A constructive algebraic analysis approach to Artstein's reduction of linear time-delay systems

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Abstract Artstein's results show that a first-order linear differential system with delayed inputs is equivalent to a first-order linear differential system without delay under an invertible transformation which includes integral and time-delay operators. Within a constructive algebraic approach, we show how this reduction can be found again, generalized and interpreted as a particular isomorphism between modules defining the two above linear systems. Moreover, we prove that Artstein's reduction can be obtained in an automatic way by means of symbolic computation techniques, and thus can be implemented in dedicated computer algebra systems.

Keywords: Algebraic analysis approach, linear differential time-delay systems, Artstein's reduction, rings of integro-differential time-delay operators, module theory, computer algebra.

1. INTRODUCTION

Artstein's famous reduction (Artstein (1982)) proves the equivalence between linear differential systems with delayed inputs and linear differential systems without time-delays. The purpose of this paper is to show how to find again the different Artstein's integral transformations in a mechanical way, i.e., without educated guess or clever thoughts. Within the *algebraic analysis approach* to linear functional systems (Chyzak et al. (2005); Fliess et al. (1998); Quadrat (2010)), we first reformulate Artstein's reduction in terms of an *isomorphism problem* (Cluzeau et al. (2008)) between two *finitely presented left modules* over a *ring of integro-differential time-delay operators*. These modules are explicitly defined by means of the matrices of functional operators defining the linear functional systems. Considering the *commutation rules* for the differential, integral, time-delay/dilation operators, we present a constructive method to find again and extend Artstein's reduction. These results advocate for a complete *algorithmic study of the noncommutative polynomial ring of integro-differential time-delay/dilation operators* (Quadrat (2015)) and for the development of dedicated packages such as `IntDiffOp` (Korporal et al. (2012)) for rings of integro-differential operators (see Korporal et al. (2012); Quadrat et al. (2013); Regensburger et al. (2009) and the references therein). Finally, this algorithmic approach can also be used to handle different computations over the ring $\mathcal{E}$ introduced in Loiseau (2000) for the study of differential time-delay systems (see Quadrat (2015)).

2. ALGEBRAIC ANALYSIS APPROACH

Within the *algebraic analysis approach*, a linear functional system is studied by means of methods of module theory, homological algebra and sheaf theory. Let us briefly review this approach. For more details, see Chyzak et al. (2005); Fliess et al. (1998); Quadrat (2010) and the references therein. In what follows, let D be a (non necessarily commutative) ring, $R \in D^{q\times p}$ a $q \times p$ matrix with entries in D, and $\mathcal{F}$ a left D-module. Then, a *behaviour* in the sense of Willems can be defined as follows:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}. \tag{1}$$

The behaviour is the analytical side of an underlying system of D-linear equations defined by means of the *left D-module finitely presented* by R, namely:

$$M := D^{1\times p}/(D^{1\times q}\,R). \tag{2}$$

Let us explain why M defines a linear system of equations. Let $\{f_j\}_{j=1,\dots,p}$ denote the *standard basis* of the *free* left D-module $D^{1\times p}$, i.e., f_j is the row vector of length p with 1 at the j^{th} entry and 0 elsewhere, $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M which sends $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda) \in M$, and $y_j := \pi(f_j)$ for $j = 1,\dots,p$. Then, $\{y_j\}_{j=1,\dots,p}$ is a set of generators of M since every $m \in M$ is the residue class $\pi(\lambda)$ of a certain $\lambda = (\lambda_1 \ \dots \ \lambda_p) \in D^{1\times p}$, which yields:

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j\, f_j\right) = \sum_{j=1}^{p} \lambda_j\, \pi(f_j) = \sum_{j=1}^{p} \lambda_j\, y_j.$$

Let us note by $R_{i\bullet}$ the i^{th} row of the matrix R. The set of generators $\{y_j\}_{j=1,\dots,p}$ of M satisfies the D-linear relations

$$\sum_{j=1}^{p} R_{ij}\, y_j = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \pi\left(\sum_{j=1}^{p} R_{ij}\, f_j\right) = \pi(R_{i\bullet}) = 0,$$

for $i = 1,\dots,q$, since $R_{i\bullet} \in D^{1\times q}\,R$. Hence, if we note $y := (y_1 \ \dots \ y_p)^T \in M^p$, then y satisfies $R\,y = 0$.

Let $\hom_D(M,\mathcal{F})$ be the abelian group formed by all the *left D-homomorphisms* (i.e., the left D-linear maps) from M to $\mathcal{F}$. If $\phi \in \hom_D(M,\mathcal{F})$, then, by definition, we have

$$\phi(d_1\, m_1 + d_2\, m_2) = d_1\, \phi_1(m_1) + d_2\, \phi(m_2),$$

for all d_1, $d_2 \in D$ and for all m_1, $m_2 \in M$. A homomorphism ϕ is an *isomorphism* if ϕ is injective and surjective.

The next result, which is a standard result in homological algebra, shows that the behaviour $\ker_{\mathcal{F}}(R.)$ can intrinsically be interpreted as the "dual" $\hom_D(M, \mathcal{F})$ of M.

Theorem 1. With the above notations, we have the following isomorphism of abelian groups

$$\begin{aligned} \chi : \hom_D(M, \mathcal{F}) &\longrightarrow \ker_{\mathcal{F}}(R.) \\ \phi &\longmapsto \eta := (\phi(y_1)\ \ldots\ \phi(y_p))^T, \end{aligned}$$

whose inverse χ^{-1} is defined by $\chi^{-1}(\eta) = \phi_\eta$, where $\phi_\eta(\pi(\lambda)) := \lambda\, \eta$ for all $\lambda \in D^{1\times p}$ and for all $\eta \in \ker_{\mathcal{F}}(R.)$.

