# Symmetries, parametrizations and potentials of multidimensional linear systems

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***Abstract*— Within the algebraic analysis approach to linear systems theory, the purpose of this paper is to study how left $D$-homomorphisms between two finitely presented left $D$-modules associated with two linear systems induce natural transformations on the autonomous elements of the two systems and on the potentials of the parametrizations of the parametrizable subsystems. Extension of these results are also considered for linear systems inducing a chain of successive parametrizations.**

## I. Homomorphisms of linear systems

Let $D$ be a ring of functional operators (e.g., ordinary or partial differential operators, time-delay operators, shift operators) and $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$) a $q \times p$ (resp., $q' \times p'$) matrix. We consider the left $D$-module finitely presented by $R$ (resp., $R'$), namely, $M = D^{1\times p}/(D^{1\times q}\, R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'}\, R')$). A *left $D$-homomorphism* $f$ (or simply *morphism*) from $M$ to $M'$ is a left $D$-linear map $f : M \longrightarrow M'$. The abelian group of all morphisms from $M$ to $M'$ is denoted $\hom_D(M, M')$. If $M = M'$, $f \in \hom_D(M, M')$ is called a left *$D$-endomorphism of $M$*. We denote by $\mathrm{end}_D(M) = \hom_D(M, M)$ the ring of all endomorphisms of $M$ also called the *endomorphism ring*.

*Lemma 1.1 ([3], Corollary 2.1):* With the previous notations, let us consider the finite presentations of $M$ and $M'$

$$\begin{array}{l} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,\\ D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0, \end{array} \tag{1}$$

where $(.R)(\lambda) = \lambda\, R$ for all $\lambda \in D^{1\times q}$ and similarly for $R'$, namely, (1) are exact sequences, i.e., $\pi$ (resp., $\pi'$) is surjective and $\ker \pi = D^{1\times q}\, R$ (resp., $\ker \pi' = D^{1\times q'}\, R'$).

1) The existence of a left $D$-morphism $f : M \longrightarrow M'$ is equivalent to the existence of two matrices

$$P \in D^{p\times p'}, \quad Q \in D^{q\times q'}$$

satisfying the commutation relation:

$$R\, P = Q\, R'.$$

Then, we have the commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0\\ \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0, \end{array}$$

where $f(\pi(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in D^{1\times p}$.

2) If we denote by $R'_2 \in D^{q'_2\times q'}$ a matrix satisfying

$$\ker_D(.R') \triangleq \{\lambda \in D^{1\times q'} \mid \lambda\, R = 0\} = D^{1\times q'_2}\, R'_2,$$

then $P$ and $Q$ are defined up to homotopy, i.e.,

$$\begin{cases} \overline{P} = P + Z_1\, R', \\ \overline{Q} = Q + R\, Z_1 + Z_2\, R'_2, \end{cases}$$

where $Z_1 \in D^{p\times q'}$ and $Z_2 \in D^{q\times q'_2}$ are two arbitrary matrices, satisfy the same relation

$$R\, \overline{P} = \overline{Q}\, R',$$

and $f(\pi(\lambda)) = \pi'(\lambda\, \overline{P})$ for all $\lambda \in D^{1\times p}$.

See [3] for algorithms which compute the matrices $P$ and $Q$ when $D$ is a commutative polynomial ring over a computable field or a noncommutative polynomial rings for which Buchberger's algorithm terminates for any admissible term order. These algorithms are implemented in the package OreMorphisms ([4]) for classes of *Ore algebras* ([1]).

In the particular case where $R' = R$, from Lemma 1.1, we obtain that the existence of a left $D$-endomorphism $f$ of $M$ is equivalent to the existence of two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying the following commutation relation:

$$R\, P = Q\, R.$$

Let $\mathcal{F}$ be a left $D$-module and consider the linear systems:

$$\begin{cases} \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}, \\ \ker_{\mathcal{F}}(R'.) = \{\zeta \in \mathcal{F}^{p'} \mid R'\,\zeta = 0\}. \end{cases}$$

The linear systems $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ are also called *behaviours*. The following lemma shows how left $D$-morphisms from $M$ to $M'$ induce abelian group morphisms between the abelian groups $\ker_{\mathcal{F}}(R')$ and $\ker_{\mathcal{F}}(R)$.

*Lemma 1.2 ([3], Corollary 2.2):* With the hypotheses and notations of Lemma 1.1, if $\mathcal{F}$ is a left $D$-module, then the *behaviour morphism* ([7]) is defined by:

$$\begin{array}{rcl} P. : \ker_{\mathcal{F}}(R') & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \zeta & \longmapsto & \eta = P\,\zeta. \end{array}$$

Lemma 1.2 illustrates how morphisms provide some kind of "Galois transformations" which send solutions of the

Dedicated to Professor Ulrich Oberst on the occasion of his 70th birthday.

second system to solutions of the first one. If $M = M'$, then Galois transformations are "Galois symmetries" of $\ker_{\mathcal{F}}(R.)$.

## II. Parametrizations of linear systems

Let us introduce a few concepts and results of module theory and homological algebra (see, e.g., [8]).

*Definition 2.1 ([8]):* Let $D$ be a left noetherian domain and $M = D^{1\times p}/(D^{1\times q}\, R)$ the left $D$-module finitely presented by the matrix $R \in D^{q\times p}$.

1) $M$ is *free of rank* $r \in \mathbb{N} = \{0, 1, \ldots\}$ if $M \cong D^{1\times r}$, where $\cong$ denotes an isomorphism.
2) $M$ is *projective* if there exist $r \in \mathbb{N}$ and a left $D$-module $P$ such that $M \oplus P \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left $D$-modules.
3) $M$ is *reflexive* if the canonical left $D$-homomorphism $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ defined by $\varepsilon(m)(f) = f(m)$ for all $f \in \hom_D(M, D)$ and all $m \in M$, is bijective, i.e., $\varepsilon$ is a left $D$-isomorphism.
4) $M$ is *torsion-free* if the torsion left $D$-submodule
$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} :\ d\, m = 0\}$$
of $M$ is reduced to 0, i.e., $t(M) = 0$.
5) $M$ is *torsion* if $t(M) = M$, i.e., every $m \in M$ is a *torsion element* of $M$, namely, $m \in t(M)$.

