Stabilization of infinite-dimensional linear systems: An algebraist's point of view

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A workshop on linear system theory,

Model reduction

Sde Boker (15-19/09/08)

Thank you very much Paul!

Outline

• The fractional representation approach to analysis and synthesis problems developed by Desoer, Callier, Vidyasagar, Francis... was successful for finite-dimensional systems.

M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.

• However, it is still in progress for infinite-dimensional systems.

R. F. Curtain, H. J. Zwart, *An Introduction to ∞-Dimensional Linear Systems Theory*, TAM 21, Springer, 1991.

• The goal of this talk is to give a simple overview of certain results recently obtained based on algebraic analysis approach.

• In particular, we shall introduce the concept of fractional ideals to study transfer functions within a module-theoretic approach.

Examples of transfer functions

- Ordinary differential equation:

$$\dot{z}(t) = z(t) + u(t), \;\; z(0) = 0 \quad \Rightarrow \; \widehat{z}(s) = \frac{1}{(s-1)}\, \widehat{u}(s).$$

- Differential time-delay equation:

$$\begin{cases} \dot{z}(t) = z(t) + u(t), \;\; x(0) = 0, \\ y(t) = \begin{cases} 0, & 0 \leq t < h, \\ z(t-h), & t \geq h, \end{cases} \end{cases} \quad \Rightarrow \; \widehat{y}(s) = \frac{e^{-h\,s}}{(s-1)}\, \widehat{u}(s).$$

- Partial differential equation:

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(x,t) - a^2\, \frac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \quad \frac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x}, t), \end{cases} \Rightarrow \; \widehat{y}(s) = \frac{\left(e^{-\frac{\overline{x}}{a}\,s} - e^{-\frac{(2\,l - \overline{x})\,s}{a}}\right)}{\left(1 - e^{-\frac{2\,a}{l}\,s}\right)}\, \widehat{u}(s).$$

Examples of transfer functions

- Heat equation:

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) - \lambda^2 \dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x}, t), \end{cases} \Rightarrow \widehat{y}(s) = \frac{\left(e^{\lambda\,(l-\overline{x})\,\sqrt{s}} - e^{-\lambda\,(l-\overline{x})\,\sqrt{s}}\right)}{\left(e^{\lambda\,l\,\sqrt{s}} - e^{-\lambda\,l\,\sqrt{s}}\right)}\,\widehat{u}(s).$$

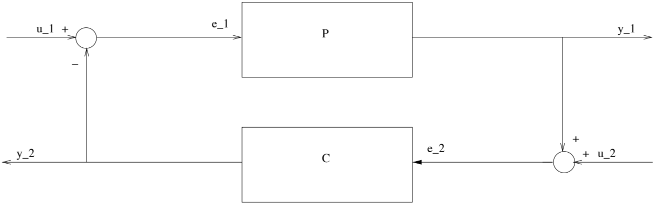
- Telegraph equation:

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2 \dfrac{\partial^2 z}{\partial x^2}(x,t) - k\,z(x,t) = 0, \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad \lim_{x \to +\infty} z(x,t) = 0, \\ y(t) = z(\overline{x}, t), \end{cases} \Rightarrow \widehat{y}(s) = e^{\frac{-\sqrt{s^2-k}}{a}\,\overline{x}}\,\widehat{u}(s).$$

Stabilization problems

- Let the open-loop $\widehat{u} \longmapsto \widehat{y} = p\,\widehat{u}$ be unstable.

- Is it possible to find a controller c such that the closed-loop is stable (e.g., for all $u_1,\ u_2 \in L^2(\mathbb{R}_+)$ or $L^\infty(\mathbb{R}_+)$)?



- Can we parametrize the set of all stabilizing controllers of p?

- Is it possible to find robust/optimal controllers c of p?

- Is it possible to find stable controllers c of p?

The fractional representation approach

• (Zames) The set of transfer functions has the structure of an algebra (parallel $+$, serie $\circ$, proportional feedback . by $\mathbb{R}$).

• (Vidyasagar) Let A be an algebra of stable transfer functions with a structure of an integral domain ($a\,b = 0,\; a \neq 0 \;\Rightarrow\; b = 0$).

$Q(A) = \{p = n/d \mid 0 \neq d,\, n \in A\}$ represents the class of systems:

$$\boxed{p \in A \;\Leftrightarrow\; p \text{ is stable}, \quad p \in K \backslash A \;\Leftrightarrow\; p \text{ is unstable}}$$

• (Zames) The algebra A should be a normed algebra so that the errors in the modelization & approximation of the real plant by the mathematical model can be considered

(e.g., Banach algebra: $\| \, a\,b \, \|_A \leq \| \, a \, \|_A \, \| \, b \, \|_A, \quad \| \, 1 \, \|_A = 1$).

Example: Hardy algebra

- Let us define the right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 0\}$.

- The Hardy algebra $H^\infty(\mathbb{C}_+)$ (Banach algebra) is defined by:

$$H^\infty(\mathbb{C}_+) = \{\text{holomorphic fcts in } \mathbb{C}_+ \,|\; \| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}.$$

- The Hardy algebra $H^\infty(\mathbb{C}_+)$ is the algebra of transfer functions of $L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+)$-stable shift-invariant ∞-dimensional systems.

- $RH_\infty = \mathbb{R}(s) \cap H^\infty(\mathbb{C}_+)$

$$= \left\{ \frac{n}{d} \;|\; 0 \neq d,\, n \in \mathbb{R}[s],\; \deg n \leq \deg d,\; d(\overline{s}) = 0 \Rightarrow \mathrm{Re}\, \overline{s} < 0 \right\}$$

is the algebra of exponentially-stable finite-dimensional plants.