Let us give a sketch of a proof of Theorem 1. Clearly, if $\eta = \chi(\phi)$ is defined as in Theorem 1, then we get

$$\forall\, i = 1, \ldots, q,\ \sum_{j=1}^{p} R_{ij}\, \phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \phi(0) = 0,$$

which shows that $\eta \in \ker_{\mathcal{F}}(R.)$. Now, ϕ_η is well-defined since $\pi(\lambda) = 0$ is equivalent to $\lambda = \mu\, R$ for a certain $\mu \in D^{1\times q}$, which yields $\lambda\, \eta = \mu\, (R\, \eta) = 0$ and shows that $\phi_\eta(0) = 0$. Moreover, ϕ_η is clearly left D-linear, and thus we have $\phi_\eta \in \hom_D(M, \mathcal{F})$. Now, if $\eta \in \ker_{\mathcal{F}}(R.)$, let us note $\sigma(\eta) := \phi_\eta$, where $\phi_\eta(\pi(\lambda)) := \lambda\, \eta$ for all $\lambda \in D^{1\times p}$. Then, we have $(\sigma \circ \chi)(\phi) = \phi_{(\phi(y_1)\ \ldots\ \phi(y_p))^T}$, i.e.

$$\begin{aligned} (\sigma \circ \chi)(\phi)(\pi(\lambda)) &= \sum_{j=1}^{p} \lambda_j\, \phi(y_j) = \phi\left(\sum_{j=1}^{p} \lambda_j\, \pi(f_j)\right) \\ &= \phi(\pi(\lambda)), \end{aligned}$$

and thus $\sigma \circ \chi = \mathrm{id}_{\hom_D(M,\mathcal{F})}$. Finally, if $\eta \in \ker_{\mathcal{F}}(R.)$, then we have $(\chi \circ \sigma)(\eta) = (\phi_\eta(y_1), \ldots, \phi_\eta(y_p))^T$, where $\phi_\eta(y_j) = f_j\, \eta = \eta_j$, i.e., $\chi \circ \sigma = \mathrm{id}_{\ker_{\mathcal{F}}(R.)}$.

Theorem 1 shows that the behaviour $\ker_{\mathcal{F}}(R.)$ can intrinsically be studied by means of the left D-modules M and $\mathcal{F}$ using module theory and homological algebra methods.

Let $R' \in D^{q'\times p'}$, $M' := D^{1\times p'}/(D^{1\times q'}\, R')$ be the left D-module finitely presented by R', $\pi' : D^{1\times p'} \longrightarrow M'$ the canonical projection onto M', and the behaviour:

$$\ker_{\mathcal{F}}(R'.) := \{\eta' \in \mathcal{F}^{p'} \mid R'\, \eta' = 0\} \cong \hom_D(M', \mathcal{F}).$$

Let us now show that $\phi \in \hom_D(M, M')$ induces a homomorphism $\phi^\star : \ker_{\mathcal{F}}(R'.) \longrightarrow \ker_{\mathcal{F}}(R.)$.

Theorem 2. (Cluzeau et al. (2008)). We have:

- Any $\phi \in \hom_D(M, M')$ is defined by
$$\phi(\pi(\lambda)) = \pi'(\lambda\, P), \quad \forall\, \lambda \in D^{1\times p}, \tag{3}$$
where $P \in D^{p\times p'}$ satisfies $D^{1\times q}\,(R\, P) \subseteq D^{1\times q'}\, R'$, i.e., P is such that there exists $Q \in D^{q\times q'}$ satisfying:
$$R\, P = Q\, R'. \tag{4}$$
The matrices P and Q are not uniquely defined since
$$\forall\, Z \in D^{p\times q'}, \quad P' := P + Z\, R', \quad Q' := Q + R\, Z, \tag{5}$$
also satisfy (4) and $\phi(\pi(\lambda)) = \pi'(\lambda\, P')$ for $\lambda \in D^{1\times p}$.
- $\phi \in \hom_D(M, M')$ induces the following homomorphism of abelian groups:
$$\begin{aligned} \phi^\star : \ker_{\mathcal{F}}(R'.) &\longrightarrow \ker_{\mathcal{F}}(R.) \\ \eta' &\longmapsto \eta := P\, \eta'. \end{aligned} \tag{6}$$

Since Theorem 2 plays a fundamental role in what follows, we give a sketch of a proof. If $\{f_j\}_{j=1,\ldots,p}$ (resp., $\{f'_k\}_{k=1,\ldots,p'}$) is the standard basis of $D^{1\times p}$ (resp., $D^{1\times p'}$), then $\{\pi(f_j)\}_{j=1,\ldots,p}$ (resp., $\{\pi'(f'_k)\}_{k=1,\ldots,p'}$) is a set of generators of M (resp., M'). Now, $\phi \in \hom_D(M, M')$ sends the generators of M to elements of M', i.e., we have

$$\forall\, j = 1, \ldots, p, \quad \phi(\pi(f_j)) = \sum_{k=1}^{p'} P_{jk}\, \pi'(f'_k),$$

where the P_{jk}'s are elements of D which must satisfy the relations $\phi(0) = 0$, i.e., ϕ maps $\sum_{j=1}^{p} R_{ij}\, \pi(f_j) = 0$ to 0. Hence, for $i = 1, \ldots, q$, we must have

$$\begin{aligned} \phi\left(\sum_{j=1}^{p} R_{ij}\, \pi(f_j)\right) &= \sum_{j=1}^{p} R_{ij}\, \phi(\pi(f_j)) \\ &= \sum_{j=1}^{p} R_{ij}\, \left(\sum_{k=1}^{p'} P_{jk}\, \pi'(f'_k)\right) \\ &= \pi'\left(\sum_{k=1}^{p'}\left(\sum_{j=1}^{p} R_{ij}\, P_{jk}\right)\, f'_k\right) = 0, \end{aligned}$$

and thus $\left(\sum_{j=1}^{p} R_{ij}\, P_{j1}\ \ldots\ \sum_{j=1}^{p} R_{ij}\, P_{jp'}\right) \in D^{1\times q'}\, R'$, i.e., there exists $Q_i \in D^{1\times q'}$ such that:

$$\forall\, i = 1, \ldots, q,\ \left(\sum_{j=1}^{p} R_{ij}\, P_{j1}\ \ldots\ \sum_{j=1}^{p} R_{ij}\, P_{jp'}\right) = Q_i\, R'.$$

If $Q := (Q_1^T \ldots Q_q^T)^T \in D^{q\times q'}$, then we obtain (4).