If $N = D^q/(R\, D^p)$ is the right $D$-module finitely presented by $R \in D^{q\times p}$, then $N$ admits a *finite free resolution*

$$0 \longleftarrow N \overset{\kappa}{\longleftarrow} D^{s_0} \overset{Q_1.}{\longleftarrow} D^{s_1} \overset{Q_2.}{\longleftarrow} D^{s_2} \overset{Q_3.}{\longleftarrow} D^{s_3} \overset{Q_4.}{\longleftarrow} \ldots, \tag{2}$$

where $s_0 = q$, $s_1 = p$, $Q_1 = R$ and $Q_i \in D^{s_{i-1}\times s_i}$ and $Q_i. : D^{s_i} \longrightarrow D^{s_{i-1}}$ is defined by $(Q_i.)(\eta) = Q_i\, \eta$ for all $\eta \in D^{s_i}$, namely, an *exact sequence*, i.e., $\kappa$ is surjective and:

$$\forall\, i \geq 1, \quad \ker_D(Q_i.) \triangleq \{\eta \in D^{s_i} \mid Q_i\, \eta = 0\} = \operatorname{im}_D(Q_{i+1}.) \triangleq Q_{i+1}\, D^{s_{i+1}}.$$

The exact sequence (2) yields the following *complex*

$$0 \longrightarrow D^{1\times s_0} \xrightarrow{.Q_1} D^{1\times s_1} \xrightarrow{.Q_2} D^{1\times s_2} \xrightarrow{.Q_3} D^{1\times s_3} \xrightarrow{.Q_4} \ldots, \tag{3}$$

namely, $\operatorname{im}_D(.Q_i) \subseteq \ker_D(.Q_{i+1})$, since $Q_i\, Q_{i+1} = 0$, for all $i \geq 1$. The *defects of exactness* of (3) are defined by

$$\begin{cases} \operatorname{ext}^0_D(N, D) = \hom_D(N, D) = \ker_D(.Q_1), \\ \operatorname{ext}^i_D(N, D) \cong \ker_D(.Q_{i+1})/\operatorname{im}_D(.Q_i), \quad i \geq 1, \end{cases}$$

where $\ker_D(.Q_{i+1}) \triangleq \{\lambda \in D^{1\times s_i} \mid \lambda\, Q_{i+1} = 0\}$ and $\operatorname{im}_D(.Q_i) \triangleq D^{1\times s_{i-1}}\, Q_i$ for all $i \geq 1$.

*Theorem 2.1 ([1]):* Let $D$ be a noetherian domain with a finite *global dimension* $\operatorname{gld}(D)$ ([8]), $M = D^{1\times p}/(D^{1\times q}\, R)$ and the *Auslander transposed* of $M$, namely, the right $D$-module $N = D^q/(R\, D^p)$ finitely presented by $R$.

1) The following left $D$-isomorphism holds:
$$t(M) \cong \operatorname{ext}^1_D(N, D). \tag{4}$$
2) $M$ is a torsion-free left $D$-module iff $\operatorname{ext}^1_D(N, D) = 0$.
3) $M$ is reflexive left $D$-module iff $\operatorname{ext}^i_D(N, D) = 0$ for $i = 1, 2$.
4) $M$ is projective left $D$-module iff $\operatorname{ext}^i_D(N, D) = 0$ for $i = 1, \ldots, \operatorname{gld}(D)$.

A left $D$-module $\mathcal{F}$ is *injective* iff for every $q \geq 1$ and every $R \in D^q$, the linear system $R\, \eta = \zeta$ admits a solution $\eta \in \mathcal{F}$, for all $\zeta \in \mathcal{F}^q$ satisfying the compatibility conditions of $R\, \eta = \zeta$, namely, $R_2\, \zeta = 0$, where $\ker_D(.R) = D^{1\times r}\, R_2$.

A left $D$-module $\mathcal{F}$ is called a *cogenerator* if, for every left $D$-module $M$ and every nonzero $m \in M$, there exists $f \in \hom_D(M, \mathcal{F})$ such that $f(m) \neq 0$. More generally, a left $D$-module $\mathcal{F}$ is *injective cogenerator* if $\mathcal{F}$ is both an injective and a cogenerator left $D$-module ([8]).

The linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left $D$-module $M$ finitely presented by the system matrix $R$ since, due to a remark of Malgrange ([6]), we have $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$. If the $D$-module $\mathcal{F}$ (i.e., also called signal space) is rich enough, i.e., is an *injective cogenerator* left $D$-module ([8]), then an exact duality exists between the systemic properties of $\ker_{\mathcal{F}}(R.)$ and the module properties of the left $D$-module $M$. For instance, the autonomous elements of $\ker_{\mathcal{F}}(R.)$ are in a 1-1 correspondence with the torsion elements of $M$. Moreover, the parametrizability property of $\ker_{\mathcal{F}}(R.)$, i.e., the existence of $Q \in D^{p\times m}$ satisfying $\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^m$, is equivalent to the torsion-freeness of the left $D$-module $M$, i.e., $t(M) = 0$. Then, $Q$ (resp., $Q\, \mathcal{F}^m$) is called a *parametrization* (resp., *an image representation*) of $\ker_{\mathcal{F}}(R.)$ and $\xi \in \mathcal{F}^m$ satisfying $\eta = Q\, \xi \in \ker_{\mathcal{F}}(R.)$ is called a *potential* of $\ker_{\mathcal{F}}(R.)$.

The next lemma and corollary, which will play important roles in what follows, generalize the above result.

*Corollary 2.1 ([1]):* Let $D$ be a noetherian domain (namely, a ring with non zero-divisors and which left and right ideals are finitely generated as left and right $D$-modules) with a finite global dimension $\operatorname{gld}(D) = n$ ([8]). Moreover, let $M = D^{1\times p}/(D^{1\times q}\, R)$ be the left $D$-module finitely presented by $R \in D^{q\times p}$. If we set $Q_1 = R$, $p_1 = p$ and $p_0 = q$, then we have the following results:

1) $M$ is a torsion-free left $D$-module iff there exists a matrix $Q_2 \in D^{p_1\times p_2}$ such that the following exact sequence of left $D$-modules holds:
$$D^{1\times p_0} \xrightarrow{.Q_1} D^{1\times p_1} \xrightarrow{.Q_2} D^{1\times p_2}.$$
2) $M$ is a reflexive left $D$-module iff there exist two matrices $Q_2 \in D^{p_1\times p_2}$ and $Q_3 \in D^{p_2\times p_3}$ such that the following exact sequence of left $D$-modules holds:
$$D^{1\times p_0} \xrightarrow{.Q_1} D^{1\times p_1} \xrightarrow{.Q_2} D^{1\times p_2} \xrightarrow{.Q_3} D^{1\times p_3}.$$
3) $M$ is a projective left $D$-module iff there exist $n$ matrices $Q_i \in D^{p_{i-1}\times p_i}$, $i = 2, \ldots, n+1$, such that the following exact sequence of left $D$-modules holds:
$$D^{1\times p_0} \xrightarrow{.Q_1} D^{1\times p_1} \xrightarrow{.Q_2} \ldots \xrightarrow{.Q_{n+1}} D^{1\times p_{n+1}}.$$

If $\mathcal{F}$ is an injective left $D$-module, then the results of Corollary 2.1 can be dualized to get the following system-theoretic interpretations of the module properties in terms of the existence of a chain of parametrizations.

*Corollary 2.2 ([1]):* Let $D$ be a noetherian domain with a finite global dimension $\mathrm{gld}(D) = n$, $M = D^{1\times p}/(D^{1\times q}\, R)$ the left $D$-module finitely presented by $R \in D^{q\times p}$ and $\mathcal{F}$ an injective left $D$-module. If we set $Q_1 = R$, $p_1 = p$ and $p_0 = q$, then we have the following results:

1) If $M$ is a torsion-free left $D$-module, then there exists a matrix $Q_2 \in D^{p_1 \times p_2}$ such that the following exact sequence of abelian groups holds
$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \mathcal{F}^{p_2},$$
i.e., $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{p_2}$. The matrix $Q_2$ is then a parametrization of the linear system $\ker_{\mathcal{F}}(Q_1.)$.
2) If $M$ is a reflexive left $D$-module, then there exist $Q_2 \in D^{p_1\times p_2}$ and $Q_3 \in D^{p_2 \times p_3}$ such that the following exact sequence of abelian groups holds
$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \mathcal{F}^{p_2} \xleftarrow{Q_3.} \mathcal{F}^{p_3},$$
i.e., $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{p_2}$ and $\ker_{\mathcal{F}}(Q_2.) = Q_3\, \mathcal{F}^{p_3}$.
3) If $M$ is a projective left $D$-module, then there exist $n$ matrices $Q_i \in D^{p_{i-1}\times p_i}$ for $i = 2, \ldots, n+1$ such that the following exact sequence of abelian groups holds
$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \;\ldots\; \xleftarrow{Q_{n+1}.} \mathcal{F}^{p_{n+1}},$$
i.e., $\ker_{\mathcal{F}}(Q_i.) = Q_{i+1}\, \mathcal{F}^{p_{i+1}}$ for $i = 1, \ldots, n$.

