Serre's reduction of linear partial differential systems based on holonomy

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***Abstract*— Given a linear functional system (e.g., ordinary/partial differential systems), Serre's reduction aims at finding an equivalent linear functional system which contains fewer equations and fewer unknowns. The purpose of this paper is to study Serre's reduction of underdetermined linear systems of partial differential equations with analytic coefficients whose formal adjoints are holonomic in the sense of algebraic analysis. In particular, we prove that every analytic linear systems of ordinary differential equations with at least one input is equivalent to a sole analytic ordinary differential equation.**

I. Algebraic analysis approach to linear systems theory

We recall that the definition of a *left D-module* (resp., *right D-module*) M is the same as the one of a k-vector space but where the field k is replaced by a ring D and the elements of D act on the left (resp., right) of M, namely, for all $m_1, m_2 \in M$ and all $d_1, d_2 \in D$, $d_1 m_1 + d_2 m_2 \in M$ (resp., $m_1 d_1 + m_2 d_2 \in M$). In particular, a k-vector space is a k-module and an abelian group is a $\mathbb{Z}$-module. For more details, see, e.g., [9], [11], [19].

We shall denote by $D^{1\times p}$ (resp., D^q) the left (resp., right) D-module formed by row (resp., column) vectors of length p (resp., q) with entries in D and by $R \in D^{q\times p}$ a $q \times p$ matrix R with entries in D. The *general linear group* of D

$$\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\; U\,V = V\,U = I_p\}$$

is the subgroup of $D^{p\times p}$ formed by invertible (*unimodular*) matrices. Moreover, we shall use the following notations:

$$\begin{array}{rcl} .R : D^{1\times q} & \longrightarrow & D^{1\times p} \\ \mu & \longmapsto & \mu\,R, \end{array} \qquad \begin{array}{rcl} R. : D^{p} & \longrightarrow & D^{q} \\ \eta & \longmapsto & R\,\eta. \end{array}$$

Within algebraic analysis (see, e.g., [4] and the references therein), a linear functional system (e.g., linear systems of ODEs or PDEs, OD time-delay equations, difference equations) can be studied by means of module theory, homological algebra and sheaf theory (see, e.g., [19]). More precisely, if D is a noncommutative polynomial ring of functional operators (e.g., OD or PD operators, time-delay operators, shift operators, difference operators), $R \in D^{q\times p}$ a $q \times p$ matrix with entries in D and $\mathcal{F}$ a left D-module, then the *linear functional system* (or *behaviour*)

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\},$$

i.e., the abelian group formed by the $\mathcal{F}$-solutions of the system $R\,\eta = 0$, can be studied by means of the left D-module $M \triangleq D^{1\times p}/(D^{1\times q}\,R)$ *finitely presented* by the matrix R. Indeed, Malgrange's remark ([14]) asserts the existence of the abelian group isomorphism (i.e., $\mathbb{Z}$-isomorphism)

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}), \tag{1}$$

where $\hom_D(M, \mathcal{F})$ is the abelian group of left D-*homomorphisms* from M to $\mathcal{F}$ (i.e., maps $f : M \longrightarrow \mathcal{F}$ satisfying $f(d_1\,m_1 + d_2\,m_2) = d_1\,f(m_1) + d_2\,f(m_2)$ for all $d_1, d_2 \in D$ and all $m_1, m_2 \in M$) and $\cong$ denotes an *isomorphism*, namely, a bijective homomorphism.

Let us describe the isomorphism (1). To do that, we first give an explicit description of M in terms of *generators and relations*. Let $\pi : D^{1\times p} \longrightarrow M = D^{1\times p}/(D^{1\times q}\,R)$ be the canonical projection onto M, namely, the left D-homomorphism which sends a row vector of $D^{1\times p}$ of length p to its residue class $\pi(\lambda)$ in M, $\{f_j\}_{j=1,\dots,p}$ the *standard basis* of $D^{1\times p}$, namely, f_j is the row vector of length p defined by 1 at the j^{th} entry and 0 elsewhere, and $y_j = \pi(f_j)$ the residue class of f_j in M for $j = 1, \dots, p$. Since every element $m \in M$ is the residue class of an element $\lambda = (\lambda_1 \;\dots\; \lambda_p) \in D^{1\times p}$, then, using the left D-linearity of the left D-homomorphism π, we get

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j\,f_j\right) = \sum_{j=1}^{p} \lambda_j\,\pi(f_j) = \sum_{j=1}^{p} \lambda_j\,y_j,$$

which shows that $\{y_j\}_{j=1,\dots,p}$ is a family of generators of the left D-module M. Moreover, if we denote by $R_{i\bullet}$ the i^{th} row of the matrix R, then $R_{i\bullet} \in D^{1\times q}\,R$, which yields $\pi(R_{i\bullet}) = 0$ and thus

$$\pi(R_{i\bullet}) = \pi\left(\sum_{j=1}^{p} R_{ij}\,f_j\right) = \sum_{j=1}^{p} R_{ij}\,\pi(f_j) = \sum_{j=1}^{p} R_{ij}\,y_j = 0, \tag{2}$$

for $i = 1, \dots, q$, which shows that the set of generators $\{y_j\}_{j=1,\dots,p}$ of M satisfies the left D-linear relations (2) and all their left D-linear combinations. Therefore, if we set $y = (y_1 \;\dots\; y_p)^T \in M^p$, then (2) becomes $R\,y = 0$.

Now, let $\chi : \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M, \mathcal{F})$ be the $\mathbb{Z}$-homomorphism defined by $\chi(\eta) = \phi_\eta$ for all $\eta \in \ker_{\mathcal{F}}(R.)$, where $\phi_\eta(\pi(\lambda)) = \lambda\,\eta \in \mathcal{F}$ for all $\lambda \in D^{1\times p}$. The $\mathbb{Z}$-homomorphism ϕ_η is well-defined since $\pi(\lambda) = \pi(\lambda')$ yields $\pi(\lambda - \lambda') = 0$, i.e., $\lambda - \lambda' = \mu\,R$ for a certain $\mu \in D^{1\times q}$, and thus $\phi_\eta(\pi(\lambda)) = \lambda\,\eta = \lambda'\,\eta + \mu\,R\,\eta = \lambda'\,\eta = \phi_\eta(\pi(\lambda'))$. Moreover, χ is injective since $\phi_\eta = 0$ yields $\lambda\,\eta = 0$ for all $\lambda \in D^{1\times p}$, and thus $\eta_j = f_j\,\eta = 0$ for all $j = 1, \ldots, p$, i.e., $\eta = 0$. It is also surjective since for all $\phi \in \hom_D(M, \mathcal{F})$, $\eta = (\phi(y_1) \; \ldots \; \phi(y_p))^T \in \mathcal{F}^p$ satisfies $\chi(\eta) = \phi$ and

$$\sum_{j=1}^{p} R_{ij}\,\eta_j = \sum_{j=1}^{p} R_{ij}\,\phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\,y_j\right) = \phi(0) = 0,$$

i.e., $\eta \in \ker_{\mathcal{F}}(R.)$. Thus, the $\mathbb{Z}$-homomorphism χ is an isomorphism and:

$$\begin{array}{rcl} \chi^{-1} : \hom_D(M, \mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \phi & \longmapsto & (\phi(y_1) \; \ldots \; \phi(y_p))^T. \end{array}$$

