathbf{R}\}\,,\end{aligned}$$

and similarly for $\mathbf{R} : (1,\, \Theta)$. □

DEFINITION 5. A fractional ideal $\mathbf{J}$ of $\mathbf{R}$ is called *divisorial* if $\mathbf{R} : (\mathbf{R} : \mathbf{J}) = \mathbf{J}$.

EXAMPLE 3.1. An invertible fractional ideal is a divisorial ideal. In particular, a principal fractional ideal is divisorial. If Θ is stabilizable or admits a coprime factorization, then $(1,\, \Theta)$ is a divisorial ideal of $\mathbf{R}$.

THEOREM 3.2. *Let Θ be a signal generator which defines a divisorial ideal $(1,\, \Theta)$ (e.g., Θ is a stabilizable plant or Θ admits a coprime factorization). Then, a stabilizing controller C is robustly regulating for a signal generator Θ if and only if we have (3.6).*

Proof. Using Theorem 3.1, we only have to prove that the conditions (3.5) and (3.6) are equivalent. In Corollary 3.2, we showed that (3.5) implies (3.6). Conversely, if we have (3.6), then we get:

$$\mathbf{R} : (\mathbf{R} : (1,\, \Theta)) \subseteq \mathbf{R} : (\mathbf{R} : (1,\, C)).$$

Since C is a stabilizing controller, 1.b of Theorem 2.1 shows that $(1,\, C)$ is invertible, and thus $(1,\, C)$ is divisorial. Using the fact that $(1,\, \Theta)$ is divisorial, the above inclusion yields (3.5).

Under the conditions made on the signal generator Θ, Theorem 3.2 shows that a stabilizing controller C is robustly regulating for the signal generator Θ if and only if for every fractional representation $C = \frac{s}{r}$ of C (non necessarily coprime), where $0 \neq r,\, s \in \mathbf{R}$, there exists $t \in \mathbf{R}$ such that:

$$\Theta = \frac{t}{r}.$$

Indeed, if $C = \frac{s}{r}$ is a fractional representation of C, then we have $r \in \mathbf{R} : (1,\, C) \subseteq \mathbf{R} : (1,\, \Theta)$, i.e., r is a denominator of the signal generator Θ.

If Θ is a stabilizable plant, then 1.b of Theorem 2.1 shows that the fractional ideal $(1,\,\Theta)$ is invertible and its inverse $\mathbf{R} : (1,\,\Theta)$ can be generated by two elements $\theta_1,\,\theta_2 \in \mathbf{R}$, i.e., $\mathbf{R} : (1,\,\Theta) = (\theta_1,\,\theta_2)$. Theorem 3.2 then shows that if $C = \frac{s}{r}$ is fractional representation of C, then $r \in (\theta_1,\,\theta_2)$, i.e., $r = \alpha_1\,\theta_1 + \alpha_2\,\theta_2$ for certain $\alpha_1,\,\alpha_2 \in \mathbf{R}$. Using the following fractional representations of Θ

$$\Theta = \frac{\gamma_1}{\theta_1} = \frac{\gamma_2}{\theta_2},$$

for certain $\gamma_1,\,\gamma_2 \in \mathbf{R}$, we obtain:

$$\begin{cases} C = \dfrac{s}{\alpha_1\,\theta_1 + \alpha_2\,\theta_2}, \\ \Theta = \dfrac{\alpha_1\,\gamma_1 + \alpha_2\,\gamma_2}{\alpha_1\,\theta_1 + \alpha_2\,\theta_2} = \dfrac{\alpha_1\,\gamma_1 + \alpha_2\,\gamma_2}{r}. \end{cases}$$

Let us now study the case where Θ admits a coprime factorization, i.e., the case where $(1,\,\Theta)$ is principal.