## III. Extension to morphisms between $\mathrm{ext}^i_D(\,\cdot\,, D)$

Let $D$ be a noetherian domain and $M$ and $M'$ two left $D$-modules respectively defined by $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$. Let $N = D^q/(R\, D^p)$ (resp., $N' = D^{q'}/(R'\, D^{p'})$) be the *Auslander transpose* of $M$ (resp., $M'$). In this section, we show that $f \in \hom_D(M, M')$ induces $f_i \in \hom_D(\mathrm{ext}^i_D(N, D), \mathrm{ext}^i_D(N', D))$ for $i \geq 0$.

From Lemma 1.1, a left $D$-morphism $f \in \hom_D(M, M')$ can be defined by means of two matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying the relation $R\, P = Q\, R'$ and the following commutative exact diagram holds:
$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. \end{array}$$

The right $D$-modules $N$ and $N'$ are respectively defined by the finite presentations
$$0 \longleftarrow N \xleftarrow{\kappa} D^q \xleftarrow{R.} D^p \xleftarrow{\iota_p \circ \pi^\star} \hom_D(M, D) \longleftarrow 0,$$
$$0 \longleftarrow N' \xleftarrow{\kappa'} D^{q'} \xleftarrow{R'.} D^{p'} \xleftarrow{\iota_{p'} \circ \pi'^\star} \hom_D(M', D) \longleftarrow 0,$$
where the right $D$-morphism $\pi^\star$ is defined by
$$\begin{array}{rcl} \pi^\star : \hom_D(M, D) & \longrightarrow & \hom_D(D^{1\times p}, D) \\ \phi & \longmapsto & \phi \circ \pi, \end{array}$$
and $\iota_p$ is the right $D$-isomorphism defined by
$$\begin{array}{rcl} \iota_p : \hom_D(D^{1\times p}, D) & \longrightarrow & D^p \\ \psi & \longmapsto & (\psi(f_1)\; \ldots\; \psi(f_p))^T, \end{array}$$
where $\{f_j\}_{j=1,\ldots,p}$ is the standard basis of $D^{1\times p}$, namely $f_j$ is the row vector of length $p$ with 1 at the $j^{\text{th}}$ entry and 0 elsewhere. The right $D$-morphisms $\pi'^\star$ and $\iota_{p'}$ can similarly be defined. If we simply denote $\hom_D(M, D)$ by $M^\star$, then $f \in \hom_D(M, M')$ induces the right $D$-morphism
$$\begin{array}{rcl} f^\star : M'^\star = \hom_D(M', D) & \longrightarrow & M^\star = \hom_D(M, D) \\ \phi & \longmapsto & \phi \circ f, \end{array}$$
and $g \in \hom_D(N', N)$ defined by the commutative exact diagram given by Figure 1, i.e., $g$ is defined by:
$$\begin{array}{rcl} g : N' & \longrightarrow & N \\ \kappa'(\lambda) & \longmapsto & \kappa(Q\, \lambda), \quad \forall\, \lambda \in D^{q'}. \end{array}$$
Hence, $f^\star$ induces the right $D$-morphism of exact sequences of right $D$-modules defined in Figure 2, with the notations $p_0 = q$, $p_1 = p$, $Q_1 = R$, $p'_0 = q'$, $p'_1 = p'$, $Q'_1 = R'$, $P_0 = Q$, $P_1 = P$, and where the matrices $Q_i \in D^{p_{i-1}\times p_i}$, $Q'_i \in D^{p'_{i-1}\times p'_i}$ and $P_i \in D^{p'_i \times p_i}$ are inductively defined by:
$$\forall\, i \geq 1, \quad \left\{ \begin{array}{l} \ker_D(Q_i.) = Q_{i+1}\, D^{p_{i+1}}, \\ \ker_D(Q'_i.) = Q'_{i+1}\, D^{p'_{i+1}}, \\ Q_i\, P_i = P_{i-1}\, Q'_i. \end{array} \right. \tag{5}$$
The matrices $P_i$'s exist for $i \geq 2$ because we have $Q_1\, P_1\, Q'_2 = P_0\, (Q'_1\, Q'_2) = 0$, which shows that $(P_1\, Q'_2)\, D^{p'_2} \subseteq \ker_D(Q_1.) = Q_2\, D^{p_2}$, and thus there exists a matrix $P_2 \in D^{p'_2 \times p_2}$ such that $P_1\, Q'_2 = Q_2\, P_2$, and similarly for the matrices $P_i$'s for $i \geq 3$.

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ ([8]) to the commutative exact diagram given by Figure 2, we obtain the morphism of complexes of left $D$-modules defined by the commutative diagram given in Figure 3, where
$$\begin{array}{rcl} \kappa^\star : N^\star & \longrightarrow & (D^{p_0})^\star \\ \phi & \longmapsto & \phi \circ \kappa, \end{array}$$
$$\begin{array}{rcl} i_{p_0} : (D^{p_0})^\star & \longrightarrow & D^{1\times p_0} \\ \varphi & \longmapsto & (\varphi(f_1)\; \ldots\; \varphi(f_{p_0})), \end{array}$$
where $\{f_j\}_{j=1,\ldots,p_0}$ is the standard basis of $D^{p_0}$. The defects of exactness of the two horizontal complexes are defined by:
$$\forall\, i \geq 1, \quad \left\{ \begin{array}{l} \mathrm{ext}^i_D(N, D) = \ker_D(.Q_{i+1})/(D^{1\times p_{i-1}}\, Q_i), \\ \mathrm{ext}^i_D(N', D) = \ker_D(.Q'_{i+1})/(D^{1\times p'_{i-1}}\, Q'_i). \end{array} \right.$$
See [8]. For every $\lambda \in \ker_D(.Q_{i+1})$, we have $\lambda\, P_i\, Q'_{i+1} = \lambda\, Q_{i+1}\, P_{i+1} = 0$, i.e., $\lambda\, P_i \in \ker_D(.Q'_{i+1})$ which yields the left $D$-morphism $.P_i : \ker_D(.Q_{i+1}) \longrightarrow \ker_D(.Q'_{i+1})$, for $i \geq 0$. Moreover, for every $\mu \in D^{1\times p_{i-1}}$, we have $(\mu\, Q_i)\, P_i = (\mu\, P_{i-1})\, Q'_i \in D^{1\times p'_{i-1}}\, Q'_i$, which yields the left $D$-morphism $.P_i : D^{1\times p_{i-1}}\, Q_i \longrightarrow D^{1\times p'_{i-1}}\, Q'_i$ for $i \geq 1$. Hence, if we denote by
$$\rho_i : \ker_D(.Q_{i+1}) \longrightarrow \mathrm{ext}^i_D(N, D),$$
$$\rho'_i : \ker_D(.Q'_{i+1}) \longrightarrow \mathrm{ext}^i_D(N', D),$$

the respective canonical projections, then the matrix $P_i$ induces the commutative exact diagram defined in Figure 4, where the left $D$-morphism $f_i$ is defined by:

$$\forall\, i \geq 1, \quad \begin{array}{rcl} f_i : \mathrm{ext}^i_D(N, D) & \longrightarrow & \mathrm{ext}^i_D(N', D) \\ \rho_i(\lambda) & \longmapsto & \rho'_i(\lambda\, P_i). \end{array} \tag{6}$$

