Yoneda product of multidimensional systems

Alban Quadrat
Inria Saclay - Île-de-France,
DISCO project, L2S, Supélec,
3 rue Joliot Curie,
91192 Gif-sur-Yvette cedex, France.
Email: alban.quadrat@inria.fr

***Abstract*—The paper aims at developing the algebraic analysis approach to multidimensional systems and behaviours. Within the algebraic analysis approach, if D is a (noncommutative) polynomial ring in n functional operators and $\mathcal{F}$ a signal space, i.e., a left D-module, then a behaviour B can be defined as $\hom_D(M, \mathcal{F})$, i.e., as the dual of a (left) D-module M finitely presented by a matrix $R \in D^{q\times p}$ defining the multidimensional system. The homomorphisms from a behaviour C to a behaviour B can be studied by means of the (left) D-homomorphisms from M to N, where M and N are respectively the (left) D-module defining B and C. Within algebraic analysis, a linear system is not only defined as a behaviour $\mathrm{ext}^0_D(M, \mathcal{F}) := \hom_D(M, \mathcal{F})$ but as the collection of $n+1$ vector spaces $\{\mathrm{ext}^i_D(M, \mathcal{F})\}_{i=0,\ldots,n}$, defining the solvability in $\mathcal{F}$ of the successive compatibility conditions induced by the multidimensional system. If the signal space $\mathcal{F}$ is rich enough (i.e., is an injective left D-module), then $\{\mathrm{ext}^i(M, \mathcal{F})\}_{i=0,\ldots,n}$ reduces to the behaviour $\mathrm{ext}^0_D(M, \mathcal{F})$. In this paper, we generalize the homomorphisms of behaviours to consider the full characterization of a linear system as $\{\mathrm{ext}^i_D(M, \mathcal{F})\}_{i=0,\ldots,n}$. To do that, we explicitly characterize the abelian group $\mathrm{ext}^i_D(M, N)$ and the maps:**

$$\forall\, i,\, j \geq 0, \quad \mathrm{ext}^i_D(M, N) \otimes_{\mathbb{Z}} \mathrm{ext}^j_D(N, \mathcal{F}) \longrightarrow \mathrm{ext}^{i+j}_D(M, \mathcal{F}).$$

The classical behaviour homomorphisms correspond to $i = j = 0$.

I. Algebraic analysis

In what follows, we assume that D is a *noetherian ring*, i.e., a ring D such that all *left/right ideals* of D are finitely generated as left/right D-modules [12]. Let $R \in D^{q\times p}$ be a $q\times p$ matrix with entries in D and $.R$ the *left D-homomorphism* (i.e., the left D-linear map) defined by:

$$\begin{array}{rcl} .R : D^{1\times q} & \longrightarrow & D^{1\times p} \\ \lambda = (\lambda_1 \, \ldots \, \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

The image $\mathrm{im}_D(.R)$ of $.R$, also denoted by $D^{1\times q}\, R$, is the left D-submodule of $D^{1\times p}$ formed by all the left D-linear combinations of the rows of R. The cokernel $\mathrm{coker}_D(.R)$ of $.R$ is the left D-module $M := D^{1\times p}/(D^{1\times q}\, R)$ formed by the residue classes $\pi(\lambda)$ of $\lambda \in D^{1\times p}$ in M, i.e.:

$$\pi(\lambda) = \pi(\lambda') \;\Leftrightarrow\; \exists\, \mu \in D^{1\times q} :\; \lambda = \lambda' + \mu\, R. \qquad (1)$$

The left D-module structure of M is defined by:

$$\forall\, \lambda_1,\, \lambda_2 \in D^{1\times p},\; \forall\, d \in D, \left\{ \begin{array}{l} \pi(\lambda_1) + \pi(\lambda_2) = \pi(\lambda_1 + \lambda_2), \\ d\, \pi(\lambda) = \pi(d\, \lambda). \end{array} \right. \qquad (2)$$

The identity (2) implies that $\pi : D^{1\times p} \longrightarrow M$ is a left D-homomorphism. The left D-module M is said to be *finitely presented* by the matrix R [12]. Let us describe M. Let $\{f_j\}_{j=1,\ldots,p}$ be the *standard basis* of $D^{1\times p}$, i.e., f_j is the row vector of length p defined by 1 at the j^{th} position and 0 anywhere else, and $y_j = \pi(f_j) \in M$ for $j = 1, \ldots, p$. Since every $m \in M$ is the residue class $\pi(\lambda)$ of a certain $\lambda \in D^{1\times p}$, writing $\lambda = \sum_{j=1}^p \lambda_j\, f_j$, we get

