Study of a spectral sequence central in the behavioural approach

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***Abstract*— Within the algebraic analysis approach to multidimensional systems, the behavioural approach can be understood as a dual theory to the module-theoretic approach. This duality is exact when the signal space is an injective cogenerator left module over the ring of functional operators. In this paper, we consider the case of a general signal space and investigate the connection between the properties of the module M defining the system and the obstruction to the existence of parametrizations of this system. To do so, we study the Cartan-Eilenberg resolution of a certain complex and a Grothendieck spectral sequence connecting the obstructions to the existence of parametrizations to the obstructions for M to be projective.**

I. ALGEBRAIC ANALYSIS APPROACH

Within *algebraic analysis* [2], [4], [6], if D denotes a ring of *functional operators* (e.g., ordinary or partial differential operators, time-delay operators, shift operators), $R \in D^{q\times p}$ a $q\times p$ matrix with entries in D and $\mathcal{F}$ a left D-module, then a *system* is defined by $\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^{p\times 1} \mid R\,\eta = 0\}$, i.e., by the kernel of the abelian group homomorphism $R.$

$$\begin{array}{rcl} R.: \mathcal{F}^p & \longrightarrow & \mathcal{F}^q \\ \eta & \longmapsto & R\,\eta, \end{array} \tag{1}$$

where $\mathcal{F}^{l\times 1}$ is simply denoted by $\mathcal{F}^l$. Within the *behavioural approach*, $\mathcal{F}$ is called a *signal space* and $\ker_{\mathcal{F}}(R.)$ a *behaviour* [4], [6]. The latter can be intrinsically studied by means of the left D-modules $\mathcal{F}$ and $M := D^{1\times p}/(D^{1\times q}\,R)$, where $D^{1\times p}$ is the left D-module formed by row vectors of length p with entries in D. Indeed, if $\hom_D(M, \mathcal{F})$ denotes the abelian group formed by all the left D-homomorphisms (i.e., left D-linear maps) from M to $\mathcal{F}$, then a standard result in module theory [5] shows that $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$.

Definition 1 ([5]):
- A sequence of left/right D-homomorphisms $d_i \in \hom_D(M_i, M_{i-1})$ is called a *complex* if $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$, or equivalently if $\operatorname{im} d_{i+1} \subseteq \ker d_i$ for all $i \in \mathbb{Z}$. It is denoted by:

$$M_\bullet : \ \ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots$$

- The *homology* of the complex $M_\bullet$ at M_i is defined by:

$$H_i(M_\bullet) := \ker d_i/\operatorname{im} d_{i+1}.$$

- The complex $M_\bullet$ is called *exact* at M_i if $H_i(M_\bullet) = 0$ and simply *exact* if $H_i(M_\bullet) = 0$ for all $i \in \mathbb{Z}$.
- A *free resolution* of a finitely generated left/right D-module M is an exact sequence of the form

$$0 \longleftarrow M \xleftarrow{\pi} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} \ldots \tag{2}$$

where the left/right D-modules F_i are *free*. Its *truncation* is the complex obtained by setting $M = 0$ in (2).

In what follows, we always assume that D is a *noetherian domain* and M is a *finitely generated* left D-module [5]. We can prove that any finitely generated left/right D-module admits a free resolution where the left/right D-modules F_i's are finitely generated left/right D-modules (see, e.g., [5]).

Let us consider a truncated free resolution of the *finitely presented* left D-module $M = D^{1\times p_0}/(D^{1\times p_1}\,R_1)$:

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0. \tag{3}$$

Applying the *contravariant left exact functor* $\hom_D(\,\cdot\,, \mathcal{F})$ to (3) and using $\hom_D(D^{1\times p_i}, \mathcal{F}) \cong \mathcal{F}^{p_i}$, we get the complex

$$\mathrm{Rhom}_D(M, \mathcal{F}) : \ \ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0,$$

whose cohomology abelian groups are defined by:

$$\begin{cases} \operatorname{ext}^0_D(M, \mathcal{F}) := \hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \operatorname{ext}^i_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/\operatorname{im}_{\mathcal{F}}(R_i.),\ i \geq 1. \end{cases} \tag{4}$$

Using (4), $\operatorname{ext}^1_D(M, \mathcal{F}) = 0$ is equivalent to $\ker_{\mathcal{F}}(R_2.) = \operatorname{im}_{\mathcal{F}}(R_1.) = R_1\,\mathcal{F}^{p_0}$, i.e., to the fact that the behaviour $\ker_{\mathcal{F}}(R_2.)$ can be parametrized by means of the matrix R_1.

Definition 2 ([5]): A left D-module $\mathcal{F}$ is *injective* if $\operatorname{ext}^i_D(M, \mathcal{F}) = 0$ for all right D-modules M and all $i \geq 1$.

Let $N := D^q/(R\,D^p)$ be the *Auslander transpose* of $M = D^{1\times p}/(D^{1\times q}\,R)$, i.e., the cokernel of the right D-homomorphism (1) where $\mathcal{F} = D$ [2]. Let $q_0 := q$, $q_1 := p$ and $R_1 := R$, and consider a truncated free resolution of N:

$$0 \longleftarrow D^{q_0} \xleftarrow{R_1.} D^{q_1} \xleftarrow{R_2.} D^{q_2} \xleftarrow{R_3.} \ldots \tag{5}$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to (5), we obtain the complex $\mathrm{Rhom}_D(N, D)$ defined by

$$0 \longrightarrow D^{1\times q_0} \xrightarrow{.R_1} D^{1\times q_1} \xrightarrow{.R_2} D^{1\times q_2} \xrightarrow{.R_3} \ldots \tag{6}$$

whose cohomology left D-modules are:

$$\begin{cases} \operatorname{ext}^0_D(N, D) := \hom_D(N, D) \cong \ker_D(.R_1), \\ \operatorname{ext}^i_D(N, D) \cong \ker_D(.R_{i+1})/\operatorname{im}_D(.R_i),\ i \geq 1. \end{cases} \tag{7}$$

Definition 3 ([5]):
- M is *projective* if there exist a left D-module P and $r \in \mathbb{Z}_{\geq 0}$ such that $M \oplus P \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of modules.
- M is *reflexive* if the canonical left D-homomorphism $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$, defined by $\varepsilon(m)(f) = f(m)$ for all $f \in \hom_D(M, D)$ and for all $m \in M$, is an isomorphism.

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- M is *torsion-free* if we have:

$$t(M) := \{m \in M \mid \exists\, d \in D \setminus \{0\} : \; d\, m = 0\} = 0.$$

Theorem 1: [2] Let assume that D has a finite *global dimension* $\mathrm{gld}(D) := n$ [5]. Then, we have:

1) $t(M) \cong \mathrm{ext}^1_D(N, D)$.
2) M is torsion-free iff $\mathrm{ext}^1_D(N, D) = 0$
3) M is reflexive iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, 2$.
4) M is projective iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, \ldots, n$.