Moreover, we define $f_0 : \ker_D(.Q_1) \longrightarrow \ker_D(.Q'_1)$ by $f_0(\lambda) = \lambda\, P_0$, for all $\lambda \in D^{1\times p_0}$. We note that the left $D$-morphisms $f_i$'s for $i \geq 1$ do depend only on $f^\star$, i.e., on $f$, because if we consider different matrices $\overline{P}_i \in D^{p'_i \times p_i}$ satisfying (5) instead of $P_i$, then, there exist $Z_i \in D^{p_{i+1}\times p'_i}$, for $i \geq 1$, such that:

$$\overline{P}_i = P_i + Z_{i-1}\, Q'_i + Q_{i+1}\, Z_i, \quad \forall\, i \geq 1.$$

Then, we get that the left $D$-morphism

$$\begin{array}{rcl} f'_i : \mathrm{ext}^i_D(N, D) & \longrightarrow & \mathrm{ext}^i_D(N, D) \\ \rho_i(\lambda) & \longmapsto & \rho'_i(\lambda\, \overline{P_i}), \end{array}$$

for all $\lambda \in \ker_D(.Q_{i+1})$, satisfies:

$$\begin{aligned} f'_i(\rho_i(\lambda)) &= \rho'_i(\lambda\, P_i) + \rho'_i((\lambda\, Z_{i-1})\, Q'_i) + \rho'_i((\lambda\, Q_{i+1})\, Z_i) \\ &= \rho'_i(\lambda\, P_i) = f_i(\rho_i(\lambda)). \end{aligned}$$

Moreover, $f_0$ only depends on $P_0$, i.e., on $f$.

*Proposition 3.1:* Let $D$ be a noetherian domain and $Q_1 \in D^{p_0\times p_1}$, $Q'_1 \in D^{p'_0 \times p'_1}$ two matrices. Consider the finitely presented left $D$-modules $M = D^{1\times p_1}/(D^{1\times p_0}\, Q_1)$, $M' = D^{1\times p'_1}/(D^{1\times p'_0}\, Q'_1)$ and their Auslander transposes, namely, the right $D$-modules $N = D^{p_0}/(Q_1\, D^{p_1})$ and $N' = D^{p'_0}/(Q'_1\, D^{p'_1})$. Then, $f \in \hom_D(M, M')$, defined by $P_1 \in D^{p_1\times p'_1}$ and $P_0 \in D^{p_0\times p'_0}$ satisfying $Q_1\, P_1 = P_0\, Q'_1$, induces left $D$-morphisms $f_i : \mathrm{ext}^i_D(N, D) \longrightarrow \mathrm{ext}^i_D(N', D)$ defined by (6), where the matrices $Q_i$'s, $Q'_i$'s and $P_i$'s are defined by (5) for $i \geq 1$.

If $R_i \in D^{q_{i-1}\times p_i}$ (resp., $R'_i \in D^{q'_{i-1}\times p'_i}$) is a matrix such that $\ker_D(.Q_{i+1}) = D^{1\times q_{i-1}}\, R_i$ (resp., $\ker_D(.Q'_{i+1}) = D^{1\times q'_{i-1}}\, R'_i$), then the left $D$-morphism $.P_i : \ker_D(.Q_{i+1}) \longrightarrow \ker_D(.Q'_{i+1})$ becomes:

$$\begin{array}{rcl} .P_i : D^{1\times q_{i-1}}\, R_i & \longrightarrow & D^{1\times q'_{i-1}}\, R'_i \\ \mu\, R_i & \longmapsto & \mu\, R_i\, P_i. \end{array}$$

Since $\mu\, R_i\, P_i \in D^{1\times q'_{i-1}}\, R'_i$ for all $\mu \in D^{1\times q_{i-1}}$, there exists $P'_{i-1} \in D^{q_{i-1}\times q'_{i-1}}$ such that:

$$\forall\, i \geq 1, \quad R_i\, P_i = P'_{i-1}\, R'_i. \tag{7}$$

Using (4), we get:

$$\left\{ \begin{array}{l} t(M) = \mathrm{ext}^1_D(N, D) = \ker_D(.Q_2)/(D^{1\times p_0}\, Q_1), \\ t(M') = \mathrm{ext}^1_D(N', D) = \ker_D(.Q'_2)/(D^{1\times p'_0}\, Q'_1). \end{array} \right.$$

Hence, Proposition 3.1 implies that $f \in \hom_D(M, M')$ induces an element of $\hom_D(t(M), t(M'))$ which is defined by the following left $D$-morphism:

$$\begin{array}{rcl} f_1 : t(M) & \longrightarrow & t(M') \\ \rho_1(\lambda) & \longmapsto & \rho'_1(\lambda\, P_1). \end{array} \tag{8}$$

We note that $\rho_1$ (resp., $\rho'_1$) is the restriction of the standard projection $\pi : D^{1\times p_1} \longrightarrow M$ (resp., $\pi' : D^{1\times p'_1} \longrightarrow M'$) to $\ker_D(.Q_2) \subseteq D^{1\times p_1}$ (resp., $\ker_D(.Q'_2) \subseteq D^{1\times p'_1}$).

Since we have $M/t(M) = D^{1\times p_1}/\ker_D(.Q_2)$ and $M'/t(M') = D^{1\times p'_1}/\ker_D(.Q'_2)$ (see Theorem 2.1), we obtain the left $D$-morphism $h_1 \in \hom_D(M/t(M), M'/t(M'))$ defined by

$$\begin{array}{ccccccc} 0 \longrightarrow & \ker_D(.Q_2) & \longrightarrow & D^{1\times p_1} & \xrightarrow{\sigma_1} & M/t(M) & \longrightarrow 0 \\ & \downarrow .P_1 & & \downarrow .P_1 & & \downarrow h_1 & \\ 0 \longrightarrow & \ker_D(.Q'_2) & \longrightarrow & D^{1\times p'_1} & \xrightarrow{\sigma'_1} & M'/t(M') & \longrightarrow 0, \end{array}$$

i.e.,

$$\begin{array}{rcl} h_1 : M/t(M) & \longrightarrow & M'/t(M') \\ \sigma_1(\lambda) & \longmapsto & \sigma'_1(\lambda\, P_1), \end{array} \tag{9}$$

for all $\lambda \in D^{1\times p_1}$, where $\sigma_1 : D^{1\times p_1} \longrightarrow M/t(M)$ (resp., $\sigma'_1 : D^{1\times p'_1} \longrightarrow M'/t(M')$) is the canonical projection.