Example: Wiener algebra

- $L^1(\mathbb{R}_+) = \{f : [0, +\infty[\to \mathbb{R} \mid \ \| f \|_1 = \int_0^{+\infty} |f(t)| \, dt < +\infty\}$,

$$l^1(\mathbb{Z}_+) = \{a : \mathbb{Z}_+ = \{0, 1, \ldots\} \to \mathbb{R} \mid \ \| (a_i)_{i \in \mathbb{Z}_+} \|_1 = \sum_{i=0}^{+\infty} |a_i| < +\infty\}.$$

- The Wiener algebra $\mathcal{A}$ is defined by:

$$\mathcal{A} = \{f = g + \textstyle\sum_{i=0}^{+\infty} a_i \, \delta_{(t-h_i)} \mid \quad g \in L^1(\mathbb{R}_+), \ (a_i)_{i \in \mathbb{Z}_+} \in l^1(\mathbb{Z}_+), \quad 0 = h_0 \leq h_1 \leq h_2 \leq \ldots\}.$$

- $\mathcal{A}$ is a Banach algebra w.r.t. $\| f \|_{\mathcal{A}} = \| g \|_1 + \| (a_i)_{i \in \mathbb{Z}_+} \|_1$.
- $\hat{\mathcal{A}} = \{\mathcal{L}(f) \mid f \in \mathcal{A}\}$, $\quad \| \widehat{f} \|_{\widehat{\mathcal{A}}} = \| f \|_{\mathcal{A}}$.
- $\mathcal{A}$ is the algebra of $L^\infty(\mathbb{R}_+) - L^\infty(\mathbb{R}_+)$-stable shift-invariant ∞-dimensional systems.

Examples

- RH_∞: algebra of exponentially-stable finite-dimensional plants:

$$p = \frac{1}{s-1} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{1}{s+1}, \quad \frac{s-1}{s+1} \in RH_\infty \;\Rightarrow\; p \in Q(RH_\infty).$$

- $\hat{\mathcal{A}}$: algebra of BIBO-stable ∞-dimensional plants:

$$p = \frac{e^{-h\,s}}{s-1} = \frac{\left(\frac{e^{-h\,s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{e^{-h\,s}}{s+1}, \quad \frac{s-1}{s+1} \in \hat{\mathcal{A}} \;\Rightarrow\; p \in Q(\hat{\mathcal{A}}).$$

- $H^\infty(\mathbb{C}_+)$: algebra of $L^2(\mathbb{R}_+)$-stable ∞-dimensional plants:

$$p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})} \in Q(H^\infty(\mathbb{C}_+)) : \; 1+e^{-2\,s}, \; 1-e^{-2\,s} \in H^\infty(\mathbb{C}_+).$$

(Weakly) coprime factorization

- Let A be an algebra of stable transfer functions and:

$$K = Q(A) = \{n/d,\ 0 \neq d,\ n \in A\}.$$

- **Definition**: $p \in K$ admits a weakly coprime factorization if:

$$\exists\, 0 \neq d,\, n \in A:\ \ p = n/d, \quad \forall\, k \in K:\ k\, n,\ k\, d \in A \ \Rightarrow\ k \in A.$$

- **Definition**: $p \in K$ admits a coprime factorization over A if:

$$\exists\, 0 \neq d,\, n,\, x,\, y \in A: \quad p = n/d, \quad d\, x + n\, y = 1.$$

- A coprime factorization is a weakly coprime factorization:

$$\forall\, k \in K:\ k\, n,\ k\, d \in A \ \Rightarrow\ k = (k\, d)\, x + (k\, n)\, y \in A.$$

Internal stabilizability (Desoer)

- Let A be an algebra of stable transfer functions, $K = Q(A)$.
- Let $p \in K$ be a plant and $c \in K$ a controler.
- The closed-loop system is defined by:

$$\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) = \left(\begin{array}{cc} 1 & c \\ -p & 1 \end{array}\right) \left(\begin{array}{c} e_1 \\ e_2 \end{array}\right), \quad \left\{\begin{array}{l} y_1 = e_2 - u_2, \\ y_2 = u_1 - e_1. \end{array}\right.$$

- **Definition**: c internally stabilizes p if we have:

$$H(p, c) = \left(\begin{array}{cc} 1 & c \\ -p & 1 \end{array}\right)^{-1} = \left(\begin{array}{cc} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1-p\,c} & \frac{1}{1+p\,c} \end{array}\right) \in A^{2\times 2}.$$

Then, c is called a stabilizing controler of p.

Strong and simultaneous stabilizabilities

- Let A be an algebra of stable transfer functions, $K = Q(A)$.

- **Definition**: $p \in K$ is strongly stabilizable if there exists a stable controller c, i.e., $c \in A$, which internally stabilizes p.

- **Definition**: The plants $p_1, \ldots, p_n \in K$ are simultaneously stabilizable if $\exists\ c \in K$ which internally stabilizes $p_1, \ldots, p_n$.

- Interests of the strong stabilizability:

Safety, good ability to track reference inputs.

- Interests of the simultaneous stabilizability:

The controller is designed to stabilize a family of plants (e.g., operating conditions, failed modes, loss of sensors/actuators).

Robust stabilizability

- Let A be a Banach algebra of stable transfer functions

$$(\text{e.g., } A = H^\infty(\mathbb{C}_+),\ \widehat{\mathcal{A}},\ A(\mathbb{D}),\ W_+).$$

- **Definition**: Let c be a stabilizing controller of $p \in Q(A)$. Then, c robustly stabilizes p if $\exists\, \epsilon > 0$ such that c internally stabilizes:

1. Additive perturbations:

$$B_1(p, \delta) = \{p + \delta \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\}.$$

2. Multiplicative perturbations:

$$B_2(p, \delta) = \{p/(1 + \delta\, p) \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\}.$$

3. Relative additive perturbations:

$$B_3(p, \delta) = \{p\,(1 + \delta) \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\}.$$

4. Relative multiplicative perturbations:

$$B_4(p, \delta) = \{p/(1 + \delta) \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\} \ldots$$

A fractional ideal approach (SCL 03)

- A is an integral domain of SISO stable plants and $K = Q(A)$.
- Let $p \in K$ be a plant and let us introduce the fractional ideal:

$$J = (1,\, p) \triangleq A + A\, p.$$

- J is defined by all the stable linear combinations of 1 and p.
- Why do we need 1? Algebraic answer: the structural properties of a plant p only depend on the system:

$$y - p\, u = 0 \;\Leftrightarrow\; (1 \quad -p) \left(\begin{array}{c} y \\ u \end{array} \right) = 0.$$

Analysis answer: the structural properties of a plant p depend on the graph of the unbounded operator:

$$u \longmapsto y = p\, u.$$

Theory of fractional ideals

"Dedekind's invention of ideals in the 1870s was a major turning point in the development of algebra", Stillvell.