Applying the *covariant right exact functor* $\cdot \otimes_D \mathcal{F}$ to (5), where $\otimes_D$ denotes a *tensor product* [5], and using the fact that $D^{q_i} \otimes_D \mathcal{F} \cong \mathcal{F}^{q_i}$ [5], we obtain the following complex

$$N \otimes^L_D \mathcal{F} : \; 0 \longleftarrow \mathcal{F}^{q_0} \xleftarrow{R_1.} \mathcal{F}^{q_1} \xleftarrow{R_2.} \mathcal{F}^{q_2} \xleftarrow{R_3.} \ldots \qquad (8)$$

whose homology abelian groups are defined by:

$$\left\{ \begin{array}{ll} \mathrm{tor}^D_0(N, \mathcal{F}) \cong \mathrm{coker}_{\mathcal{F}}(R_1.) = \mathcal{F}^{q_0} / \mathrm{im}_{\mathcal{F}}(R_1.), & \\ \mathrm{tor}^D_i(N, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_i.) / \mathrm{im}_{\mathcal{F}}(R_{i+1}.), & i \geq 1. \end{array} \right. \qquad (9)$$

Note that $\mathrm{tor}^D_1(N, \mathcal{F}) = 0$ yields $\ker_{\mathcal{F}}(R_1.) = \mathrm{im}_{\mathcal{F}}(R_2.)$, i.e., $\ker_{\mathcal{F}}(R_1.)$ can be parametrized by means of R_2.

Definition 4 ([5]): A left D-module $\mathcal{F}$ is said to be *flat* if $\mathrm{tor}^D_i(N, \mathcal{F}) = 0$ for all right D-modules N and all $i \geq 1$.

An important problem in mathematical systems theory is to parametrize systems [2], [6]. This problem is called *image representation* in the behavioural approach. If the functional space $\mathcal{F}$ is a flat left D-module, then $\mathrm{tor}^D_1(N, \mathcal{F}) = 0$, and thus $\ker_{\mathcal{F}}(R_1.)$ can be parametrized by means of R_2 defined by $\ker_D(R_1.) = R_2\, D^{q_2}$ (see (5)). Thus, any behaviour is parametrizable over a flat left D-module $\mathcal{F}$. Unfortunately, few standard functional spaces $\mathcal{F}$ are flat left D-modules [6].

If M is a projective left D-module and $\mathcal{F}$ any left D-module, then we can show that $\ker_{\mathcal{F}}(R_1.)$ can be parametrized by R_2 defined by $\ker_D(R_1.) = R_2\, D^{q_2}$ [2]. Unfortunately, few systems define a projective module M.

Another approach is to note that if M is torsion-free, i.e., $t(M) \cong \mathrm{ext}^1_D(N, D) = 0$, then applying the contravariant functor $\hom_D(\cdot\,, \mathcal{F})$ to the exact sequence $D^{1\times q_0} \xrightarrow{.R_1} D^{1\times q_1} \xrightarrow{.R_2} D^{1\times q_2}$, we get the complex $\mathcal{F}^{q_0} \xleftarrow{R_1.} \mathcal{F}^{q_1} \xleftarrow{R_2.} \mathcal{F}^{q_2}$ whose cohomology abelian group at $\mathcal{F}^{q_1}$ is $\mathrm{ext}^1_D(N_2, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.)/\mathrm{im}_{\mathcal{F}}(R_2.)$, where $N_2 := \mathrm{coker}_D(.R_2) = D^{1\times q_2}/(D^{1\times q_1}\, R_2)$. In particular, if $\mathcal{F}$ is an injective left D-module (see Definition 2), then $\mathrm{ext}^1_D(N_2, \mathcal{F}) = 0$, i.e., $\ker_{\mathcal{F}}(R_1.) = \mathrm{im}_{\mathcal{F}}(R_2.)$, which shows that the linear system $\ker_{\mathcal{F}}(R_1.)$ can be parametrized when M is torsion-free and $\mathcal{F}$ is an injective left D-module. The hypothesis of an injective module $\mathcal{F}$ is standard in the behavioural approach [2], [4], [6]. Before studying the general case, let us study the injective case in more detail.

Lemma 1: [5] Let us consider the following complex:

$$M_\bullet : \; \ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots$$

Then, we have the following short exact sequences:

$$\forall\, i \in \mathbb{Z}, \quad 0 \longrightarrow H_i(M_\bullet) \longrightarrow \mathrm{coker}\, d_{i+1} \longrightarrow \mathrm{im}\, d_i \longrightarrow 0.$$

Theorem 2: Let $N = D^q/(R\, D^p)$ be the Auslander transpose right D-module of $M = D^{1\times p}/(D^{1\times q}\, R)$ and $\mathcal{F}$ an injective left D-module. Then, we have:

$$\forall\, i \geq 1, \quad \mathrm{tor}^D_i(N, \mathcal{F}) \cong \hom_D(\mathrm{ext}^i_D(N, D), \mathcal{F}). \qquad (10)$$

Proof: Let us consider a free resolution (5) of N. Applying the contravariant left exact functor $\hom_D(\cdot\,, D)$ to (5), we get the complex $\mathrm{Rhom}_D(N, D)$ defined by (6). Let $R_{ii} := R_i$, $q_{ii} := q_i$. Then, (6) becomes:

$$0 \longrightarrow D^{1\times q_{00}} \xrightarrow{.R_{11}} D^{1\times q_{11}} \xrightarrow{.R_{22}} D^{1\times q_{22}} \xrightarrow{.R_{33}} \ldots$$

Let $R_{(i-1)i} \in D^{q_{(i-2)i}\times q_{(i-1)i}}$ be a matrix such that $\ker_D(.R_{ii}) = \mathrm{im}_D(.R_{(i-1)i})$ for $i \geq 2$. Then we get the following diagram formed by horizontal exact sequences:

$$\begin{array}{ccccc} & & & & \| \;\; \ldots \\ & & D^{1\times q_{13}} \xrightarrow{.R_{23}} & D^{1\times q_{23}} & \xrightarrow{.R_{33}} D^{1\times q_{33}} \\ & & & \| & \\ D^{1\times q_{02}} & \xrightarrow{.R_{12}} & D^{1\times q_{12}} \xrightarrow{.R_{22}} & D^{1\times q_{22}} & \\ & & \| & & \\ D^{1\times q_{01}} & \xrightarrow{.R_{11}} & D^{1\times q_{11}}. & & \end{array} \qquad (11)$$

Let $M_{(i-j)i} := \mathrm{coker}_D(.R_{(i-j)i})$ for $i \geq 1$ and $i \geq j$. Applying Lemma 1 to the complex $\mathrm{Rhom}_D(N, D)$ defined by (6) and using $\mathrm{im}_D(.R_{(i+1)(i+1)}) \cong M_{i(i+1)}$, we get the following short exact sequences for $i \geq 1$:

$$0 \longrightarrow \mathrm{ext}^i_D(N, D) \longrightarrow M_{ii} \longrightarrow M_{i(i+1)} \longrightarrow 0. \qquad (12)$$

We then have $\hom_D(M_{ii}, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{ii}.)$ and $\hom_D(M_{i(i+1)}, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i(i+1)}.)$. Since $\mathcal{F}$ is injective, we get the following short exact sequence (see, e.g., [5])

$$\begin{array}{rl} 0 \longleftarrow & \hom_D(\mathrm{ext}^i_D(N, D), \mathcal{F}) \\ & \longleftarrow \ker_{\mathcal{F}}(R_{ii}.) \longleftarrow \ker_{\mathcal{F}}(R_{i(i+1)}.) \longleftarrow 0, \end{array}$$

i.e., $\hom_D(\mathrm{ext}^i_D(N, D), \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{ii}.)/\ker_{\mathcal{F}}(R_{i(i+1)}.)$.