*Corollary 3.1:* With the previous hypotheses and notations, $f \in \hom_D(M, M')$ induces the two left $D$-morphisms (8) and (9).

*Example 3.1:* Let us consider the tank model studied in [5] obtained by linearizing the Saint-Venant equations around the Riemann invariants. If $D = \mathbb{R}[\partial, \delta]$, where $\partial = \frac{d}{dt}$ and $\delta$ is the shift operator, then the system matrix is given by:

$$R = \left( \begin{array}{ccc} \delta^2 & 1 & -2\,\delta\,\partial \\ 1 & \delta^2 & -2\,\delta\,\partial \end{array} \right) \in D^{2\times 3}.$$

Let $M = D^{1\times 3}/(D^{1\times 2}\, R)$ be the left $D$-module finitely presented by the matrix $R$ and $N = D^2/(R\, D^3)$ its Auslander transpose. Let us consider a $D$-endomorphism $f$ of $M$ defined by two matrices $P \in D^{3\times 3}$ and $Q \in D^{2\times 2}$ satisfying $R\, P = Q\, R$. With the notations $P_0 = Q$, $P_1 = P$, $Q_1 = R$, then $\ker_D(Q_1.) = Q_2\, D$, where:

$$Q_2 = (2\,\partial\,\delta \quad 2\,\partial\,\delta \quad \delta^2 + 1)^T.$$

Then, $f$ induces the commutative exact diagram of $D$-modules defined in Figure 5. Applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to the previous commutative exact diagram, we obtain the morphism of complexes:

$$\begin{array}{ccccccc} 0 \longrightarrow & D^{1\times 2} & \xrightarrow{.Q_1} & D^{1\times 3} & \xrightarrow{.Q_2} & D & \longrightarrow 0 \\ & \downarrow .P_0 & & \downarrow .P_1 & & \downarrow .P_2 & \\ 0 \longrightarrow & D^{1\times 2} & \xrightarrow{.Q_1} & D^{1\times 3} & \xrightarrow{.Q_2} & D & \longrightarrow 0. \end{array}$$

The defects of exactness of the horizontal complexes are

$$\left\{ \begin{array}{l} \mathrm{ext}^1_D(N, D) = \ker_D(.Q_2)/(D^{1\times 2}\, Q_1) \\ \qquad\qquad = (D^{1\times 2} R_1)/(D^{1\times 2}\, Q_1) = t(M), \\ \mathrm{ext}^2_D(N, D) = D/(D^{1\times 3}\, Q_2) = D/(2\,\partial\,\delta, \delta^2 + 1), \end{array} \right.$$

where the matrix $R_1 \in D^{2\times 3}$ is defined by:

$$R_1 = \left( \begin{array}{ccc} 1 & -1 & 0 \\ 0 & -1 - \delta^2 & 2\,\partial\,\delta \end{array} \right).$$

Hence, the $D$-module $t(M)$ is generated by

$$\begin{cases} \tau_1 = y_1 - y_2, \\ \tau_2 = -(1+\delta^2)\, y_2 + 2\,\partial\,\delta\, y_3, \end{cases}$$

which satisfy $(\delta^2 - 1)\,\tau_i = 0$ for $i = 1, 2$. If $\{e_i\}_{i=1,2,3}$ is the standard basis of $D^{1\times 3}$ and $\sigma_1 : D^{1\times 3} \longrightarrow M/t(M) = D^{1\times 3}/(D^{1\times 2}\, R_1)$ the canonical projection onto $M/t(M)$, then the $z_i = \sigma_1(e_i)$'s generate $M/t(M)$ and satisfy:

$$\begin{cases} z_1 - z_2 = 0, \\ -(1+\delta^2)\, z_2 + 2\,\partial\,\delta\, z_3 = 0. \end{cases}$$

Using (6), (7) and Corollary 3.1, we obtain the $D$-morphisms

$$\begin{array}{rcl} (D^{1\times 2}\, R_1)/(D^{1\times 2}\, Q_1) & \xrightarrow{f_1} & (D^{1\times 2}\, R_1)/(D^{1\times 2}\, Q_1) \\ \pi(\mu\, R_1) & \longmapsto & \pi(\mu\, R_1\, P_1) = \pi(\mu\, P_0'\, R_1), \end{array}$$

$$\begin{array}{rcl} D/(D^{1\times 3}\, Q_2) & \xrightarrow{f_2} & D/(D^{1\times 3}\, Q_2) \\ \rho_2(\lambda) & \longmapsto & \rho_2(\lambda\, P_2), \end{array}$$

the commutative exact diagram of $D$-modules defined in Figure 6, where $P_0' \in D^{2\times 2}$ satisfies $R_1\, P_1 = P_0'\, R_1$ and:

$$\begin{array}{rcl} h_1 : M/t(M) & \longrightarrow & M/t(M) \\ \sigma_1(\lambda) & \longmapsto & \sigma_1(\lambda\, P_1). \end{array}$$

We find that $f \in \mathrm{end}_D(M)$ defined by

$$P_0 = \begin{pmatrix} -2\,\partial & 2\,\partial \\ 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2\,\partial & -2\,\partial & 0 \\ \delta & -\delta & 0 \end{pmatrix}$$

yields

$$P_2 = 0, \quad P_0' = \begin{pmatrix} -2\,\partial & 0 \\ -2\,\partial & 0 \end{pmatrix},$$

and induces that the $D$-morphisms $f_1$, $f_2$ and $h_1$ defined by:

$$\begin{cases} f_1(\tau_1) = -2\,\partial\,\tau_1, \\ f_1(\tau_2) = -2\,\partial\,\tau_1, \end{cases} \quad f_2 = 0, \quad \begin{cases} h_1(z_1) = 0, \\ h_1(z_2) = 2\,\partial\,(z_1 - z_2) = 0, \\ h_1(z_3) = \delta\,(z_1 - z_2) = 0. \end{cases}$$

Finally, if we consider another $f \in \mathrm{end}_D(M)$ defined by

$$P_0 = 0, \quad P_1 = \begin{pmatrix} 0 & 0 & 2\,\partial\,\delta \\ 0 & 0 & 2\,\partial\,\delta \\ 0 & 0 & 1+\delta^2 \end{pmatrix},$$

then

$$P_2 = \delta^2 + 1, \quad P_0' = 0,$$

which induces the $D$-morphisms $f_1$, $f_2$ and $h_1$ defined by:

$$\begin{cases} f_1(\tau_1) = 0, \\ f_1(\tau_2) = 0, \end{cases} \quad f_2(\rho_2(1)) = (\delta^2+1)\,\rho_2(1) = 0,$$

$$\begin{cases} h_1(z_1) = 2\,\partial\,\delta\, z_3 = (1+\delta^2)\, z_2, \\ h_1(z_2) = 2\,\partial\,\delta\, z_3 = (1+\delta^2)\, z_2, \\ h_1(z_3) = (1+\delta^2)\, z_3. \end{cases}$$

## IV. Morphisms, parametrizations and potentials

Corollary 3.1 is a well-known result in module theory, which can directly be proved without using the extension modules $\mathrm{ext}^1_D(N, D)$ and $\mathrm{ext}^1_D(N', D)$. However, we recall that the left $D$-modules $\mathrm{ext}^i_D(N, D)$'s for $i \geq 1$, characterize whether or not the left $D$-module $M$ is torsion-free, reflexive or projective (see Theorem 2.1 and [1]). Hence, for every $f \in \hom_D(M, M')$, Proposition 3.1 shows that the left $D$-morphisms $f_i$'s defined by (6) send the obstructions $\mathrm{ext}^i_D(N, D)$'s for $i \geq 1$ for $M$ to be projective to the obstructions $\mathrm{ext}^i_D(N', D)$'s for $i \geq 1$ for $M'$ to be projective. Hence, Corollary 3.1 is just a particular case of the previous remark.