- **Definition:** An A-submodule J of $K = Q(A)$ is a fractional ideal of A if $\exists \, 0 \neq d \in A$ such that $(d)\, J = \{d\, j \mid j \in J\} \subseteq A$.

- A fractional ideal $J \subseteq A$ is called an ideal of A.

- A fractional ideal J is principal if $\exists \, k \in K$ s.t. $J = A\, k = (k)$.

- $I\, J = \{\sum_{i=1}^{n} a_i\, b_i \mid a_i \in I, \; b_i \in J\}$, $A : J = \{k \in K \mid (k)\, J \subseteq A\}$.

- A fractional ideal J is invertible if $\exists \, I \in \mathcal{F}(A)$ such that $I\, J = A$.

If J is invertible then its inverse J^{-1} is unique and defined by $A : J$.

Main results (SCL 03)

- Let A be a ring of stable transfer functions and $K = Q(A)$.
- Let $p \in K$ be a transfer function.
- Let $J = (1,\ p)$ be a fractional ideal, $A : J = \{d \in A \mid d\ p \in A\}$.
- **Theorem**: 1. p is stable iff $J = (1) = A$ iff $A : J = (1) = A$.

2. p admits a weakly coprime factorization iff:

$$\exists\, 0 \neq d \in A : \quad A : J = (d).$$

Then, $p = n/d$, $(n = d\, p \in A)$, is a weakly coprime factorization.

3. p is internally stabilizable iff J is invertible, i.e., iff:

$$\exists\, a,\ b \in A, \quad a + b\, p = 1, \quad a\, p \in A.$$

If $a \neq 0$, then $c = b/a$ is a stabilizing controller of p and:

$$J^{-1} = (a,\ b), \quad a = 1/(1 + p\, c), \quad b = c/(1 + p\, c).$$

Main results (SCL 03)

4. $c \in K$ internally stabilizes $p \in K$ iff we have:

$$(1,\ p)\,(1,\ c) = (1 + p\,c).$$

5. $c \in K$ externally stabilizes $p \in K$, i.e., $p\,c/(1 - p\,c) \in A$, iff:

$$(1,\ p\,c) = (1 + p\,c).$$

6. p is strongly stabilizable iff there exists $c \in A$ such that:

$$(1,\ p) = (1 + p\,c).$$

7. p admits a coprime factorization iff J is principal.

Then, there exists $0 \neq d \in A$ such that $(1,\ p) = (1/d)$ and $p = n/d$ is a coprime factorization of p ($n = d\,p \in A$).

Proof 1

- $J = (1,\ p)$, $\ A : J = \{d \in A \mid d\, p \in A\}$. If J is invertible, then

$$1 \in J\, J^{-1} = (1,\ p)\,(A : J) = \{\alpha + \beta\, p \mid \alpha,\ \beta \in A : J\}$$

$$\Leftrightarrow\ \exists\ a,\ b \in A : \ \begin{cases} a + b\, p = 1, \\ a\, p \in A, \quad b\, p \in A. \end{cases}$$

If $a \neq 0$, then $c = b/a \in K$ satisfies:

$$H(p,\ c) = \begin{pmatrix} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1+p\,c} & \frac{1}{1+p\,c} \end{pmatrix} = \begin{pmatrix} a & -b \\ a\,p & a \end{pmatrix} \in A^{2\times 2},$$

$\Rightarrow c = b/a$ internally stabilizes p $\quad (a = 0 \Rightarrow c = 1 + b$ IS $p)$.

- If p is internally stabilizable, then there exists $c \in K$ s.t.:

$$a = 1/(1+p\,c) \in A, \quad a\,p = p/(1+p\,c) \in A, \quad b = c/(1+p\,c) \in A.$$

Let $I = (a,\ b)$. Then, $a + b\,p = 1 \in I\,J \Rightarrow I\,J = A \Rightarrow I = J^{-1}$.

Example

- $A = H^{\infty}(\mathbb{C}_+), \quad p = \frac{e^{-s}}{(s-1)} \in Q(A), \quad J = (1,\, p).$

$$\gcd\left(\frac{e^{-s}}{s+1}, \frac{s-1}{s+1}\right) = 1 \;\Rightarrow\; A : J = \{d \in A \mid d\, p \in A\} = \left(\frac{s-1}{s+1}\right).$$

- p is internally stabilizable iff

$$\exists\, a,\, b \in (A : J) :\; a + b\, p = 1 \quad \Leftrightarrow \quad \exists\, x,\, y \in A : \begin{cases} a = \left(\frac{s-1}{s+1}\right) x, \\ b = \left(\frac{s-1}{s+1}\right) y, \\ a + b\, p = 1. \end{cases}$$

$$a + b\, p = 1 \;\Leftrightarrow\; \left(\frac{s-1}{s+1}\right)(x + p\, y) = 1 \;\;\Leftrightarrow x = \frac{s+1}{s-1} - p\, y$$
$$\Leftrightarrow x = \frac{(s+1) - e^{-s}\, y}{s-1}.$$

- $x \in A \;\Rightarrow\; ((s+1) - e^{-s}\, y(s))(1) = 0 \;\Rightarrow\; y(1) = 2\, e.$

Example

- Hence, we can take:

$$y(s) = 2\,e \;\Rightarrow\; x(s) = 1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right) \in A.$$

- Therefore, we get:

$$\begin{cases} a = \left(\frac{s-1}{s+1}\right) x = \left(\frac{s-1}{s+1}\right)\left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right), \\ b = \left(\frac{s-1}{s+1}\right) y = 2\,e\left(\frac{s-1}{s+1}\right), \\ a + b\,p = 1. \end{cases}$$