Applying the exact functor $\hom_D(\cdot\,, \mathcal{F})$ to (11), we get $\mathrm{im}_{\mathcal{F}}(R_{(i+1)(i+1)}.) = \ker_{\mathcal{F}}(R_{i(i+1)}.)$. By definition (see (9)), we have $\mathrm{tor}^D_i(N, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{ii}.)/\mathrm{im}_{\mathcal{F}}(R_{(i+1)(i+1)}.)$ for $i \geq 1$, and thus we finally obtain (10). ■

Note that $\hom_D(\mathrm{ext}^1_D(N, D), \mathcal{F})$ defines the *autonomous elements* of $\ker_{\mathcal{F}}(R.)$ [2], [4], [6]. Theorem 2 connects the obstructions for getting a long chain of successive parametrizations of the form (8) to the behaviours associated with the obstructions $\mathrm{ext}^i_D(N, D)$'s for M to be projective.

II. CARTAN-EILENBERG RESOLUTION

The rest of the paper aims at extending Theorem 2 in the case where $\mathcal{F}$ is no longer an injective left D-module.

The goal of this section is to construct a *free resolution of the complex* $\mathrm{Rhom}_D(N, D)$ (6) called a *Cartan-Eilenberg resolution* [5]. It plays a central role in the rest of the paper.

In what follows, we shall use the following notations: a complex is always oriented from left to right and from bottom to up. We first need to rewrite the truncated free resolution (5) of N according to the notational conventions

$$\ldots \xrightarrow{R_{-3}.} D^{r_{-2}} \xrightarrow{R_{-2}.} D^{r_{-1}} \xrightarrow{R_{-1}.} D^{r_0} \longrightarrow 0, \qquad (13)$$

where $R_{-1} = R$, $r_0 = q$ and $r_{-1} = p$. Moreover, applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to (13), we obtain the complex $\mathrm{Rhom}_D(N, D)$ rewritten as follows

$$0 \longrightarrow D^{1\times r^0} \xrightarrow{.R^0} D^{1\times r^1} \xrightarrow{.R^1} D^{1\times r^2} \xrightarrow{.R^2} \ldots \quad (14)$$

where $R^i := R_{-i-1}$ for $i \geq 0$ and $r^i = r_{-i}$ for $i \geq 0$. The complex (14) can be decomposed as a sequence of the following short exact sequences:

$$0 \longrightarrow \ker_D(.R^i) \longrightarrow D^{1\times r^i} \xrightarrow{.R^i} \mathrm{im}_D(.R^i) \longrightarrow 0, \quad (15)$$

$$0 \longrightarrow \mathrm{im}_D(.R^i) \longrightarrow \ker_D(.R^{i+1}) \xrightarrow{\kappa^{i+1}} \mathrm{ext}_D^{i+1}(N, D) \longrightarrow 0. \quad (16)$$

Let us now consider a free resolution of $\ker_D(.R^0)$:

$$\ldots \xrightarrow{.S^{0,-2}} D^{1\times s^{0,-1}} \xrightarrow{.S^{0,-1}} D^{1\times s^{0,0}} \xrightarrow{.S^{0,0}} \ker_D(.R^0) \longrightarrow 0. \quad (17)$$

Combining (17) with the short exact sequence (15), i.e.,

$$0 \longrightarrow \ker_D(.R^0) \longrightarrow D^{1\times r^0} \xrightarrow{.R^0} \mathrm{im}_D(.R^0) \longrightarrow 0, \quad (18)$$

we obtain the following free resolution of $\mathrm{im}_D(.R_0)$:

$$\ldots \xrightarrow{.U^{1,-2}} D^{1\times u^{1,-1}} \xrightarrow{.U^{1,-1}} D^{1\times u^{1,0}} \xrightarrow{.R^0} \mathrm{im}_D(.R^0) \longrightarrow 0, \quad (19)$$

$$u^{1,0} = r^0, \quad \forall\, i \geq 1, \quad u^{1,-i} = s^{0,1-i}, \quad U^{1,-i} = S^{0,1-i}. \quad (20)$$

Lemma 2: With the notations of (17), (19), (20), and

$$\forall\, i \geq 0, \quad t^{0,-i} = s^{0,-i} + u^{1,-i} = s^{0,-i} + s^{0,1-i},$$
$$X^{0,-i} = (I_{s^{0,-i}} \quad 0),$$
$$Y^{0,-i} = \begin{pmatrix} 0 \\ I_{u^{1,-i}} \end{pmatrix}, \quad T^{0,0} = \begin{pmatrix} S^{0,0} \\ I_{u^{1,0}} \end{pmatrix},$$
$$T^{0,-i} = \begin{pmatrix} S^{0,-i} & 0 \\ (-1)^i\, I_{s^{0,1-i}} & U^{1,-i} \end{pmatrix},$$

the diagram (21) is commutative and exact. It defines a *free resolution of the short exact sequence* (18).

Proof: The first and third vertical sequences of (21) are exact since they are respectively a free resolution of $\ker_D(.R^0)$ and of $\mathrm{im}_D(.R^0)$. All the horizontal short sequences are exact and starting from the second one, they *split* [5]. We can easily check that all the squares commute. Since $T^{0,-i-1}\, T^{0,-i} = 0$ for $i \geq 0$, the second vertical sequence is a complex, i.e., $\mathrm{im}_D(.T^{0,-i-1}) \subseteq \ker_D(.T^{0,-i})$. If $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.T^{0,0})$, i.e., $\lambda_1\, S^{0,0} + \lambda_2 = 0$, then $\lambda = \lambda_1\, (I_{s^{0,0}} \quad -S^{0,0})$, which yields $\ker_D(.T^{0,0}) \subseteq \mathrm{im}_D(.T^{0,-1})$, i.e., $\ker_D(.T^{0,0}) = \mathrm{im}_D(.T^{0,-1})$, and shows that the second vertical sequence is exact at $D^{1\times t^{0,0}}$. Now, let $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.T^{0,-i})$, i.e.:

$$\lambda_1\, S^{0,-i} + (-1)^i\, \lambda_2 = 0, \quad \lambda_2\, U^{1,-i} = 0. \quad (22)$$

Since $\ker_D(.U^{1,-i}) = \mathrm{im}_D(.U^{1,-i-1})$, where by definition $U^{1,-i-1} = S^{0,-i}$, there exists $\mu_2 \in D^{1\times u^{1,-i-1}}$ such that $\lambda_2 = \mu_2\, S^{0,-i}$. Substituting it in the first equation of (22), we obtain $\lambda_1 + (-1)^i\, \mu_2 \in \ker_D(.S^{0,-i}) = \mathrm{im}_D(.S^{0,-i-1})$, and thus $\mu_1 \in D^{1\times s^{0,-i-1}}$ exists such that $\lambda_1 + (-1)^i\, \mu_2 = \mu_1\, S^{0,-i-1}$, which yields $(\lambda_1 \quad \lambda_2) = \big(\mu_1\, S^{0,-i-1} + (-1)^{i+1}\, \mu_2 \quad \mu_2\, S^{0,-i}\big) = (\mu_1 \quad \mu_2)\, T^{0,-i-1}$ and shows that $\ker_D(.T^{0,-i}) \subseteq \mathrm{im}_D(.T^{0,-i-1})$, and thus $\ker_D(.T^{0,-i}) = \mathrm{im}_D(.T^{0,-i-1})$. The second vertical sequence of (21) is then exact at $D^{1\times t^{0,-i}}$, which finally proves the exactness of the commutative diagram (21). ■