Theorem 1.1 ([14]): Let D be a ring, $R \in D^{q\times p}$ a matrix, $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by R, $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M, $\{f_j\}_{j=1,\ldots,p}$ the standard basis of $D^{1\times p}$, $y_j = \pi(f_j)$ for $j = 1, \ldots, p$, and $\mathcal{F}$ a left D-module. Then, we have the following abelian group isomorphism:

$$\begin{array}{rcl} \hom_D(M, \mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \\ \phi & \longmapsto & \eta = (\phi(y_1) \; \ldots \; \phi(y_p))^T. \end{array} \quad (3)$$

Hence, there is a one-to-one correspondence between the elements of $\hom_D(M, \mathcal{F})$ and the elements of $\ker_{\mathcal{F}}(R.)$.

Remark 1.1: Theorem 1.1 shows that $\ker_{\mathcal{F}}(R.)$ can be studied by means of the finitely presented left D-module M and the left D-module $\mathcal{F}$: $M = D^{1\times p}/(D^{1\times q}\,R)$ intrinsically defines the linear system of equations defined by the matrix $R \in D^{q\times p}$ and $\mathcal{F}$ is the functional space where we seek the solutions of the linear functional system. In this paper, we shall study the linear system $\ker_{\mathcal{F}}(R.)$ by means of the module properties of the finitely presented left D-module M.

In what follows, D will denote a *noncommutative noetherian domain*, namely, a unital ring satisfying that $d\,d'$ is not necessarily equal to $d'\,d$ for $d, d' \in D$, containing no nontrivial zero-divisors, i.e., $d\,d' = 0$ yields $d = 0$ or $d' = 0$, and every left (resp., right) ideal of D is finitely generated, i.e., can be generated by a finite family of elements of D as a left (resp., right) D-module ([19]).

A *differential ring* $(A, \{\delta_1, \ldots, \delta_n\})$ is a commutative ring A equipped with n *commuting derivations* $\delta_i : A \longrightarrow A$ for $i = 1, \ldots, n$, namely, maps satisfying

$$\forall\, a_1, a_2 \in A, \quad \left\{ \begin{array}{l} \delta_i \circ \delta_j = \delta_j \circ \delta_i, \\ \delta_i(a_1 + a_2) = \delta_i(a_1) + \delta_i(a_2), \\ \delta_i(a_1\,a_2) = \delta_i(a_1)\,a_2 + a_1\,\delta_i(a_2), \end{array} \right.$$

for all $i, j = 1, \ldots, n$. If we take $a_1 = a_2 = 1$, then the above equality yields $\delta_i(1) = 2\,\delta_i(1)$, i.e., $\delta_i(1) = 0$. If A is a field and $a \in A \setminus \{0\}$, then

$$\delta_i(a)\,a^{-1} + a\,\delta_i(a^{-1}) = \delta_i(a\,a^{-1}) = \delta_i(1) = 0,$$

which yields $\delta_i(a^{-1}) = -a^{-2}\,\delta_i(a)$ and A is then called a *differential field*. In what follows, we shall mainly focus on the differential ring $\left(A, \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}\right)$, where $A = k[x_1, \ldots, x_n]$, $k[[x_1, \ldots, x_n]]$ (i.e., the ring of formal power series at 0 with coefficients in k), where k is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $k\{x_1, \ldots, x_n\}$ where $k = \mathbb{R}$ or $\mathbb{C}$ (i.e., the ring of locally convergent power series at 0 or the ring of germs of real analytic or holomorphic functions at 0) or the differential fields $A = k$ and $k(x_1, \ldots, x_n)$.

The *ring of PD operators* in $\partial_1, \ldots, \partial_n$ with coefficients in the differential ring $(A, \{\delta_1, \ldots, \delta_n\})$, simply denoted by $D = A\langle \partial_1, \ldots, \partial_n \rangle$, is the noncommutative polynomial ring in the ∂_i's with coefficients in the ring A satisfying

$$\partial_i\,\partial_j = \partial_j\,\partial_i, \quad \partial_i\,a = a\,\partial_i + \delta_i(a),$$

for all $a \in A$ and all $i, j = 1, \ldots, n$. An element $d \in D$ can be written as $d = \sum_{0 \leq |\nu| \leq r} a_\nu\,\partial^\nu$, where $a_\nu \in A$, $\nu = (\nu_1 \; \ldots \; \nu_n)^T \in \mathbb{N}^n$, $|\nu| = \nu_1 + \cdots + \nu_n$ and $\partial^\nu = \partial_1^{\nu_1} \ldots \partial_n^{\nu_n}$. If $n = 1$, then we shall simply use the notations $\delta = \frac{d}{dt}$ instead of δ_1, ∂ instead of ∂_1 and $k[t]$, $k(t)$, $k[[t]]$ and $k\{t\}$ instead of $k[x_1]$, $k(x_1)$, $k[[x_1]]$ and $k\{x_1\}$.

The *first* and the *second Weyl algebra* are defined by:

$$\left\{ \begin{array}{l} A_n(k) = k[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle, \\ B_n(k) = k(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle. \end{array} \right.$$

The ring $D = A\langle \partial_1, \ldots, \partial_n \rangle$, where $A = k$, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[[x_1, \ldots, x_n]]$, and k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, $k = \mathbb{R}$ or $\mathbb{C}$, is noetherian.

Let us recall a few definitions of module theory.

Definition 1.1 ([11], [19]): Let D be a left noetherian domain and M a *finitely generated* left D-module, namely, M can be generated by a finite family of elements of M as a left D-module.

1) M is *free* if there exists $r \in \mathbb{N} = \{0, 1, \ldots\}$ such that $M \cong D^{1\times r}$. Then, r is called the *rank* of the free left D-module M and is denoted by $\mathrm{rank}_D(M)$.
2) M is *stably free* if there exist $r, s \in \mathbb{N}$ such that:
$$M \oplus D^{1\times s} \cong D^{1\times r}.$$
Then, $r - s$ is called the *rank* of the stably free left D-module M.
3) M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module N such that $M \oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules.
4) M is *torsion-free* if the *torsion left D-submodule of* M, $t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} : \; d\,m = 0\}$, is reduced to 0, i.e., if $t(M) = 0$. The elements of $t(M)$ are the *torsion elements* of M.
5) M is *cyclic* if M is generated by $m \in M$, i.e.:
$$M = D\,m \triangleq \{d\,m \mid d \in D\}.$$

A free module is clearly stably free (take $s = 0$ in 2 of Definition 1.1), a stably free module is projective (take $P = D^{1\times s}$ in 3 of Definition 1.1) and a projective module is torsion-free (since it can be embedded into a free, and thus, into a torsion-free module).