$\Rightarrow$ A stabilizing controller c of p is defined by:

$$c = \frac{b}{a} = \frac{2\,e\,(s-1)}{(s-1) + 2\,(1 - e^{-(s-1)})} = \frac{2\,e\,(s-1)}{s + 1 - 2\,e^{-(s-1)}}.$$

Proof 2

• $J = (1,\ p)$ is principal iff there exists $0 \neq k \in K$ s.t. $J = (k)$, i.e., iff there exist $0 \neq d,\ n,\ x,\ y \in A$ s.t.:

$$\left\{ \begin{array}{l} 1 = d\,k, \\ p = n\,k, \\ k = x + y\,p \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} k = 1/d, \\ p = n/d, \\ 1/d = x + y\,(n/d), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} k = 1/d, \\ p = n/d, \\ d\,x + n\,y = 1. \end{array} \right.$$

$\Rightarrow$ p admits a coprime factorization $p = n/d$ iff $J = (1/d)$.

• A principal fractional ideal $J = (k) \neq 0$ is invertible:

$$J^{-1} = (1/k).$$

$\Rightarrow$ if p admits a coprime factorization then p is stabilizable.

• $(d\,x) + (d\,y)\,p = 1$, i.e., $a = d\,x,\ b = d\,y \in J^{-1} = (d)$,

$$\Rightarrow c = b/a = y/x \in \mathrm{Stab}(p).$$

Example

- Let $A = H^{\infty}(\mathbb{C}_+)$ and $p = \frac{e^{-s}}{(s-1)} \in K = Q(A)$.
- Let $J = (1,\, p)$ be the fractional ideal of A generated by 1 and p.
- $J = \left(\frac{s+1}{s-1}\right)$ is a principal ideal as we have:

$$\begin{cases} 1 = \left(\frac{s-1}{s+1}\right)\,\left(\frac{s+1}{s-1}\right), \\ \frac{e^{-s}}{(s-1)} = \left(\frac{e^{-s}}{s+1}\right)\,\left(\frac{s+1}{s-1}\right), \\ \frac{(s+1)}{(s-1)} = \left(1 + 2\,\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + 2\,e\,\frac{e^{-s}}{(s-1)} \qquad (\star). \end{cases}$$

$p = \dfrac{n}{d}$, $d = \frac{(s-1)}{(s+1)}$, $n = \frac{e^{-s}}{(s+1)}$, is a coprime factorization of p:

$$(\star) \Leftrightarrow \left(\frac{s-1}{s+1}\right)\,\left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(\frac{e^{-s}}{s+1}\right)\,(2\,e) = 1.$$

Example

- Let A be the Wiener algebra W_+ of holomorphic functions in the unit disc $\mathbb{D}$ whose Taylor series converge absolutely:

$$W_+ = \left\{ f(z) = \sum_{i=0}^{+\infty} a_i\, z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty \right\}.$$

- A is the algebra of the BIBO-stable causal filters.
- Let us consider $J = (1,\ p)$ where $p = e^{-\left(\frac{1+z}{1-z}\right)}$

$$\left\{ \begin{array}{l} n = (1-z)^3\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \\ d = (1-z)^3 \in A, \end{array} \right. \quad \Rightarrow \quad p = \frac{n}{d} \in Q(A).$$

- The ideal $A : J = \{d \in A \mid d\, p \in A\}$ of A is not finitely generated

(R. Mortini & M. Von Renteln, "Ideals in Wiener algebra", J. Austral. Math. Soc., 46 (1989), 220-228).

$\Rightarrow$ p does not admit a (weakly) coprime factorization and p is not internally stabilizable.

Example

- Let A be the disc algebra $A(\mathbb{D})$ of holomorphic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ which are continuous on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

- We have $n = (1 - z)\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \quad d = (1 - z) \in A,$

$$\Rightarrow p = \frac{n}{d} = e^{-\left(\frac{1+z}{1-z}\right)} \in Q(A), \quad J = (1,\, p).$$

- The ideal $A : J = \{d \in A \mid d\, p \in A\} = \{d \in A \mid d(1) = 0\}$ of A is maximal and is not finitely generated

(R. Mortini, "Finitely generated prime ideals in H^∞ and $A(\mathbb{D})$", Math. Z., 191 (1986), 297-302).

$\Rightarrow$ p does not admit a (weakly) coprime factorization and p is not internally stabilizable.

Proof 3

- If p is strongly stabilizable then there exists $c \in A$ such that:

$$a = \frac{1}{1 + p\,c} \in A, \quad a\,p = \frac{p}{1 + p\,c} \in A, \quad b = \frac{c}{1 + p\,c} = c\,a \in A.$$

Using the fact that $c \in A$, we obtain:

$$J^{-1} = (a,\, b) = (a) = ((1 + p\,c)^{-1}) \;\Rightarrow\; J = (J^{-1})^{-1} = (1 + p\,c).$$

- We suppose that there exists $c \in A$ such that $(1,\, p) = (1 + p\,c)$

$$\Rightarrow\; \exists\, 0 \neq d,\, n \in A : \left\{ \begin{array}{l} 1 = d\,(1 + p\,c), \\ p = n\,(1 + p\,c), \end{array} \right. \;\Rightarrow\; \left\{ \begin{array}{l} p = \dfrac{n}{d}, \\ d + n\,c = 1, \end{array} \right.$$

$$\Rightarrow \left(\begin{array}{cc} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1+p\,c} & \frac{1}{1+p\,c} \end{array} \right)^{-1} = \left(\begin{array}{cc} d & -d\,c \\ n & d \end{array} \right) \in A^{2 \times 2}$$

$\Rightarrow$ $c \in A$ internally stabilizes p, i.e., p is strongly stabilizable.