The P_{jk}'s are not uniquely defined by $\phi \in \hom_D(M, M')$. Indeed, if $\phi(\pi(f_j)) = \sum_{k=1}^{p'} \overline{P}_{jk}\, \pi'(f'_k)$ for $j = 1, \ldots, p$, where the $\overline{P}_{jk}$'s are elements of D, then we have

$$\pi'\left(\sum_{k=1}^{p'} (\overline{P}_{jk} - P_{jk})\, f'_k\right) = \sum_{k=1}^{p'} \left(\overline{P}_{jk} - P_{jk}\right)\, \pi'(f'_k) = 0,$$

i.e., $\overline{P}_{j\bullet} - P_{j\bullet} = (\overline{P}_{j1} - P_{j1}, \ldots, \overline{P}_{jp'} - P_{jp'})$ belongs to $D^{1\times q'}\, R'$, and thus there exists $Z_j \in D^{1\times q'}$ satisfying $\overline{P}_{j\bullet} - P_{j\bullet} = Z_j\, R'$. Hence, we obtain $\overline{P} - P = Z\, R'$, where $Z := (Z_1^T \ldots Z_p^T)^T \in D^{p\times q'}$. Moreover, using (4), we get

$$R\, \overline{P} = R\, P + R\, Z\, R' = Q\, R' + R\, Z\, R' = (Q + R\, Z)\, R',$$

which proves that $R\, \overline{P} = \overline{Q}\, R'$ where $\overline{Q} := Q + R\, Z$. Finally, 2 is a direct consequence of (4), i.e.:

$$\forall\, \eta' \in \ker_{\mathcal{F}}(R'.),\ R\,(P\, \eta') = Q\,(R'\, \eta') = 0.$$

3. TRANSFORMATIONS OF LINEAR DTD SYSTEMS

Let us now consider the following linear differential system

$$\dot{z}(t) = E\, z(t) + F\, v(t), \tag{7}$$

where $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times m}$, and the following linear differential time-delay (DTD) system

$$\dot{x}(t) = A\, x(t) + B_0\, u(t) + B_1\, u(t-h), \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$, B_0, $B_1 \in \mathbb{R}^{n\times m}$, and:

$$h \in \mathbb{R}_{\geq 0} := \{t \in \mathbb{R} \mid t \geq 0\}.$$

Let $D := \mathbb{R}[\partial, \delta]$ be the polynomial ring of DTD operators, where $(\partial\, y)(t) := \dot{y}(t)$ and $(\delta\, y)(t) := y(t-h)$. The ring D is commutative since we have:

$$((\partial \circ \delta)\, y)(t) = \partial(y(t-h)) = \dot{y}(t-h) = ((\delta \circ \partial)\, y)(t).$$

The composition of operators $\circ$ will simply be denoted by the standard product. An element $d \in D$ is of the form $d = \sum_{0 \le i \le r,\, 0 \le j \le s} a_{ij}\, \partial^i\, \delta^j$, where $a_{ij} \in \mathbb{R}$ and ∂^i is the i^{th} composition of ∂, i.e., the i^{th} derivative, and:

$$\forall\, j \in \mathbb{N}, \quad (\delta^j\, y)(t) = y(t - j\, h).$$

The matrices of DTD operators associated with (7) and (8) are respectively defined by:

$$R = (\partial\, I_n - E \quad -F) \in D^{n \times (n+m)},$$
$$R' = (\partial\, I_n - A \quad -B_0 - B_1\, \delta) \in D^{n \times (n+m)}.$$

Let us consider the D-module $M := D^{1 \times (n+m)}/(D^{1 \times n}\, R)$ (resp., $M' := D^{1 \times (n+m)}/(D^{1 \times n}\, R')$) finitely presented by the matrix R (resp., R'). Let $\pi : D^{1 \times (n+m)} \longrightarrow M$ (resp., $\pi' : D^{1 \times (n+m)} \longrightarrow M'$) be the canonical projections onto M (resp., M'). By Theorem 2, $\phi \in \hom_D(M, M')$ is defined by (3), where the matrix

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in D^{(n+m) \times (n+m)} \tag{9}$$

satisfies (4) for a certain $Q \in D^{n \times n}$.

Using (5) and the fact that R' is a matrix of first-order operators in ∂, without loss of generality, we can assume that P_{11} and P_{21} depend only on δ. If we note $C := \mathbb{R}[\delta]$, then $P_{11} \in C^{n \times n}$, $P_{21} \in C^{m \times n}$, $P_{12} \in D^{n \times m}$ and $P_{22} \in D^{m \times m}$, and thus (4) yields:

$$\begin{cases} (\partial\, I_n - E)\, P_{11}(\delta) - F\, P_{21}(\delta) = Q(\partial, \delta)\, (\partial\, I_n - A), \\ (\partial\, I_n - E)\, P_{12}(\partial, \delta) - F\, P_{22}(\partial, \delta) = -Q(\partial, \delta)\, (B_0 + B_1\, \delta). \end{cases} \tag{10}$$

The first equation of (10) implies that:

$$(P_{11}(\delta) - Q(\partial, \delta))\, \partial + Q(\partial, \delta)\, A - E\, P_{11}(\delta) - F\, P_{21}(\delta) = 0.$$

Considering the degree in ∂ of the above element, this last identity can only hold if $\deg_{\partial} Q(\partial, \delta) = 0$, i.e., Q depends only on δ, which yields:

$$\begin{cases} Q = P_{11}(\delta), \\ P_{11}(\delta)\, A - E\, P_{11}(\delta) - F\, P_{21}(\delta) = 0. \end{cases} \tag{11}$$

For the study of the second equation of (10), see Quadrat (2015). To simplify, we suppose here that $P_{12} \in C^{n \times m}$ and $P_{22} \in C^{m \times m}$, i.e., the transformation (6) defined by P, i.e.

$$\begin{cases} z = P_{11}\, x + P_{12}\, u, \\ v = P_{21}\, x + P_{22}\, u, \end{cases} \tag{12}$$

does not contain derivatives of u. Then, we get:

$$(\partial\, I_n - E)\, P_{12}(\delta) - F\, P_{22}(\delta) + P_{11}(\delta)\, (B_0 + B_1\, \delta) = 0$$
$$\Leftrightarrow\ \partial\, P_{12}(\delta) + P_{11}(\delta)\, (B_0 + B_1\, \delta) - E\, P_{12}(\delta) - F\, P_{22}(\delta) = 0. \tag{13}$$

The identity only holds if we have:

$$\begin{cases} P_{12} = 0, \\ P_{11}(\delta)\, (B_0 + B_1\, \delta) - F\, P_{22}(\delta) = 0. \end{cases} \tag{14}$$

To simplify again, we consider the particular case where the matrix P is defined by $P_{11} \in \mathrm{GL}_n(\mathbb{R})$, $P_{22} \in \mathrm{GL}_m(\mathbb{R})$ and $P_{21} = 0$, where $\mathrm{GL}_n(\mathbb{R})$ is the group of invertible matrices of $\mathbb{R}^{n \times n}$. Note that P_{21} corresponds to a feedback (see (12)). P is invertible and its inverse P^{-1} is defined by:

$$P^{-1} = \begin{pmatrix} P_{11}^{-1} & -P_{11}^{-1}\, P_{12}\, P_{22}^{-1} \\ 0 & P_{22}^{-1} \end{pmatrix}. \tag{15}$$