Corollary 3.3. *If $\Theta = \frac{\gamma}{\theta}$, where γ and θ are coprime stable elements, then a stabilizing controller C solves the robust regulation problem if and only if $(\theta^{-1}) \subseteq (1,\,C)$ or equivalently if and only if $\mathbf{R} : (1,\,C) \subseteq (\theta)$, i.e., if and only if for every fractional representation $C = \frac{s}{r}$, $0 \neq r,\,s \in \mathbf{R}$, we have $r = \alpha\,\theta$ for a certain $\alpha \in \mathbf{R}$. Then, we have:*

$$C = \frac{s}{\alpha\,\theta}, \quad \Theta = \frac{\alpha\,\gamma}{r}.$$

Proof. By Proposition 2.1, we get $(1,\,\Theta) = (\theta^{-1})$. The condition $(1,\,\Theta) \subseteq (1,\,C)$ becomes $(\theta^{-1}) \subseteq (1,\,C)$. This condition is equivalent to the condition $\mathbf{R} : (1,\,C) \subseteq (\theta)$ since C is stabilizable and $(1,\,\Theta)$ is divisorial. □

Theorem 3.2 and Corollary 3.3 are generalizations of the classical formulation of the frequency domain internal model principle [24]: conditions on the existence of coprime factorizations for P or C are not required.

Example 3.2. If C admits a coprime factorization $C = \frac{s}{r}$, then we have $\mathbf{R} : (1,\,C) = (r)$, which yields $(r) \subseteq (\theta)$, i.e., $r = \alpha\,\theta$ for a certain $\alpha \in \mathbf{R}$.

1 of Theorem 2.1 shows that $C \in \mathbf{F}$ stabilizes $P \in \mathbf{F}$ if and only if we have

$$(1,\,P)\,(S)\,(1,\,C) = \mathbf{R} \;\Leftrightarrow\; (1,\,P)\,(1,\,C) = (1 - P\,C),$$

where $S := (1 - P\,C)^{-1}$. For more details, see [22]. Using Proposition 2.1, P admits a coprime fractorization if and only if C does. If $P \in \mathbf{R}$, then we get $(1,\,C) = (1 - P\,C)$.

We end this section by showing that a robustly regulating controller of a stable plant necessarily contains the denominator of some factorization of the generator as an internal model.

Theorem 3.3. *If $P \in \mathbf{R}$, then a stabilizing controller C is robustly regulating if and only if the generator Θ has a factorization $\Theta = \frac{\gamma}{\theta}$ such that $(\theta^{-1}) \subseteq (1, C)$.*

Proof. Using $(1,\,C) = (1 - P\,C)$ (see above), (3.5) becomes $(1,\,\Theta) \subseteq (1 - P\,C)$, which is equivalent to the existence of $\gamma,\,\theta \in \mathbf{R}$ such that

$$\begin{cases} \Theta = \gamma\,(1 - P\,C), \\ 1 = \theta\,(1 - P\,C), \end{cases} \Leftrightarrow \begin{cases} \theta^{-1} = (1 - P\,C) \in (1,\,C), \\ \Theta = \frac{\gamma}{\theta}, \end{cases}$$

which shows the necessity. Now, since $(\Theta) \subseteq (\theta^{-1})$ for all fractional representations $\Theta = \frac{\gamma}{\theta}$, $(\theta^{-1}) \subseteq (1, C)$ is a sufficient condition for (3.5).

□

4 Solvability of the Robust Regulation Problem

In this section, we give necessary and sufficient conditions for the solvability of the robust regulation problem. The first lemma gives a solvability condition for stable plants.

Lemma 4.1. *If $P \in \mathbf{R}$, then the robust regulation problem is solvable if and only if $\mathbf{R} \subseteq (\Theta^{-1},\,P)$.*

Proof. In order to show necessity, assume that C is a robustly regulating controller. 1 of Theorem 2.1 shows that $C = U\,S^{-1}$, where $S,\,U \in \mathbf{R}$ satisfy (2.1). Since C is regulating $S\,\Theta = (1 - P\,C)^{-1}\,\Theta \in \mathbf{R}$. This implies:

$$\mathbf{R} = (1) = (S - U\,P) = ((S\,\Theta)\,\Theta^{-1} - U\,P) \subseteq (\Theta^{-1},\,P).$$

For the necessity part, assume that $\mathbf{R} \subseteq (\Theta^{-1}, P)$. Now there exist $\alpha,\,\beta \in \mathbf{R}$ such that:

$$1 = \alpha\,\Theta^{-1} - \beta\,P. \tag{4.7}$$

If $\alpha = 0$, then we get $1 = -\beta\,P$, and thus

$$1 = h\,\Theta^{-1} + (h\,\Theta^{-1} - 1)\,\beta\,P \in (\Theta^{-1},\,P),$$

where $0 \neq h \in \mathbf{R}$ is chosen so that $h\,\Theta^{-1} \in \mathbf{R}$. Thus, without restricting generality we can assume that $\alpha \neq 0$.