The converses of the previous results are generally not true. However, they hold in particular interesting situations.

Theorem 1.2 ([11], [15], [17], [18], [19]): 1) If D is a *principal left ideal domain*, namely, every left ideal of the domain D is cyclic (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field A such as $A = k$, $k(t)$ and $k[\![t]\!][t^{-1}]$, where k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), or $k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$), then every finitely generated torsion-free left D-module is free.

2) If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring with coefficients in a field k, then every finitely generated projective D-module is free (*Quillen-Suslin theorem*).

3) If D is the Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), then every finitely generated projective left D-module is stably free and every finitely generated stably free left D-module of rank at least 2 is free (Stafford's theorem).

4) If $D = A\langle\partial\rangle$ is the ring of OD operators with coefficients in a differential field $A = k[\![t]\!]$, where k is a field of characteristic 0, or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then every finitely generated projective left D-module is stably free and every finitely generated stably free left D-module of rank at least 2 is free.

If the matrix R has *full row rank*, namely,

$$\ker_D(.R) \triangleq \{\mu \in D^{1\times q} \mid \mu\, R = 0\} = 0,$$

then the next proposition characterizes when the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is a stably free or free module.

Theorem 1.3 ([4], [10], [17]): Let D be a noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, and the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ finitely presented by R.

1) M is a projective left D-module iff M is a stably free left D-module.

2) M is a stably free left D-module of rank $p - q$ iff R admits a *right-inverse* over D, namely, iff there exists a matrix $S \in D^{p\times q}$ satisfying $R\, S = I_q$.

3) M is a free left D-module of rank $p-q$ iff there exists $U \in \mathrm{GL}_p(D)$ such that $R\, U = (I_q \quad 0)$. If we write $U = (S \quad Q)$, $S \in D^{p\times q}$ and $Q \in D^{p\times(p-q)}$, then

$$\begin{array}{rcl} \psi : M & \longrightarrow & D^{1\times(p-q)} \\ \pi(\lambda) & \longmapsto & \lambda\, Q, \end{array}$$

is a left D-isomorphism and ψ^{-1} is defined by:

$$\begin{array}{rcl} \psi^{-1} : D^{1\times(p-q)} & \longrightarrow & M \\ \mu & \longmapsto & \pi(\mu\, T), \end{array}$$

where the matrix $T \in D^{(p-q)\times p}$ is defined by:

$$U^{-1} = \begin{pmatrix} R \\ T \end{pmatrix} \in D^{p\times p}.$$

Then, $M \cong D^{1\times p}\, Q = D^{1\times(p-q)}$ and the matrix Q is called an *injective parametrization* of M. Finally, $\{\pi(T_{i\bullet})\}_{i=1,\ldots,p-q}$ defines a basis of the free left D-module M of rank $p - q$.

The Quillen-Suslin theorem (resp., Stafford's theorem) has recently been implemented in the package QUILLENSUSLIN ([10]) (resp., STAFFORD ([17])).

II. HOLONOMIC D-MODULES

In this section, we consider the ring $D = A\langle\partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficients in the differential ring $A = k$, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$. The ring D has a natural *order filtration* defined by

$$D_r = \left\{ \sum_{0\le|\alpha|\le r} a_\alpha\, \partial^\alpha \mid a_\alpha \in A \right\},$$

for all $r \in \mathbb{N}$. Then, we can check that the following filtration conditions hold:

1) $\forall\, r,\, s \in \mathbb{N},\, r \le s \;\Rightarrow\; D_r \subseteq D_s$.
2) $D = \bigcup_{r\in\mathbb{N}} D_r$.
3) $\forall\, r,\, s \in \mathbb{N},\, D_r\, D_s \subseteq D_{r+s}$.

The ring D is then called a *filtered ring* and an element of D_r is said to have a *degree* less or equal to r. We can easily check that $D_0 = A$ and D_r is a finitely generated A-module.

If $d_1,\, d_2 \in D$, then we can define the *bracket* of d_1 and d_2 by $[d_1, d_2] = d_1\, d_2 - d_2\, d_1$. Now, if $d_1 \in D_r$ and $d_2 \in D_s$, then $d_1\, d_2$ and $d_2\, d_1$ belong to D_{r+s} since $D_r\, D_s \subseteq D_{r+s}$ and $D_s\, D_r \subseteq D_{r+s}$. Moreover, we can check that we have $[d_1, d_2] \in D_{r+s-1}$, i.e., $[D_r, D_s] \subseteq D_{r+s-1}$.

Let us now introduce the following A-module

$$\mathrm{gr}(D) = \bigoplus_{r\in\mathbb{N}} D_r/D_{r-1},$$

where we set $D_{-1} = 0$. If $\pi_r : D_r \longrightarrow D_r/D_{r-1}$ is the canonical projection, then the A-module $\mathrm{gr}(D)$ inherits a ring structure defined by

$$\begin{cases} \pi_r(d_1) + \pi_s(d_2) \triangleq \pi_t(d_1 + d_2) \in D_t/D_{t-1}, \\ \pi_r(d_1)\, \pi_s(d_2) \triangleq \pi_{r+s}(d_1\, d_2) \in D_{r+s}/D_{r+s-1}, \end{cases}$$

where $t = \max(r, s)$ and for all $d_1 \in D_r$ and all $d_2 \in D_s$. The ring $\mathrm{gr}(D)$ is called the *graded ring* associated with the order filtration of D.

Let $\chi_i \triangleq \pi_1(\partial_i) \in D_1/D_0$ for all $i = 1, \ldots, n$. Then, $\pi_1([\partial_i, \partial_j]) = 0$ and $\pi_1([\partial_i, a]) = 0$ for all $a \in A$ and all $i, j = 1, \ldots, n$ since $[\partial_i, \partial_j] = 0$ and $[\partial_i, a] \in D_0$, which shows that

$$\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$$

is the commutative polynomial ring in $\chi_1, \ldots, \chi_n$ with coefficients in the commutative noetherian ring A.

We can now generalize the concepts of filtered and graded rings for modules.

Definition 2.1 ([1], [7], [13]): Let M be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module.