Hence, the matrix P defines an isomorphism (Cluzeau et al. (2008)), i.e., $M \cong M'$. Then, (11) and (14) yield:

$$\begin{cases} Q = P_{11}, \\ P_{12} = 0, \\ E = P_{11}\, A\, P_{11}^{-1}, \\ P_{11}\, B_1\, \delta + P_{11}\, B_0 - F\, P_{22} = 0. \end{cases} \tag{16}$$

Considering the degree of the last equation in δ, we get $P_{11}\, B_1 = 0$ and $P_{11}\, B_0 - F\, P_{22} = 0$, which yields $B_1 = 0$ since $P_{11} \in \mathrm{GL}_n(\mathbb{R})$ and $F = P_{11}\, B_0\, P_{22}^{-1}$. If B_1 is assumed to be nonzero, then (10) has no solutions in D, and thus (8) cannot be equivalent to (7) under a transformation P satisfying $P_{11} \in \mathrm{GL}_n(\mathbb{R}), P_{22} \in \mathrm{GL}_m(\mathbb{R})$ and $P_{21} = 0$.

4. RINGS OF INTEGRO-DIFFERENTIAL DELAY OPERATORS AND ARTSTEIN'S REDUCTION

Analyzing the arguments of Section 3, the fact that (10) does not admit a solution of the form $P_{11} \in \mathrm{GL}_n(\mathbb{R})$, $P_{22} \in \mathrm{GL}_m(\mathbb{R})$ and $P_{21} = 0$ comes from the fact that $\deg_{\partial} \partial\, P_{12}(\delta) = 1$ (see (13)). Now, if P_{12} contains a right inverse of ∂, i.e., an *integral operator*, then the term ∂ can be cancelled in $\partial\, P_{12}$.

Let I be the integral operator defined by the $\mathbb{R}$-linear map which maps a function y to its integral z, i.e.:

$$I : y(\cdot) \longmapsto z(\cdot), \quad z(t) := \int_0^t y(\tau)\, d\tau.$$

We recall that the differential operator ∂, the multiplication operator defined by an $a \in \mathcal{A} := C^{\infty}(\mathbb{R}_{\ge 0})$ and the TD operator are respectively defined by:

$$\partial : y(\cdot) \longmapsto \dot{y}(\cdot), \qquad a(\cdot) : y(\cdot) \longmapsto a(\cdot)\, y(\cdot),$$
$$\delta : y(\cdot) \longmapsto y(\,\cdot - h).$$

The operators I and ∂ satisfy the following relations

$$\partial\, I = 1, \quad I\, \partial = 1 - e_0, \tag{17}$$

where 1 is the identity operator and e_0 is the *evaluation character*, i.e., $e_0(y) = y(0)$ (Regensburger et al. (2009)). If $a \in \mathcal{A}$, then the composition of I with a is defined by

$$I\, a : y(\cdot) \longmapsto z(\cdot), \quad z(t) = \int_0^t a(\tau)\, y(\tau)\, d\tau, \tag{18}$$

i.e., a corresponds to the *kernel* of the integral operator $I\, a$. The composition of a with I is defined by:

$$a\, I : y(\cdot) \longmapsto z(\cdot), \quad z(t) = a(t) \int_0^t y(\tau)\, d\tau. \tag{19}$$

Since we now suppose that the *support* of the function of y is included in $\mathbb{R}_{\ge 0}$, i.e., $y(t) = 0$ for $t < 0$, then we have

$$(\delta\, I)(y)(t) = \delta \int_0^t y(\tau)\, d\tau = \int_0^{t-h} y(\tau)\, d\tau = \int_h^t y(s - h)\, ds$$
$$= \int_0^t y(s - h)\, ds = \int_0^t (\delta\, y)(s)\, ds,$$

i.e., the following relation between I and δ holds:

$$\delta\, I = I\, \delta. \tag{20}$$

If P_{12} is supposed to depend also on I, then (13) becomes:

$$(\partial\, I_n - E)\, P_{12}(\delta, I) + P_{11}\, (B_0 + B_1\, \delta) - F\, P_{22} = 0. \tag{21}$$

In order that (21) holds, using (17), $P_{12}(\delta, I)$ must contain I so that the operator ∂ can be cancelled in the term $\partial\, P_{12}(\delta, I)$. Now, considering the degree of (21) in δ, we also get that $P_{12}(\delta, I)$ contains δ. Hence, let us consider the following ansatz for P_{12}

$$P_{12} = a_0\, \delta\, I\, a_1 + a_2\, I\, a_3 + a_4\, \delta + a_5, \tag{22}$$

where the a_i's belong to $\mathcal{A}$. Let us also note:

$$\Delta := (\partial\, I_n - E)\, P_{12} + P_{11}\,(B_0 + B_1\,\delta) - F\, P_{22}.$$

Substituting (22) into (21) and using (17), (18), (19), (20), $\partial\, a = a\,\partial + \dot{a}$ and $\delta\, a = a(\,\cdot - h)\,\delta$ for all $a \in \mathcal{A}$ (Chyzak et al. (2005)), we obtain:

$$\begin{aligned}\Delta &= a_0\,(\partial\, I)\,\delta\, a_1 + \dot{a}_0\,\delta\, I\, a_1 + a_2\,(\partial\, I)\, a_3 + \dot{a}_2\, I\, a_3 + a_4\,\partial\,\delta \\ &\quad + \dot{a}_4\,\delta + a_5\,\partial + \dot{a}_5 - E\,(a_0\,\delta\, I\, a_1 + a_2\, I\, a_3 + a_4\,\delta + a_5) \\ &\quad + P_{11}\,(B_0 + B_1\,\delta) - F\, P_{22} \\ &= (\dot{a}_0 - E\, a_0)\,\delta\, I\, a_1 + (\dot{a}_2 - E\, a_2)\, I\, a_3 + a_4\,\partial\,\delta + a_0\,\delta\, a_1 \\ &\quad + (\dot{a}_4 - E\, a_4 + P_{11}\, B_1)\,\delta + a_5\,\partial + \dot{a}_5 - E\, a_5 + a_2\, a_3 \\ &\quad + P_{11}\, B_0 - F\, P_{22} \\ &= (\dot{a}_0 - E\, a_0)\,\delta\, I\, a_1 + (\dot{a}_2 - E\, a_2)\, I\, a_3 + a_4\,\partial\,\delta \\ &\quad + (a_0\, a_1(\,\cdot - h) + \dot{a}_4 - E\, a_4 + P_{11}\, B_1)\,\delta + a_5\,\partial \\ &\quad + \dot{a}_5 - E\, a_5 + a_2\, a_3 + P_{11}\, B_0 - F\, P_{22}.\end{aligned}$$