Since $\beta\,P \in \mathbf{R}$, we see that $\alpha\,\Theta^{-1} \in \mathbf{R}$. The fact that $P\,\alpha\,\Theta^{-1} \in \mathbf{R}$ and the equation (4.7) imply that $C := \frac{\beta}{\alpha\,\Theta^{-1}}$ stabilizes P by Theorem 2.1. Furthermore, we have $(1 - P\,C)^{-1}\,\Theta = \alpha\,\Theta^{-1}\,\Theta = \alpha \in \mathbf{R}$ and $(1 - P\,C)^{-1}\,P\,\Theta \in \mathbf{R}$ since $P \in \mathbf{R}$, so C is regulating.□

The main results of this section are two necessary and sufficient solvability conditions for the robust regulation problem. In the next theorem, we state the solvability of the robust regulation problem as a robust regulation problem of a certain stable plant. A checkable condition of Corollary 4.1 for the solvability follows.

THEOREM 4.1. *The following assertions are equivalent:*

1. *The robust regulation problem is solvable.*
2. *There exists a stabilizing controller $C = \frac{U}{S}$ such that (2.1) holds and $\mathbf{R} \subseteq (\Theta^{-1}, U\,P)$.*
3. *There exists a stabilizing controller $C = \frac{U}{S}$ such that (2.1) holds and $(1, \Theta) \subseteq (1, U\,P\,\Theta)$, i.e., the fractional ideal associated with Θ is contained in the fractional ideal associated with $U\,P\,\Theta = (1-S)\,\Theta$.*

Proof. $1 \Rightarrow 2$. If $C = \frac{U}{S}$ is robustly regulating, then we have $1 = S - U\,P = \Theta\,S\,\Theta^{-1} - U\,P$, where $\Theta\,S \in \mathbf{R}$.

$2 \Rightarrow 1$. Since C is stabilizing, we get $U\,P \in \mathbf{R}$. There exists a robustly regulating controller C_i for $U\,P$ by Lemma 4.1. Next we show that

$$C_r := C + (U - C\,U\,P)\,C_i = C\,(1 + C_i), \tag{4.8}$$

is robustly regulating, which shows the claim.

First, we note that we have:

$$\frac{1}{1 - P\,C_r} = \frac{1}{(1 - P\,C)\,(1 - U\,P\,C_i)}. \tag{4.9}$$

We can easily verify that the proposed controller is stabilizing by using the assumptions that C stabilizes P, that C_i stabilizes $U\,P$ and equation (4.9).

Furthermore, since C_i is robustly regulating, we have that $\frac{\Theta}{1 - U\,P\,C_i} \in \mathbf{R}$ which, together with (4.9), shows that the proposed controller is regulating.

$2 \Leftrightarrow 3$. The condition $(1, \Theta) \subseteq (1, U\,P\,\Theta)$ is equivalent to the existence of $\alpha, \beta \in \mathbf{R}$ such that:

$$\begin{aligned} \Theta = \alpha + \beta\,U\,P\,\Theta \quad &\Leftrightarrow \quad 1 = \alpha\,\Theta^{-1} + \beta\,U\,P \\ &\Leftrightarrow \quad \mathbf{R} \subseteq (\Theta^{-1}, U\,P). \end{aligned}$$

□

EXAMPLE 4.1. It is well-known that plant transmission zeros at the natural frequencies of the signals to be tracked make robustness unachievable [3, 14, 15]. We show that this result follows by Theorem 4.1. Let us consider the ring $\mathbf{R} := RH_\infty$. We first note that $RH_\infty \subseteq (\Theta^{-1}, U\,P)$ is equivalent to $1 = \alpha\,\Theta^{-1} + \beta\,U\,P$, for some $\alpha, \beta \in RH_\infty$. If P has a zero on an unstable pole of Θ, we see that Θ^{-1} and P share a common zero. The above condition cannot be satisfied since stable α and β do not have unstable poles.