Lemma 3 ([3]): Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p}$ be such that $\mathrm{im}_D(.R) \subseteq \ker_D(.R')$. Moreover, let $R'' \in D^{q\times q'}$ and $R'_2 \in D^{r'\times q'}$ be such that $R = R''\, R'$ and $\ker_D(.R') = \mathrm{im}_D(.R'_2)$. If $L = D^{1\times q'}/\Big(D^{1\times (q+r')}\, (R''^T \quad R_2'^T)^T\Big)$, $Q := \ker_D(.R')/\mathrm{im}_D(.R)$ $\sigma : D^{1\times q'} \longrightarrow L$ and $\kappa : \ker_D(.R') \longrightarrow Q$ the canonical projections onto respectively L and Q, then we have the following isomorphism:

$$\begin{array}{ccc} L & \longrightarrow & Q \\ \sigma(\nu) & \longmapsto & \kappa(\nu\, R'). \end{array}$$

Using (16), we have the following short exact sequence:

$$0 \longrightarrow \mathrm{im}_D(.R^0) \longrightarrow \ker_D(.R^1) \xrightarrow{\kappa^1} \mathrm{ext}_D^1(N, D) \longrightarrow 0. \quad (23)$$

The beginning of a free resolution of $\ker_D(.R^1)$ is then:

$$D^{1\times s'^{1,-1}} \xrightarrow{.S'^{1,-1}} D^{1\times s'^{1,0}} \xrightarrow{.S'^{1,0}} \ker_D(.R^1) \longrightarrow 0.$$

Hence, we get $\mathrm{ext}_D^1(N, D) \cong \mathrm{im}_D(.S'^{1,0})/\mathrm{im}_D(.R^0)$. Using $\mathrm{im}_D(.R^0) \subseteq \mathrm{im}_D(.S'^{1,0})$, there exists $S''^{1,0} \in D^{u^{1,0}\times s'^{1,0}}$ such that $R^0 = S''^{1,0}\, S'^{1,0}$. If $v^{1,0} = s'^{1,0}$,

$$v^{1,-1} = u^{1,0} + s'^{1,-1}, \quad V^{1,-1} = \begin{pmatrix} S''^{1,0} \\ S'^{1,-1} \end{pmatrix} \in D^{v^{1,-1}\times s^{1,0}},$$

then Lemma 3 yields the following isomorphism

$$\begin{array}{rcl} L = D^{1\times v^{1,0}}/(D^{1\times v^{1,-1}}\, V^{1,-1}) & \cong & \mathrm{ext}_D^1(N, D) \\ \sigma(\nu) & \longmapsto & \kappa^1(\nu\, S'^{1,0}), \end{array}$$

where are σ and κ^1 are the two canonical projections:

$$\sigma : D^{1\times v^{1,0}} \longrightarrow L, \quad \kappa^1 : \mathrm{im}_D(.S'^{1,0}) \longrightarrow \mathrm{ext}_D^1(N, D).$$

Consider a free resolution of $\mathrm{ext}_D^1(N, D)$ of the form:

$$\ldots \xrightarrow{.V^{1,-1}} D^{1\times v^{1,0}} \xrightarrow{\kappa^1 \circ .S'^{1,0}} \mathrm{ext}_D^1(N, D) \longrightarrow 0. \quad (24)$$

Now, let $V^{1,-2} = (X \quad -Y)$, where $X \in D^{v^{1,-2}\times u^{1,0}}$ and $Y \in D^{v^{1,-2}\times s'^{1,-1}}$. Then, we have $X\, S''^{1,0} = Y\, S'^{1,-1}$ and using $R^0 = S''^{1,0}\, S'^{1,0}$, we then get

$$X\, R^0 = X\, S''^{1,0}\, S'^{1,0} = Y\, S'^{1,-1}\, S'^{1,0} = 0,$$

i.e., $\mathrm{im}_D(.X) \subseteq \ker_D(.R^0) = \mathrm{im}_D(.S^{0,0}) = \mathrm{im}_D(.U^{1,-1})$, and thus there exists $L^{1,-1} \in D^{v^{1,-2}\times u^{1,-1}}$ such that:

$$X = L^{1,-1}\, U^{1,-1}. \quad (25)$$

Using $V^{1,-3}\, V^{1,-2} = (V^{1,-3}\, X \quad -V^{1,-3}\, Y) = (0 \quad 0)$, we get $V^{1,-3}\, L^{1,-1}\, U^{1,-1} = V^{1,-3}\, X = 0$, i.e., $\mathrm{im}_D(.(V^{1,-3}\, L^{1,-1})) \subseteq \ker_D(.U^{1,-1}) = \mathrm{im}_D(.U^{1,-2})$, and thus there exists a matrix $L^{1,-2} \in D^{v^{1,-3}\times u^{1,-2}}$ such that

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 \longrightarrow & & \ker_D(.R^0) & \longrightarrow & D^{1\times r^0} & \xrightarrow{.R^0} & \mathrm{im}_D(.R^0) & \longrightarrow 0 & \\
 & & \uparrow .S^{0,0} & & \uparrow .T^{0,0} & & \uparrow .R^0 & & \\
0 \longrightarrow & & D^{1\times s^{0,0}} & \xrightarrow{.X^{0,0}} & D^{1\times t^{0,0}} & \xrightarrow{.Y^{0,0}} & D^{1\times u^{1,0}} & \longrightarrow 0 & \\
 & & \uparrow .S^{0,-1} & & \uparrow .T^{0,-1} & & \uparrow .U^{1,-1} & & \\
0 \longrightarrow & & D^{1\times s^{0,-1}} & \xrightarrow{.X^{0,-1}} & D^{1\times t^{0,-1}} & \xrightarrow{.Y^{0,-1}} & D^{1\times u^{1,-1}} & \longrightarrow 0 & \\
 & & \uparrow .S^{0,-2} & & \uparrow .T^{0,-2} & & \uparrow .U^{1,-2} & & \\
0 \longrightarrow & & D^{1\times s^{0,-2}} & \xrightarrow{.X^{0,-2}} & D^{1\times t^{0,-2}} & \xrightarrow{.Y^{0,-2}} & D^{1\times u^{1,-2}} & \longrightarrow 0 & \\
 & & \uparrow .S^{0,-3} & & \uparrow .T^{0,-3} & & \uparrow .U^{1,-3} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\qquad (21)
$$

$V^{1,-3}\,L^{1,-1} = L^{1,-2}\,U^{1,-2}$. Similarly, we can prove that there exist matrices $L^{1,-i} \in D^{v^{1,-i-1}\times u^{1,-i}}$ such that:

$$\forall\, i \geq 2, \quad V^{1,-i-1}\,L^{1,1-i} = L^{1,-i}\,U^{1,-i}. \qquad (26)$$