1) A *filtration* of M is a sequence $\{M_q\}_{q \in \mathbb{N}}$ of A-submodules of M (with the convention that $M_{-1} = 0$) such that:
   a) For all $q, r \in \mathbb{N}$, $q < r$ implies that $M_q \subseteq M_r$.
   b) $M = \bigcup_{q \in \mathbb{N}} M_q$.
   c) For all $q, r \in \mathbb{N}$, we have $D_r \, M_q \subseteq M_{q+r}$.

   The left D-module M is then called a *filtered module*
2) The associated *graded* $\mathrm{gr}(D)$-module $\mathrm{gr}(M)$ is:
   a) $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q / M_{q-1}$.
   b) For every $d \in D_r$ and every $m \in M_q$, we set $\pi_r(d) \, \sigma_q(m) \triangleq \sigma_{q+r}(d\, m) \in M_{q+r}/M_{q+r-1}$, where $\sigma_q : M_q \longrightarrow M_q/M_{q-1}$ is the canonical projection for all $q \in \mathbb{N}$.
3) A filtration $\{M_q\}_{q \in \mathbb{N}}$ is called a *good filtration* if it satisfies one of the equivalent conditions:
   a) M_q is a finitely generated A-module for all $q \in \mathbb{N}$ and there exists $p \in \mathbb{N}$ such that $D_r \, M_p = M_{p+r}$ for all $r \in \mathbb{N}$.
   b) $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q/M_{q-1}$ is a finitely generated $\mathrm{gr}(D)$-module.

Example 2.1: Let M be a finitely generated left D-module defined by a family of generators $\{y_1, \ldots, y_p\}$. Then, the filtration $M_q = \sum_{i=1}^{p} D_q \, y_i$ is a good filtration of M since we then have $\mathrm{gr}(M) = \sum_{i=1}^{p} \mathrm{gr}(D) \, y_i$, which proves that $\mathrm{gr}(M)$ is a finitely generated left $\mathrm{gr}(D)$-module.

If M is a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module, then $\mathrm{gr}(M)$ is a finitely generated module over the commutative polynomial ring $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$. Hence, we are back to the realm of commutative algebra. Based on techniques of algebraic geometry and commutative algebra, we can then characterize invariants of $\mathrm{gr}(M)$ which are important invariants of the differential module M.

Definition 2.2 ([9]): A *proper prime ideal* of a commutative ring A is an ideal $\mathfrak{p} \subsetneq A$ which satisfies that $a\, b \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. The set of all the proper prime ideals of A is denoted by $\mathrm{spec}(A)$ and is a topological space endowed with the *Zariski topology* defined by the Zariski-closed sets $V(I) = \{\mathfrak{p} \in \mathrm{spec}(A) \mid I \subseteq \mathfrak{p}\}$, where I is an ideal of A.

Example 2.2: If $(a_1, \ldots, a_n) \in \mathbb{C}^n$, then the finitely generated ideal $\mathfrak{m} = (x - a_1, \ldots, x_n - a_n)$ of the ring $D = \mathbb{C}[x_1, \ldots, x_n]$ is a *maximal ideal* of D, namely, $\mathfrak{m}$ is not contained in any proper ideal of D different from $\mathfrak{m}$. A maximal ideal $\mathfrak{m}$ is a prime ideal. Indeed, if we have $x \notin \mathfrak{m}$ and $x\, y \in \mathfrak{m}$, then, since $\mathfrak{m}$ is maximal, we get $A\, x + \mathfrak{m} = A$, and thus, there exist $a \in A$ and $b \in \mathfrak{m}$ such that $a\, x + b = 1$. Then, we obtain $y = a\,(x\, y) + (y\, b) \in \mathfrak{m}$, which proves that $\mathfrak{m}$ is prime. For instance, the twisted cubic is defined by the prime ideal $\mathfrak{p} = (x_2 - x_1^2, x_3 - x_1^2)$ of $\mathbb{C}[x_1, x_2, x_3]$.

We can now introduce the important concept of a *characteristic variety* of a differential module.

Proposition 2.1 ([1], [7], [13]): Let M be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module and $G = \mathrm{gr}(M)$ the associated graded $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$-module for a good filtration of M. Then, the *characteristic ideal* $I(M)$ of M is the ideal of $\mathrm{gr}(D)$ defined by:

$$I(M) = \sqrt{\mathrm{ann}(G)} \triangleq \{a \in \mathrm{gr}(D) \mid \exists\, n \in \mathbb{N} : \; a^n \, G = 0\}.$$

It does not depend on the good filtration of M. The *characteristic variety* of M is then the subset of $\mathrm{spec}(\mathrm{gr}(D))$ defined by:

$$\mathrm{char}_D(M) = \{\mathfrak{p} \in \mathrm{spec}(\mathrm{gr}(D)) \mid \sqrt{\mathrm{ann}(G)} \subseteq \mathfrak{p}\}.$$

According to Example 2.1, every finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module M admits a good filtration and thus a characteristic variety. The *dimension* of the left D-module M can then be defined as the geometric dimension of the characteristic variety $\mathrm{char}_D(M)$ of M.

Definition 2.3 ([1], [7], [9], [13]): Let M be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module. Then, the *dimension* of M is the supremum of the lengths of the chains

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_d$$

of distinct proper prime ideals in the commutative ring $\mathrm{gr}(D)/I(M) = A[\chi_1, \ldots, \chi_n]/I(M)$. If $M = 0$, then we set $\dim_D(M) = -1$.

We shall simply write $\dim(D)$ instead of $\dim_D(D)$.

Example 2.3 ([1], [7], [13]): We have

$$\left\{ \begin{array}{l} \dim(k[x_1, \ldots, x_n]) = n, \\ \dim(B_n(k')) = n, \\ \dim(A\langle \partial_1, \ldots, \partial_n \rangle) = 2\, n, \end{array} \right.$$

where k is a field, k' is a field of characteristic 0 and $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$.

Example 2.4: Let us consider the linear PD system:

$$\left\{ \begin{array}{l} \Phi_1 = (\partial_4 - x_3\, \partial_2 - 1)\, y = 0, \\ \Phi_2 = (\partial_3 - x_4\, \partial_1)\, y = 0. \end{array} \right. \quad (4)$$

We can check that (4) is not *formally integrable* since

$$(\partial_4 - x_3\, \partial_2 - 1)\, \Phi_2 + (x_4\, \partial_1 - \partial_3)\, \Phi_1 = (\partial_2 - \partial_1)\, y = 0$$

is a new non-trivial first order PDE which does not appear in (4). Adding this new equation to (4), then we can check that the new linear PD system defined by

$$\left\{ \begin{array}{l} (\partial_4 - x_3\, \partial_2 - 1)\, y = 0, \\ (\partial_3 - x_4\, \partial_1)\, y = 0, \\ (\partial_2 - \partial_1)\, y = 0, \end{array} \right. \quad (5)$$

is formally integrable and *involutive* ([16]). Therefore, using the Cartan-Kähler-Janet's theorem (see [16]), we can obtain a formal power series (analytic) solution of (5) in a neighbourhood of $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ which satisfies an appropriate set of initial conditions.