Example

- $A = H^\infty(\mathbb{C}_+), \quad K = Q(A), \quad p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})} \in K.$
- We have $J = (1,\ p) = \left(\frac{1}{1-e^{-2\,s}}\right)$ because:

$$\left\{\begin{array}{l} 1 = (1 - e^{-2\,s})\, \dfrac{1}{(1-e^{-2\,s})}, \\[2ex] p = \dfrac{(1+e^{-2\,s})}{(1-e^{-2\,s})} = (1+e^{-2\,s})\, \dfrac{1}{(1-e^{-2\,s})}, \\[2ex] \dfrac{1}{(1-e^{-2\,s})} = \dfrac{1}{2} \times 1 + \dfrac{1}{2}\, \dfrac{(1+e^{-2\,s})}{(1-e^{-2\,s})}. \end{array}\right.$$

$\Rightarrow$ coprime factorization $\left\{\begin{array}{l} p = \dfrac{(1+e^{-2\,s})}{(1-e^{-2\,s})}, \\[2ex] \frac{1}{2}\,(1-e^{-2\,s}) + \frac{1}{2}\,(1+e^{-2\,s}) = 1. \end{array}\right.$

$\Rightarrow$ $c = 1$ is a stable stabilizing controller of p.

- **Check:** $1 + p\,c = \frac{2}{(1-e^{-2\,s})}$ and $J = \left(\frac{1}{1-e^{-2\,s}}\right) = (1 + p\,c)$.

Bass stable rank (Feintuch, Blondel, SIAM 04)

- **Definition**: The stable rank of a ring A equals 1, denoted by $\mathrm{sr}(A) = 1$, if, for every d and n of A satisfying $d\,x + n\,y = 1$ for certain x and y of A, then there exists $c \in A$ such that:

$$d + n\,c \in \mathrm{U}(A) = \{a \in A \mid \exists\, b \in A : \; a\,b = b\,a = 1\}.$$

- **Theorem**: Let A be a ring of transfer functions and $K = Q(A)$. Then, every transfer function $p \in K$ which admits a coprime factorization is strongly stabilizable iff $\mathrm{sr}(A) = 1$.

- **Example**: The following Banach algebras $H^\infty(\mathbb{C}_+)$,

$$H^\infty(\mathbb{D}),\; A(\mathbb{D}),\; W_+,\; L_1(\mathbb{R}_+) + \mathbb{C}\,\delta,\; L^\infty(i\,\mathbb{R}),\; \mathbb{C}(s) \cap H^\infty(\mathbb{C}_+)$$

have a stable rank equal to 1

(Treil 92, Jones/Marshall/Wolff 86, Rupp 90, Mikkola/Sasane 07, Handelman 79).

Topological stable rank (SIAM 04)

- $U_n(A) = \{a = (a_1, \dots a_n) \in A^n \mid \exists\, b = (b_1, \dots, b_n) :\; a\, b^T = 1\}$.

- **Definition**: If A is a Banach algebra, then the topological stable range $\mathrm{tsr}(A)$ of A is the least integer n such that $\mathrm{U}_n(A)$ is dense in $A^{1\times n}$ for the product topology (else we set $\mathrm{tsr}(A) = \infty$).

- **Proposition** (Rieffel, 83): $\mathrm{sr}(A) \leq \mathrm{tsr}(A)$.

- **Theorem**: Let A be a Banach algebra such that $\mathrm{tsr}(A) = 2$, then every system − defined by a transfer function $p = n/d$ $(0 \neq d,\ n \in A)$ − is as close as we want to a plant which admits a coprime factorization, i.e.:

$$\forall\, \epsilon > 0,\ \exists\, (d_\epsilon,\ n_\epsilon) \in \mathrm{U}_2(A) : \begin{cases} \| \, d - d_\epsilon \, \|_A \leq \epsilon, \\ \| \, n - n_\epsilon \, \|_A \leq \epsilon. \end{cases}$$

- **Theorem:** $\mathrm{tsr}(H^\infty(\mathbb{D})) = 2$ (Suárez 96), $\mathrm{tsr}(A(\mathbb{D})) = 2$ (Rieffel 83), $\mathrm{tsr}(W_+) = 2$ (Sasane).

Robust stabilization

- Let $p \in Q(A)$ and $\delta \in A$.
- $c \in Q(A)$ internally stabilizes p and $p + \delta$ iff we have:

$$\left\{ \begin{array}{l} (1,\ p)\,(1,\ c) = (1 + p\,c), \\ (1,\ p + \delta)\,(1,\ c) = (1 + (p + \delta)\,c), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (1,\ p)\,(1,\ c) = (1 + p\,c), \\ (1,\ p)\,(1,\ c) = (1 + (p + \delta)\,c), \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} (1,\ p)\,(1,\ c) = (1 + p\,c), \\ \left(\dfrac{1 + (p + \delta)\,c}{1 + p\,c}\right) = \left(1 + \dfrac{\delta\,c}{1 + p\,c}\right) = A, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} c \text{ IS } p, \\ 1 + \dfrac{\delta\,c}{(1 + p\,c)} \in \mathrm{U}(A). \end{array} \right.$$

- If A is a Banach algebra, then (small gain theorem):

$$\| \, 1 - a \, \|_A < \ 1 \ \Rightarrow \ a \in \mathrm{U}(A) = \{a \in A \mid \exists\, b \in A : \ a\,b = b\,a = 1\}.$$

$\Rightarrow$ A sufficient condition for robust stabilization is:

$$\| \, \delta \, \|_A < \| \, c/(1 + p\,c) \, \|_A^{-1} \, .$$

Robust stabilization

- Let A be a Banach algebra, $p \in Q(A)$ and $\delta \in A$.
- c internally stabilizes p and $p/(1+\delta\, p)$ iff we have:

$$\left\{\begin{array}{l} (1,\ p)\,(1,\ c) = (1+p\,c), \\ \left(1,\ \dfrac{p}{(1+\delta\, p)}\right)\,(1,\ c) = \left(1+\dfrac{p\,c}{(1+\delta\, p)}\right), \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (1,\ p)\,(1,\ c) = (1+p\,c), \\ (1+\delta\, p,\ p)\,(1,\ c) = (1+p\,c+\delta\, p), \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (1,\ p)\,(1,\ c) = (1+p\,c), \\ \left(\dfrac{1+p\,c+\delta\, p}{1+p\,c}\right) = \left(1+\dfrac{\delta\, p}{1+p\,c}\right) = A, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} c \text{ IS } p, \\ 1+\dfrac{\delta\, p}{(1+p\,c)} \in \mathrm{U}(A). \end{array}\right.$$