Lemma 4: The diagram (27) is commutative and exact, where $s^{1,-i} = u^{1,-i} + v^{1,-i}$, $i \geq 0$,

$$S^{1,0} = \begin{pmatrix} R^0 \\ S'^{1,0} \end{pmatrix}, \quad S^{1,-1} = \begin{pmatrix} U^{1,-1} & 0 \\ -I_{u^{1,0}} & S''^{1,0} \\ 0 & S'^{1,-1} \end{pmatrix},$$

$$\forall\, i \geq 2, \quad S^{1,-i} = \begin{pmatrix} U^{1,-i} & 0 \\ (-1)^i\,L^{1,1-i} & V^{1,-i} \end{pmatrix}.$$

$$Z^{1,-i} = (I_{u^{1,-i}} \quad 0), \quad W^{1,-i} = \begin{pmatrix} 0 \\ I_{v^{1,-i}} \end{pmatrix}.$$

Proof: Combining the free resolutions (19) and (24), we obtain the diagram (27). By construction, the first and third vertical sequences of (27) are exact and we can check that all the horizontal sequences are exact. Starting from the second one, they split. Moreover, every square commutes. The only thing left to prove is that the second vertical sequence is exact. We can first check that $S^{1,-i-1}\,S^{1,-i} = 0$ for $i \geq 0$, which shows that the second vertical sequence is a complex, i.e., $\mathrm{im}_D(.S^{1,-i-1}) \subseteq \ker_D(.S^{1,-i})$. Now, let $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.S^{1,0})$, i.e., $\lambda_1\,R^0 + \lambda_2\,S'^{1,0} = 0$, and using $R^0 = S''^{1,0}\,S'^{1,0}$, we get $(\lambda_1\,S''^{1,0} + \lambda_2)\,S'^{1,0} = 0$, i.e., $\lambda_1\,S'^{1,0} + \lambda_2 \in \ker_D(.S'^{1,0}) = \mathrm{im}_D(.S'^{1,-1})$, i.e., $\mu \in D^{1\times s'^{1,-1}}$ exists such that $\lambda_1\,S''^{1,0} + \lambda_2 = \mu\,S'^{1,-1}$, which shows that

$$(\lambda_1 \quad \lambda_2) = (\lambda_1 \quad \mu\,S'^{1,-1} - \lambda_1\,S''^{1,0}) = (0 \quad -\lambda_1 \quad \mu_1)\,S^{1,-1},$$

i.e., $\ker_D(.S^{1,0}) \subseteq \mathrm{im}_D(.S^{1,-1})$ and proves that $\ker_D(.S^{1,0}) = \mathrm{im}_D(.S^{1,-1})$, i.e., the exactness of the second vertical sequence of (27) at $D^{1\times s^{1,0}}$. Now, let us consider $\lambda = (\lambda_1 \quad \lambda_2 \quad \lambda_3) \in \ker_D(.S^{1,-1})$, i.e.:

$$\begin{cases} \lambda_1\,U^{1,-1} - \lambda_2 = 0, \\ (\lambda_2 \quad \lambda_3) \in \ker_D(.V^{1,-1}) = \mathrm{im}_D(.V^{1,-2}). \end{cases} \qquad (28)$$

Using $V^{1,-2} = (X \quad -Y)$, there exists $\mu \in D^{1\times v^{1,-2}}$ such that $(\lambda_2 \quad \lambda_3) = \mu\,(X \quad -Y)$. Using (25) and (28), we get $\lambda_2 = \lambda_1\,U^{1,-1} = \mu\,X = \mu\,L^{1,-1}\,U^{1,-1}$, i.e., $\lambda_1 - \mu\,L^{1,-1} \in \ker_D(.U^{1,-1}) = \mathrm{im}_D(.U^{1,-2})$, there exists $\nu \in D^{1\times u^{1,-2}}$ such that $\lambda_1 - \mu\,L^{1,-1} = \nu\,U^{1,-2}$, and thus:

$$\lambda = (\nu\,U^{1,-2} + \mu\,L^{1,-1} \quad \mu\,V^{1,-2}) = (\nu \quad \mu)\,S^{1,-2}.$$

Hence, we obtain $\ker_D(.S^{1,-1}) \subseteq \mathrm{im}_D(.S^{1,-2})$, i.e., $\ker_D(.S^{1,-1}) = \mathrm{im}_D(.S^{1,-2})$, which shows that the second vertical sequence of (27) is exact at $D^{1\times s^{1,-1}}$. Finally, let us consider $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.S^{1,-i})$, i.e.:

$$\lambda_1\,U^{1,-i} + \lambda_2\,(-1)^i\,L^{1,1-i} = 0, \quad \lambda_2\,V^{1,-i} = 0.$$

Now, using $\ker_D(.V^{1,-i}) = \mathrm{im}_D(.V^{1,-i-1})$, there exists $\mu \in D^{1\times v^{1,-i-1}}$ such that $\lambda_2 = \mu\,V^{1,-i-1}$, which yields $\lambda_1\,U^{1,-i} + \mu\,(-1)^i\,V^{1,-i-1}\,L^{1,1-i} = 0$, and thus, using (26), $(\lambda_1 + \mu\,(-1)^i\,L^{1,-i})\,U^{1,-i} = 0$, i.e., $\lambda_1 + \mu\,(-1)^i\,L^{1,-i} \in \ker_D(.U^{1,-i}) = \mathrm{im}_D(.U^{1,-i-1})$. Then, $\nu \in D^{1\times u^{1,-i-1}}$ exists such that $\lambda_1 + \mu\,(-1)^i\,L^{1,-i} = \nu\,U^{1,-i-1}$, and thus, $\lambda = (\nu\,U^{1,-i-1} + \mu\,(-1)^{i+1}\,L^{1,-i} \quad \mu\,V^{1,-i-1}) = (\nu \quad \mu)\,S^{1,-i-1}$, which shows that $\ker_D(.S^{1,-i}) \subseteq \mathrm{im}_D(.S^{1,-i-1})$, i.e., $\ker_D(.S^{1,-i}) = \mathrm{im}_D(.S^{1,-i-1})$. Hence, the second vertical sequence of (27) is exact at $D^{1\times s^{1,-i}}$. ■

We can repeat what has been done for (21) and (27) with the short exact sequences (15) and (16) for $i \geq 1$. Using the short exact sequences (15) and (16), we get the complex

$$\begin{array}{ccccccc}
\ldots \rightarrowtail & \ker_D(.R^i) & \rightarrowtail & D^{1\times r^i} & \twoheadrightarrow & \mathrm{im}_D(.R^i) & \\
\rightarrowtail & \ker_D(.R^{i+1}) & \rightarrowtail & D^{1\times r^{i+1}} & \twoheadrightarrow & \mathrm{im}_D(.R^{i+1}) & \\
 & & & & & & \rightarrowtail \ldots
\end{array}$$

where $\rightarrowtail$ denotes a monomorphism and $\twoheadrightarrow$ an epimorphism. Combining the obtained complexes, we get the complex (29) with exact vertical sequences and $i, j \geq 0$:

$$\Delta^{i,-j} = Y^{i,-j}\,Z^{i+1,-j}\,X^{i+1,-j} = \begin{pmatrix} 0 & 0 & 0 \\ I_{u^{i+1,-j}} & 0 & 0 \end{pmatrix}.$$

The complex we finally obtain is called a *Cartan-Eilenberg resolution* of $\mathrm{Rhom}_D(N, D)$ of the form (29) [5].