Using (5), the characteristic variety of the left $D = A_4(\mathbb{C})$-module $M = D/(D^{1\times 2}\, R)$ finitely presented by the matrix $R = (\partial_4 - x_3\, \partial_2 - 1 \quad \partial_3 - x_4\, \partial_1)^T$ is defined by the ideal

$$I(M) = (\chi_4 - x_3\, \chi_2,\ \chi_3 - x_4\, \chi_1,\ \chi_2 - \chi_1)$$

of $\mathrm{gr}(D) = \mathbb{C}[x_1, x_2, x_3, x_4, \chi_1, \chi_2, \chi_3, \chi_4]$. The characteristic variety $\mathrm{char}_D(M)$ of M is then the affine algebraic variety of $\mathbb{C}^8$ defined by the ideal $I(M)$ of $\mathrm{gr}(D)$:

$$\mathrm{char}_D(M) = \{(x_1, x_2, x_3, x_4, \chi_1, \chi_1, x_4\, \chi_1, x_3\, \chi_1) \mid \chi_1,\ x_i \in \mathbb{C},\ i = 1, \ldots, 4\}.$$

The Krull dimension of $\mathrm{char}_D(M)$ is 5, i.e., $\dim_D(M) = 5$.

Definition 2.4 ([1], [7], [13]): Let M be a non-zero finitely generated left $D = A\langle \partial_1, \ldots, \partial_n\rangle$-module. If $\dim_D(M) = n$ then M is called a *holonomic* left D-module.

Example 2.5: The ODE $t\, \dot{y} - y = 0$ defines the holonomic left $D = A_1(\mathbb{C})$-module $M = D/D\,(t\, \partial - 1)$. Indeed, the characteristic variety $\mathrm{char}_D(M)$ of M is defined by the characteristic ideal $I(M) = (t\, \chi)$ of the commutative polynomial ring $\mathrm{gr}(D) = \mathbb{C}[t, \chi]$, which implies that

$$\mathrm{char}_D(M) = \{(t, 0) \mid t \in \mathbb{C}\} \cup \{(0, \chi) \mid \chi \in \mathbb{C}\}$$

is a 1-dimensional affine algebraic variety of $\mathbb{C}^2$, and thus:

$$\dim_D(M) = 1.$$

Theorem 2.1 ([1], [7], [13]): If $D = A\langle \partial\rangle$ is the ring of OD operators with coefficients in $A = k[t]$, $k[\![t]\!]$, where k is a field of characteristic 0, or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then a left (resp., right) D-module M is holonomic iff M is a torsion left (resp., right) D-module.

Theorem 2.2 ([1], [7], [13]): A holonomic left $D = A\langle \partial_1, \ldots, \partial_n\rangle$-module M is cyclic, i.e., M can be generated by one element as a left D-module. More precisely, if $\{y_j\}_{j=1,\ldots,p}$ is a set of generators of the holonomic left D-module M, then there exist $d_2, \ldots, d_p \in D$ such that M is generated by $z = y_1 + d_2\, y_2 + \cdots + d_p\, y_p$. Similar results hold for holonomic right D-modules.

Remark 2.1: For $D = A_n(k)$, where k is a computable field of characteristic 0 (e.g., $k = \mathbb{Q}$), a constructive algorithm for the computation of a cyclic element of a holonomic left D-module M is given in [12]. The corresponding algorithm is implemented in the package SERRE ([20]) built upon OREMODULES ([5]).

III. SERRE'S REDUCTION

Let us now recall a few results on *Serre's reduction* ([20]).

Theorem 3.1 ([2], [3]): Let D be a noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, namely, $\ker_D(.R) = 0$, $\Lambda \in D^{q\times(q-r)}$, $P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}$ and $M = D^{1\times p}/(D^{1\times q}\, R)$ (resp., $E = D^{1\times(p+q-r)}/(D^{1\times q}\, P)$) the left D-module finitely presented by R (resp., P) which defines the following *short exact sequence*

$$0 \longrightarrow D^{1\times(q-r)} \stackrel{\alpha}{\longrightarrow} E \stackrel{\beta}{\longrightarrow} M \longrightarrow 0,$$

namely, α is injective, β is surjective and $\ker\beta = \mathrm{im}\,\alpha$. Then, the following results are equivalent:

1) The left D-module E is stably free of rank $p - r$.
2) The matrix $P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}$ admits a *right-inverse*, namely, there exists $S \in D^{(p+q-r)\times q}$ such that $P\, S = I_q$.
3) $\mathrm{ext}^1_D(E, D) \triangleq D^q/\left(P\, D^{(p+q-r)}\right) = 0$.
4) $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$ generates the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$, where the right D-homomorphism $\tau : D^q \longrightarrow D^q/(R\, D^p)$ is the canonical projection and $\Lambda_{\bullet i}$ denotes the i^{th} column of the matrix Λ.

Finally, the previous results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in the right D-module

$$\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right) \triangleq D^{q\times(q-r)}/\left(R\, D^{p\times(q-r)}\right), \quad (6)$$

i.e., they depend only on the row vector

$$(\tau(\Lambda_{\bullet 1}) \ \ldots\ \tau(\Lambda_{\bullet (q-r)})) \in \mathrm{ext}^1_D(M, D)^{1\times(q-r)}.$$

Remark 3.1: If we take $r = q - 1$, i.e., $\Lambda \in D^q$, then Theorem 3.1 shows that $E = D^{1\times(p+1)}/(D^{1\times q}\, P))$, where $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$, is a stably free left D-module of rank $p - q + 1$ iff $\tau(\Lambda)$ generates the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^{p+1})$, i.e., iff $\mathrm{ext}^1_D(M, D)$ is a cyclic right D-module ([20]).

