Constructive computation of flat outputs of a class of multidimensional linear systems with variable coefficients

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Abstract— **The purpose of this paper is to give a constructive algorithm for the computation of bases of finitely presented free modules over the Weyl algebras of differential operators with polynomial or rational coefficients. In particular, we show how to use these results in order to recognize when a multidimensional linear system defined by partial differential equations with polynomial or rational coefficients is flat and, if so, to compute flat outputs and the injective image representations of the system. These new results are based on recent constructive proofs of a famous result in non-commutative algebra due to J. T. Stafford [27]. The different algorithms have been implemented in the package STAFFORD [25] based on OREMODULES [2]. These results allow us to achieve the general solution of the so-called Monge problem for multidimensional linear systems defined by partial differential equations with polynomial or rational coefficients. Finally, we constructively answer an open question posed by Datta [5] on the possibility to generalize the results of [13] to multi-input multi-output polynomial time-varying controllable linear systems. We show that every controllable ordinary differential linear system with at least two inputs and polynomial coefficients is flat.**

Keywords— **Flat multidimensional linear systems, injective image representation, constructive computation of bases of free modules, Stafford's results, non-commutative algebra.**

I. A PEDESTRIAN INTRODUCTION TO THE MONGE PROBLEM

A. Introduction

Let us introduce the so-called *Monge problem* (1784). We refer the reader to [29] and the references therein for historical details. Let D be a *ring of differential operators* (e.g., the Weyl algebra $A_n(k) = k[x_1, \ldots, x_n][d_1, \ldots, d_n]$ of differential operators in $d_i = \partial/\partial x_i$ with polynomial coefficients in x_j) and $\mathcal{F}$ a *functional space* which satisfies:

$$\forall\, P_1,\ P_2 \in D, \quad \forall\, y_1,\ y_2 \in \mathcal{F}: \quad P_1\, y_1 + P_2\, y_2 \in \mathcal{F}. \tag{1}$$

For instance, if D is the Weyl algebra $A_n(k)$, we can take $\mathcal{F} = C^\infty(\mathbb{R}^n)$. In terms of module theory, property (1) means that $\mathcal{F}$ has a left D-module structure [26]. Let us consider $R \in D^{q\times p}$ and the linear system of PDEs (or *behaviour* [15], [16], [21], [28], [30]) defined by:

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

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The Monge problem questions the existence of a matrix of differential operators $Q \in D^{p\times m}$ such that we have:

$$\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.) \triangleq Q\,(\mathcal{F}^m).$$

If such a matrix Q exists, we then say that Q is a *parametrization* of the system $\ker_{\mathcal{F}}(R.)$. In the behavioural approach to multidimensional linear systems, we say that the behaviour $\ker_{\mathcal{F}}(R.)$ admits an *image representation* [15], [16], [21], [28], [30]. Let us give a few examples.

Example 1: 1) We consider the ring $D = \mathbb{R}(t)\left[\frac{d}{dt}\right]$ of differential operators in d/dt with rational coefficients in t, $\mathcal{F} = C^\infty(\mathbb{R})$ and the following matrix of differential operators

$$R = \left(\frac{d^2}{dt^2} + \alpha(t)\,\frac{d}{dt} + 1,\ -\frac{d}{dt} - \alpha(t)\right) \in D^{1\times 2},$$

where α denotes a time-varying parameter which belongs to $\mathbb{R}(t)$. Then, we get the following system:

$$\ker_{\mathcal{F}}(R.) = \{(y,\ u)^T \in \mathcal{F}^2 \mid \ddot{y}(t) + \alpha(t)\,\dot{y}(t) + y(t) - \dot{u}(t) - \alpha(t)\,u(t) = 0\}.$$

It was proved in [17] that we have the following parametrization of the system $\ker_{\mathcal{F}}(R.)$:

$$\begin{cases} y(t) = \dot{\xi}(t) + \alpha(t)\,\xi(t), \\ u(t) = \ddot{\xi}(t) + \alpha(t)\,\dot{\xi}(t) + (\dot{\alpha}(t) + 1)\,\xi(t). \end{cases}$$

This parametrization is *injective* as we then have:

$$\xi = -\dot{y} + u.$$

2) Let us consider the ring $D = \mathbb{R}[d_1, d_2, d_3]$ of differential operators with constant coefficients, the D-module $\mathcal{F} = C^\infty(\mathbb{R}^3)$ and the system defined by the divergence operator in $\mathbb{R}^3$, namely:

$$\ker_{\mathcal{F}}(\mathrm{div.}) = \{\vec{A} = (A_1,\ A_2,\ A_3)^T \in \mathcal{F}^3 \mid d_1\, A_1 + d_2\, A_2 + d_3\, A_3 = 0\}.$$

In mathematical physics, it is well-known that the divergence operator is parametrized by the curl operator, namely, the operator defined by the matrix

$$\mathrm{curl} = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \in D^{3\times 3},$$

i.e., we have $\ker_{\mathcal{F}}(\mathrm{div.}) = \mathrm{curl}\,(\mathcal{F}^3)$. Let us check whether or not this parametrization is injective, i.e., whether or not $\mathrm{curl}\,\vec{B} = \vec{0}$ implies $\vec{B} = \vec{0}$. It is also well-known in mathematical physics that the curl operator is parametrized by the gradient operator defined by $\mathrm{grad} = (d_1,\ d_2,\ d_3)^T$. In other words, we have the following equality:

$$\{\vec{B} = (B_1,\ B_2,\ B_3)^T \in \mathcal{F}^3 \mid \mathrm{curl}\,\vec{B} = \vec{0}\} \\ = \mathrm{grad}\,(\mathcal{F}).$$

Hence, the parametrization of the divergence operator by means of the curl operator is not injective because the curl operator is parametrized by the gradient operator.

B. Systems & Modules

Before giving necessary and sufficient conditions for parametrizability, we need to introduce some notations and results obtained by B. Malgrange [12]. Let us consider a matrix $R \in D^{q\times p}$ of differential operators and let us define the *finitely presented left D-module*

$$M = D^{1\times p}/(D^{1\times q}\,R), \tag{2}$$

where $D^{1\times p}$ (resp., D^p) denotes the left (resp., right) D-module of row (resp., column) vectors of length p with entries in D. By convention, we set $D^{1\times 0} = 0$. The introduction of the previous left D-module M is very natural as it generalizes well-known algebras which play central roles in algebraic geometry and number theory.

Example 2: 1) Cauchy's construction of the field $\mathbb{C}$ of complex numbers was $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$, i.e., $\mathbb{C}$ can be defined as the ring of real polynomials in x modulo the relation x^2+1. If we consider $D = \mathbb{R}[x]$ and $R = (x^2+1) \in D$, then we obtain that:

$$M = D/(D\,R) = \mathbb{R}[x]/(\mathbb{R}[x]\,(x^2+1)) = \mathbb{C}.$$

2) The rings of numbers such as

$$A = \mathbb{Z}[i\sqrt{5}]/(\mathbb{Z}[i\sqrt{5}]\,(1+i\sqrt{5}) + \mathbb{Z}[i\sqrt{5}]\,2)$$

appear everywhere in the literature of algebraic number theory. Hence, if we consider $D = \mathbb{Z}[i\sqrt{5}]$ and $R = (1+i\sqrt{5},\quad 2)^T \in D^2$, then we get:

$$M = D/(D^{1\times 2}\,R) = A.$$

3) In algebraic geometry, we associate with any affine algebraic variety defined by the complex solutions of a set of polynomials $P_1,\ldots,P_q \in \mathbb{R}[x_1,\ldots,x_n]$ the algebra $A = \mathbb{R}[x_1,\ldots,x_n]/I$, where I denotes the ideal of $\mathbb{R}[x_1,\ldots,x_n]$ generated by $P_1,\ldots,P_m$, i.e., $I = \sum_{i=1}^m D\,P_i$. Hence, if we consider the algebra $D = \mathbb{R}[x_1,\ldots,x_n]$ and $R = (P_1,\ldots,P_q)^T \in D^q$, we then obtain:

$$M = D/(D^{1\times q}\,R) = \mathbb{R}[x_1,\ldots,x_n]/I = A.$$

Hence, we see that the left D-module M defined by (2) extends some well-known algebraic objects encountered in the algebra literature to general linear systems. See [3], [17], [18], [19], [20], [21], [23], [24] for more details.

Let us introduce a few definitions of homological algebra [26] that will be useful in what follows.