III. GROTHENDIECK SPECTRAL SEQUENCE

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the truncated Cartan-Eilenberg resolution of $\mathrm{Rhom}_D(N, D)$ obtained by removing the first horizontal complex of (29), we obtain the double complex (30), denoted by $\mathcal{F}^{\bullet,\bullet}$, where for $i, j \geq 0$:

$$T_{-i,j} = T^{i,-j-1}, \quad t_{-i,j} = t^{i,-j}, \quad \Delta_{-i,j} = \Delta^{i-1,-j}.$$

Let $\mathrm{tot}(\mathcal{F}^{\bullet,\bullet})$ be the sequence formed by

$$\mathcal{G}^n := \bigoplus_{-p+q=n} \mathcal{F}^{t_{-p,q}} = \bigoplus_{-p\leq 0} \mathcal{F}^{t_{-p,n+p}} = \bigoplus_{q\geq 0} \mathcal{F}^{t_{n-q,q}},$$

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathrm{im}_D(.R^0) \arrow[r] \arrow[u] & \ker_D(.R^1) \arrow[r, "\kappa_1"] \arrow[u] & \mathrm{ext}^1_D(N,D) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & D^{1\times u^{1,0}} \arrow[r, ".Z^{1,0}"] \arrow[u, ".R^0"'] & D^{1\times s^{1,0}} \arrow[r, ".W^{1,0}"] \arrow[u, ".S^{1,0}"'] & D^{1\times v^{1,0}} \arrow[r] \arrow[u, "\kappa^1 \circ .S'^{1,0}"'] & 0 \\
0 \arrow[r] & D^{1\times u^{1,-1}} \arrow[r, ".Z^{1,-1}"] \arrow[u, ".U^{1,-1}"'] & D^{1\times s^{1,-1}} \arrow[r, ".W^{1,-1}"] \arrow[u, ".S^{1,-1}"'] & D^{1\times v^{1,-1}} \arrow[r] \arrow[u, ".V^{1,-1}"'] & 0 \\
0 \arrow[r] & D^{1\times u^{1,-2}} \arrow[r, ".Z^{1,-2}"] \arrow[u, ".U^{1,-2}"'] & D^{1\times s^{1,-2}} \arrow[r, ".W^{1,-2}"] \arrow[u, ".S^{1,-2}"'] & D^{1\times v^{1,-2}} \arrow[r] \arrow[u, ".V^{1,-2}"'] & 0 \\
 & \vdots \arrow[u, ".U^{1,-3}"'] & \vdots \arrow[u, ".S^{1,-3}"'] & \vdots \arrow[u, ".V^{1,-3}"'] &
\end{tikzcd} \tag{27}$$

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & D^{1\times r^0} \arrow[r, ".R^0"] \arrow[u] & D^{1\times r^1} \arrow[r, ".R^1"] \arrow[u] & D^{1\times r^2} \arrow[r, ".R^3"] \arrow[u] & \ldots \\
0 \arrow[r] & D^{1\times t^{0,0}} \arrow[r, ".\Delta^{0,0}"] \arrow[u, ".T^{0,0}"'] & D^{1\times t^{1,0}} \arrow[r, ".\Delta^{1,0}"] \arrow[u, ".T^{1,0}"'] & D^{1\times t^{2,0}} \arrow[r, ".\Delta^{2,0}"] \arrow[u, ".T^{2,0}"'] & \ldots \\
0 \arrow[r] & D^{1\times t^{0,-1}} \arrow[r, ".\Delta^{0,-1}"] \arrow[u, ".T^{0,-1}"'] & D^{1\times t^{1,-1}} \arrow[r, ".\Delta^{1,-1}"] \arrow[u, ".T^{1,-1}"'] & D^{1\times t^{2,-1}} \arrow[r, ".\Delta^{2,-1}"] \arrow[u, ".T^{2,-1}"'] & \ldots \\
0 \arrow[r] & D^{1\times t^{0,-2}} \arrow[r, ".\Delta^{0,-2}"] \arrow[u, ".T^{0,-2}"'] & D^{1\times t^{1,-2}} \arrow[r, ".\Delta^{1,-2}"] \arrow[u, ".T^{1,-2}"'] & D^{1\times t^{2,-2}} \arrow[r, ".\Delta^{2,-2}"] \arrow[u, ".T^{2,-2}"'] & \ldots \\
 & \vdots \arrow[u, ".T^{0,-3}"'] & \vdots \arrow[u, ".T^{1,-3}"'] & \vdots \arrow[u, ".T^{2,-3}"'] &
\end{tikzcd} \tag{29}$$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\ldots \arrow[r, "\Delta_{-4,3}\cdot"] & \mathcal{F}^{t_{-3,3}} \arrow[r, "\Delta_{-3,3}\cdot"] \arrow[u] & \mathcal{F}^{t_{-2,3}} \arrow[r, "\Delta_{-2,3}\cdot"] \arrow[u] & \mathcal{F}^{t_{-1,3}} \arrow[r, "\Delta_{-1,3}\cdot"] \arrow[u] & \mathcal{F}^{t_{0,3}} \arrow[r] \arrow[u] & 0 \\
\ldots \arrow[r, "\Delta_{-4,2}\cdot"] & \mathcal{F}^{t_{-3,2}} \arrow[r, "\Delta_{-3,2}\cdot"] \arrow[u, "T_{-3,2}\cdot"'] & \mathcal{F}^{t_{-2,2}} \arrow[r, "\Delta_{-2,2}\cdot"] \arrow[u, "T_{-2,2}\cdot"'] & \mathcal{F}^{t_{-1,2}} \arrow[r, "\Delta_{-1,2}\cdot"] \arrow[u, "T_{-1,2}\cdot"'] & \mathcal{F}^{t_{0,2}} \arrow[r] \arrow[u, "T_{0,2}\cdot"'] & 0 \\
\ldots \arrow[r, "\Delta_{-4,1}\cdot"] & \mathcal{F}^{t_{-3,1}} \arrow[r, "\Delta_{-3,1}\cdot"] \arrow[u, "T_{-3,1}\cdot"'] & \mathcal{F}^{t_{-2,1}} \arrow[r, "\Delta_{-2,1}\cdot"] \arrow[u, "T_{-2,1}\cdot"'] & \mathcal{F}^{t_{-1,1}} \arrow[r, "\Delta_{-1,1}\cdot"] \arrow[u, "T_{-1,1}\cdot"'] & \mathcal{F}^{t_{0,1}} \arrow[r] \arrow[u, "T_{0,1}\cdot"'] & 0 \\
\ldots \arrow[r, "\Delta_{-4,0}\cdot"] & \mathcal{F}^{t_{-3,0}} \arrow[r, "\Delta_{-3,0}\cdot"] \arrow[u, "T_{-3,0}\cdot"'] & \mathcal{F}^{t_{-2,0}} \arrow[r, "\Delta_{-2,0}\cdot"] \arrow[u, "T_{-2,0}\cdot"'] & \mathcal{F}^{t_{-1,0}} \arrow[r, "\Delta_{-1,0}\cdot"] \arrow[u, "T_{-1,0}\cdot"'] & \mathcal{F}^{t_{0,0}} \arrow[r] \arrow[u, "T_{0,0}\cdot"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{30}$$

for $n \in \mathbb{Z}$, and the $\mathbb{Z}$-homomorphisms δ^n defined by:

$$\mathcal{G}^n = \bigoplus\nolimits_{-p+q=n} \mathcal{F}^{t_{-p,q}} \xrightarrow{\delta^n} \mathcal{G}^{n+1} = \bigoplus\nolimits_{-p+q=n+1} \mathcal{F}^{t_{-p,q}}$$
$$\delta^n_{|\mathcal{F}^{t_{-p,q}}}(\eta_{-p,q}) = (\ldots, T_{-p,q}\,\eta_{-p,q}, (-1)^n\,\Delta_{-p,q}\,\eta_{-p,q}, \ldots),$$
$$T_{-p,q}\,\eta_{-p,q} \in \mathcal{F}^{t_{-p,q+1}}, \quad (-1)^n\,\Delta_{-p,q}\,\eta_{-p,q} \in \mathcal{F}^{t_{-p+1,q}}.$$