$\Rightarrow$ A sufficient condition for robust stabilization is:

$$\|\ \delta\ \|_A < \|\ p/(1+p\,c)\ \|_A^{-1}.$$

General Q-parametrization (SCL 03)

- **Theorem**: Let c be a stabilizing controller of $p \in Q(A)$,

$$a = 1/(1 + p\,c), \quad b = c/(1 + p\,c), \quad J = (1,\ p).$$

Then, all stabilizing controllers of p are

$$c(q_1,\ q_2) = \frac{b + a^2\,q_1 + b^2\,q_2}{a - a^2\,p\,q_1 - b^2\,p\,q_2}, \quad (\star)$$

where q_1 and q_2 any element of A: $a - a^2\,p\,q_1 - b^2\,p\,q_2 \neq 0$.

1. $(\star)$ depends on only one free parameter

$$\Leftrightarrow\ p^2 \text{ admits a coprime factorization } p^2 = s/r.$$

$$(\star)\ \Leftrightarrow\ c(q) = \frac{b + r\,q}{a - r\,p\,q}, \quad \forall\, q \in A:\ a - r\,p\,q \neq 0.$$

2. If p admits a coprime factorization $p = n/d$, $d\,x + n\,y = 1$:

$$(\star)\ \Leftrightarrow\ c(q) = \frac{y + d\,q}{x - n\,q}, \quad \forall\, q \in A:\ x - n\,q \neq 0.$$

Solving Zames-Francis' conditions

- Let $p \in Q(A)$ be an internally stabilizable plant

$$\Leftrightarrow \ \exists \, a, \, b \in A, \quad a + b\, p = 1, \quad a\, p \in A, \quad (\star)$$

$$\Leftrightarrow \ \exists \, b \in A, \quad b\, p, \quad p\, (1 + b\, p) \in A. \quad \text{(Zames-Francis)}$$

- If $a \neq 0$, then $c = b/a = b/(1 + b\, p)$ internally stabilizes p.

- $J = (1, \, p)$ is invertible and $J^{-1} = (a, \, b) \Rightarrow J^2 = (1, \, p, \, p^2)$

$$\Rightarrow J^{-2} = (J^2)^{-1} = \{\alpha \in A \,|\, \alpha\, p, \ \alpha\, p^2 \in A\}.$$

$$\Rightarrow J^{-2} = (J^{-1})^2 = (a, \, b)^2 = (a, \, a\, b, \, b^2).$$

- Using $(\star)$, we get $a\, b = (b)\, a^2 + (a\, p)\, b^2 \in (a^2, \, b^2)$

$$\Rightarrow J^{-2} = (a^2, a\, b, \, b^2) = (a^2, \, b^2).$$

Solving Zames-Francis' conditions

- Let us find all the possible a' and b' satisfying:

$$\exists\, a',\, b' \in A, \quad a' + b'\,p = 1, \quad a'\,p \in A, \quad (1)$$

- Using the fact that $a,\ b,\ a\,p \in A$ and $a + b\,p = 1$, we get:

$$(b' - b)\,p = a - a' \in A, \quad (b' - b)\,p^2 = (a - a')\,p \in A,$$

$$\Rightarrow\ b' - b \in \{\alpha \in A \,|\, \alpha\,p,\ \alpha\,p^2 \in A\} = (a^2,\ b^2),$$

$$\Rightarrow\ \exists\, q_1,\ q_2 \in A: \quad \begin{cases} b' = b + q_1\,a^2 + q_2\,b^2, \\ a' = a - (q_1\,a^2 + q_2\,b^2)\,p, \end{cases} \quad (2)$$

$$\Rightarrow\ c' = \frac{b'}{a'} = \frac{b + q_1\,a^2 + q_2\,b^2}{a - (q_1\,a^2 + q_2\,b^2)\,p} \in \mathrm{Stab}(p).$$

- We can check that, for all q_1 and $q_2 \in A$, (2) satisfies (1).

Zames-Francis Q-parametrization

- Let $p = \dfrac{n}{d}$ be a coprime factorization of p over A:

$$d\,x + n\,y = 1.$$

$$\Rightarrow \; J = (1/d) \;\Rightarrow\; J^{-2} = (d^2),$$

$$\Rightarrow \; a = d\,x, \; b = d\,y \in J^{-1} = (d): \; a + b\,p = 1, \quad a\,p \in A.$$

$$\Rightarrow \; c(q) = \frac{b + q\,d^2}{a - q\,d^2\,p} = \frac{d\,y + q\,d^2}{d\,x - d\,n\,q} = \frac{y + q\,d}{x - n\,q}.$$

- **Conclusion:** We have just found again Zames-Francis and Youla-Kučera parametrizations of all stabilizing controllers of p.

Distributed delay

- $p = e^{-s}/(s-1)$ is stabilized by the controller (distributed delay):

$$c = 2\,e\,(s-1)/\left(s+1-2\,e^{-(s-1)}\right).$$

$$\begin{cases} a = \frac{1}{(1+p\,c)} = \frac{(s+1-2\,e^{-(s-1)})}{(s+1)} \in H^\infty(\mathbb{C}_+), \\ b = \frac{c}{1+p\,c} = \frac{2\,e\,(s-1)}{(s+1)} \in H^\infty(\mathbb{C}_+), \\ a\,p = \frac{p}{(1+p\,c)} = \frac{e^{-s}}{(s+1)}\,\frac{(s+1-2\,e^{-(s-1)})}{(s-1)} \in H^\infty(\mathbb{C}_+). \end{cases}$$

- We obtain that all stabilizing controllers of p have the form:

$$c(q) = \frac{2\,e+q\,\frac{(s-1)}{(s+1)}}{1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)-q\,\frac{e^{-s}}{(s+1)}},\; q = \left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right)^2 q_1 + 4\,e^2\,q_2.$$

$\Rightarrow$ the Youla-Kučera parametrization for the coprime factorization:

$$p = \frac{n}{d},\; n = \frac{e^{-s}}{(s+1)},\; d = \frac{(s-1)}{(s+1)},\; d\,\left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + n\,(2\,e) = 1.$$

K. Mori, CDC 1999, 973-975

- Let $A = \mathbb{R}[x^2, x^3]$ be the ring of discrete time delay systems without the unit delay.