EXAMPLE 4.2. Choose $\mathbf{R} := RH_\infty$ and $\Theta := \frac{1}{s}$. Consider the plant $P := \frac{s+1}{s^2+1}$. In Example 2.2, we found the parameterization (2.3) of all the stabilizing controllers of P. Using (2.3) and $S = \frac{s^2+1}{(s+2)\,(s+3)}$, Theorem 4.1 shows that the robust regulation problem is solvable if (and only if) there exist $\alpha, \beta, Q \in \mathbf{R}$ such that:

$$\alpha\,\Theta^{-1} + \beta\,(-5 + Q\,S)\,S\,P = 1. \tag{4.10}$$

For instance, if we choose

$$\alpha := \frac{s-1}{(s+2)\,(s+3)}, \quad \beta := -\frac{6}{5}, \quad Q := 0, \tag{4.11}$$

then we have

$$\begin{aligned} &\alpha\,\Theta^{-1} + \beta\,(-5 + Q\,S)\,S\,P \\ &\quad = \frac{s(s-1)}{(s+2)\,(s+3)} - \frac{6}{5}\,\frac{-5\,(s^2+1)}{(s+2)\,(s+3)}\,\frac{s+1}{s^2+1} = 1, \end{aligned} \tag{4.12}$$

i.e., (4.10) holds, which shows that $C = -5$ is a robustly regulating controller for P and Θ.

In Example 4.2, we used the parameterization of all the stabilizing controllers of Theorem 2.1 and Theorem 4.1 to check the solvability of the robust regulation problem. We can formulate this idea in the general case which gives a checkable condition for the solvability.

COROLLARY 4.1. *Let $C = \frac{U}{S}$ be a stabilizing controller, where $S, U \in \mathbf{R}$ satisfy (2.1). Then, the robust regulation problem is solvable if and only if there exist $\beta, Q_1, Q_2 \in \mathbf{R}$ such that:*

$$\Theta\,(1 + \beta\,(U + Q_1\,S^2 + Q_2\,U^2)\,P) \in \mathbf{R}. \tag{4.13}$$

Proof. Theorems 2.1 and 4.1 imply that the robust regulation problem is solvable if and only if there exist $\alpha, \beta, Q_1, Q_2 \in \mathbf{R}$ such that:

$$1 = \alpha\,\Theta^{-1} - \beta\,(U + Q_1\,S^2 + Q_2\,U^2)\,P \;\Leftrightarrow\; (4.13).$$

□

REMARK 4.1. The above results give a design procedure for robustly regulating controllers. One first finds a stabilizing controller $C_0 := \frac{U}{S}$ of the plant P. If (4.13) holds, then a stabilizing controller $C(Q_1, Q_2)$ of P that satisfies the condition of Theorem 4.1 is given by $C(Q_1, Q_2) := \frac{U + Q_1\,S^2 + Q_2\,S^2}{S + Q_1\,P\,S^2 + Q_2\,P\,U^2}$. Then, following the proof of Theorem 4.1, we have to find a robustly regulating controller C_i of $(U + Q_1\,S^2 + Q_2\,U^2)\,P \in \mathbf{R}$ (i.e., by replacing U by $U + Q_1\,S^2 + Q_2\,U^2$ in (4.8)). The controller C_i is defined by $C_i := \frac{\beta}{\alpha\,\Theta^{-1}}$ (see the proof of Lemma 4.1) and is directly related to (4.13).