Using $\Delta_{-p,q+1}\,T_{-p,q} = T_{-p+1,q}\,\Delta_{-p,q}$, $T_{-p+1,q}\,T_{-p,q} = 0$ and $\Delta_{-p+1,q}\,\Delta_{-p,q} = 0$, we can easily check that $\mathrm{tot}(\mathcal{F}^{\bullet,\bullet})$ is a complex, i.e., $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{Z}$:

$$(\delta^{n+1} \circ \delta^n)_{|\mathcal{F}^{t_{-p,q}}}(\eta_{-p,q}) = (\ldots, T_{-p,q+1}\,T_{-p,q}\,\eta_{-p,q},$$
$$(-1)^{n+1}\,\Delta_{-p,q+1}\,T_{-p,q}\,\eta_{-p,q} + (-1)^n\,T_{-p+1,q}\Delta_{-p,q}\,\eta_{-p,q},$$
$$\Delta_{-p+1,q}\,\Delta_{-p,q}\,\eta_{-p,q}, \ldots) = 0.$$

The *spectral sequence* is a technique which aims at computing $H^n(\mathrm{tot}(\mathcal{F}^{\bullet,\bullet}))$ at $\mathcal{G}^n$. See, e.g., [1], [5].

Theorem 3: [1], [5] With the above notations, we have

$${}_2E_2^{p,-q} \cong \mathrm{ext}^p_D(\mathrm{ext}^q_D(N,D), \mathcal{F}) \underset{p}{\Rightarrow} \mathrm{tor}^D_{-p+q}(N, \mathcal{F}),$$

which means that there exists an ascending filtration of $\mathrm{tor}^D_p(N, \mathcal{F})$ whose graded part is isomorphic to a subfactor of $\mathrm{ext}^p_D(\mathrm{ext}^q_D(N,D), \mathcal{F})$.

Proof: Since the vertical exact sequences of (29) end with the free left D-modules $D^{1\times r^i}$, they split (see, e.g., [5]), which shows that the cohomology

$$H^q(\mathcal{F}^{-p,\bullet}) \cong \ker_{\mathcal{F}}(T_{-p,q}.)/\mathrm{im}_{\mathcal{F}}(T_{-p,q-1}.)$$

of the p^{th} vertical complex $\mathcal{F}^{-p,\bullet}$ of $\mathcal{F}^{\bullet,\bullet}$ at $\mathcal{F}^{t_{-p,q}}$ is

$$\forall\, p \geq 0, \quad \begin{cases} H^0(\mathcal{F}^{-p,\bullet}) \cong \ker_{\mathcal{F}}(T_{-p,0}.) \cong \mathcal{F}^{r_{-p}}, \\ H^q(\mathcal{F}^{-p,\bullet}) = 0, \quad q \geq 1, \end{cases}$$

where $r_{-p} = r^p$. Moreover, we can easily check that we have the following commutative diagram with exact columns:

$$\begin{tikzcd}
 & 0 & 0 & \\
\ldots \arrow[r, "\Delta_{-2,0}\cdot"] & H^0\left(\mathcal{F}^{-1,\bullet}\right) \arrow[r, "\Delta_{-1,0}\cdot"] \arrow[u] & H^0\left(\mathcal{F}^{0,\bullet}\right) \arrow[r] \arrow[u] & 0 \\
\ldots \arrow[r, "R_{-2}\cdot"] & \mathcal{F}^{r_{-1}} \arrow[r, "R_{-1}\cdot"] \arrow[u, "T^{1,0}."'] & \mathcal{F}^{r_0} \arrow[r] \arrow[u, "T^{00}."'] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Applying the covariant right exact functor $\cdot \otimes_D \mathcal{F}$ to the complex (13), we obtain the following complex

$$\ldots \xrightarrow{R_{-3}.} \mathcal{F}^{r_{-2}} \xrightarrow{R_{-2}.} \mathcal{F}^{r_{-1}} \xrightarrow{R_{-1}.} \mathcal{F}^{r_0} \longrightarrow 0,$$

whose homologies are defined by:

$$\left\{ \begin{array}{l} \mathrm{tor}_0^D(N, \mathcal{F}) = N \otimes_D \mathcal{F} \cong \mathrm{coker}_{\mathcal{F}}(R_{-1}.), \\ \mathrm{tor}_p^D(N, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{-p}.)/\mathrm{im}_{\mathcal{F}}(R_{-p-1}.), \quad p \geq 1. \end{array} \right. \tag{31}$$

The homologies $H^{-p}(H^0(\mathcal{F}^{\bullet,\bullet}))$ of the following complex

$$H^0(\mathcal{F}^{\bullet,\bullet}) : \ \ldots \xrightarrow{\Delta_{-2,0}.} H^0\left(\mathcal{F}^{-1,\bullet}\right) \xrightarrow{\Delta_{-1,0}.} H^0\left(\mathcal{F}^{0,\bullet}\right) \longrightarrow 0$$

are then defined by (31). According to [5], it shows that the spectral sequence associated with the *first filtration* ${}_1F_i(\mathcal{G}^n) = \bigoplus_{-p \leq i} \mathcal{F}^{t_{-p,n+p}}$ of $\mathrm{tot}(\mathcal{F}^{\bullet,\bullet})$ is defined by:

$$\left\{ \begin{array}{l} {}_1E_0^{-p,q} = \mathcal{F}^{t_{-p,q}}, \\ {}_1E_1^{-p,q} = H^q(\mathcal{F}^{-p,\bullet}) = \left\{ \begin{array}{ll} \mathcal{F}^{r_{-p}}, & q = 0, \\ 0, & q \geq 1, \end{array} \right. , \\ {}_1E_2^{-p,q} = H^{-p}(H^q(\mathcal{F}^{\bullet,\bullet})) = \left\{ \begin{array}{ll} \mathrm{tor}_p^D(N, \mathcal{F}), & q = 0, \\ 0, & q \geq 1. \end{array} \right. \end{array} \right.$$

Using ${}_1E_2^{-p,q} = 0$, $q \geq 1$, the spectral sequence *collapses* on the p-axis and $H^{-p}(\mathrm{tot}(\mathcal{F}^{\bullet,\bullet})) \cong {}_1E_2^{-p,0} = \mathrm{tor}_p^D(N, \mathcal{F})$ [5]. For more details, see [5].

Let us now characterize the cohomology

$$H^{-q}(\mathcal{F}^{\bullet,p}) \cong \ker_{\mathcal{F}}(\Delta_{-q,p}.)/\mathrm{im}_{\mathcal{F}}(\Delta_{-q-1,p}.)$$

of the p^{th} horizontal complex $\mathcal{F}^{\bullet,p}$ of $\mathcal{F}^{\bullet,\bullet}$ at $\mathcal{F}^{-q,p}$.