Let us suppose that $a_1 \neq 0$ and $a_3 \neq 0$. Then, $\Delta = 0$ if:

$$\left\{\begin{array}{l}\dot{a}_0 - E\, a_0 = 0, \\ \dot{a}_2 - E\, a_2 = 0, \\ a_4 = 0, \\ a_0\, a_1(\,\cdot - h) + \dot{a}_4 - E\, a_4 + P_{11}\, B_1 = 0, \\ a_5 = 0, \\ \dot{a}_5 - E\, a_5 + a_2\, a_3 + P_{11}\, B_0 - F\, P_{22} = 0,\end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l}\dot{a}_0 - E\, a_0 = 0, \\ \dot{a}_2 - E\, a_2 = 0, \\ a_4 = 0, \\ a_5 = 0, \\ a_0\, a_1(\,\cdot - h) + P_{11}\, B_1 = 0, \\ a_2\, a_3 + P_{11}\, B_0 - F\, P_{22} = 0.\end{array}\right. \tag{23}$$

Integrating the first two equations of (23), we get:

$$\left\{\begin{array}{l}a_0 = e^{E\,t}\, c_0,\; c_0 \in \mathbb{R}^{n\times n}, \\ a_2 = e^{E\,t}\, c_2,\; c_2 \in \mathbb{R}^{n\times n}.\end{array}\right. \tag{24}$$

Substituting (24) into (23), we obtain the following equations on the initial conditions c_0 and c_2

$$\begin{aligned}&\left\{\begin{array}{l}e^{E\,t}\, c_0\, a_1(\,\cdot - h) + P_{11}\, B_1 = 0, \\ e^{E\,t}\, c_2\, a_3 + P_{11}\, B_0 - F\, P_{22} = 0,\end{array}\right. \\ \Leftrightarrow &\left\{\begin{array}{l}c_0\, a_1 = -e^{-E\,(t+h)}\, P_{11}\, B_1, \\ c_2\, a_3 = e^{-E\,t}\,(F\, P_{22} - P_{11}\, B_0).\end{array}\right.\end{aligned} \tag{25}$$

Hence, (22) becomes:

$$\begin{aligned}P_{12} &= e^{E\,t}\, c_0\,\delta\, I\, a_1 + e^{E\,t}\, c_2\, I\, a_3 = e^{E\,t}\,(\delta\, I\,(c_0\, a_1) + I\,(c_2\, a_3)) \\ &= e^{E\,t}\,(-\delta\, I\, e^{-E\,(t+h)}\, P_{11}\, B_1 + I\, e^{-E\,t}\,(F\, P_{22} - P_{11}\, B_0)) \\ &= -e^{E\,t}\,\delta\, I\, e^{-E\,(t+h)}\, P_{11}\, B_1 + e^{E\,t}\, I\, e^{-E\,t}\,(F\, P_{22} - P_{11}\, B_0).\end{aligned}$$

In other words, the operator P_{12} is defined by

$$\begin{aligned}(P_{12}\, u)(t) &= -\,e^{E\,t}\int_0^{t-h} e^{-E\,(\tau+h)}\, P_{11}\, B_1\, u(\tau)\, d\tau \\ &\quad + e^{E\,t}\int_0^{t} e^{-E\,\tau}\,(F\, P_{22} - P_{11}\, B_0)\, u(\tau)\, d\tau \\ &= -\int_0^{t-h} e^{E\,(t-(\tau+h))}\, P_{11}\, B_1\, u(\tau)\, d\tau \\ &\quad + \int_0^{t} e^{E\,(t-\tau)}\,(F\, P_{22} - P_{11}\, B_0)\, u(\tau)\, d\tau,\end{aligned}$$

where $P_{11} \in \mathrm{GL}_n(\mathbb{R})$ and $P_{22} \in \mathrm{GL}_m(\mathbb{R})$, and (12) yields:

$$\left\{\begin{aligned}z(t) &= P_{11}\, x(t) - \int_0^{t-h} e^{E\,(t-(\tau+h))}\, P_{11}\, B_1\, u(\tau)\, d\tau \\ &\quad + \int_0^{t} e^{E\,(t-\tau)}\,(F\, P_{22} - P_{11}\, B_0)\, u(\tau)\, d\tau, \\ v(t) &= P_{22}\, u(t).\end{aligned}\right.$$

Now, using the third identity of (16), i.e., $E = P_{11}\, A\, P_{11}^{-1}$, we get $e^{E\,t} = P_{11}\, e^{A\,t}\, P_{11}^{-1}$, which yields:

$$\begin{aligned}z(t) = P_{11}\,\Big(x(t) &- \int_0^{t-h} e^{A\,(t-(\tau+h))}\, B_1\, u(\tau)\, d\tau \\ &+ \int_0^{t} e^{A\,(t-\tau)}\,(P_{11}^{-1}\, F\, P_{22} - B_0)\, u(\tau)\, d\tau\Big).\end{aligned}$$

Now, note that if we set $P_{11}^{-1}\, F\, P_{22} - B_0 = e^{-A\,h}\, B_1$, i.e., if $F = P_{11}\,(B_0 + e^{-A\,h}\, B_1)\, P_{22}^{-1}$, then we obtain:

$$\begin{aligned}z(t) &= P_{11}\,\Big(x(t) - \int_0^{t-h} e^{A\,(t-(\tau+h))}\, B_1\, u(\tau)\, d\tau \\ &\qquad + \int_0^{t} e^{A\,(t-(\tau+h))}\, B_1\, u(\tau)\, d\tau\Big) \\ &= P_{11}\,\Big(x(t) + \int_{t-h}^{t} e^{A\,(t-(\tau+h))}\, B_1\, u(\tau)\, d\tau\Big).\end{aligned}$$

We find again *Artstein's transformation* Artstein (1982).