We obtain the straightforward corollary of Proposition 3.1 and of the previous remark.

*Corollary 4.1:* With the previous hypotheses and notations, if $M$ and $M'$ are both torsion-free left $D$-modules, then every $f \in \hom_D(M, M')$ induces the following commutative exact diagram

$$\begin{array}{ccccccccc} D^{1\times p_0} & \xrightarrow{.Q_1} & D^{1\times p_1} & \xrightarrow{.Q_2} & D^{1\times p_2} & \xrightarrow{\pi_2} & M_2 & \longrightarrow & 0 \\ \downarrow .P_0 & & \downarrow .P_1 & & \downarrow .P_2 & & \downarrow h_2 & & \\ D^{1\times p_0'} & \xrightarrow{.Q_1'} & D^{1\times p_1'} & \xrightarrow{.Q_2'} & D^{1\times p_2'} & \xrightarrow{\pi_2'} & M_2' & \longrightarrow & 0, \end{array}$$

where $\pi_2 : D^{1\times p_2} \longrightarrow M_2$ (resp., $\pi_2' : D^{1\times p_2'} \longrightarrow M_2'$) is the canonical projection onto $M_2 = D^{1\times p_2}/(D^{1\times p_1}\, Q_2)$ (resp., $M_2' = D^{1\times p_2'}/(D^{1\times p_1'}\, Q_2')$) and:

$$\forall\, \lambda \in D^{1\times q_2}, \quad h_2(\pi_2(\lambda)) = \pi_2'(\lambda\, P_2).$$

If $M$ and $M'$ are both reflexive left $D$-modules, then we get the commutative exact diagram of left $D$-modules defined in Figure 7, where $\pi_3 : D^{1\times p_3} \longrightarrow M_3$ (resp., $\pi_3' : D^{1\times p_3'} \longrightarrow M_3'$) is the canonical projection onto the finitely presented left $D$-module $M_3 = D^{1\times p_3}/(D^{1\times p_2}\, Q_3)$ (resp., $M_3' = D^{1\times p_3'}/(D^{1\times p_2'}\, Q_3')$) and:

$$\forall\, \lambda \in D^{1\times q_3}, \quad h_3(\pi_3(\lambda)) = \pi_3'(\lambda\, P_3).$$

If $M$ and $M'$ are both projective left $D$-modules and $D$ has a finite global dimension $\mathrm{gld}(D) = n$ ([8]), then we get the commutative exact diagram of left $D$-modules defined in Figure 8, where $\pi_n : D^{1\times p_n} \longrightarrow M_n$ (resp., $\pi_n' : D^{1\times p_n'} \longrightarrow M_n'$) is the canonical projection onto the left $D$-module $M_n = D^{1\times p_n}/(D^{1\times p_{n-1}}\, Q_n)$ (resp., $M_n' = D^{1\times p_n'}/(D^{1\times p_{n-1}'}\, Q_n')$) and:

$$\forall\, \lambda \in D^{1\times q_n}, \quad h_n(\pi_n(\lambda)) = \pi_n'(\lambda\, P_n).$$

Finally, if $M$ and $M'$ are both free left $D$-modules, then there exist two matrices $Q_2 \in D^{p_1\times p_2}$ and $Q_2' \in D^{p_1'\times p_2'}$ such that the following commutative exact diagram holds:

$$\begin{array}{ccccccc} D^{1\times p_0} & \xrightarrow{.Q_1} & D^{1\times p_1} & \xrightarrow{.Q_2} & D^{1\times p_2} & \longrightarrow & 0 \\ \downarrow .P_0 & & \downarrow .P_1 & & \downarrow .P_2 & & \\ D^{1\times p_0'} & \xrightarrow{.Q_1'} & D^{1\times p_1'} & \xrightarrow{.Q_2'} & D^{1\times p_2'} & \longrightarrow & 0. \end{array}$$

The next corollary directly follows from Corollary 4.1.

$$\begin{array}{ccccccccc}
0 \longleftarrow & N & \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{\iota_p \circ \pi^{\star}} & M^{\star} & \longleftarrow 0 \\
& \uparrow g & & \uparrow Q. & & \uparrow P. & & \uparrow f^{\star} & \\
0 \longleftarrow & N' & \xleftarrow{\kappa'} & D^{q'} & \xleftarrow{R'.} & D^{p'} & \xleftarrow{\iota_{p'} \circ \pi'^{\star}} & M'^{\star} & \longleftarrow 0
\end{array}$$

Fig. 1. Figure 1

$$\begin{array}{ccccccccc}
0 \longleftarrow & N & \xleftarrow{\kappa} & D^{p_0} & \xleftarrow{Q_1.} & D^{p_1} & \xleftarrow{Q_2.} & D^{p_2} & \xleftarrow{Q_3.} \ldots \\
& \uparrow g & & \uparrow P_0. & & \uparrow P_1. & & \uparrow P_2. & \\
0 \longleftarrow & N' & \xleftarrow{\kappa'} & D^{p'_0} & \xleftarrow{Q'_1.} & D^{p'_1} & \xleftarrow{Q'_2.} & D^{p'_2} & \xleftarrow{Q'_3.} \ldots
\end{array}$$

Fig. 2. Figure 2

$$\begin{array}{ccccccccc}
0 \longrightarrow & N^{\star} & \xrightarrow{i_{p_0} \circ \kappa^{\star}} & D^{1\times p_0} & \xrightarrow{.Q_1} & D^{1\times p_1} & \xrightarrow{.Q_2} & D^{1\times p_2} & \xrightarrow{.Q_3} \ldots \\
& \downarrow g^{\star} & & \downarrow .P_0 & & \downarrow .P_1 & & \downarrow .P_2 & \\
0 \longrightarrow & N'^{\star} & \xrightarrow{i_{p'_0} \circ \kappa'^{\star}} & D^{1\times p'_0} & \xrightarrow{.Q'_1} & D^{1\times p'_1} & \xrightarrow{.Q'_2} & D^{1\times p'_2} & \xrightarrow{.Q'_3} \ldots
\end{array}$$

Fig. 3. Figure 3

$$\begin{array}{ccccccc}
0 \longrightarrow & D^{1\times p_{i-1}}\, Q_i & \longrightarrow & \ker_D(.Q_{i+1}) & \xrightarrow{\rho_i} & \mathrm{ext}^i_D(N,D) = \ker_D(.Q_{i+1})/(D^{1\times p_{i-1}}\, Q_i) & \longrightarrow 0 \\
& \downarrow .P_i & & \downarrow .P_i & & \downarrow f_i & \\
0 \longrightarrow & D^{1\times p'_{i-1}}\, Q'_i & \longrightarrow & \ker_D(.Q'_{i+1}) & \xrightarrow{\rho'_i} & \mathrm{ext}^i_D(N',D) = \ker_D(.Q'_{i+1})/(D^{1\times p'_{i-1}}\, Q'_i) & \longrightarrow 0
\end{array}$$