Theorem 3.2 ([2], [3]): Let D be a noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, $0 \leq r \leq q - 1$ and $\Lambda \in D^{q\times(q-r)}$ such that there exists $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying $(R \quad -\Lambda)\, U = (I_q \quad 0)$. If we decompose the unimodular matrix U as follows

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \quad (7)$$

where $S_1 \in D^{p\times q}$, $S_2 \in D^{(q-r)\times q}$, $Q_1 \in D^{p\times(p-r)}$ and $Q_2 \in D^{(q-r)\times(p-r)}$, and if $L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2)$ is the left D-module finitely presented by the full row rank matrix Q_2, i.e., defined by the following short exact sequence

$$0 \longrightarrow D^{1\times(q-r)} \stackrel{.Q_2}{\longrightarrow} D^{1\times(p-r)} \stackrel{\kappa}{\longrightarrow} L \longrightarrow 0, \quad (8)$$

then we have:

$$M = D^{1\times p}/(D^{1\times q}\, R)\ \cong\ L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2). \quad (9)$$

Conversely, if M is isomorphic to a left D-module L defined by the short exact sequence (8), then there exist two matrices $\Lambda \in D^{q\times(q-r)}$ and $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying:

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

Corollary 3.1 ([2], [3]): With the notations of Theorem 3.2, the left D-isomorphism (9) obtained in Theorem 3.2 is defined by

$$\begin{array}{rcl} M = D^{1\times p}/(D^{1\times q}\, R) & \stackrel{\varphi}{\longrightarrow} & L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2) \\ \pi(\lambda) & \longmapsto & \kappa(\lambda\, Q_1), \end{array}$$

and its inverse $\varphi^{-1} : L \longrightarrow M$ is defined by $\varphi^{-1}(\kappa(\mu)) = \pi(\mu\, T_1)$, where

$$U^{-1} = \left(\begin{array}{cc} R & -\Lambda \\ T_1 & -T_2 \end{array}\right) \in \mathrm{GL}_{p+q-r}(D),$$

$T_1 \in D^{(p-r)\times p}$ and $T_2 \in D^{(p-r)\times(q-r)}$. These results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in the right D-module $\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right)$ defined by (6).

A straightforward consequence of Corollary 3.1 is the following result.

Corollary 3.2 ([2], [3]): Let $\mathcal{F}$ be a left D-module and:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\},$$
$$\ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{(p-r)} \mid Q_2\,\zeta = 0\}.$$

Then, we have $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$ and:

$$\ker_{\mathcal{F}}(R.) = Q_1\, \ker_{\mathcal{F}}(Q_2.), \quad \ker_{\mathcal{F}}(Q_2.) = T_1\, \ker_{\mathcal{F}}(R.).$$

Corollary 3.3 ([2], [3]): Let $R \in D^{q\times p}$ be a full row rank matrix and $\Lambda \in D^{q\times(q-r)}$ such that $P = (R \quad -\Lambda)$ admits a right-inverse over D. Then, Theorem 3.2 holds when D satisfies one of the following properties:

1) D is a left principal ideal domain (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field A such as k or $k(t)$, where k is a field),
2) $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k,
3) D is either $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - r \geq 2$.
4) $D = A\langle\partial\rangle$ is the ring of OD operators, where $A = k[\![t]\!]$ and k is a field of characteristic 0, or $k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, and $p - r \geq 2$.

Corollary 3.4 ([2], [3]): With the notations of Theorem 3.2 and Corollary 3.1, if the matrix $\Lambda \in D^{q\times(q-r)}$ admits a left-inverse $\Gamma \in D^{(q-r)\times q}$, i.e., $\Gamma\,\Lambda = I_{q-r}$, then Q_1 admits the left-inverse $T_1 - T_2\,\Gamma\, R \in D^{(p-r)\times p}$ and the left D-module $\ker_D(.Q_1)$ is stably free of rank r.

Moreover, if the left D-module $\ker_D(.Q_1)$ is free of rank r, then there exists a matrix $Q_3 \in D^{p\times r}$ such that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D)$. If we write $W^{-1} = (Y_3^T \quad Y_1^T)^T$, where $Y_3 \in D^{r\times p}$ and $Y_1 \in D^{(p-r)\times p}$, then the matrix $X = (R\,Q_3 \quad \Lambda)$ is unimodular, i.e., $X \in \mathrm{GL}_q(D)$ and:

$$V = X^{-1} = \left(\begin{array}{c} Y_3\, S_1 \\ Q_2\, Y_1\, S_1 - S_2 \end{array}\right).$$

The matrix R is then equivalent to the matrix $X\,\mathrm{diag}(I_r, Q_2)\, W^{-1}$, or equivalently:

$$V\,R\,W = \left(\begin{array}{cc} I_r & 0 \\ 0 & Q_2 \end{array}\right).$$

Finally, the left D-module $\ker_D(.Q_1)$ is free when D satisfies 1 or 2 of Corollary 3.3 or if $D = A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $r \geq 2$ or if $D = A\langle\partial\rangle$ is the ring of OD operators with coefficients in the differential ring $A = k[\![t]\!]$, where k a field of characteristic 0, or in $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, and $r \geq 2$.

IV. Main results

Theorem 4.1: Let $D = A\langle\partial_1, \ldots, \partial_n\rangle$ be the ring of PD operators with coefficients in $A = k[x_1, \ldots, x_n]$ or $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, $R \in D^{q\times p}$ a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module finitely presented by R. If $\mathrm{ext}^1_D(M, D) = D^q/(R\,D^p)$ is a holonomic right D-module, then Theorem 3.1 holds and we can choose a column vector $\Lambda \in D^q$ which admits a left-inverse over D and which is such that $\tau(\Lambda)$ generates the right D-module $D^q/(R\,D^p)$, where $\tau : D^q \longrightarrow D^q/(R\,D^p)$ is the canonical projection onto $\mathrm{ext}^1_D(M, D)$. If $A = k[x_1, \ldots, x_n]$ and $p - q \geq 1$, then Theorem 3.2 and Corollaries 3.1 and 3.2 hold, i.e., $M \cong L = D^{1\times p-q+1}/(D\,Q_2)$, where $Q_2 \in D^{1\times(p-q+1)}$. Finally, if $q \geq 3$, then Corollary 3.4 holds, i.e., the matrix R is equivalent to $\mathrm{diag}(I_{q-1}, Q_2)$.

Proof: Since by hypothesis, $\mathrm{ext}^1_D(M, D)$ is a holonomic right D-module, Theorem 2.2 proves that $\mathrm{ext}^1_D(M, D)$ is cyclic and it can be generated by $\tau(\Lambda)$, where $\Lambda = (1 \ \ d_2 \ \ \ldots \ d_p)$, for certain d_i's in D. Using Remark 3.1, we obtain that $E = D^{1\times(p+1)}/(D^{1\times q}\, P)$, where $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$, is stably free of rank $p+1-q$. If $A = k[x_1, \ldots, x_n]$, i.e., $D = A_n(k)$, and $p + 1 - q \geq 2$, i.e., $p - q \geq 1$, then 3 of Theorem 1.2 shows that E is a free left D-module of rank $p + 1 - q$, and using 3 of Theorem 1.3, Theorem 3.2 holds. Moreover, $\Gamma = (1\ 0\ \ldots\ 0)$ is a left-inverse of Λ, and thus Corollary 3.4 holds. Finally, if $r = q - 1 \geq 2$, i.e., $q \geq 3$, then the stably free left D-module $\ker_D(.Q_1)$ of rank r is free by Stafford's theorem (see 3 of Theorem 1.2) and Corollary 3.4 proves that R is equivalent to $\mathrm{diag}(I_{q-1}, Q_2)$ for a certain matrix $Q_2 \in D^{1\times(p-q+1)}$. ■