EXAMPLE 4.3. Let us continue Example 4.2. A robustly regulating controller can be constructed as in the proof of Theorem 4.1 using the defined stabilizing $C = -5$ and the robustly regulating controller C_i of $U + Q_1\,S^2 + Q_2\,U^2 = (Q\,S - 5)\,S$, where $S = \frac{s+1}{(s+2)\,(s+3)}$

and $Q = 0$. The desired controller C_r is of the form $C_r = C\,(1 + C_i) = -5\,(1 + C_i)$, where C_i can be constructed as in the proof of Lemma 4.1. Using α and β defined in (4.11), we find the controller

$$C_i = \frac{-\beta}{\alpha\,\Theta^{-1}} = \frac{\frac{6}{5}}{s\,\frac{s-1}{(s+2)\,(s+3)}} = \frac{6\,(s+2)\,(s+3)}{5\,s\,(s-1)}$$

that robustly regulates $U\,P = -5\,\frac{s+1}{(s+2)\,(s+3)} \in \mathbf{R}$. This follows by (4.12), Theorem 2.1, and Lemma 4.1. Thus, the robustly regulating controller is finally defined by:

$$C_r = -\frac{11\,s^2 + 25\,s + 36}{s\,(s-1)}.$$

If the signal generator Θ has a coprime factorization then we can simplify Theorem 4.1.

Theorem 4.2. *If $\Theta = \frac{\gamma}{\theta}$, where γ and θ are coprime stable elements, then the robust regulation problem is solvable if and only if the plant P is stabilizable and*

$$\mathbf{R} \subseteq (\theta,\, P),$$

or equivalently if and only if P is stabilizable and:

$$\left(1,\, \theta^{-1}\right) \subseteq \left(1,\, \theta^{-1}\,P\right).$$

Proof. In order to show necessity, assume that C is a robustly regulating controller. Since C is stabilizing, there exits $S, U \in \mathbf{R}$ such that $C = \frac{U}{S}$ and (2.1) holds. By Corollary 3.3, i.e., $(\theta^{-1}) \subseteq (1,\, C)$, there exist $\alpha,\, \beta \in \mathbf{R}$ such that $\theta^{-1} = \alpha + \beta\,C$. Using (2.1), we get:

$$\begin{aligned}1 &= \alpha\,\theta + \theta\,\beta\,C\\ &= \alpha\,\theta + \theta\,\beta\,C\,(S - U\,P)\\ &= (\alpha + \beta\,U)\,\theta - (\beta\,\theta\,(C\,S))\,P \in (\theta,\, P).\end{aligned}$$

Next, we show sufficiency. Assume that $\mathbf{R} \subseteq (\theta,\, P)$ and that there exist $S,\, U \in \mathbf{R}$ such that (2.1) holds and $S \neq 0$. By assumption, there exist $\alpha,\, \beta \in \mathbf{R}$ such that:

$$1 = \alpha\,\theta + \beta\,P = \alpha\,\theta\,(S - U\,P) + \beta\,P = \alpha\,S\,\theta + (\beta - \alpha\,\theta\,U)\,P.$$

Without restricting generality we can assume that we have $\alpha \neq 0$. Since $\alpha\,\theta\,S\,P \in \mathbf{R}$, $C := \frac{-\beta + \alpha\,\theta\,U}{\alpha\,\theta\,S}$ is then a stabilizing controller of P by Theorem 2.1. The claim follows by Theorem 4.1. Finally, $\left(1,\, \theta^{-1}\right) \subseteq \left(1,\, \theta^{-1}\,P\right)$ is equivalent to the existence of $\delta,\, \sigma \in \mathbf{R}$ such that $\theta^{-1} = \delta + \sigma\,\theta^{-1}\,P \Leftrightarrow 1 = \delta\,\theta + \sigma\,P$, i.e., to $\mathbf{R} \subseteq (\theta,\, P)$. □