Definition 1: 1) A sequence $(\delta_i : M_i \longrightarrow M_{i-1})_{i\in\mathbb{Z}}$ of morphisms $\delta_i : M_i \longrightarrow M_{i-1}$ between left D-modules is *a complex* if we have:

$$\forall\, i \in \mathbb{Z}, \quad \mathrm{im}\,\delta_i \subseteq \ker\delta_{i-1}.$$

We denote the previous complex by:

$$\ldots \xrightarrow{\delta_{i+2}} M_{i+1} \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \xrightarrow{\delta_{i-1}} \ldots \tag{3}$$

2) The *defect of exactness of the complex (3) at M_i* is:

$$H(M_i) = \ker\delta_i/\mathrm{im}\,\delta_{i+1}.$$

3) The complex (3) is *exact at M_i* if we have:

$$H(M_i) = 0 \quad\Longleftrightarrow\quad \ker\delta_i = \mathrm{im}\,\delta_{i+1}.$$

4) The complex (3) is *exact* if:

$$\forall\, i \in \mathbb{Z}, \quad \ker\delta_i = \mathrm{im}\,\delta_{i+1}.$$

5) The complex (3) is a *split exact sequence* if it is exact and there exist morphisms $s_i : M_{i-1} \longrightarrow M_i$ satisfying the following conditions:

$$\forall\, i \geq 0, \quad \begin{cases} s_{i+1}\circ s_i = 0, \\ s_i\circ\delta_i + \delta_{i+1}\circ s_{i+1} = id_{M_i}. \end{cases}$$

6) A *finite free resolution* of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{4}$$

where $p_i \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$, $R_i \in D^{p_i\times p_{i-1}}$,

$$\begin{array}{rcl} (.R_i) : D^{1\times p_i} & \longrightarrow & D^{1\times p_{i-1}} \\ \lambda & \longmapsto & (.R_i)(\lambda) = \lambda\,R_i, \end{array}$$

and $R_m = 0$ for a certain $m \geq 1$.

Example 3: The following sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact if f is injective, i.e., $\ker f = 0$, $\ker g = \mathrm{im}\,f$ and g is surjective, i.e., $\mathrm{coker}\,g \triangleq M''/\mathrm{im}\,g = 0$.

We have the following important result and definitions.

Theorem 1: [26] Let $\mathcal{F}$ be a left D-module, M a left D-module and (4) a finite free resolution of M. Then, the defects of exactness of the following complex

$$\ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0,$$

where $(R_i.) : \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by $(R_i.)\,\eta = R_i\,\eta$, for all $\eta \in \mathcal{F}^{p_{i-1}}$, only depend on the left D-modules M

and $\mathcal{F}$. Up to an isomorphism, we denote these defects of exactness by:

$$\begin{cases} \mathrm{ext}^0_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \mathrm{ext}^i_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/(R_i\,(\mathcal{F}^{p_i})), \quad i \geq 1. \end{cases}$$

Finally, we have $\mathrm{ext}^0_D(M,\mathcal{F}) = \hom_D(M,\mathcal{F})$, where $\hom_D(M,\mathcal{F})$ denotes the abelian group of D-morphisms (namely, D-linear maps) from M to $\mathcal{F}$.

Using the previous result, B. Malgrange made the remark that we then have ($R_1 = R$, $p_1 = p$ and $p_2 = q$)

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M,\mathcal{F}), \tag{5}$$

where $\cong$ denotes an isomorphism of abelian groups (k-vector spaces if $\mathcal{F}$ has the structure of a k-vector space) [12]. This idea was developed by the Japanese school of M. Sato (in particular, M. Sato, M. Kashiwara, T. Kawai) [8]. In particular, (5) gives an intrinsic formulation of the system $\ker_{\mathcal{F}}(R.)$, as the right hand side of (5) only depends on the left D-modules M and $\mathcal{F}$ and we can prove that M is intrinsically defined, the equality $M = D^{1\times p}/(D^{1\times q}\,R)$ being nothing else than a particular representation of the system (i.e., the beginning of a particular finite free resolution of the left D-module M). We refer the reader to [4], [20] for more details concerning equivalences of linear systems within module theory and homological algebra.

Before recalling the first main result concerning the Monge problem, let us introduce a few more definitions.

Definition 2:

1) [26] A left D-module $\mathcal{F}$ is called *injective* if, for every left D-module M, and, for all $i \geq 1$, we have $\mathrm{ext}^i_D(M,\mathcal{F}) = 0$.
2) [26] A left D-module $\mathcal{F}$ is called *cogenerator* if, for every left D-module M, we have:

$$\hom_D(M,\mathcal{F}) = 0 \quad \Longrightarrow \quad M = 0.$$

Theorem 2: [26] An injective cogenerator left D-module $\mathcal{F}$ exists for every ring D.

We give examples of modules which are injective cogenerators.

Example 4:

1) If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$) of smooth functions (resp., distributions) on Ω is an injective cogenerator module over the ring $\mathbb{R}[d_1,\ldots,d_n]$ of differential operators with coefficients in $\mathbb{R}$ [12].
2) [30] If $\mathcal{F}$ denotes the set of all functions that are smooth on $\mathbb{R}$ except for a finite number of points, then $\mathcal{F}$ is an injective cogenerator left $\mathbb{R}(t)\left[\frac{d}{dt}\right]$-module.

Let us recall the concept of *formal adjoint* of a matrix R of differential operators.

Definition 3: [3], [19] Let $\mathbb{Q} \subseteq k$ be a field and D one of the two following *Weyl algebras*:

$$\begin{array}{l} A_n(k) = k[x_1,\ldots,x_n][d_1,\ldots,d_n], \\ B_n(k) = k(x_1,\ldots,x_n)[d_1,\ldots,d_n]. \end{array} \tag{6}$$

1) An *involution* θ of D is a k-linear map $\theta : D \longrightarrow D$ satisfying the following two conditions:

$$\forall\, P,\, Q \in D, \quad \begin{cases} \theta \circ \theta = \mathrm{id}_D, \\ \theta(P\,Q) = \theta(Q)\,\theta(P). \end{cases}$$

2) Let θ be the *involution* of D defined by:

$$\forall\, a \in k, \quad \begin{cases} \theta(d_i) = -d_i, \\ \theta(x_i) = x_i, \\ \theta(a) = a. \end{cases}$$

If $R \in D^{q\times p}$ is a matrix of differential operators, then the *formal adjoint* of R is defined by:

$$\widetilde{R} = (\theta(R_{ij}))^T.$$

Example 5: Let us consider $D = A_3(\mathbb{Q})$ and the matrix $R = -(d_1 - x_3, \quad d_2, \quad d_3) \in D^{1\times 3}$ of differential operators. Then, the formal adjoint $\widetilde{R}$ of R is defined by:

$$\begin{array}{rcl} \widetilde{R} & = & -(\theta(d_1 - x_3), \quad \theta(d_2), \quad \theta(d_3))^T \\ & = & (d_1 + x_3, \quad d_2, \quad d_3)^T. \end{array}$$

We are now in position to state the first main result concerning the Monge problem.

Theorem 3: Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$ and $\mathcal{F}$ be an injective cogenerator left D-module. Then, the following statements are equivalent:

1) There exists $Q \in D^{p\times m}$ such that we have:

$$\ker_{\mathcal{F}}(R.) = Q\,(\mathcal{F}^m).$$

2) There exists $Q \in D^{p\times m}$ such that we have:

$$\ker_D(.Q) \triangleq \{\lambda \in D^{1\times p} \mid \lambda\,Q = 0\} = D^{1\times q}\,R.$$

3) The left D-module M is *torsion-free*, namely, the torsion submodule of M defined by

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D :\ P\,m = 0\}$$

is trivial, i.e., $t(M) = 0$.
4) $\mathrm{ext}^1_D(\widetilde{N}, D) = 0$, where $\widetilde{N}$ is the left D-module defined by the formal adjoint $\widetilde{R}$ of the matrix R:

$$\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R}).$$

We refer the reader to [3], [15], [19], [21] for the proofs. General algorithms for computing $\widetilde{R}$, $\mathrm{ext}^1_D(\widetilde{N}, D)$, $t(M)$ and Q as in the previous theorem are developed in [3], [17], [19], [21]. These algorithms have been implemented in the package OREMODULES [2] and they have been illustrated in the library of examples of OREMODULES containing more than 30 examples. In particular, the parametrizations given in Example 1 can be obtained by using the constructive algorithms developed in [3], [17], [19], [21].

We note that the concept of torsion-free module is only a particular one in a long list of possible properties of modules developed in homological algebra. Let us recall some of them.

Definition 4: [26] Let us consider $R \in D^{q\times p}$ and the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$.

1) M is said to be *free* if there exists a non-negative integer $r \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ such that:
$$M \cong D^{1\times r}.$$
2) M is said to be *stably free* if there exist $r,\, s \in \mathbb{Z}_+$ such that:
$$M \oplus D^{1\times s} \cong D^{1\times r}.$$
3) M is said to be *projective* if there exist $r \in \mathbb{Z}_+$ and a left D-module P such that:
$$M \oplus P \cong D^{1\times r}.$$
4) M is said to be *reflexive* if the morphism
$$\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D),$$
defined by
$$\forall\, m \in M,\ \forall\, f \in \hom_D(M, D),\ \varepsilon(m)(f) = f(m),$$
is an isomorphism of left D-modules.
5) M is *torsion-free* if we have:
$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

We have the following important results [14], [26].