Theorem 3. Let $P_{11} \in \mathrm{GL}_n(\mathbb{R})$ and $P_{22} \in \mathrm{GL}_m(\mathbb{R})$. Then, the following two linear systems

$$\dot{x}(t) = A\, x(t) + B_0\, u(t) + B_1\, u(t-h),\;\; \dot{z}(t) = E\, z(t) + F\, v(t),$$

where $A,\, E \in \mathbb{R}^{n\times n}$ and $B_0,\, B_1,\, F \in \mathbb{R}^{n\times m}$ are such that $E = P_{11}\, A\, P_{11}^{-1}$, are equivalent under the following invertible transformation:

$$\left\{\begin{aligned}z(t) = P_{11}\,\Big(x(t) &- \int_0^{t-h} e^{A\,(t-(\tau+h))}\, B_1\, u(\tau)\, d\tau \\ &+ \int_0^{t} e^{A\,(t-\tau)}\,(P_{11}^{-1}\, F\, P_{22} - B_0)\, u(\tau)\, d\tau\Big), \\ v(t) = P_{22}\, u(t).&\end{aligned}\right.$$

We note that Theorem 3 also holds when $B_1 = 0$, i.e., the following two linear differential systems

$$\dot{x}(t) = A\, x(t) + B_0\, u(t), \quad \dot{z}(t) = E\, z(t) + F\, v(t),$$

are equivalent under the invertible transformation

$$\left\{\begin{array}{l}z(t) = P_{11}\Big(x(t) + \displaystyle\int_0^{t} e^{A\,(t-\tau)}\,(P_{11}^{-1}\, F\, P_{22} - B_0)\, u(\tau)\, d\tau\Big), \\ v(t) = P_{22}\, u(t),\end{array}\right.$$

where $P_{11} \in \mathrm{GL}_n(\mathbb{R})$ and $P_{22} \in \mathrm{GL}_m(\mathbb{R})$. Indeed, we have to solve (13) where $B_1 = 0$ and P_{12} does not depend on δ. Hence, we can consider (22) where $a_0 = 0$ and $a_4 = 0$, and the result directly follows from the above computations.

Corollary 4. Let $P_{11} \in \mathrm{GL}_n(\mathbb{R})$ and $P_{22} \in \mathrm{GL}_m(\mathbb{R})$. Then, the two following two linear systems

$$\dot{x}(t) = A\, x(t) + B_0\, u(t) + B_1\, u(t-h),\;\; \dot{z}(t) = E\, z(t) + F\, v(t),$$

where $A,\, E \in \mathbb{R}^{n\times n}$ and $B_0,\, B_1,\, F \in \mathbb{R}^{n\times m}$ are such that

$$\left\{\begin{array}{l}E = P_{11}\, A\, P_{11}^{-1}, \\ F = P_{11}\,(B_0 + e^{-A\,h}\, B_1)\, P_{22}^{-1},\end{array}\right.$$

are equivalent under the invertible transformation:

$$\begin{cases} z(t) = P_{11} \left(x(t) + \int_{t-h}^{t} e^{A\,(t-(\tau+h))}\, B_1\, u(\tau)\, d\tau \right), \\ v(t) = P_{22}\, u(t). \end{cases}$$

See Quadrat (2015) for the case of multi-delays.

5. TIME-VARYING CASE

If E, F, A, B_0 and B_1 now depend on the time t, then the same computations as the ones done in Section 4 yield (10), and using $\partial\, a = a\, \partial + \dot{a}$ for all $a \in \mathcal{A}$, we obtain:

$$\begin{cases} Q = P_{11}, \\ \dot{P}_{11} = E\, P_{11} - P_{11}\, A, \\ \partial\, P_{12}(\delta) - E\, P_{12}(\delta) + P_{11}\,(B_0 + B_1\, \delta) - F\, P_{22} = 0. \end{cases}$$

Then, repeating the same computations as in Section 4 with the last equation of the above system, we get:

$$\begin{cases} \dot{a}_0(t) - E(t)\, a_0 = 0, \\ \dot{a}_2(t) - E(t)\, a_2 = 0, \\ a_4 = 0, \\ a_5 = 0, \\ a_0(t)\, a_1(t-h) + P_{11}(t)\, B_1(t) = 0, \\ a_2(t)\, a_3(t) + P_{11}(t)\, B_0(t) - F(t)\, P_{22}(t) = 0. \end{cases}$$

If Φ is a *fundamental matrix* of $\dot{a} = E(t)\, a$, then the first two equations of the above system yield $a_0 = \Phi(t, t_0)\, c_0$ and $a_2 = \Phi(t, t_0)\, c_2$, where $c_0,\, c_2 \in \mathbb{R}^{n\times n}$ and $t_0 \in \mathbb{R}_{\geq 0}$. Substituting them into the last two equations, we obtain:

$$\begin{cases} c_0\, a_1(t) = -\Phi^{-1}(t+h, t_0)\, P_{11}(t+h)\, B_1(t+h), \\ c_2\, a_3(t) = \Phi^{-1}(t, t_0)(F(t)\, P_{22}(t) - P_{11}(t)\, B_0(t)). \end{cases}$$

Hence, (22) becomes:

$$\begin{aligned} P_{12} &= \Phi(t, t_0)\, c_0\, \delta\, I\, a_1 + \Phi(t, t_0)\, c_2\, I\, a_3 \\ &= \Phi(t, t_0)\, (\delta\, I\, c_0\, a_1 + I\, c_2\, a_3) \\ &= \Phi(t, t_0)\, \big(-\delta\, I\, \Phi^{-1}(t+h, t_0)\, P_{11}(t+h)\, B_1(t+h) \\ &\quad + I\, \Phi^{-1}(t, t_0)(F(t)\, P_{22}(t) - P_{11}(t)\, B_0(t))\big). \end{aligned}$$

In other words, the operator P_{12} is defined by

$$\begin{aligned} &(P_{12}\, u)(t) = \\ &\Phi(t, t_0) \left(-\int_{t_0}^{t-h} \Phi^{-1}(\tau+h, t_0)\, P_{11}(\tau+h)\, B_1(\tau+h)\, u(\tau)\, d\tau \right. \\ &\left. + \int_{t_0}^{t} \Phi^{-1}(\tau, t_0)(F(\tau)\, P_{22}(\tau) - P_{11}(\tau)\, B_0(\tau))\, u(\tau)\, d\tau \right). \end{aligned} \tag{26}$$