Fig. 4. Figure 4

$$\begin{array}{ccccccccc}
0 \longleftarrow & N & \xleftarrow{\kappa} & D^{2} & \xleftarrow{Q_1.} & D^{3} & \xleftarrow{Q_2.} & D & \longleftarrow 0 \\
& \uparrow g & & \uparrow P_0. & & \uparrow P_1. & & \uparrow P_2. & \\
0 \longleftarrow & N & \xleftarrow{\kappa} & D^{2} & \xleftarrow{Q_1.} & D^{3} & \xleftarrow{Q_2.} & D & \longleftarrow 0
\end{array}$$

Fig. 5. Figure 5

$$\begin{array}{ccccccccc}
0 \longrightarrow & D^{1\times 2} & \xrightarrow{.R_1} & D^{1\times 3} & \xrightarrow{.Q_2} & D & \longrightarrow & \mathrm{ext}^2_D(N,D) & \longrightarrow 0 \\
& \downarrow .P'_0 & & \downarrow .P_1 & & \downarrow .P_2 & & \downarrow f_2 & \\
0 \longrightarrow & D^{1\times 2} & \xrightarrow{.R_1} & D^{1\times 3} & \xrightarrow{.Q_2} & D & \longrightarrow & \mathrm{ext}^2_D(N,D) & \longrightarrow 0
\end{array}$$

Fig. 6. Figure 6

$$\begin{array}{cccccccccc}
D^{1\times p_0} & \xrightarrow{.Q_1} & D^{1\times p_1} & \xrightarrow{.Q_2} & D^{1\times p_2} & \xrightarrow{.Q_3} & D^{1\times p_3} & \xrightarrow{\pi_3} & M_3 & \longrightarrow 0 \\
\downarrow .P_0 & & \downarrow .P_1 & & \downarrow .P_2 & & \downarrow .P_3 & & \downarrow h_3 & \\
D^{1\times p'_0} & \xrightarrow{.Q'_1} & D^{1\times p'_1} & \xrightarrow{.Q'_2} & D^{1\times p'_2} & \xrightarrow{.Q'_3} & D^{1\times p'_3} & \xrightarrow{\pi'_3} & M'_3 & \longrightarrow 0
\end{array}$$

Fig. 7. Figure 7

$$\begin{array}{cccccccccc}
D^{1\times p_0} & \xrightarrow{.Q_1} & D^{1\times p_1} & \xrightarrow{.Q_2} \ldots \xrightarrow{.Q_{n-1}} & D^{1\times p_{n-1}} & \xrightarrow{.Q_n} & D^{1\times p_n} & \xrightarrow{\pi_n} & M_n & \longrightarrow 0 \\
\downarrow .P_0 & & \downarrow .P_1 & & \downarrow .P_{n-1} & & \downarrow .P_n & & \downarrow h_n & \\
D^{1\times p'_0} & \xrightarrow{.Q'_1} & D^{1\times p'_1} & \xrightarrow{.Q'_2} \ldots \xrightarrow{.Q'_{n-1}} & D^{1\times p'_{n-1}} & \xrightarrow{.Q'_n} & D^{1\times p'_n} & \xrightarrow{\pi'_n} & M'_n & \longrightarrow 0
\end{array}$$

Fig. 8. Figure 8

*Corollary 4.2:* With the previous hypotheses and notations, if $M$ and $M'$ are both torsion-free left $D$-modules and $\mathcal{F}$ an injective left $D$-module, then every $f \in \hom_D(M, M')$ induces the commutative exact diagram of abelian groups defined in Figure 9, where $k_2 : \ker_{\mathcal{F}}(Q'_2.) \longrightarrow \ker_{\mathcal{F}}(Q_2.)$ is defined by $k_2(\zeta') = P_2\, \zeta'$ for all $\zeta' \in \ker_{\mathcal{F}}(Q'_2.)$.

Now, if $M$ and $M'$ are two reflexive left $D$-modules and $\mathcal{F}$ an injective left $D$-module, then $f \in \hom_D(M, M')$ induces the commutative exact diagram of abelian groups defined in Figure 10, where $k_3 : \ker_{\mathcal{F}}(Q'_3.) \longrightarrow \ker_{\mathcal{F}}(Q_3.)$ is defined by $k_3(\zeta') = P_3\, \zeta'$ for all $\zeta' \in \ker_{\mathcal{F}}(Q'_3.)$.

Moreover, if $M$ and $M'$ are two projective left $D$-modules and $\mathcal{F}$ is a left $D$-module, then $f \in \hom_D(M, M')$ induces the following commutative exact diagram of abelian groups defined in Figure 11, where $k_n : \ker_{\mathcal{F}}(Q'_n.) \longrightarrow \ker_{\mathcal{F}}(Q_n.)$ is defined by $k_n(\zeta') = P_n\, \zeta'$ for all $\zeta' \in \ker_{\mathcal{F}}(Q'_n.)$.

Finally, if $M$ and $M'$ are both free left $D$-modules and $\mathcal{F}$ is a left $D$-module, then there exist two matrices $Q_2 \in D^{p_1 \times p_2}$ and $Q'_2 \in D^{p'_1 \times p'_2}$ such that the following commutative exact diagram of left $D$-modules holds:

$$\begin{array}{ccccccc}
\mathcal{F}^{p_0} & \xleftarrow{Q_1.} & \mathcal{F}^{p_1} & \xleftarrow{Q_2.} & \mathcal{F}^{p_2} & \longleftarrow 0 \\
\uparrow P_0. & & \uparrow P_1. & & \uparrow P_2. & \\
\mathcal{F}^{p'_0} & \xleftarrow{Q'_1.} & \mathcal{F}^{p'_1} & \xleftarrow{Q'_2.} & \mathcal{F}^{p'_2} & \longleftarrow 0.
\end{array}$$

*Example 4.1:* Let $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$ and consider the left $D$-module $M = D^{1\times 3}/(D\, Q_1)$ finitely presented by the divergence operator $Q_1 = (\partial_1 \quad \partial_2 \quad \partial_3)$ in $\mathbb{R}^3$. The left $D$-module $M$ is reflexive ([1]) and $M$ can be parametrized by the matrix

$$Q_2 = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \in D^{3\times 3}$$

defining the curl operator, i.e., $\ker_D(.Q_2) = D\, Q_1$. Moreover, the matrix $Q_3 = Q_1^T$ defining the gradient operator parametrizes the $D$-module $D^{1\times 3}/(D^{1\times 3}\, Q_2)$, i.e., $\ker_D(.Q_3) = D^{1\times 3}\, Q_2$. Using OREMORPHISMS ([4]), we obtain that an element $f \in \mathrm{end}_D(M)$ is defined by $f(\pi(\lambda)) = \pi(\lambda\, P_1)$, where $\pi : D^{1\times 3} \longrightarrow M$ is the canonical projection onto $M$, $\lambda \in D^{1\times 3}$ and $P_1 \in D^{3\times 3}$ is defined by