- A is used for high-speed circuits, computer memory devices. . .

- $p = (1 - x^3)/(1 - x^2) \in Q(A), \quad J = (1, p)$.

- Using $(1 - x^3)(1 + x^3) = (1 - x^2)(1 + x^2 + x^4)$, we get

$$p = \frac{(1 - x^3)}{(1 - x^2)} = \frac{(1 + x^2 + x^4)}{(1 + x^3)}.$$

$A : J = (1 - x^2, 1 + x^3)$ is not principal because $(x + 1) \notin A$.

$\Rightarrow$ p does not admit a (weakly) coprime factorization.

- $J\,(A : J) = (1-x^2,\, 1+x^3,\, 1-x^3,\, 1+x^2+x^4)$.
- $(1+x^3)/2 + (1-x^3)/2 = 1 \in J\,(A : J)$

$$\Rightarrow \begin{cases} a = (1+x^3)/2 \in A : J, \\ b = (1-x^2)/2 \in A : J, \\ a + b\,p = 1, \end{cases}$$

$\Rightarrow \; c = b/a = (1-x^2)/(1+x^3)$ internally stabilizes p.

- $J^{-1} = (1-x^2,\, 1+x^3) \Rightarrow J^{-2} = ((1-x^2)^2,\, (1+x^3)^2)$.
- $(x+1) \notin A \;\Rightarrow\; J^{-2}$ is not principal ideal of A.

$\Rightarrow$ All stabilizing controllers of p have the form

$$c(q_1, q_2) = \frac{2\,(1-x^2) + (1+x^3)^2\,q_1 + (1-x^2)^2\,q_2}{2\,(1+x^3) - (1+x^3)\,(1+x^2+x^4)\,q_1 - (1-x^2)\,(1-x^3)\,q_2},$$

for all $q_1,\, q_2 \in A$ such that the denominator exists.

Convexity of $H(p, c)$

- Let $c_\star$ be a stabilizing controller of $p \in Q(A)$.
- All stabilizing controllers of p are parametrized by

$$c(q_1, q_2) = \frac{(1 + p\, c_*)\, c_* + q_1 + q_2\, {c_*}^2}{(1 + p\, c_*) - q_1\, p - q_2\, p\, {c_*}^2}$$

$$\forall\, q_1,\, q_2 \in A : \ (1 + p\, c_*) - q_1\, p - q_2\, p\, {c_*}^2 \neq 0.$$

- Then, the closed-loop system

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1+p\,c} & \frac{1}{1+p\,c} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ is affine/convex in the } q_i\text{'s}$$

$$\begin{pmatrix} \frac{1}{1+p\,c_*} - q_1\, \frac{p}{(1+p\,c_*)^2} - q_2\, \frac{p\,{c_*}^2}{(1+p\,c_*)^2} & -\frac{c_*}{1+p\,c_*} - q_1\, \frac{1}{(1+p\,c_*)^2} - q_2\, \frac{{c_*}^2}{(1+p\,c_*)^2} \\ \frac{p}{1+p\,c_*} - q_1\, \frac{p^2}{(1+p\,c_*)^2} - q_2\, \frac{(p\,c_*)^2}{(1+p\,c_*)^2} & \frac{1}{1+p\,c_*} - q_1\, \frac{p}{(1+p\,c_*)^2} - q_2\, \frac{p\,{c_*}^2}{(1+p\,c_*)^2} \end{pmatrix}$$

i.e., $\quad \forall\, \lambda \in A : \quad H(p,\, c(\lambda\, q_1 + (1 - \lambda)\, q_1',\, \lambda\, q_2 + (1 - \lambda)\, q_2'))$

$$= \lambda\, H(p,\, c(q_1,\, q_2)) + (1 - \lambda)\, H(p,\, c(q_1',\, q_2')).$$

Sensitivity minimization

- Let A be a Banach algebra (e.g., $H^\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}$, $W_+ \ldots$).
- Let $c_\star$ be a stabilizing controller of $p \in Q(A)$ and:

$$a = 1/(1 + p\, c_\star), \quad b = c_\star/(1 + p\, c_\star) \in A.$$

- Let $w \in A$ be a weighted function. Then, we have:

$$\inf_{c \in \mathrm{Stab}(p)} \| \, w/(1+p\, c) \, \|_A = \inf_{q_1,\, q_2 \in A} \| \, w\,(a - a^2\, p\, q_1 - b^2\, p\, q_2) \, \|_A \quad (\star)$$

i.e., $(\star)$ is a convex problem in the q_i's!

- If $p = n/d$ is a coprime factorization $(x,\, y \in A, d\, x + n\, y = 1)$

$$\Rightarrow \; a - a^2\, p\, q_1 - b^2\, p\, q_2 = d\,(x - q\, n).$$

$\forall\, q \in A, \quad \exists\, q_1,\; q_2 \in A : \quad q = x^2\, q_1 + y^2\, q_2, \quad$ where:

$$q_1 = d^2\,(1 + 2\, n\, y)\, q, \quad q_2 = n^2\,(1 + 2\, d\, x)\, q.$$

$$(\star) \;\Leftrightarrow\; \inf_{q \in A} \| \, w\, d\,(x - n\, q) \, \|_A \,.$$

Classification of the rings A (SIAM 03)

• **Theorem**: Let A be a integral domain of stable transfer functions and $K = Q(A)$.