Theorem 4: 1) We have the following implications among the module properties:
$$\text{free} \Longrightarrow \text{stably free} \Longrightarrow \text{projective} \Longrightarrow$$
$$\text{reflexive} \Longrightarrow \text{torsion-free}.$$
2) If D is a *left principal ideal domain*, namely, every left ideal of D can be generated by means of an element of D (e.g., $\mathbb{Q}(t)\left[\frac{d}{dt}\right]$), then every torsion-free left D-module is free.
3) If D is a *left hereditary ring*, namely, every left ideal of D is projective (e.g., $\mathbb{Q}[t]\left[\frac{d}{dt}\right]$), then every torsion-free left D-module is projective.
4) (Quillen-Suslin theorem) If $D = k[d_1, \ldots, d_n]$, where k is a field of constants, namely, $d_i\, a = 0$ for all $a \in k$ and $i = 1, \ldots, n$, then every projective D-module is free.

We can now state the following important theorem in the behavioural approach to multidimensional linear systems defined by PDEs with polynomial or rational coefficients. In particular, it explains the meaning of the concepts of free / stably free / projective / reflexive / torsion-free modules in systems theory and in the parametrizability problem.

Theorem 5: [3], [21] Let D be one of the Weyl algebras defined in (6) and let us consider a matrix $R \in D^{q\times p}$ of differential operators, an injective cogenerator left D-module $\mathcal{F}$, $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}$ and the following left D-modules
$$M = D^{1\times p}/(D^{1\times q}\, R), \quad \widetilde{N} = D^{1\times q}/(D^{1\times p}\, \widetilde{R}),$$
where $\widetilde{R}$ is the formal adjoint of R. We then have the equivalences presented in Fig. 1.

Constructive algorithms have been given in [3], [17], [19] for computing the extension modules $\mathrm{ext}^i_D(\widetilde{N}, D)$. Therefore, we can constructively check whether or not the left D-module M admits some torsion elements, or is torsion-free, reflexive, projective or stably free. Moreover, these algorithms allow us to compute the different matrices $Q_i \in D^{m_{i-1}\times m_i}$ ($m_0 \triangleq p$). We refer the reader to OREMODULES [2] for implementations of these algorithms and its library of examples illustrating Theorem 5. Finally, we note that it was proved in [20] that the left D-module $\widetilde{N}$ only depends on M up to a *projective equivalence* [26], which shows the intrinsicness of the statements given in Theorem 5.

The parametrizability/image representation problem has important applications in the study of controllability of multidimensional linear systems in terms of the possibility to patch the solutions of the systems [15], [16], [28] and in optimal control [22], Diophantine equations [17], motion planning and tracking [6]. See [28] for a nice survey on the behavioural approach to multidimensional linear systems.

Finally, we note that "?" in Fig. 1 means that no simple characterization of freeness is known in homological algebra. The purpose of this paper is to study such a characterization based on one of J. T. Stafford's results [27] and to obtain a constructive algorithm for computing bases of free left D-modules, where D is a Weyl algebra as in (6). We first complete Fig. 1 given in Theorem 5, achieving the previous characterizations and concluding the parametrizability problem (image representation problem). Moreover, we recall that a multidimensional linear system $\ker_{\mathcal{F}}(R.)$ is said to be *flat* if there exists an injective parametrization, and thus, by Theorem 5, if and only if the corresponding left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is free [3], [6], [17]. Hence, if we can effectively decide freeness, we can then test whether or not a multidimensional linear system defined by PDEs with polynomial or rational coefficients is flat. To finish, we also note that there is a one-to-one correspondence between the bases of the free left D-module M and the so-called *flat outputs* of $\ker_{\mathcal{F}}(R.)$. Therefore, the knowledge of a constructive algorithm which computes bases of a free left module M over a Weyl algebra D will give us a way to compute the corresponding flat outputs. We point out that

Module M	**Homological algebra**	**Parametrizations**
with torsion	$t(M) \cong \mathrm{ext}^1_D(\widetilde{N}, D)$	$\emptyset$
torsion-free	$\mathrm{ext}^1_D(\widetilde{N}, D) = 0$	$\exists\, Q_1 \in D^{p \times m_1} :\ \ker_{\mathcal{F}}(R.) = Q_1\,(\mathcal{F}^{m_1})$
reflexive	$\mathrm{ext}^i_D(\widetilde{N}, D) = 0,$ $i = 1, 2$	$\exists\, Q_1 \in D^{p \times m_1},\ Q_2 \in D^{m_1 \times m_2} :$ $\ker_{\mathcal{F}}(R.) = Q_1\,(\mathcal{F}^{m_1}),$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\,(\mathcal{F}^{m_2})$
projective = stably free	$\mathrm{ext}^i_D(\widetilde{N}, D) = 0,$ $1 \leq i \leq n$	$\exists\, Q_1 \in D^{p \times m_1},\ Q_i \in D^{m_{i-1} \times m_i},\ i = 2, \ldots, n :$ $\ker_{\mathcal{F}}(R.) = Q_1\,(\mathcal{F}^{m_1}),$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\,(\mathcal{F}^{m_2}),$ $\ldots$ $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n\,(\mathcal{F}^{m_n})$
free	?	$\exists\, Q_1 \in D^{p \times m},\ T_1 \in D^{m \times p} :$ $\ker_{\mathcal{F}}(R.) = Q_1\,(\mathcal{F}^{m}),$ $T_1\,Q_1 = I_m$

Fig. 1.

this problem was still open even for 1-D linear systems defined by ordinary differential equations with polynomial coefficients. See [13], [23] for more details.

C. *Stably free modules & Projective dimension*

The purpose of this section is to give a characterization of stably free modules which will be used in what follows. Let us start with the following result.

Proposition 1: Let us consider a finite free resolution of a left D-module M of the form:

$$0 \longrightarrow D^{1 \times p_m} \xrightarrow{.R_m} \ldots \xrightarrow{.R_1} D^{1 \times p_0} \xrightarrow{\pi} M \longrightarrow 0. \quad (7)$$

1) If $m \geq 3$ and there exists $S_m \in D^{p_{m-1} \times p_m}$ such that $R_m\,S_m = I_{p_m}$, then we have the finite free resolution of M

$$0 \longrightarrow D^{1 \times p_{m-1}} \xrightarrow{.T_{m-1}} D^{1 \times (p_{m-2}+p_m)} \xrightarrow{.T_{m-2}} D^{1 \times p_{m-3}} \xrightarrow{.R_{m-3}} \ldots \xrightarrow{\pi} M \longrightarrow 0, \quad (8)$$

with the following notations:

$$T_{m-1} = (R_{m-1}, \quad S_m), \quad T_{m-2} = \begin{pmatrix} R_{m-2} \\ 0 \end{pmatrix}.$$

2) If $m = 2$ and there exists $S_2 \in D^{p_1 \times p_2}$ such that $R_2\,S_2 = I_{p_2}$, then we have the finite free resolution

$$0 \longrightarrow D^{1 \times p_1} \xrightarrow{.T_1} D^{1 \times (p_0+p_2)} \xrightarrow{\tau} M \longrightarrow 0, \quad (9)$$

with the notations $T_1 = (R_1 \quad S_2)$ and:

$$\begin{array}{rcl} \tau = \pi \oplus 0 : D^{1 \times (p_0+p_2)} & \longrightarrow & M \\ \lambda = (\lambda_1 \quad \lambda_2) & \longmapsto & \tau(\lambda) = \pi(\lambda_1). \end{array}$$

Proof: 1. We suppose that $m \geq 3$. Let us prove that (8) is an exact sequence. Using the fact that (7) is an exact sequence, and thus, $R_{m-1}\,R_{m-2} = 0$, we obtain

$$\begin{array}{rcl} T_{m-1}\,T_{m-2} & = & (R_{m-1}, \quad S_m) \begin{pmatrix} R_{m-2} \\ 0 \end{pmatrix} \\ & = & R_{m-1}\,R_{m-2} = 0, \end{array}$$

which proves that $(D^{1 \times p_{m-1}}\,T_{m-1}) \subseteq \ker_D(.T_{m-2})$.