Since P_{11} and Φ are non-singular matrices, so is $P_{11}^{-1}\, \Phi$. Now, using $\dot{P}_{11} = E\, P_{11} - P_{11}\, A$, we get

$$\begin{aligned} &\frac{d}{dt}\,(P_{11}^{-1}\, \Phi) - A\,(P_{11}^{-1}\, \Phi) = \frac{dP_{11}^{-1}}{dt}\, \Phi + P_{11}^{-1}\, \dot{\Phi} - A\, P_{11}^{-1}\, \Phi \\ &= -P_{11}^{-1} \dot{P}_{11}\, P_{11}^{-1}\, \Phi + P_{11}^{-1}\, E\, \Phi - A\, P_{11}^{-1}\, \Phi \\ &= -P_{11}^{-1}\,((\dot{P}_{11} - E\, P_{11} + P_{11}\, A\,) P_{11}^{-1})\, \Phi = 0, \end{aligned}$$

which shows that $\Psi := P_{11}^{-1}\, \Phi$ is a fundamental matrix of $\dot{a} = A\, a$. Then, (26) can be rewritten as follows:

$$\begin{aligned} &(P_{12}\, u)(t) \\ &= P_{11}(t)\, \Psi(t, t_0) \left(-\int_{t_0}^{t-h} \Psi^{-1}(\tau+h, t_0)\, B_1(\tau+h)\, u(\tau)\, d\tau \right. \\ &\left. + \int_{t_0}^{t} \Psi^{-1}(\tau, t_0)\, (P_{11}(\tau)^{-1}\, F(\tau)\, P_{22}(\tau) - B_0(\tau))\, u(\tau)\, d\tau \right) \\ &= P_{11}(t) \left(-\int_{t_0}^{t-h} \Psi(t, t_0)\, \Psi^{-1}(\tau+h, t_0)\, B_1(\tau+h)\, u(\tau)\, d\tau \right. \\ &+ \int_{t_0}^{t} \Psi(t, t_0)\, \Psi^{-1}(\tau, t_0)\, (P_{11}(\tau)^{-1}\, F(\tau)\, P_{22}(\tau) - B_0(\tau)) \\ &\left. \qquad u(\tau)\, d\tau \right). \end{aligned} \tag{27}$$

Using the properties of the fundamental matrix Ψ, i.e.,

$$\begin{aligned} \forall\, t_1,\, t_2,\, t_3 \in \mathbb{R}_{\geq 0},\ & \Psi(t_3, t_2)\, \Psi(t_2, t_1) = \Psi(t_3, t_1), \\ \forall\, t \in \mathbb{R}_{\geq 0},\ & \det \Psi(t, t_0) \neq 0, \\ & \Psi^{-1}(t_2, t_1) = \Psi(t_1, t_2), \end{aligned}$$

we get $\Psi(t, t_0)\, \Psi^{-1}(\tau+h, t_0) = \Psi(t, \tau+h)$. Thus, (27) finally becomes:

$$\begin{aligned} (P_{12}\, u)(t) = P_{11}(t) &\left(-\int_{t_0}^{t-h} \Psi(t, \tau+h)\, B_1(\tau+h)\, u(\tau)\, d\tau \right. \\ &\left. + \int_{t_0}^{t} \Psi(t, \tau)\, (P_{11}(\tau)^{-1}\, F(\tau)\, P_{22}(\tau) - B_0(\tau))\, u(\tau)\, d\tau \right). \end{aligned}$$

Theorem 5. Let $P_{11} \in \mathrm{GL}_n(\mathcal{A})$ and $P_{22} \in \mathrm{GL}_m(\mathcal{A})$. Then, the following two linear systems

$$\begin{aligned} \dot{x}(t) &= A(t)\, x(t) + B_0(t)\, u(t) + B_1(t)\, u(t-h), \\ \dot{z}(t) &= E(t)\, z(t) + F(t)\, v(t), \end{aligned}$$

where $A,\, E \in \mathcal{A}^{n\times n}$ and $B_0,\, B_1,\, F \in \mathcal{A}^{n\times m}$ are such that $E = \dot{P}_{11}\, P_{11}^{-1} + P_{11}\, A\, P_{11}^{-1}$, are equivalent under the following invertible transformation:

$$\begin{cases} z(t) = P_{11}(t) \left(x(t) - \int_{t_0}^{t-h} \Psi(t, \tau+h)\, B_1(\tau+h)\, u(\tau)\, d\tau \right. \\ \qquad \left. + \int_{t_0}^{t} \Psi(t, \tau)\, (P_{11}(\tau)^{-1}\, F(\tau)\, P_{22}(\tau) - B_0(\tau))\, u(\tau)\, d\tau \right), \\ v(t) = P_{22}(t)\, u(t). \end{cases}$$

With the above notations, if we set

$$P_{11}(t)^{-1}\, F(t)\, P_{22}(t) - B_0(t) = \Psi(t+h, t)^{-1}\, B_1(t+h),$$

then we obtain:

$$z(t) = P_{11}(t) \left(x(t) + \int_{t-h}^{t} \Psi(t, \tau+h)\, B_1(\tau+h)\, u(\tau)\, d\tau \right).$$

Corollary 6. Let $P_{11} \in \mathrm{GL}_n(\mathcal{A})$ and $P_{22} \in \mathrm{GL}_m(\mathcal{A})$. Then, the following two linear systems

$$\begin{aligned} \dot{x}(t) &= A(t)\, x(t) + B_0(t)\, u(t) + B_1(t)\, u(t-h), \\ \dot{z}(t) &= E(t)\, z(t) + F(t)\, v(t), \end{aligned}$$

where $A,\, E \in \mathcal{A}^{n\times n}$ and $B_0,\, B_1,\, F \in \mathcal{A}^{n\times m}$ are such that

$$\begin{cases} E(t) = \dot{P}_{11}(t)\, P_{11}^{-1}(t) + P_{11}(t)\, A(t)\, P_{11}^{-1}(t), \\ F(t) = P_{11}(t)\, \left(B_0(t) + \Psi(t, t+h)^{-1}\, B_1(t+h)\right)\, P_{22}(t)^{-1}, \end{cases}$$

where Ψ is a fundamental matrix of $\dot{a} = A\, a$, are equivalent under the following invertible transformation:

$$\begin{cases} z(t) = P_{11}(t) \left(x(t) + \int_{t-h}^{t} \Psi(t, \tau+h)\, B_1(\tau+h)\, u(\tau)\, d\tau \right), \\ v(t) = P_{22}(t)\, u(t). \end{cases}$$

6. THE DILATION CASE

The following system is considered in Artstein (1982):

$$\dot{x}(t) = A\, x(t) + B_0\, u(t) + B_1\, u(t/2).$$

Let us suppose that $q \in \mathbb{R} \setminus \{0\}$ is not a root of unity. Let $\delta_q : y(t) \longmapsto y(q\, t)$ be the *dilation operator*. We then get:

$$\begin{cases} (\partial\, \delta_q)(y(t)) = \dfrac{d}{dt}\, y(q\, t) = q\, \dot{y}(q\, t) = q\, (\delta_q\, \partial)(y(t)), \\ (\delta_q\, I)(y(t)) = \delta_q \displaystyle\int_0^t y(\tau)\, d\tau = \int_0^{q\,t} y(\tau)\, d\tau \\ \qquad\qquad = q \displaystyle\int_0^t y(q\, s)\, ds = q\, (I\, \delta_q)(y(t)). \end{cases} \tag{28}$$