Since the following exact sequence splits

$$0 \longrightarrow D^{1 \times s^{q,-p}} \xrightarrow{.X^{q,-p}} D^{1 \times t^{q,-p}} \xrightarrow{.Y^{q,-p}} D^{1 \times u^{q+1,-p}} \longrightarrow 0,$$

we obtain the following split exact sequence:

$$0 \longleftarrow \mathcal{F}^{s^{q,-p}} \xleftarrow{X^{q,-p}.} \mathcal{F}^{t^{q,-p}} \xleftarrow{Y^{q,-p}.} \mathcal{F}^{u^{q+1,-p}} \longleftarrow 0.$$

Similarly, the following split exact sequence

$$0 \longrightarrow D^{1 \times u^{q,-p}} \xrightarrow{.Z^{q,-p}} D^{1 \times s^{q,-p}} \xrightarrow{.W^{q,-p}} D^{1 \times v^{q,-p}} \longrightarrow 0$$

yields the following split exact sequence:

$$0 \longleftarrow \mathcal{F}^{u^{q,-p}} \xleftarrow{Z^{q,-p}.} \mathcal{F}^{s^{q,-p}} \xleftarrow{W^{q,-p}.} \mathcal{F}^{v^{q,-p}} \longleftarrow 0.$$

Hence, we get $\ker_{\mathcal{F}}(Y^{q,-p}.) = 0$, $\mathrm{im}_{\mathcal{F}}(X^{q,-p}.) = \mathcal{F}^{s^{q,-p}}$, $\mathrm{im}_{\mathcal{F}}(Z^{q,-p}.) = \mathcal{F}^{u^{q,-p}}$. Now, since by definition, we have

$$\left\{ \begin{array}{l} \Delta_{-q,p} = \Delta^{q-1,-p} = Y^{q-1,-p}\, Z^{q,-p}\, X^{q,-p}, \\ \Delta_{-q-1,p} = \Delta^{q,-p} = Y^{q,-p}\, Z^{q+1,-p}\, X^{q+1,-p}, \end{array} \right.$$

we get $\mathrm{im}_{\mathcal{F}}(\Delta_{-q-1,p}.) = \mathrm{im}_{\mathcal{F}}(Y^{q,-p}.) = \ker_{\mathcal{F}}(X^{q,-p}.)$. We have $\ker_{\mathcal{F}}(\Delta_{-q,p}.) = \ker_{\mathcal{F}}((Z^{q,-p}\, X^{q,-p}).)$, and thus:

$$H^{-q}(\mathcal{F}^{\bullet,p}) \cong \ker_{\mathcal{F}}((Z^{q,-p}\, X^{q,-p}).)/\ker_{\mathcal{F}}(X^{q,-p}.). \tag{32}$$

If $f \in \hom_D(M', M)$, $g \in \hom_D(M, M'')$ and f is surjective, then we can show that $\ker g \cong \ker(g \circ f)/\ker f$. Since $Z^{q,-p}.$ is surjective, we then obtain:

$$H^{-q}(\mathcal{F}^{\bullet,p}) \cong \ker_{\mathcal{F}}(Z^{q,-p}.) = \mathrm{im}_{\mathcal{F}}(W^{q,-p}.) \cong \mathcal{F}^{v^{q,-p}}.$$

Hence, we get the following commutative diagram

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ & \uparrow V_{-q,p+1}. & & \uparrow S_{-q,p+1}. & \\ 0 \longrightarrow & \mathcal{F}^{v_{-q,p+1}} & \xrightarrow{W_{-q,p+1}.} & H^{-q}(\mathcal{F}^{\bullet,p+1}) & \longrightarrow 0 \\ & \uparrow V_{-q,p}. & & \uparrow S_{-q,p}. & \\ 0 \longrightarrow & \mathcal{F}^{v_{-q,p}} & \xrightarrow{W_{-q,p}.} & H^{-q}(\mathcal{F}^{\bullet,p}) & \longrightarrow 0 \\ & \uparrow V_{-q,p-1}. & & \uparrow S_{-q,p-1}. & \\ 0 \longrightarrow & \mathcal{F}^{v_{-q,p-1}} & \xrightarrow{W_{-q,p-1}.} & H^{-q}(\mathcal{F}^{\bullet,p-1}) & \longrightarrow 0, \\ & \uparrow V_{-q,p-2}. & & \uparrow S_{-q,p-2}. & \\ & \vdots & & \vdots & \end{array} \tag{33}$$

with exact horizontal sequences, where $W_{-q,p} = W^{q,-p}$, $V_{-q,p} = V^{q,-p-1}$, $S_{-q,p} = S^{q,-p-1}$, $v_{-q,p} = v^{q,-p}$, $p \geq 0$.

Now, applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the free resolution of $\mathrm{ext}_D^q(N, D)$ (see (24))

$$\ldots \xrightarrow{.V^{q,-1}} D^{1 \times v^{q,0}} \xrightarrow{\kappa^q \circ S'^{q,0}} \mathrm{ext}_D^q(N, D) \longrightarrow 0,$$

we obtain the following complex

$$\ldots \xleftarrow{V_{-q,2}.} \mathcal{F}^{v_{-q,2}} \xleftarrow{V_{-q,1}.} \mathcal{F}^{v_{-q,1}} \xleftarrow{V_{-q,0}.} \mathcal{F}^{v_{-q,0}} \longleftarrow 0,$$

whose cohomologies are defined by:

$$\left\{ \begin{array}{l} \mathrm{ext}_D^0(\mathrm{ext}_D^q(N, D), \mathcal{F}) \cong \ker_{\mathcal{F}}(V_{-q,0}.), \\ \mathrm{ext}_D^p(\mathrm{ext}_D^q(N, D), \mathcal{F}) \cong \ker_{\mathcal{F}}(V_{-q,p}.)/\mathrm{im}_{\mathcal{F}}(V_{-q,p-1}.). \end{array} \right.$$

Hence, the beginning of the spectral sequence associated with the second filtration of the complex $\mathrm{tot}(\mathcal{F}^{\bullet,\bullet})$ is defined by:

$$\left\{ \begin{array}{l} {}_2E_0^{p,-q} = \mathcal{F}^{t_{-q,p}}, \\ {}_2E_1^{p,-q} = H^{-q}(\mathcal{F}^{\bullet,p}) \cong \mathcal{F}^{v^{q,-p}}, \\ {}_2E_2^{p,-q} = H^p(H^{-q}(\mathcal{F}^{\bullet,\bullet})) \cong \mathrm{ext}_D^p(\mathrm{ext}_D^q(N, D), \mathcal{F}). \end{array} \right.$$

A standard result for bounded double complexes then shows that the two spectral sequences ${}_1E_r^{-p,q}$ and ${}_2E_s^{p,-q}$ abut at $H^{-p}(\mathrm{tot}(\mathcal{F}^{\bullet,\bullet}))$ [5], which finally proves the result. ■

Theorem 3 shows that a method exists which converges to the computation of the $\mathrm{tor}_i^D(N, \mathcal{F})$'s, i.e., of the defects of parametrizability of (8), by means of the $\mathrm{ext}_D^i(\mathrm{ext}_D^j(N, D), \mathcal{F})$'s, i.e., of the functional obstructions formed by the algebraic obstruction for M to be projective. Making this construction explicit is a fundamental issue in the behavioural approach. See [1] for the concept of *generalized morphisms* which can be used for this issue.

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