Let us now consider $(\lambda, \quad \mu) \in \ker_D(.T_{m-2})$. We then have $(\lambda, \quad \mu)\,T_{m-2} = \lambda\,R_{m-2} = 0$ and using the fact that (7) is an exact sequence, there exists $\nu \in D^{1 \times p_{m-1}}$ such that $\lambda = \nu\,R_{m-1}$. Let us define:

$$\zeta = \nu\,(I_{p_{m-1}} - S_m\,R_m) + \mu\,R_m \in D^{1 \times p_{m-1}}.$$

Using the relations $R_m R_{m-1} = 0$ and $R_m S_m = I_{p_m}$, we then get

$$\begin{array}{rcl} \zeta T_{m-1} & = & \zeta\,(R_{m-1}, \quad S_m) \\ & = & (\nu\,(I_{p_{m-1}} - S_m R_m)\,R_{m-1} + \mu\,R_m R_{m-1}, \\ & & \quad \nu\,(I_{p_{m-1}} - S_m R_m)\,S_m + \mu\,R_m S_m) \\ & = & (\nu\,R_{m-1}, \quad \mu) = (\lambda, \quad \mu), \end{array}$$

which proves that $\ker_D(.T_{m-2}) \subseteq (D^{1\times p_{m-1}}\,T_{m-1})$, and thus, the exactness of (8) at $D^{1\times(p_{m-2}+p_m)}$.

Moreover, using the fact that (7) is an exact sequence, we then have

$$D^{1\times(p_{m-2}+p_m)}\,T_{m-2} = D^{1\times p_{m-2}}\,R_{m-2} = \ker_D(.R_{m-3}),$$

which proves that (8) is exact at $D^{1\times p_{m-3}}$.

Finally, using again the fact that R_m admits a right-inverse S_m, we obtain that the exact sequence

$$0 \to D^{1\times p_m} \xrightarrow{.R_m} D^{1\times p_{m-1}} \xrightarrow{.R_{m-1}} D^{1\times p_{m-1}}\,R_{m-1} \to 0$$

splits, i.e., there exists a morphism

$$\varphi : (D^{1\times p_{m-1}}\,R_{m-1}) \longrightarrow D^{1\times p_{m-1}}$$

such that we have [3], [17], [26]:

$$(.R_m)\circ(.S_m) + \varphi\circ(.R_{m-1}) = id_{D^{1\times p_{m-1}}}.$$

Hence, if $\lambda \in \ker_D(.T_{m-1})$, we then get

$$\begin{array}{c} (\lambda\,R_{m-1}, \quad \lambda\,S_m) = (0, \quad 0) \\ \Rightarrow \lambda = (\lambda\,S_m)\,R_m + \varphi(\lambda\,R_{m-1}) = 0, \end{array}$$

which proves that the morphism $(.T_{m-1})$ is injective.

2 can be proved similarly. ■

Let us illustrate Proposition 1 by means of an example.

Example 6: We consider the ordinary differential linear system whose solution in $\mathcal{D}'(\mathbb{R})$ is $y = \dot{\delta}$, namely, the derivative of the Dirac distribution δ at $t = 0$:

$$\left\{ \begin{array}{l} t^2\,y(t) = 0, \\ t\,\dot{y}(t) + 2\,y(t) = 0. \end{array} \right.$$

If we consider the ring $D = A_1(\mathbb{Q})$ of differential operators in $\frac{d}{dt}$ with polynomial coefficients in t over $\mathbb{Q}$, $R = \left(t^2, \quad t\,\frac{d}{dt} + 2\right)^T \in D^2$ and the left D-module

$$M = D/(D^{1\times 2}\,R) = D/\left(D\,t^2 + D\,\left(t\,\frac{d}{dt} + 2\right)\right),$$

then a finite free resolution of M is defined by

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_2 = \left(\frac{d}{dt}, \quad -t\right) \in D^{1\times 2}$ (see [3] for more details). We easily check that $S_2 = \left(t, \quad \frac{d}{dt}\right)^T \in D^2$ is a right-inverse of R_2. Hence, using Proposition 1, we obtain the following finite free resolution of M

$$0 \longrightarrow D^{1\times 2} \xrightarrow{.T_1} D^{1\times 2} \xrightarrow{\tau} M \longrightarrow 0,$$

with the following notations:

$$T_1 = \left(\begin{array}{cc} t^2 & t \\ t\,\frac{d}{dt} + 2 & \frac{d}{dt} \end{array}\right) \in D^{2\times 2}, \quad \tau = \pi \oplus 0.$$

Let us state two useful results.

Proposition 2:
1) [26] Let M be a projective left D-module defined by a finite free resolution of the form (7). Then, the exact sequence (7) splits.
2) [26] If $\mathcal{F}$ is a left D-module, then the functor $\hom_D(\cdot\,, \mathcal{F})$ transforms split exact sequences of left D-modules into split exact sequences of abelian groups.

We have the following important characterization of stably free left D-modules.

Proposition 3: A left D-module M is stably free iff there exist two matrices $R' \in D^{q'\times p'}$ and $S' \in D^{p'\times q'}$ satisfying the following two conditions:

$$\left\{ \begin{array}{l} M \cong D^{1\times p'}/(D^{1\times q'}\,R'), \\ R'\,S' = I_{q'}. \end{array} \right. \qquad (10)$$

Proof: If M is a stably free left D-module, then there exist $p',\, q' \in \mathbb{Z}_+$ such that $M \oplus D^{1\times q'} \cong D^{1\times p'}$. Let us denote by $\psi : D^{1\times p'} \longrightarrow M \oplus D^{1\times q'}$ the above isomorphism and by $\pi_1 : M \oplus D^{1\times q'} \longrightarrow M$ the canonical projection onto M. Hence, we obtain the following commutative exact diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \ker_D(\pi_1\circ\psi) & & D^{1\times q'} & & \\ & & \downarrow & & \downarrow{\scriptstyle i_1} & & \\ 0 \longrightarrow & & D^{1\times p'} & \xrightarrow{\psi} & M \oplus D^{1\times q'} & & \longrightarrow 0 \\ & & \downarrow{\scriptstyle \pi_1\circ\psi} & & \downarrow{\scriptstyle \pi_1} & & \\ 0 \longrightarrow & & M & \xrightarrow{id} & M & & \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

which shows that we have:

$$\psi(\ker_D(\pi_1\circ\psi)) = 0 \oplus D^{1\times q'} = i_1(D^{1\times q'}).$$

Therefore, the first vertical exact sequence becomes the following exact sequence

$$0 \longrightarrow D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi_1\circ\psi} M \longrightarrow 0, \qquad (11)$$

where $R' \in D^{q'\times p'}$ is the matrix representing the morphism $\psi^{-1}\circ i_1 : D^{1\times q'} \longrightarrow D^{1\times p'}$ with respect to the standard bases of $D^{1\times q'}$ and $D^{1\times p'}$. If we denote by $\pi_2 : M \oplus D^{1\times q'} \longrightarrow D^{1\times q'}$ the canonical projection onto $D^{1\times q'}$, we then have:

$$\pi_2 \circ i_1 = id_{D^{1\times q'}}.$$

Hence, the morphism $\pi_2 \circ \psi : D^{1\times p'} \longrightarrow D^{1\times q'}$, represented by $S' \in D^{p'\times q'}$ with respect to the standard bases of $D^{1\times p'}$ and $D^{1\times q'}$, satisfies that

$$(\pi_2 \circ \psi) \circ (\psi^{-1} \circ i_1) = id_{D^{1\times q'}},$$

i.e., $R'\,S' = I_{q'}$, which proves the result.

Conversely, if the left D-module M is the cokernel of the D-morphism $.R' : D^{1\times q'} \longrightarrow D^{1\times p'}$, where the matrix R' admits a right-inverse S', then we obtain

$$\ker_D(.R') = \{\lambda \in D^{1\times q'} \mid \lambda\, R' = 0\} = 0$$

as $\lambda = (\lambda\, R')\, S' = 0$. Using the fact that a stably free module is projective, by 1 of Proposition 2, the exact sequence (11) splits and we obtain $M \oplus D^{1\times q'} \cong D^{1\times p'}$, which shows that M is a stably free left D-module.

■

Using the fact that a projective left D-module is a stably free left D-module and we can always construct a finite free resolution of a finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ [3], we obtain that if M is a stably free left D-module, then, by Proposition 1, (8) is a shorter finite free resolution of M. By induction on the length of the finite free resolutions of M, we finally obtain a short finite free resolution of M of the form (9), where the matrix T_1 admits a right-inverse. Hence, in what follows, we can always suppose that a stably free left D-module M can be defined by $M = D^{1\times p}/(D^{1\times q}\, R)$, where $R \in D^{q\times p}$ admits a right-inverse $S \in D^{p\times q}$. The corresponding algorithm has been implemented in OREMODULES [2].

Let us illustrate this result by means of an example.