Let us consider the following two linear systems $\dot{z}(t) = E(t)\, z(t) + F(t)\, v(t)$, $\dot{x}(t) = A\, x(t) + B_0\, u(t) + B_1\, u(q\, t)$, the following matrices of functional operators

$$R = (\partial\, I_n - E \quad -F),\ R' = (\partial\, I_n - A \quad -B_0 - B_1\, \delta_q)$$

with entries in the ring D of integro-differential dilation operators and the two finitely presented left D-modules:

$$M = D^{1\times(n+m)}/(D^{1\times n}\, R),\ M' = D^{1\times(n+m)}/(D^{1\times n}\, R').$$

Let $\phi \in \hom_D(M, M')$ be an isomorphism defined by (9) where $P_{11} \in \mathrm{GL}_n(\mathbb{R})$, $P_{21} = 0$ and $P_{22} \in \mathrm{GL}_m(\mathbb{R})$. Then, repeating what is done in Sections 3 and 4, we get

$$\begin{cases} Q = P_{11}, \\ E = P_{11}\, A\, P_{11}^{-1}, \\ \Delta := (\partial\, I_n - E)\, P_{12} + P_{11}\, (B_0 + B_1\, \delta_q) - F\, P_{22} = 0, \end{cases}$$

where $P_{12} = a_0\, \delta_q\, I\, a_1 + a_2\, I\, a_3$. Then, using the identities $\partial\, I = 1$, $\delta_q\, a = a(q\, \cdot)\, \delta_q$ for all $a \in \mathcal{A}$, and (28), we obtain:

$$\begin{aligned} \Delta &= a_0\, \partial\, \delta_q\, I\, a_1 + \dot{a}_0\, \delta_q\, I\, a_1 + a_2\, \partial\, I\, a_3 + \dot{a}_2\, I\, a_3 \\ &\quad - E\, (a_0\, \delta_q\, I\, a_1 + a_2\, I\, a_3) + P_{11}\, (B_0 + B_1\, \delta_q) - F\, P_{22} \\ &= (\dot{a}_0 - E\, a_0)\, \delta_q\, I\, a_1 + (q\, a_0)\, \delta_q\, a_1 + P_{11}\, B_1\, \delta_q \\ &\quad + (\dot{a}_2 - E\, a_2)\, I\, a_3 + a_2\, a_3 + P_{11}\, B_0 - F\, P_{22} \\ &= (\dot{a}_0 - E\, a_0)\, \delta_q\, I\, a_1 + (q\, a_0\, a_1(q\, \cdot) + P_{11}\, B_1)\, \delta_q \\ &\quad + (\dot{a}_2 - E\, a_2)\, I\, a_3 + a_2\, a_3 + P_{11}\, B_0 - F\, P_{22}. \end{aligned}$$

Let us suppose that $a_1 \neq 0$ and $a_3 \neq 0$. Then, $\Delta = 0$ if:

$$\begin{cases} \dot{a}_0 - E\, a_0 = 0, \\ \dot{a}_2 - E\, a_2 = 0, \\ q\, a_0\, a_1(q\, \cdot) + P_{11}\, B_1 = 0, \\ a_2\, a_3 + P_{11}\, B_0 - F\, P_{22} = 0. \end{cases} \tag{29}$$

Solving the first two equations of (29), we get $a_0 = e^{E\, t}\, c_0$ and $a_2 = e^{E\, t}\, c_2$, where $c_0,\, c_2 \in \mathbb{R}^{n\times n}$. Substituting these solutions into the last two equations of (29), we obtain:

$$\begin{cases} c_0\, a_1(t) = -q^{-1}\, e^{-q^{-1}\, E\, t}\, P_{11}\, B_1, \\ c_2\, a_3(t) = e^{-E\, t}\, (F\, P_{22} - P_{11}\, B_0), \end{cases}$$

$$\begin{aligned} P_{12} &= e^{E\, t}\, c_0\, \delta_q\, I\, a_1 + e^{E\, t}\, c_2\, I\, a_3 = e^{E\, t}\, (\delta_q\, I\, c_0\, a_1 + I\, c_2\, a_3) \\ &= e^{E\, t}\, (\delta_q\, I\, (-q^{-1}\, e^{-q^{-1}\, E\, t}\, P_{11}\, B_1) \\ &\qquad\qquad + I\, e^{-E\, t}\, (F\, P_{22} - P_{11}\, B_0)). \end{aligned}$$

Hence, we obtain:

$$\begin{aligned} (P_{12}\, u)(t) = &- q^{-1} \int_0^{q\,t} e^{E\,(t - q^{-1}\, \tau)}\, P_{11}\, B_1\, u(\tau) \\ &+ \int_0^t e^{E\,(t-\tau)}\, (F\, P_{22} - P_{11}\, B_0)\, u(\tau)\, d\tau. \end{aligned}$$

Now, using $E = P_{11}\, A\, P_{11}^{-1}$, we obtain:

$$\begin{cases} z(t) = P_{11}\, \Big(x(t) + \displaystyle\int_0^t e^{A\,(t-\tau)}\, (P_{11}^{-1}\, F\, P_{22} - B_0)\, u(\tau)\, d\tau \\ \qquad\qquad - q^{-1} \displaystyle\int_0^{q\,t} e^{A\,(t - q^{-1}\,\tau)}\, B_1\, u(\tau)\Big), \\ v(t) = P_{22}(t)\, u(t). \end{cases}$$

If we set $F\, P_{22} - P_{11}\, B_0 = q^{-1}\, e^{(1-q^{-1})\, E\, \tau}\, P_{11}\, B_1$, i.e.,

$$F = P_{11}\, \Big(B_0 + q^{-1}\, e^{(1-q^{-1})\, E\, t}\, B_1\Big)\, P_{22}^{-1},$$

then we obtain

$$P_{12}\, u(t) = q^{-1} \int_{q\,t}^t e^{E\,(t - q^{-1}\,\tau)}\, P_{11}\, B_1\, u(\tau)\, d\tau,$$

and thus:

$$\begin{cases} z(t) = P_{11}(t)\, \Big(x(t) + q^{-1} \displaystyle\int_{q\,t}^t e^{A\,(t - q^{-1}\,\tau)}\, B_1\, u(\tau)\, d\tau\Big), \\ v(t) = P_{22}(t)\, u(t). \end{cases}$$

For similar results on other classes of linear functional systems and more results on rings of integro-differential delay operators, see Quadrat (2015).

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