$$P_1 = \begin{pmatrix} \alpha_8 & -\alpha_3\, \partial_3 - \alpha_7\, \partial_2 & -\alpha_4\, \partial_3 - \alpha_6\, \partial_2 \\ -\alpha_5\, \partial_3 & \alpha_8 + \alpha_7\, \partial_1 + (\alpha_1 - \alpha_9)\, \partial_3 & -\alpha_2\, \partial_3 + \alpha_6\, \partial_1 \\ \alpha_5\, \partial_2 & \alpha_3\, \partial_1 + (-\alpha_1 + \alpha_9)\, \partial_2 & \alpha_2\, \partial_2 + \alpha_4\, \partial_1 + \alpha_8 \end{pmatrix},$$

where the $\alpha_i$'s are arbitrary elements of $D$ for $i = 1, \ldots, 9$. We can check that $Q_1\, P_1 = \alpha_8\, Q_1$. According to Corollary 4.1, there exist two matrices

$$P_2 = \begin{pmatrix} \alpha_8 + \alpha_2\, \partial_2 + (\alpha_1 - \alpha_9)\, \partial_3 & \alpha_5\, \partial_3 & -\alpha_5\, \partial_2 \\ -\alpha_4\, \partial_2 + \alpha_3\, \partial_3 & \alpha_2\, \partial_2 + \alpha_4\, \partial_1 + \alpha_8 & -\alpha_3\, \partial_1 + (\alpha_1 - \alpha_9)\, \partial_2 \\ \alpha_6\, \partial_2 - \alpha_7\, \partial_3 & \alpha_2\, \partial_3 - \alpha_6\, \partial_1 & \alpha_8 + \alpha_7\, \partial_1 + (\alpha_1 - \alpha_9)\, \partial_3 \end{pmatrix}$$

$$P_3 = \alpha_8 + \alpha_2\, \partial_2 + (\alpha_1 - \alpha_9)\, \partial_3$$

such that the commutative exact diagram defined in Figure 7 holds, i.e., such that $Q_2\, P_2 = P_1\, Q_2$ and $Q_3\, P_3 = P_2\, Q_3$.

If $\mathcal{F}$ is a $D$-module, then the endomorphism $f$ of $M$ defined by the matrix $P_1$ induces the Galois transformation:

$$\begin{array}{rcl} k_1 : \ker_{\mathcal{F}}(Q_1.) & \longrightarrow & \ker_{\mathcal{F}}(Q_1.) \\ \eta & \longmapsto & \overline{\eta} = P_1\, \eta. \end{array}$$

Now, if $\mathcal{F}$ is an injective $D$-module (e.g., $\mathcal{F} = C^\infty(\Omega)$, where $\Omega$ is an open convex subset of $\mathbb{R}^3$) and $\eta \in \ker_{\mathcal{F}}(Q_1.)$ is parametrized by a potential $\xi \in \mathcal{F}^3$, i.e., $\eta = Q_2\, \xi$, then the Galois transformation $P_1.$ induces a transformation on $\xi$ defined by $\overline{\xi} = P_2\, \xi$ which satisfies $\overline{\eta} = Q_2\, \overline{\xi}$.

This result can be checked again as follows: combining $\overline{\eta} = P_1\, \eta$, $\eta = Q_2\, \xi$ and $Q_2\, P_2 = P_1\, Q_2$, we get

$$\overline{\eta} = P_1\, (Q_2\, \xi) = Q_2\, (P_2\, \xi) = Q_2\, \overline{\xi},$$

where $\overline{\xi} = P_2\, \xi$.

In its turn, the transformation $P_2.$ induces the following Galois transformation of $\ker_{\mathcal{F}}(Q_2.)$:

$$\begin{array}{rcl} k_2 : \ker_{\mathcal{F}}(Q_2.) & \longrightarrow & \ker_{\mathcal{F}}(Q_2.) \\ \xi & \longmapsto & \overline{\xi} = P_2\, \xi. \end{array}$$

If $\xi \in \ker_{\mathcal{F}}(Q_2.)$ is parametrized by a potential $\theta \in \mathcal{F}$, i.e., $\xi = Q_3\, \theta$, then the Galois transformation $P_2.$ induces the transformation on $\theta$ defined by $\overline{\theta} = P_3\, \theta$, which is such that $\overline{\xi} = Q_3\, \overline{\theta}$. Indeed, combining $\overline{\xi} = P_2\, \xi$, $\xi = Q_3\, \theta$ and $Q_3\, P_3 = P_2\, Q_3$, we finally obtain

$$\overline{\xi} = P_2\, (Q_3\, \theta) = Q_3\, (P_3\, \theta) = Q_3\, \overline{\theta},$$

where $\overline{\theta} = P_3\, \theta$.

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$$\begin{array}{ccccccccc}
\mathcal{F}^{p_0} & \xleftarrow{Q_1.} & \mathcal{F}^{p_1} & \xleftarrow{Q_2.} & \mathcal{F}^{p_2} & \longleftarrow & \ker_{\mathcal{F}}(Q_2.) & \longleftarrow & 0 \\
\uparrow P_0. & & \uparrow P_1. & & \uparrow P_2. & & \uparrow k_2 & & \\
\mathcal{F}^{p_0'} & \xleftarrow{Q_1'.} & \mathcal{F}^{p_1'} & \xleftarrow{Q_2'.} & \mathcal{F}^{p_2'} & \longleftarrow & \ker_{\mathcal{F}}(Q_2'.) & \longleftarrow & 0
\end{array}$$

Fig. 9. Figure 9

$$\begin{array}{ccccccccccc}
\mathcal{F}^{p_0} & \xleftarrow{Q_1.} & \mathcal{F}^{p_1} & \xleftarrow{Q_2.} & \mathcal{F}^{p_2} & \xleftarrow{Q_3.} & \mathcal{F}^{p_3} & \longleftarrow & \ker_{\mathcal{F}}(Q_3.) & \longleftarrow & 0 \\
\uparrow P_0. & & \uparrow P_1. & & \uparrow P_2. & & \uparrow P_3. & & \uparrow k_3 & & \\
\mathcal{F}^{p_0'} & \xleftarrow{Q_1'.} & \mathcal{F}^{p_1'} & \xleftarrow{Q_2'.} & \mathcal{F}^{p_2'} & \xleftarrow{Q_3'.} & \mathcal{F}^{p_3'} & \longleftarrow & \ker_{\mathcal{F}}(Q_3'.) & \longleftarrow & 0
\end{array}$$

Fig. 10. Figure 10

$$\begin{array}{ccccccccccc}
\mathcal{F}^{p_0} & \xleftarrow{Q_1.} & \mathcal{F}^{p_1} & \xleftarrow{Q_2.} \dots \xleftarrow{Q_{n-1}.} & \mathcal{F}^{p_{n-1}} & \xleftarrow{Q_n.} & \mathcal{F}^{p_n} & \longleftarrow & \ker_{\mathcal{F}}(Q_n.) & \longleftarrow & 0 \\
\uparrow P_0. & & \uparrow P_1. & & \uparrow P_{n-1}. & & \uparrow P_n. & & \uparrow k_n & & \\
\mathcal{F}^{p_0'} & \xleftarrow{Q_1'.} & \mathcal{F}^{p_1'} & \xleftarrow{Q_2'.} \dots \xleftarrow{Q_{n-1}'.} & \mathcal{F}^{p_{n-1}'} & \xleftarrow{Q_n'.} & \mathcal{F}^{p_n'} & \longleftarrow & \ker_{\mathcal{F}}(Q_n'.) & \longleftarrow & 0
\end{array}$$

Fig. 11. Figure 11