Example 7: Let us consider $D = A_1(\mathbb{Q})$ and the left D-module $M = D^{1\times 2}/(D^{1\times 2}\, R)$, where R is defined by:

$$R = \begin{pmatrix} -t^2 & t\,\frac{d}{dt} - 1 \\ -t\,\frac{d}{dt} - 2 & \frac{d^2}{dt^2} \end{pmatrix} \in D^{2\times 2}.$$

We can check that M has the free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D^{1\times 2} \xrightarrow{\pi} M \longrightarrow 0,$$

with the notation $R_2 = \left(\frac{d}{dt}, \quad -t\right) \in D^{1\times 2}$. Moreover, the matrix $S_2 = \left(t, \quad \frac{d}{dt}\right)^T$ is a right-inverse of R_2. Hence, if we denote by $T_1 = (R, \quad S_2)$, then, by Proposition 1, we obtain the finite free resolution of M:

$$0 \longrightarrow D^{1\times 2} \xrightarrow{.T_1} D^{1\times 3} \xrightarrow{\tau} M \longrightarrow 0. \qquad (12)$$

We can check that T_1 admits the following right-inverse:

$$S_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ \frac{d}{dt} & -t \end{pmatrix} \in D^{3\times 2}.$$

Therefore, the exact sequence (12) splits, and thus, M is a stably free left D-module of rank 1 and (12) is a minimal free resolution of M.

II. CONSTRUCTIVE COMPUTATION OF FLAT OUTPUTS

A. Introduction

Let us start by explaining what are the main difficulties of testing freeness for a left D-module M.

Let us consider the k-vector space (e.g., $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$):

$$V = \{(x,\ y,\ z)^T \in k^3 \mid 2\,x + 3\,y + 5\,z = 0\}.$$

If we want to compute a basis of V, we usually do the following computations:

$$2\,x + 3\,y + 5\,z = 0 \quad \Longrightarrow \quad x = -\frac{3}{2}\,y - \frac{5}{2}\,z$$

$$\Longrightarrow \quad \begin{cases} x = -\frac{3}{2}\,y - \frac{5}{2}\,z, \\ y = y, \quad \forall\, y,\, z \in k. \\ z = z, \end{cases}$$

Therefore, we obtain the following basis

$$\left\{ \left(-\frac{3}{2}, 1, 0\right)^T, \ \left(-\frac{5}{2}, 0, 1\right)^T \right\}$$

of the k-vector space V, i.e., we have:

$$V = k\left(-\frac{3}{2},\ 1,\ 0\right)^T + k\left(-\frac{5}{2},\ 0,\ 1\right)^T.$$

Let us now consider the $\mathbb{Z}$-module defined by

$$P = \{(x,\ y,\ z)^T \in \mathbb{Z}^3 \mid 2\,x + 3\,y + 5\,z = 0\}$$

obtained by taking the ring $\mathbb{Z}$ instead of the field k. We note that we cannot repeat the same computations as $1/2$ does not belong to $\mathbb{Z}$. However, we have the following characterization of the fact that $\{(\alpha_1,\ \beta_1,\ \gamma_1)^T, (\alpha_2,\ \beta_2,\ \gamma_2)^T\}$ is a family of generators of the $\mathbb{Z}$-module P:

$$\begin{aligned} &\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P = \mathbb{Z}\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \\ &\Longleftrightarrow \ \exists\ t_1,\ t_2 \in \mathbb{Z}, \quad \begin{cases} x = \alpha_1\, t_1 + \alpha_2\, t_2, \\ y = \beta_1\, t_1 + \beta_2\, t_2, \\ z = \gamma_1\, t_1 + \gamma_2\, t_2. \end{cases} \end{aligned} \qquad (13)$$

Moreover, $\{(\alpha_i,\ \beta_i,\ \gamma_i)^T\}_{1\leq i\leq 2}$ is a basis of P iff (13) is injective, i.e., iff there exist $a_{ij} \in \mathbb{Z}$, $i = 1,\ 2$, $j = 1, 2, 3$, such that:

$$(13) \quad \Longrightarrow \quad t_i = a_{i1}\, x + a_{i2}\, y + a_{i3}\, z, \quad i = 1, 2.$$

Hence, we find again the fact that freeness is equivalent to the existence of an injective parametrization of the linear system P (see Theorem 5). The Hermite canonical form of the vector $(2,\ 3,\ 5)^T$ is $(1,\ 0,\ 0)^T$, and thus, we obtain that P is a free $\mathbb{Z}$-module and

$$P = \mathbb{Z}\,(9,\ -11,\ 3)^T + \mathbb{Z}\,(7,\ -8,\ 2)^T,$$

i.e., we have the following injective parametrization of P:

$$\left\{\begin{array}{l} x = 9\,t_1 + 7\,t_2, \\ y = -11\,t_1 - 8\,t_2, \\ z = 3\,t_1 + 2\,t_2, \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} t_1 = -2\,x - 2\,y - z, \\ t_2 = 3\,x + 3\,y + 2\,z. \end{array}\right.$$

Finally, we note that no canonical form such as Hermite, Smith or Jacobson forms exists over the Weyl algebras $A_n(k)$ for $n \geq 1$ and $B_n(k)$ for $n \geq 2$ because they are not left principal ideal domains. Hence, we need to pursue another way that we are going to describe now.

B. Computation of bases over the Weyl algebras

In what follows, we shall use the notation D for the Weyl algebras $A_n(k)$ or $B_n(k)$ defined in (6), where k is a field containing $\mathbb{Q}$. Let us recall a famous result in non-commutative algebra due to J. T. Stafford.

Theorem 6: [27] Let a_1, a_2, $a_3 \in D$ and the left ideal $I = D\,a_1 + D\,a_2 + D\,a_3$ of D generated by a_1, a_2 and a_3. Then, there exist λ, $\mu \in D$ such that we have:

$$I = D\,(a_1 + \lambda\,a_3) + D\,(a_2 + \mu\,a_3).$$

A direct consequence of Theorem 6 is that any left ideal of D can be generated by two elements of D.

Example 8: Let us consider $D = A_3(\mathbb{Q})$ and the left ideal $I = D\,(d_1 + x_3) + D\,d_2 + D\,d_3$ of D. Then, we have $I = D\,(d_1 + x_3) + D\,(d_2 + d_3)$ as:

$$\left\{\begin{array}{lcl} d_2 & = & (d_2\,(d_2 + d_3))\,(d_1 + x_3) \\ & & -(d_2\,(d_1 + x_3))\,(d_2 + d_3), \\ d_3 & = & (d_3\,(d_2 + d_3))\,(d_1 + x_3) \\ & & -(d_3\,(d_1 + x_3))\,(d_2 + d_3). \end{array}\right.$$

Therefore, we can take $\lambda = 0$ and $\mu = 1$ in Theorem 6.

Two constructive proofs of Theorem 6 have recently been developed in [9], [11]. They have been implemented in the package STAFFORD [25] using OREMODULES [2].

The following important corollary of Theorem 6 is also due to J. T. Stafford.

Corollary 1: [27] A stably free left D-module M with $\mathrm{rank}_D(M) \geq 2$ is free.

The purpose of this paper is to give a constructive proof of this corollary (contrary to the original one). In particular, it will give us an effective algorithm for the computation of bases of the free left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$, and thus, for the flat outputs of the corresponding system $\ker_{\mathcal{F}}(R.)$ (for any left D-module $\mathcal{F}$). We also note that another algorithm for the computation of bases of free modules over the Weyl algebras was given in [7]. However, we believe that our algorithm is simpler than the one developed in [7] as it is conceptually nothing else than a Gaussian elimination.

Let us introduce a few definitions.

Definition 5:

1) The *general linear group* $\mathrm{GL}_m(D)$ is the group of invertible matrices with entries in D:

$$\mathrm{GL}_m(D) = \{\,U \in D^{m\times m} \mid \exists\, V \in D^{m\times m} : \; U\,V = V\,U = I_m\,\}.$$

2) The *elementary group* $\mathrm{EL}_m(D)$ is the subgroup of $\mathrm{GL}_m(D)$ generated by all matrices of the form

$$I_m + r\,E_{ij}, \quad r \in D, \quad i \neq j,$$

where E_{ij} denotes the matrix with 1 at position (i, j) and 0 elsewhere.

3) A column vector $a = (a_1, \ldots, a_m)^T \in D^m$ is said to be *unimodular* if it admits a *left-inverse* $b = (b_1, \ldots, b_m) \in D^{1\times m}$, namely, if we have:

$$b\,a = \sum_{i=1}^{m} b_i\,a_i = 1.$$

4) We denote by $\mathrm{U}_m(D)$ the set of all unimodular vectors of D^m.

The next proposition will play an important role in what follows.