Example 4.1: Let us consider the commutative polynomial ring $D = \mathbb{Q}\,[\partial_x, \partial_y]$ of PD operators and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ finitely presented by R defined by:

$$R = \left(\begin{array}{ccc} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{array}\right) \in D^{2\times 3}. \tag{10}$$

The matrix R defines the equation $R\,\sigma = 0$ of the equilibrium of the *stress tensor* in $\mathbb{R}^2$, namely:

$$\left\{\begin{array}{l} \partial_x\, \sigma^{11} + \partial_y\, \sigma^{12} = 0, \\ \partial_x\, \sigma^{12} + \partial_y\, \sigma^{22} = 0. \end{array}\right. \tag{11}$$

We can check that $\mathrm{ext}^1_D(M, D) = D^2/\left(R\,D^3\right)$ is a $\mathbb{Q}$-vector space of dimension 3 and a basis of $\mathrm{ext}^1_D(M, D)$ is defined by the vectors $\tau((1 \quad 0)^T)$, $\tau((0 \quad 1)^T)$ and $\tau((0 \quad \partial_x)^T)$, where $\tau : D^2 \longrightarrow D^2/(R\,D^3)$ is the canonical projection. Hence, without loss of generality, we can assume that Λ has the form $\Lambda = (a \quad b + c\,\partial_x)^T$, where a, b and c are three arbitrary constants. Considering the new ring $D' = \mathbb{Q}[a, b, c]\,[\partial_x, \partial_y]$, $P = (R \quad -\Lambda)$ and the D'-module $E = {D'}^{1\times 4}/({D'}^{1\times 2}\, P)$, then, using Gröbner basis techniques, we can check that the matrix P does not admit a right-inverse with entries in D'. According to Theorem 1.3, we obtain that the A-module E is not a stably free D'-module, which proves that (11) cannot be defined by a sole

PDE with constant coefficients, and the minimal number of generators $\mu(M)$ of the D-module M is 3.

Let $M' = B^{1\times 3}/(B^{1\times 2}\,R)$ be the left $B = A_2(\mathbb{Q})$-module finitely presented by R. The right B-module $\mathrm{ext}^1_B(M', B) = B^2/(R\,B^3)$ is holonomic and thus cyclic by Proposition 2.2. The element $\tau(\Lambda)$ of $\mathrm{ext}^1_B(M', B)$, where $\Lambda = (1 \quad x)^T$, generates $\mathrm{ext}^1_B(M', B)$ since the matrix $P = (R \quad -\Lambda) \in B^{2\times 4}$ admits the following right-inverse:

$$T = \begin{pmatrix} -x & 1 \\ -x^2 & x \\ -x^3 & x^2 \\ -x\,(x\,\partial_y + \partial_x) - 2 & \partial_x + x\,\partial_y \end{pmatrix}.$$

The left B-module $E' = B^{1\times 4}/(B^{1\times 2}\,P)$ is then stably free of rank 2 (see Remark 3.1), i.e., free by Stafford's theorem (see 3 of Theorem 1.2). Using the package STAFFORD ([17]), an injective parametrization of E' is defined by

$$Q = \begin{pmatrix} \partial_y & \partial_x \\ x\,\partial_y & x\,\partial_x - 1 \\ x^2\,\partial_y - 1 & x\,\partial_x - x \\ (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which yields:

$$M' \cong B^{1\times 2}/(B\,((\partial_x + x\,\partial_y)\,\partial_y \quad (\partial_x + x\,\partial_y)\,\partial_x - \partial_y)).$$

Since $\Gamma = (1 \quad 0)$ is a left-inverse of Λ, using Corollary 3.4, we obtain the following unimodular matrices:

$$W = \begin{pmatrix} -1 & \partial_y & \partial_x \\ -x & x\,\partial_y & x\,\partial_x - 1 \\ -x^2 & x^2\,\partial_y - 1 & x\,(x\,\partial_x - 1) \end{pmatrix},$$

$$W^{-1} = \begin{pmatrix} x\,\partial_x & x\,\partial_y - \partial_x & -\partial_y \\ 0 & x & -1 \\ x & -1 & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} -(\partial_x + x\,\partial_y) & 1 \\ -x\,(\partial_x + x\,\partial_y) - 1 & x \end{pmatrix},$$

$$X^{-1} = \begin{pmatrix} x & -1 \\ x^2\,\partial_y + x\,\partial_x + 2 & -(\partial_x + x\,\partial_y) \end{pmatrix}.$$

Then, R defined by (10) is equivalent to $\overline{R} = X^{-1}\,R\,W$,

$$\overline{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which proves that (11) is equivalent to the following PDE

$$(\partial_x + x\,\partial_y)\,\partial_y\,\tau_2 + (\partial_x + x\,\partial_y)\,\partial_x\,\tau_3 - \partial_y\,\tau_3 = 0,$$

under the following invertible transformations:

$$\begin{cases} \sigma^{11} = \partial_y\,\tau_2 + \partial_x\,\tau_3, \\ \sigma^{12} = x\,\partial_y\,\tau_2 + x\,\partial_x\,\tau_3 - \tau_3, \\ \sigma^{22} = x^2\,\partial_y\,\tau_2 - \tau_2 + x^2\,\partial_x\,\tau_3 - x\,\tau_3, \end{cases}$$

$$\begin{cases} \tau_1 = x\,(\partial_x\,\sigma^{11} + \partial_y\,\sigma^{12}) - (\partial_x\,\sigma^{12} + \partial_y\,\sigma^{22}) = 0, \\ \tau_2 = x\,\sigma^{12} - \sigma^{22}, \\ \tau_3 = x\,\sigma^{11} - \sigma^{12}. \end{cases}$$

If D is a domain and $M = D^{1\times p}/(D^{1\times q}\,R)$ a left D-module finitely presented by a full row rank matrix R, then we can prove that the right D-module $D^q/(R\,D^p)$ is torsion ([4]). Then, using Theorem 2.1 and 4 of Corollary 3.3, we obtain the following corollary of Theorem 4.1.

Corollary 4.1: Let $D = A\langle\partial\rangle$ be the ring of OD operators with coefficients in $A = k[t]$ or $k[\![t]\!]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, $R \in D^{q\times p}$ a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by R. Then, Theorem 3.1 holds and $\Lambda \in D^q$ can be chosen so that it admits a left-inverse over D and $\tau(\Lambda)$ generates the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(R\,D^p)$. Moreover, if $p - q \geq 1$, then Theorem 3.2 and Corollaries 3.1 and 3.2 hold. Finally, if $q \geq 3$, then Corollary 3.4 holds.