Example 4.4. If the plant P has a coprime factorization $P = \frac{n}{d}$, then there exist $x,\, y \in \mathbf{R}$ such that $x\,n + y\,d = 1$. Let $\theta \in \mathbf{R}$ be the denominator of a coprime factorization of Θ, so Theorem 4.2 shows that the robust regulation problem is solvable if and only if $\mathbf{R} \subseteq (\theta,\, P)$, i.e., if and only if there exist $v,\, w \in \mathbf{R}$ such that $1 = v\,\theta + w\,P = v\,\theta + w\,P\,(x\,n + y\,d) = v\,\theta + (x\,w\,P + y\,w)\,n$. Since $w\,P = 1 - v\,\theta \in \mathbf{R}$, this means that θ and n are coprime. Conversely, using $(\theta,\, n) \subseteq (\theta,\, P)$, if θ and n are coprime, i.e., $(\theta,\, n) = \mathbf{R}$, then we get $\mathbf{R} \subseteq (\theta,\, P)$, which proves that the robust regulation problem is solvable if and only if n and θ are coprime (i.e., the classical solvability condition of [24]).

Our last example is a bit more involved. The given plant does not have a coprime factorization and we see that an arbitrarily chosen stabilizing controller does not necessarily satisfy the condition of Theorem 4.1.

Example 4.5. Recall [22, Example 4] that originates from [18]. There $\mathbf{R} := \mathbb{R}[x^2, x^3]$ and $P := \frac{1-x^3}{1-x^2} \in \mathbf{F}$. The original motivation of this example is in modeling high speed electronic circuits. It was shown that this plant does not admit a coprime factorization over $\mathbf{R}$ and that $C := \frac{x^2-1}{1+x^3}$ is a stabilizing controller. In addition, a factorization $C = \frac{U}{S}$ that satisfies (2.1) is given by $S := \frac{1+x^3}{2}$ and $U := \frac{x^2-1}{2}$. Let us now consider robust regulation with generator $\Theta := x^{-2} \in \mathbf{F}$. If $Q_1 := -2$, $Q_2 := 0$ and $\beta := 1 - x^2$, then (4.13) takes the form:

$$\begin{aligned}(4.14)\quad & x^{-2}\left(1 + (1 - x^2)\left(\frac{x^2-1}{2} - 2\frac{(1+x^3)^2}{2^2}\right)\frac{1-x^3}{1-x^2}\right)\\ &= x^{-2}\left(1 + \frac{x^9 + x^6 - x^5 + x^2 - 2}{2}\right)\\ &= \frac{x^7 + x^4 - x^3 + 1}{2} \in \mathbb{R}[x^2, x^3].\end{aligned}$$

By Corollary 4.1, the robust regulation problem is solvable. Let us now construct a robustly regulating controller. By Remark 4.1, a robustly regulating controller is given by (4.8) if we can find a robustly regulating controller C_i for $(U + Q_1\,S^2)\,P$. To this end, we choose α to be the stable element of (4.14). Following the proof of Lemma 4.1, we find out that $C_i := \frac{\beta}{\alpha\,\Theta^{-1}}$ robustly regulates $(U + Q_1\,S^2)\,P \in \mathbf{R}$. The robustly regulating controller C_r is then defined by:

$$C_r = C\,(1 + C_i) = \frac{(x^2-1)\,(x^9 + x^6 - x^5 + x^2 - 2)}{x^2\,(x^3+1)\,(x^7 + x^4 - x^3 + 1)}.$$

Note that we cannot choose $Q_1 = Q_2 = 0$ because, if we do, we then need to find $\alpha,\, \beta \in \mathbb{R}[x^2, x^3]$ such that:

$$1 = \alpha\,\Theta^{-1} + \beta\,U\,P = \alpha\,x^2 + \beta\,\frac{x^3 - 1}{2}.$$

We must choose $\beta = p - 2$, where p is a polynomial of order greater than or equal to 2. But then we cannot get rid of $-x^3$ since the lowest order terms $\alpha\,x^2$ can contain are x^2 and x^4 when $\alpha \in \mathbb{R}[x^2, x^3]$. Thus, the controller C in Theorem 4.1 cannot be chosen totally arbitrarily.

5 Concluding Remarks

In this article, we have developed frequency domain robust regulation theory that uses no coprime factorizations for SISO systems. The results extend the class of systems we can deal with, and give new formulations for the internal model principle and generalize the solvability condition presented in [24] for rational plants. Our next step is to find a parameterization for all robustly regulating controllers and to generalize the presented theory to multi-input multi-output systems.

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