Proposition 4: Let us consider $m \geq 3$ and a unimodular vector $a = (a_1, \ldots, a_m)^T \in \mathrm{U}_m(D)$. Then, there exists a matrix $E \in \mathrm{EL}_m(D)$ such that:

$$E\,a = (1,\ 0, \ldots, 0)^T.$$

Proof: Applying Theorem 6 to the left ideal

$$I = D\,a_1 + D\,a_2 + D\,a_m$$

of D, there exist λ, $\mu \in D$ such that:

$$I = D\,(a_1 + \lambda\,a_m) + D\,(a_2 + \mu\,a_m).$$

Using the fact that $a \in \mathrm{U}_m(D)$, we then obtain $\sum_{i=1}^m D\,a_i = D$, and thus, we have:

$$D\,(a_1 + \lambda\,a_m) + D\,(a_2 + \mu\,a_m) + \sum_{i=3}^{m-1} D\,a_i = D.$$

Hence, we get:

$$a' = (a_1 + \lambda\,a_m,\ a_2 + \mu\,a_m,\ a_3, \ldots, a_{m-1})^T \in \mathrm{U}_{m-1}(D).$$

Let us define $a'_1 = a_1 + \lambda\,a_m$, $a'_2 = a_2 + \mu\,a_m$ and $a'_i = a_i$, $i \geq 3$, and the following matrix:

$$E_1 = \left(\begin{array}{cccccc} 1 & 0 & 0 & \ldots & 0 & \lambda \\ 0 & 1 & 0 & \ldots & 0 & \mu \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{array}\right) \in \mathrm{EL}_m(D).$$

We then have:

$$E_1\,a = (a'_1,\ a'_2, \ldots, a'_{m-1},\ a_m)^T.$$

Now, using the fact that $a' \in \mathrm{U}_{m-1}(D)$, there exist $b_1, \ldots, b_{m-1} \in D$ such that:

$$\sum_{i=1}^{m-1} b_i\, a'_i = 1. \tag{14}$$

Multiplying (14) by $a'_1 - 1 - a_m$, we obtain:

$$\sum_{i=1}^{m-1} (a'_1 - 1 - a_m)\, b_i\, a'_i = (a'_1 - 1 - a_m).$$

Let us denote by $a''_i = (a'_1 - 1 - a_m)\, b_i$, $i \geq 1$, and define the following matrix:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ a''_1 & a''_2 & a''_3 & \ldots & a''_{m-1} & 1 \end{pmatrix} \in \mathrm{EL}_m(D).$$

We then have:

$$E_2\,(a'_1, \ldots, a'_{m-1}, a_m)^T = (a'_1, \ldots, a'_{m-1}, a'_1 - 1)^T.$$

Hence, if we define the following elementary matrix

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathrm{EL}_m(D),$$

then we get:

$$E_3\,(a'_1, \ldots, a'_{m-1}, a'_1 - 1)^T = (1,\ a'_2, \ldots, a'_{m-1},\ a'_1 - 1)^T.$$

Finally, if we introduce the matrix

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -a'_2 & 1 & 0 & \ldots & 0 & 0 \\ -a'_3 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a'_{m-1} & 0 & 0 & \ldots & 1 & 0 \\ -a'_1 + 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathrm{EL}_m(D),$$

we then obtain:

$$E_4\,(1,\ a'_2, \ldots, a'_{m-1},\ a'_1 - 1)^T = (1,\ 0, \ldots,\ 0)^T.$$

Hence, the matrix $E = E_4\, E_3\, E_2\, E_1 \in \mathrm{EL}_m(D)$ satisfies:

$$E\,(a_1, \ldots, a_m)^T = (1,\ 0, \ldots,\ 0)^T.$$

■

Example 9: Let us consider the algebra $D = A_3(\mathbb{Q})$ and the column vector $a = (d_1 + x_3,\ d_2,\ d_3)^T$. We easily check that $b = (d_3, \quad 0, \quad -(d_1 + x_3))$ is a left-inverse of a, i.e., $a \in \mathrm{U}_3(D)$. Therefore, by Proposition 4, there exists a matrix $E \in \mathrm{EL}_3(D)$ such that $E\,a = (1,\ 0,\ 0)^T$. Let us compute such a matrix. We first need to apply Theorem 6 to the left ideal $I = D\,(d_1 + x_3) + D\,d_2 + D\,d_3$. Using Example 8, we can take $\lambda = 0$ and $\mu = 1$. If we define

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{EL}_3(D),$$

we then obtain $E_1\,a = (d_1 + x_3, \quad d_2 + d_3, \quad d_3)^T$. Now, we can check that we have the Bézout identity:

$$(d_2 + d_3)\,(d_1 + x_3) - (d_1 + x_3)\,(d_2 + d_3) = 1.$$

Therefore, if we define $a''_1 = (d_1 + x_3 - 1 - d_3)\,(d_2 + d_3)$, $a''_2 = -(d_1 + x_3 - 1 - d_3)\,(d_1 + x_3)$ and

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a''_1 & a''_2 & 1 \end{pmatrix} \in \mathrm{EL}_3(D),$$

we then get:

$$\begin{aligned} & E_2\ (d_1 + x_3, \quad d_2 + d_3, \quad d_3)^T \\ & = (d_1 + x_3, \quad d_2 + d_3, \quad d_1 + x_3 - 1)^T. \end{aligned}$$

Then, if we denote by

$$E_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{EL}_3(D),$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ -(d_2 + d_3) & 1 & 0 \\ -(d_1 + x_3 - 1) & 0 & 1 \end{pmatrix} \in \mathrm{EL}_3(D),$$

and $E = E_4\, E_3\, E_2\, E_1 \in \mathrm{EL}_3(D)$, we finally obtain:

$$E\,a = (1, \quad 0, \quad 0)^T.$$

We now state the main result of the paper.

Theorem 7: Let $R \in D^{q \times p}$ be a matrix which admits a right-inverse $S \in D^{p \times q}$, namely, $R\,S = I_q$, and satisfies $p \geq q + 2$. Then, $M = D^{1 \times p}/(D^{1 \times q}\,R)$ is a free left D-module with $\mathrm{rank}(M) = p - q \geq 2$.

Proof: Let us define the formal adjoint $\widetilde{R} \in D^{p \times q}$ of $R \in D^{q \times p}$ (see Definition 3). Taking the formal adjoint on both sides of the equality $R\,S = I_q$, we then get $\widetilde{S}\,\widetilde{R} = I_q$, which shows that $\widetilde{R}$ admits the left-inverse $\widetilde{S}$. In particular, the first column of $\widetilde{R}$ is a unimodular vector of D^p and $p \geq q + 2 \geq 3$. Hence, by applying Proposition 4 to the first column of $\widetilde{R}$, we obtain $E_1 \in \mathrm{EL}_p(D)$ such that:

$$E_1\,\widetilde{R} = \begin{pmatrix} 1 & \star \\ 0 & \\ \vdots & \widetilde{R_2} \\ 0 & \end{pmatrix}, \quad \widetilde{R_2} \in D^{(p-1) \times (q-1)}.$$

If $q \geq 2$, then we can easily check that the first column of the matrix $\widetilde{R_2} \in D^{(p-1)\times(q-1)}$ is unimodular and we have $p-1 \geq q+1 \geq 3$. Applying Proposition 4, we obtain $F_2 \in \mathrm{EL}_{p-1}(D)$ such that:

$$F_2\,\widetilde{R_2} = \left(\begin{array}{cc} 1 & \star \\ 0 & \\ \vdots & \widetilde{R_3} \\ 0 & \end{array}\right), \quad \widetilde{R_3} \in D^{(p-2)\times(q-2)}.$$

Hence, if we denote by

$$E_2 = \left(\begin{array}{cc} 1 & 0 \\ 0 & F_2 \end{array}\right) \in \mathrm{EL}_p(D),$$

we then obtain:

$$(E_2\,E_1)\,\widetilde{R} = \left(\begin{array}{ccc} 1 & \star & \star \\ 0 & 1 & \star \\ \vdots & 0 & \\ \vdots & \vdots & \widetilde{R_3} \\ 0 & 0 & \end{array}\right).$$

If $q \geq 3$, then we can also check that the first column of $\widetilde{R_3} \in D^{(p-2)\times(q-2)}$ is unimodular and $p-2 \geq q \geq 3$. By induction, we finally obtain $E_q \in \mathrm{EL}_p(D)$ such that:

$$(E_q \cdots E_1)\,\widetilde{R} = \left(\begin{array}{ccccc} 1 & \star & \star & \cdots & \star \\ 0 & 1 & \star & \cdots & \star \\ 0 & 0 & 1 & \cdots & \star \\ 0 & 0 & 0 & \ddots & \star \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right).$$