Corollary 4.1 shows that every analytic linear OD system with at least one input is isomorphic to an analytic linear control OD system defined by a sole equation. Moreover, if the system has at least 3 equations, then the system is equivalent to a sole ODE.

Since the rings $D = B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$, where k is a field of characteristic 0, or $k\{t\}[t^{-1}]\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, are simple principal left ideal domains (see, e.g., [1], [13]), using the concept of *Jacobson normal form*, namely, a generalization of the Smith normal form to principal left or right ideal domains (see, e.g., [8], [22]), one can prove that for every matrix $R \in D^{q\times p}$, there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $d \in D$ such that $V\,R\,W = \mathrm{diag}(1, \ldots, 1, d, 0, \ldots, 0)$, i.e., R is equivalent to the diagonal matrix $\overline{R} = \mathrm{diag}(1, \ldots, 1, d, 0, \ldots, 0)$, for a certain $d \in D$. In particular, if R has full row rank, i.e., $\ker_D(.R) = 0$, then R is equivalent to $\mathrm{diag}(1, \ldots, 1, d)$.

Now, if $D = A_1(k)$, $k[\![t]\!]\langle\partial\rangle$, where k is a field of characteristic 0, or $k\{t\}\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $R \in D^{q\times p}$, then the Jacobson normal form of R can be computed by considering the injection of D into the simple principal left ideal domain D', where D' is respectively $B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$ and $k\{t\}[t^{-1}]\langle\partial\rangle$. Therefore, there exist $V \in \mathrm{GL}_q(D')$, $W \in \mathrm{GL}_p(D')$ and $e \in D'$ such that $V\,R\,W = \mathrm{diag}(1, \ldots, 1, e, 0, \ldots, 0)$. However, artificial singularities may have been introduced in e, V and W. Corollary 4.1 shows that there always exist three matrices $Q_2 \in D^{1\times(p-q+1)}$, $X \in \mathrm{GL}_q(D)$ and $Y \in \mathrm{GL}_p(D)$ such that $X\,R\,Y = \mathrm{diag}(I_{q-1}, Q_2)$. The entries of Q_2, X, Y, X^{-1} and Y^{-1} belong to D, i.e., do not contain singularities.

Example 4.2: Let $M = D^{1\times 4}/(D^{1\times 3}\,R)$ be the left $D = k[t]\langle\partial\rangle$-module finitely presented by the following matrix:

$$R = \begin{pmatrix} 1 & 0 & 0 & \partial \\ \partial & 1 & 1 & t \\ 0 & 0 & t\,\partial & t\,\partial^2 - t \end{pmatrix}.$$

Using Remark 2.1, the vector $\Lambda = (0 \quad 1 \quad 1)^T$ is such that the matrix $P = (R \quad -\Lambda)$ admits the following right-inverse:

$$S = \begin{pmatrix} 1 & -\partial & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \end{pmatrix}^T.$$

Therefore, the right D-module $\mathrm{ext}^1_D(M, D) = D^3/(R\, D^4)$ is cyclic and is generated by $\tau(\Lambda)$, and thus the left D-module $E = D^{1\times 5}/(D^{1\times 3}\, P)$ is stably free of rank 2, i.e., is free of rank 2 by Stafford's theorem (see 3 of Theorem 1.2). Computing an injective parametrization of E, we obtain that the matrix $Q = (Q_1^T \quad Q_2^T) \in D^{5\times 2}$, where

$$Q_1 = \begin{pmatrix} \partial & 0 \\ -\partial^2 - \partial + 2\,t & t\,\partial - 1 \\ \partial & 1 \\ -1 & 0 \end{pmatrix}, \quad Q_2 = (t \quad t\,\partial),$$

satisfies $\ker_D(.Q) = D^{1\times 3}\, P$ and $T\, Q = I_2$, where:

$$T = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, we obtain $M \cong D^{1\times 2}/(D\, Q_2)$. Moreover, since Λ admits the left-inverse $\Gamma = (0 \quad 0 \quad 1)$, we obtain that R is equivalent to $\mathrm{diag}(I_2, Q_2)$. More precisely, we have $\ker_D(.Q_1) = D^{1\times 2}\, K$, where

$$K = \begin{pmatrix} 1 & 0 & 0 & \partial \\ (t+1)\,\partial & 1 & -t\,\partial + 1 & 2\,t \end{pmatrix},$$

and the matrix Q_3 defined by

$$Q_3 = \begin{pmatrix} 1 & -\partial - 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^T$$

is a right-inverse of K, i.e., $W = (Q_3 \quad Q_1) \in \mathrm{GL}_4(D)$. Then, $X = (R\, Q_3 \quad \Lambda)$ and $V = X^{-1}$ are defined by:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ t\,\partial & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ t\,\partial & 1 & -1 \\ -t\,\partial & 0 & 1 \end{pmatrix}.$$

Finally, we obtain:

$$\overline{R} = V\, R\, W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & t\,\partial \end{pmatrix}.$$

We state applications of Theorem 3.1 to controllability.

Proposition 4.1: Let $D = k\{t\}\langle\partial\rangle$, $R_1 \in D^{q\times q}$, $\ker_D(.R_1) = 0$, $R_2 \in D^{q\times r}$, $R = (R_1 \quad -R_2) \in D^{q\times(q+r)}$, $M = D^{1\times(q+r)}/(D^{1\times q}\, R)$, $N = D^{1\times q}/(D^{1\times q}\, R_1)$ and $P = D^q/(R_1\, D^q)$. The analytic OD system $R_1\, y = R_2\, u$ is controllable in a neighbourhood of $t = 0$ iff M is a stably free left D-module of rank r or iff $\{\tau((R_2)_{\bullet i})\}_{i=1,\ldots,r}$ generates the right D-module $P = \mathrm{ext}^1_D(N, D)$, where $\tau : D^q \longrightarrow P$ is the canonical projection onto P.

Proposition 4.1 directly follows from Theorem 3.1 and the well-known fact that $R_1\, y = R_2\, u$ is controllable in a neighbourhood of $t = 0$ iff R admits a right-inverse, i.e., $R_1\, S_1 - R_2\, S_2 = I_q$ for certain $S_1 \in D^{q\times q}$ and $S_2 \in D^{r\times q}$ (see, e.g., [21], [18], [22] and the references therein). A dual statement for observability can similarly be given.

Corollary 4.2: Let $A = \mathbb{R}\{t\}$ the ring of germs of analytic functions at 0, $F \in A^{n\times n}$ and $G \in A^{n\times m}$. Then, the analytic linear OD system $\dot{x} = F\,x + G\,u$ is controllable in a neighbourhood of $t = 0$ iff $\{\tau(G_{\bullet i})\}_{i=1,\ldots,m}$ generates the right D-module $P = D^n/((\partial\, I_n - F)\, D^n)$.

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