1. Every transfer function $p \in K$ admits a weakly coprime factorization iff A is a GCDD, i.e., any two elements of A admits a greatest common divisor (coherent Sylvester domain).

2. Every transfer function $p \in K$ is internally stabilizable iff A is a Prüfer domain, i.e., any f.g. ideal of A is invertible.

3. Every transfer function $p \in K$ admits a coprime factorization iff A is a Bézout domain, i.e., any f.g. ideal of A is principal (Vidya.).

• RH_∞ is a PID $\Rightarrow$ GCD, Prüfer and Bézout domains.

• $H^\infty(\mathbb{C}_+)$ is a GCDD but is not a Prüfer and a Bézout domain.

Pre-Bézout rings

• **Definition**: An integral domain A is a pre-Bézout ring if, for every pair (d, n) of elements of A which admits a greatest common divisor $\gcd(d,\, n)$, then there exist $x,\, y \in A$ satsifying:

$$d\,x + n\,y = \gcd(d,\, n).$$

• **Example**: $A(\mathbb{D})$ and W_+ are pre-Bézout ring.

• **Proposition**: Let A be a pre-Bézout ring. Then, $1 \Leftrightarrow 2$:

1. $p \in Q(A)$ admits a weakly coprime factorization.
2. $p \in Q(A)$ admits a coprime factorization.

• Some of the previous results are nicely explained in the booklet:

A. Sasane, Algebras of Holomorphic Functions and Control Theory.

- Let $\mathcal{F}$ be a A-module, $p \in Q(A)$ and $u \in \mathcal{F}$.
- What is the meaning of $p\,u$?

$$p\,u = y \;\Leftrightarrow\; \forall\, d \in A : J,\; (d\,p)\,u = d\,y \quad (\star).$$

$$\Rightarrow \begin{cases} \mathrm{ann}_{\mathcal{F}}(A : J) = \{z \in \mathcal{F} \mid \forall\, d \in A : J,\; d\,z = 0\}, \\ \overline{\mathcal{F}} = \mathcal{F}/\mathrm{ann}_{\mathcal{F}}(A : J), \end{cases}$$

and we denote by $\overline{y}$ the residue class of y in $\overline{\mathcal{F}}$ satisfying $(\star)$.

- We can introduce the following two A-modules:

$$\begin{cases} \mathrm{dom}_{\mathcal{F}}(p) = \{u \in \mathcal{F} \mid p\,u \in \overline{\mathcal{F}}\}, \\ \mathrm{graph}_{\mathcal{F}}(p) = \{(u \quad p\,u)^T \in \mathcal{F} \times \overline{\mathcal{F}} \mid u \in \mathcal{F}\}. \end{cases}$$

- If $\mathcal{F}$ be a torsion-free A-module, i.e.,

$$t(\mathcal{F}) = \{z \in \mathcal{F} \mid \exists\, 0 \neq d \in A,\; d\,z = 0\},$$

then we have $\overline{\mathcal{F}} = \mathcal{F}$ (e.g., H^2 is a torsion-free (flat) H^∞-module).

Operator-theoretical approach (Acta 05)

- We have the exact sequence of A-module:

$$0 \longleftarrow J \xleftarrow{f} A^{1\times 2} \xleftarrow{g} A : J \longleftarrow 0, \quad (\square)$$

$$\begin{array}{rcl} A^{1\times 2} & \xrightarrow{f} & J \\ (a \quad b) & \longmapsto & a + b\,p, \end{array} \qquad \begin{array}{rcl} A : J & \xrightarrow{g} & A^{1\times 2} \\ d & \longmapsto & (-d\,p \quad d). \end{array}$$

- Applying $\hom_A(\,\cdot\,, \mathcal{F})$ to $(\square)$ gives the exact sequence

$$0 \longrightarrow \hom_A(J, \mathcal{F}) \xrightarrow{f^\star} \mathcal{F}^2 \xrightarrow{g^\star} \hom_A(A : J, \mathcal{F}) \longrightarrow \mathrm{ext}^1_A(J, \mathcal{F}) \longrightarrow 0,$$

which yields, after some computations, to the exact sequence

$$0 \longrightarrow \mathrm{dom}_{\mathcal{F}}(p) \xrightarrow{\left(\begin{array}{c} 1 \\ p \end{array}\right).} \mathcal{F} \times \overline{\mathcal{F}} \xrightarrow{\sigma} \hom_A(A : J, \mathcal{F}) \longrightarrow \mathrm{ext}^1_A(J, \mathcal{F}) \longrightarrow 0,$$

$$\forall\, u \in \mathcal{F}, \quad \forall\, \overline{y} \in \overline{\mathcal{F}}, \quad \sigma((u \quad \overline{y})) = y - p\,u.$$

Conclusion (MCSS06 $\times 2$)

• The extension of the previous results to MIMO systems uses the so-called theory of lattices .

• If $P \in K^{q\times r}$, then we have the lattices:

$$\mathcal{L} = (I_q \quad -P)\, A^{q+r},\ \mathcal{M} = A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix},$$

$$A : \mathcal{L} = \{\lambda \in A^{1\times q} \,|\, \lambda\, P \in A^{1\times r}\},\ A : \mathcal{M} = \{\lambda \in A^r \,|\, P\, \lambda \in A^q\},$$

• Moreover, if $P = D^{-1}\, N = \widetilde{N}\, \widetilde{D}^{-1}$, $R = (D \quad -N) \in A^{q\times(q+r)}$ and $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in A^{(q+r)\times r}$, then we have the lattices:

$$\mathcal{P} = R\, A^{q+r},\ \mathcal{Q} = A^{1\times(q+r)}\, \widetilde{R},$$

$$A : \mathcal{P} = \{\lambda \in K^{1\times q} \,|\, \lambda\, R \in A^{1\times(q+r)}\},\ A : \mathcal{Q} = \left\{\lambda \in K^r \,|\, \widetilde{R}\, \lambda \in A^{q+r}\right\}$$

• Certain of these lattices were already used by Paul in his famous paper "Algebraic system theory: an analyst's point of view" .

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