Lattices, operators and duality

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Abstract

The purpose of this communication is to explain how the operator-theoretic approach to stabilizability of infinite-dimensional linear systems developed in [2, 4, 11] can be interpreted as the dual theory of the lattices approach to stabilization problems developed in [7, 9, 10]. This allows us to develop a unified mathematical approach to the results developed, in literature, for different classes on infinite-dimensional linear systems. In particular, we extend the results obtained in [8] for SISO plants to MIMO plants.

Keywords

Lattices, fractional ideals, stabilization problems, operator-theoretic approach to stabilizability, operators, domains, graphs, duality, MIMO systems.

Let A be an integral domain of stable single-input single-output (SISO) plants (e.g., RH_∞, $H_\infty(\mathbb{C}_+)$, $\mathcal{A}$, $\hat{\mathcal{A}}$, W_+) and $K = Q(A) \triangleq \{p = n/d \mid 0 \neq d,\, n \in A\}$ its quotient field. Moreover, let $P \in K^{q \times r}$ be a transfer matrix of a plant. Then, P admits *fractional representations* of the form $P = D^{-1}\, N = \widetilde{N}\, \widetilde{D}^{-1}$, where:

$$\left\{ \begin{array}{ll} R = (D \quad -N) \in A^{q \times (q+r)}, & \det D \neq 0, \\ \widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in A^{(q+r) \times r}, & \det \widetilde{D} \neq 0. \end{array} \right.$$

A transfer matrix P always admits fractional representations because, e.g., we can take $D = d\, I_q$, $N = d\, P$, $\widetilde{D} = d\, I_r$ and $\widetilde{N} = P\, d$, where d denotes the product of the denominators of the entries of P. A fractional representation $P = D^{-1}\, N$ of P is said to be a *weakly left coprime* of P if, for all $\lambda \in K^{1 \times q}$, $\lambda\, R \in A^{1 \times (q+r)}$ yields $\lambda \in A^{1 \times q}$. Moreover, $P = D^{-1}\, N$ is a *left-coprime factorization* of P if there exist $X \in A^{q \times q}$ and $Y \in A^{r \times q}$ so that the Bézout identity $D\, X - N\, Y = I_q$ holds, and similarly for (weakly) right-coprime factorizations. Moreover, a plant P is said to be *internally stabilizable* if there exists a controller $C \in K^{r \times q}$ such that $\det(I_q - P\, C) \neq 0$ and

$$H(P, C) = \begin{pmatrix} (I_q - P\, C)^{-1} & (I_q - P\, C)^{-1}\, P \\ C\, (I_q - P\, C)^{-1} & I_r + C\, (I_q - P\, C)^{-1}\, P \end{pmatrix} \in A^{(q+r) \times (q+r)},$$

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i.e., if all the entries of the closed-loop transfer matrix $H(P, C)$ are stable. If so, C is then called a *stabilizing controller* of P. The search for (weakly) left/right/doubly coprime factorizations of P and for the stabilizing controllers of P are important issues in the study of stabilization problems of infinite-dimensional systems (e.g., internal/strong/simultaneous/robust/optimal stabilization problems). See [2, 3, 11].

In [7, 9, 10], the algebraic concept of a *lattice L of a finite-dimensional K-vector space V* (see [1]), namely, an A-submodule L of V contained in a finitely generated A-submodule of V and satisfying

$$K\,L = \left\{ \sum_{i=1}^{n} k_i\, l_i \mid k_i \in K,\, l_i \in L,\, n \in \mathbb{Z}_+ \right\} = V,$$

was introduced in the study of transfer matrices. For instance, $\mathcal{L} = (I_q \quad -P)\, A^{(q+r)}$ and $\mathcal{P} = R\, A^{(q+r)}$ are two examples of lattices of K^q, and $\mathcal{M} = A^{1\times(q+r)}\,(P^T \quad I_r^T)$ and $\mathcal{Q} = A^{1\times(q+r)}\,\widetilde{R}$ are two lattices of $K^{1\times r}$. Necessary and sufficient conditions for existence of (weakly) left/right/doubly coprime factorizations and internal stabilizing controllers of P were obtained in [9, 10] based on the lattices $\mathcal{L}$, $\mathcal{M}$, $\mathcal{P}$ and $\mathcal{Q}$ and their algebraic *duals*:

$$\begin{array}{ll} A : \mathcal{L} = \left\{\lambda \in A^{1\times q} \mid \lambda\, P \in A^{1\times r}\right\}, & A : \mathcal{M} = \{\lambda \in A^r \mid P\,\lambda \in A^q\}, \\ A : \mathcal{P} = \left\{\lambda \in K^{1\times q} \mid \lambda\, R \in A^{1\times(q+r)}\right\}, & A : \mathcal{Q} = \left\{\lambda \in K^r \mid \widetilde{R}\,\lambda \in A^{q+r}\right\}. \end{array}$$

Moreover, a general parametrization of all the stabilizing controllers of an internal stabilizable plant P was obtained. If P admits a doubly coprime factorization, then this parametrization reduces to the classical Youla-Kučera parametrization. These results extend the ones developed in [7] for SISO plants (i.e., $P \in K$) based on the lattices of K called the *fractional ideals* of K ([1]). See [5, 6] for related results using indirectly lattices.

If $\mathcal{F}$ denotes an A-module (e.g., $\mathcal{F} = H_2(\mathbb{C}_+)$ and $A = H_\infty(\mathbb{C}_+)$, $\hat{\mathcal{A}}$ or RH_∞; $L^p(\mathbb{R}_+)$ with $p \in [1, +\infty]$ and $A = \mathcal{A}$), then, it was shown in [8] how the module-theoretic duality induced by the *contravariant left exact functor* $\hom_A(\,\cdot\,, \mathcal{F})$ − which assigns an A-module M to the A-module of A-linear applications from M to $\mathcal{F}$ − allowed us to translate the fractional ideal approach to SISO plants into the operator-theoretic approach. In particular, we can find again and generalize the concepts of operators, domains and graphs even in the case where $\mathcal{F}$ admits *torsion elements*, namely, non-trivial elements of the following A-submodule of $\mathcal{F}$:

$$t(\mathcal{F}) = \{f \in \mathcal{F} \mid \exists\, 0 \neq a \in A :\; a\, f = 0\}.$$

Within a common mathematical framework, this new approach unifies different results obtained, in literature, for different classes of linear systems (e.g., [2, 4, 11]) and, particularly, the characterization of internal stabilizability by means of the splitting of the space $\mathcal{F}^{(q+r)}$ into the direct sum of the graph of P and the graph of a stabilizing controller C ([4, 11]). Finally, it shows that the operator-theoretic approach of linear systems can be interpreted as a *behavioural approach* ([9]).

The purpose of this communication is to explain how the previous results can be extended to multi-input multi-output (MIMO) plants by means of the lattice approach and the module-theoretic duality induced by the functor $\hom_A(\cdot, \mathcal{F})$. This new approach gives a unified viewpoint on classical questions on stabilization and robust control problems (e.g., internal stabilization, stabilizing controllers, graph topology, gap or ν-gap metrics).

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