Hence, if we define the matrix $E = E_q \cdots E_1 \in \mathrm{EL}_p(D)$, then we easily get that every row vector

$$v = (v_1, \ldots, v_p) \in \ker_D(.(E\,\widetilde{R}))$$

satisfies $v_i = 0$ for $i = 1, \ldots, q$ and $v_{q+1}, \ldots, v_p$ are arbitrary elements in D. Therefore, we have:

$$\ker_D(.(E\,\widetilde{R})) = D^{1\times(p-q)}\,(0 \quad I_{p-q}).$$

Using the fact that E is invertible over D, we can check that $\ker_D(.\widetilde{R}) = \ker_D(.(E\,\widetilde{R}))\,E$, and thus,

$$\ker_D(.\widetilde{R}) = D^{1\times(p-q)}\,((0 \quad I_{p-q})\,E) = D^{1\times(p-q)}\,F,$$

where $F \in D^{(p-q)\times p}$ denotes the matrix formed by the last $p-q$ rows of E. By taking the last $p-q$ columns of the inverse E^{-1} of E, we obtain a matrix $G \in D^{p\times(p-q)}$ which satisfies $F\,G = I_{p-q}$. Using the identities

$$\widetilde{S}\,\widetilde{R} = I_q, \quad F\,\widetilde{R} = 0, \quad F\,G = I_{p-q},$$

we obtain:

$$\left(\begin{array}{c} \widetilde{S} - \widetilde{S}\,G\,F \\ F \end{array}\right)(\widetilde{R} \quad G) = \left(\begin{array}{cc} I_q & 0 \\ 0 & I_{p-q} \end{array}\right) = I_p. \quad (15)$$

If we define the matrices

$$\left\{\begin{array}{l} Q = \widetilde{F} \in D^{p\times(p-q)}, \\ T = \widetilde{G} \in D^{(p-q)\times p}, \\ S' = S - Q\,T\,S \in D^{p\times q}, \end{array}\right.$$

and apply the involution θ to (15), we finally obtain the following Bézout identity [17]:

$$\left(\begin{array}{c} R \\ T \end{array}\right)(S' \quad Q) = I_p. \quad (16)$$

The fact that the Weyl algebras $A_n(k)$ and $B_n(k)$ are left and right noetherian rings [14] implies that they are *stably finite* [10], namely, for all $U \in D^{p\times p}$ such that $U\,V = I_p$, for a certain $V \in D^{p\times p}$, then satisfies $V\,U = I_p$, i.e., $U \in \mathrm{GL}_p(D)$. Applying this result to (16), we obtain the new Bézout identity:

$$(S' \quad Q)\left(\begin{array}{c} R \\ T \end{array}\right) = I_p \quad \Longleftrightarrow \quad S'\,R + Q\,T = I_p. \quad (17)$$

The matrix $(R^T \quad T^T)^T$ is then invertible over D, which, in algebra, is well-known to be equivalent to the fact that the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is free [17], [26]. Let us give the complete proof.

The condition $R\,Q = 0$ implies that:

$$(D^{1\times q}\,R) \subseteq \ker_D(.Q).$$

Moreover, if $v \in \ker_D(.Q)$, then from (17) we obtain $v = (v\,S')\,R$, which shows that $v \in (D^{1\times q}\,R)$ and $\ker_D(.Q) = (D^{1\times q}\,R)$. Moreover, $\ker_D(.R) = 0$ as S is a right-inverse of R, i.e., $R\,S = I_q$, and $w \in \ker_D(.R)$ implies $w = (w\,R)\,S = 0$. Finally, $(D^{1\times p}\,Q) = D^{1\times(p-q)}$ because $(D^{1\times p}\,Q) \subseteq D^{1\times(p-q)}$ and, for all $u \in D^{1\times(p-q)}$, we have $u = (u\,T)\,Q \in (D^{1\times p}\,Q)$. Therefore, we obtain the following split short exact sequence [26]:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times(p-q)} \longrightarrow 0. \quad (18)$$

Then, a standard argument in homological algebra shows that $M = \mathrm{coker}(.R) \cong (D^{1\times p}\,Q) = D^{1\times(p-q)}$, proving that M is a free left D-module of rank $p-q$ and a basis is given by the columns of Q.

■

Let us illustrate Theorem 7 and its constructive proof.

Example 10: Let us consider $D = A_3(\mathbb{Q})$,

$$R = -(d_1 - x_3, \quad d_2, \quad d_3) \in D^{1\times 3},$$

the left D-module $M = D^{1\times 3}/(D\,R)$, any left D-module $\mathcal{F}$ (e.g., $\mathcal{F} = C^\infty(\mathbb{R}^3)$) and the system:

$$\begin{array}{l} \ker_{\mathcal{F}}(R.) = \{(y_1,\ y_2,\ y_3)^T \in \mathcal{F}^3 \mid \\ d_1\,y_1(x) + d_2\,y_2(x) + d_3\,y_3(x) - x_3\,y_1(x) = 0\}. \end{array} \quad (19)$$

We note that if we remove the last term $x_3\,y_1$ in the previous equation, then we obtain the divergence operator in $\mathbb{R}^3$ studied in Example 1. As was recalled in Example 1, if $\mathcal{F} = C^\infty(\mathbb{R}^3)$, the divergence operator is parametrized by the curl operator, but the curl operator is not an injective parametrization because the system formed by the curl operator is parametrized by the gradient operator. Moreover, using Theorem 5 and the fact that the gradient operator cannot be parametrized, we obtain that the $D = \mathbb{Q}[d_1, d_2, d_3]$-module associated with the divergence operator is reflexive but not free. Hence, it does not admit a basis.

However, we can check that the matrix R admits the right-inverse $S = (-d_3, \quad 0, \quad d_1 - x_3)^T$. By Theorem 7, we then obtain that the left D-module $M = D^{1\times 3}/(D\, R)$ is free of rank 2. By following the constructive proof of Theorem 7 we can compute a basis of M and an injective parametrization of (19).

The formal adjoint $\widetilde{R} = (d_1 + x_3, \quad d_2, \quad d_3)^T$ of R has already been computed in Example 5. Now, we need to apply Proposition 4 to $\widetilde{R}$. The computations were done in Example 9 and the matrix $E \in \mathrm{EL}_3(D)$ defined there satisfies $E\, \widetilde{R} = (1, \quad 0, \quad 0)^T$. Hence, taking the last two columns of $\theta(E)$, we obtain a basis of M or, equivalently, the following parametrization of $\ker_{\mathcal{F}}(R.)$

$$\left\{\begin{array}{lcl} y_1(x) & = & (1 - L_1)\,(d_2 + d_3)\,\xi_1(x) \\ & & +((1 - L_1)\,(d_1 - x_3) + 1)\,\xi_2(x), \\ y_2(x) & = & (-L_2\,(d_2 + d_3) + 1)\,\xi_1(x) \\ & & -L_2\,(d_1 - x_3)\,\xi_2(x), \\ y_3(x) & = & (-(1 + L_2)\,(d_2 + d_3) + 1)\,\xi_1(x) \\ & & -(1 + L_2)\,(d_1 - x_3)\,\xi_2(x), \end{array}\right. \tag{20}$$

with the following notations:

$$\left\{\begin{array}{l} L_1 = (d_2 + d_3)\,(d_1 - d_3 - x_3 + 1), \\ L_2 = -(d_1 - x_3)\,(d_1 - d_3 - x_3 + 1). \end{array}\right.$$

We can check that (20) is injective as we have

$$\left\{\begin{array}{lcl} \xi_1(x) & = & (-d_1^2 + d_1\, d_3 - x_3\, d_3 + (2\, x_3 - 1)\, d_1 \\ & & -x_3^2 + x_3 + 1)\, y_2(x) \\ & & +(d_1^2 - d_1\, d_3 + x_3\, d_3 - (2\, x_3 - 1)\, d_1 \\ & & +x_3^2 - x_3)\, y_3(x), \\ \xi_2(x) & = & y_1(x) + (-d_3^2 + d_1\, d_2 - d_2\, d_3 + d_1\, d_3 \\ & & +d_2 - (x_3 - 1)\, d_3 - x_3\, d_2 - 2)\, y_2(x) \\ & & +(d_3^2 - d_1\, d_2 + d_2\, d_3 - d_1\, d_3 \\ & & +(x_3 - 1)\, d_3 + (x_3 - 1)\, d_2 + 2)\, y_3(x), \end{array}\right.$$

which proves that $\{\xi_1, \xi_2\}$ is a flat output of $\ker_{\mathcal{F}}(R.)$.

Finally, we point out that the fact that (20) parametrizes $\ker_{\mathcal{F}}(R.)$ for any left D-module directly follows from (16) and (17) or, equivalently, from the fact that (18) is a so-called split short exact sequence and the functor $\hom_D(\,\cdot\,, \mathcal{F})$ transforms split exact sequences of left D-modules into split exact sequences of abelian groups/k-vector spaces (see 2 of Proposition 2).

Finally, combining Proposition 1 and Theorem 7, we obtain a constructive algorithm for the computation of bases of a stably free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ of rank at least 2. Indeed, using Proposition 1, we can compute two matrices $R' \in D^{q'\times p'}$ and $S' \in D^{p'\times q'}$ such that $M \cong D^{1\times p'}/(D^{1\times q'}\, R')$ and $R'\, S' = I_{q'}$. Now, the rank of a module being an intrinsic property, we obtain that $p' - q' \geq 2$. Hence, using Theorem 7, we can compute a basis of the left D-module $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ and we obtain the matrices $S' \in D^{p'\times q'}$, $Q' \in D^{p'\times (p'-q')}$ and $T' \in D^{(p'-q')\times p'}$ such that we have the following split short exact sequence:

$$0 \longrightarrow D^{1\times q'} \underset{\xleftarrow{.S'}}{\xrightarrow{.R'}} D^{1\times p'} \underset{\xleftarrow{.T'}}{\xrightarrow{.Q'}} D^{1\times (p'-q')} \longrightarrow 0.$$

Finally, we easily check that an injective parametrization Q of M can be obtained by removing the last $p' - p$ (zero) rows of Q'. A basis of M is then defined by $\{\pi(e_i\, T)\}_{i=1,\ldots,(p'-q')}$, where $T \in D^{(p'-q')\times p}$ is the matrix obtained by removing the last $p' - p$ columns of the matrix T' and $\{e_i\}_{i=1,\ldots,p'-q'}$ denotes the standard basis of $D^{1\times (p'-q')}$.

C. A constructive answer to Datta's question

In [23], the following result was proved.

Corollary 2: Every controllable ordinary differential linear system with polynomial coefficients and at least two inputs is flat.

Corollary 2 answers an open question posed by Datta [5] on the possibility to generalize the results of [13] for multi-input multi-output time-varying controllable linear systems. We point out that no effective algorithms for the computation of the corresponding flat outputs were known. Theorem 7 then solves the problem by giving a constructive answer to Datta's question in the case of polynomial coefficients. The corresponding algorithm has been implemented in STAFFORD. Let us illustrate Corollary 2 by means an example.

Example 11: We consider the following time-varying linear control system:

$$\left\{\begin{array}{l} \dot{x}_2(t) - u_2(t) = 0, \\ \dot{x}_1(t) - t\, u_1(t) = 0. \end{array}\right. \tag{21}$$

Let us consider the Weyl algebra $D = A_1(\mathbb{Q})$ and the left D-module $M = D^{1\times 4}/(D^{1\times 2}\, R)$, where R is defined by:

$$R = \left(\begin{array}{cccc} 0 & \frac{d}{dt} & 0 & -1 \\ \frac{d}{dt} & 0 & -t & 0 \end{array}\right) \in D^{2\times 4}.$$

We denote by $\ker_{\mathcal{F}}(R.)$ the corresponding system, where $\mathcal{F}$ is any left D-module (e.g., $\mathcal{F} = C^\infty(\mathbb{R})$).

If we consider for the moment the left $B_1(\mathbb{Q})$-module $P = B_1(\mathbb{Q})^{1\times 4}/(B_1(\mathbb{Q})^{1\times 2}\, R)$, then by using a Jacobson form, we can easily check that we have the following injective parametrization of $\ker_{\mathcal{F}}(R.)$ for any left $B_1(\mathbb{Q})$-module $\mathcal{F}$ (e.g., $\mathcal{F} = \mathbb{Q}(t)$):

$$\begin{cases} x_1(t) = \xi_1(t), \\ x_2(t) = \xi_2(t), \\ u_1(t) = \dfrac{1}{t}\,\dot{\xi}_1(t), \\ u_2(t) = \dot{\xi}_2(t). \end{cases}$$

We note that the previous parametrization of $\ker_{\mathcal{F}}(R.)$ is singular at $t = 0$.

However, we easily check that

$$S = \begin{pmatrix} 0 & 0 & 0 & -1 \\ t & 0 & \frac{d}{dt} & 0 \end{pmatrix}^T$$

is a right-inverse of R, i.e., $R\,S = I_2$ and $p - q = 2$. Therefore, by Theorem 7, we obtain that the left D-module $M = D^{1\times 4}/(D^{1\times 2}\,R)$ is free. Let us compute a basis of M, i.e., a non-singular parametrization of the system (21). The formal adjoint of R is defined by:

$$\widetilde{R} = \begin{pmatrix} 0 & -\frac{d}{dt} & 0 & -1 \\ -\frac{d}{dt} & 0 & -t & 0 \end{pmatrix}^T.$$

Considering the first unimodular column of $\widetilde{R}$ and applying Proposition 4, we obtain:

$$\begin{aligned} &D\,0 + D\left(-\tfrac{d}{dt}\right) + D\,(-1) \\ &= D\,(0 + 1\cdot(-1)) + D\left(-\tfrac{d}{dt} + 0\cdot(-1)\right). \end{aligned}$$

Hence, taking $\lambda = 1$ and $\mu = 0$, we define the following matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{d}{dt} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

Defining $E = E_4\,E_3\,E_2\,E_1$, we then get:

$$E\,\widetilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -t & -\frac{d}{dt} \end{pmatrix}^T.$$

Now, we consider the second column of $E\,\widetilde{R}$ and, in particular, the vector $\left(0, \quad -t, \quad -\frac{d}{dt}\right)^T$. We can check that this vector is unimodular. Applying Proposition 4, we obtain:

$$D\,0 + D\,(-t) + D\left(-\frac{d}{dt}\right) = D\left(0 - \frac{d}{dt}\right) + D\,(-t).$$

Then, taking $\lambda = 1$ and $\mu = 0$, we define the following elementary matrices:

$$E_1' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & \frac{d}{dt} & 1 \end{pmatrix},$$

$$E_3' = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4' = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{d}{dt}+1 & 0 & 1 \end{pmatrix}.$$

Defining the matrix $E' = E_4'\,E_3'\,E_2'\,E_1' \in \mathrm{EL}_3(D)$ and $E'' = \mathrm{diag}(1, E') \in \mathrm{EL}_4(D)$, we then get:

$$(E''\,E)\,\widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The last two columns of the formal adjoint of $E''\,E$ form the following matrix in $D^{4\times 2}$:

$$Q = \begin{pmatrix} t^2 & -t\,\frac{d}{dt} + 1 \\ t\,(t+1) & -(t+1)\,\frac{d}{dt} + 1 \\ t\,\frac{d}{dt} + 2 & -\frac{d^2}{dt^2} \\ t\,(t+1)\,\frac{d}{dt} + 2\,t + 1 & -(t+1)\,\frac{d^2}{dt^2} \end{pmatrix}.$$

We then check that Q admits the left-inverse

$$T = \begin{pmatrix} 0 & 0 & t+1 & -1 \\ t+1 & -t & 0 & 0 \end{pmatrix} \in D^{2\times 4}.$$

Hence, the time-varying linear control system (21) is injectively parametrized by

$$\begin{cases} x_1(t) = t^2\,\xi_1(t) - t\,\dot{\xi}_2(t) + \xi_2(t), \\ x_2(t) = t\,(t+1)\,\xi_1(t) - (t+1)\,\dot{\xi}_2(t) + \xi_2(t), \\ u_1(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) - \ddot{\xi}_2(t), \\ u_2(t) = t\,(t+1)\,\dot{\xi}_1(t) + (2\,t+1)\,\xi_1(t) - (t+1)\,\ddot{\xi}_2(t) \end{cases}$$

and a flat output $\{\xi_1,\, \xi_2\}$ of $\ker_{\mathcal{F}}(R.)$ is defined by:

$$\begin{cases} \xi_1(t) = (t+1)\,u_1(t) - u_2(t), \\ \xi_2(t) = (t+1)\,x_1(t) - t\,x_2(t). \end{cases}$$

We do not know whether or not Corollary 2 can be extended to the case of real analytic coefficients. If we consider the ring of differential operators $D = \mathbb{C}\{t\}\left[\frac{d}{dt}\right]$ with coefficients in the ring $\mathbb{C}\{t\}$ of convergent power series, it is well known that every left ideal of D can be generated by means of two elements of D [1]. Two such generators can be found by means of a computation of a standard basis as it is explained in [1]. However, we do not know if D is *strongly simple*, namely, if, for every a_1, a_2 and $a_3 \in D$, there exist λ and $\mu \in D$ satisfying:

$$D\,a_1 + D\,a_2 + D\,a_3 = D\,(a_1 + \lambda\,a_3) + D\,(a_2 + \mu\,a_3).$$

If so, then Proposition 4 and Theorem 7 also hold for this particular ring $D = \mathbb{C}\{t\}\left[\frac{d}{dt}\right]$. This question will be studied in the future as well as the case of real analytic coefficients.

III. CONCLUSION

Based on new constructive proofs of one of J. T. Stafford's results, we have given in this paper a constructive algorithm which computes bases of free modules over the Weyl algebras (whose ground fields contain $\mathbb{Q}$). Using a dictionary existing between system and module theories, we are now able to constructively compute the flat outputs of flat multidimensional linear systems defined by PDEs with polynomial or rational coefficients. The extension of the results of this paper to other classes of multidimensional linear systems such as differential time-delay or discrete systems will be studied in the future. Finally, as the constructive proofs of Theorem 6 developed in [9], [11] use computations of many time-consuming Gröbner bases, we can only handle relative small examples up to now. Optimization of the different algorithms will be studied in forthcoming publications.

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