$$\pi(\lambda) = \pi\left(\sum_{j=1}^p \lambda_j\, f_j\right) = \sum_{j=1}^p \lambda_j\, \pi(f_j) = \sum_{j=1}^p \lambda_j\, y_j,$$

which shows that M is finitely generated by $\{y_j\}_{j=1,\ldots,p}$ as a left D-module, i.e., $\{y_j\}_{j=1,\ldots,p}$ is a family of generators of the left D-module M. If $R_{i\bullet}$ denotes the i^{th} row of the matrix R, i.e., $R_{i\bullet} = \sum_{j=1}^p R_{ij}\, f_j \in D^{1\times q}\, R$, then we have $\pi(R_{i\bullet}) = 0$ for $i = 1, \ldots, q$, which yields

$$\pi\left(\sum_{j=1}^p R_{ij}\, f_j\right) = \sum_{j=1}^p R_{ij}\, \pi(f_j) = \sum_{j=1}^p R_{ij}\, y_j = 0, \qquad (3)$$

for $i = 1, \ldots, q$. Hence, the generators $\{y_j\}_{j=1,\ldots,p}$ of M satisfy (3) and all their the left D-linear combinations.

If $\mathcal{F}$ is a left D-module and $\mathcal{F}^p := \mathcal{F}^{p\times 1}$, then the *linear system* or *behaviour* is defined by:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}.$$

Let us now explain the links between $\ker_{\mathcal{F}}(R.)$ and M. Let $\hom_D(M, \mathcal{F})$ be the *abelian group*, i.e., the $\mathbb{Z}$-module, formed by all the left D-homomorphisms from M to $\mathcal{F}$. We can check that the following $\mathbb{Z}$-homomorphism

$$\begin{array}{rcl} \chi : \hom_D(M, \mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \phi & \longmapsto & (\phi(y_1) \, \ldots \, \phi(y_p))^T \end{array} \qquad (4)$$

is an isomorphism and, for $\eta = (\eta_1 \, \ldots \, \eta_p)^T \in \ker_{\mathcal{F}}(R.)$, $\phi_\eta := \chi^{-1}(\eta) \in \hom_D(M, \mathcal{F})$ is defined by:

$$\forall\, \lambda_1, \ldots, \lambda_p \in D, \quad \phi_\eta\left(\sum_{j=1}^p \lambda_j\, y_j\right) = \sum_{j=1}^p \lambda_j\, \eta_j.$$

We note that (4) shows that $\ker_{\mathcal{F}}(R.)$ depends only on M and $\mathcal{F}$ up to isomorphism, i.e., depends on the *isomorphism type* of M and of $\mathcal{F}$. Two isomorphic finitely presented left D-modules $M \cong M'$ yield two isomorphic behaviours

$\hom_D(M,\mathcal{F}) \cong \hom_D(M',\mathcal{F})$. We get an intrinsic formulation of the behaviour which is independent of the particular embedding $\ker_{\mathcal{F}}(R.) \subseteq \mathcal{F}^p$. In the sixties, this remark was the starting point of the development of a mathematical theory called *algebraic analysis* [8]. This theory studies the behaviour $\ker_{\mathcal{F}}(R.)$ by means of the properties of the left D-modules M and $\mathcal{F}$. For more details, see [2], [9], [10].

II. Homological algebra

Let us review a few concepts of homological algebra [6], [12]. A sequence of left/right D-modules M_i and left/right D-homomorphisms $\delta_{i+1} \in \hom_D(M_{i+1}, M_i)$ is called a *complex* if $\delta_i \circ \delta_{i+1} = 0$ for all $i \in \mathbb{Z}$, i.e., if:

$$\forall\, i \in \mathbb{Z}, \quad \operatorname{im} \delta_{i+1} \subseteq \ker \delta_i.$$

A complex $(\delta_{i+1} : M_{i+1} \longrightarrow M_i)_{i \in \mathbb{Z}}$ is denoted by:

$$M_\bullet \;\ldots \xrightarrow{\delta_{i+2}} M_{i+1} \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \xrightarrow{\delta_{i-2}} \ldots$$

The *defect of exactness* of the complex $M_\bullet$ at M_i is the left/right D-module defined by $H_i(M_\bullet) = \ker \delta_i / \operatorname{im} \delta_{i+1}$. The complex $M_\bullet$ is said to be *exact at* M_i (resp., *exact*) if $H_i(M_\bullet) = 0$ (resp., $H_i(M_\bullet) = 0$ for all $i \in \mathbb{Z}$), i.e., $\ker \delta_i = \operatorname{im} \delta_{i+1}$ (resp., $\ker \delta_i = \operatorname{im} \delta_{i+1}$ for all $i \in \mathbb{Z}$). For instance, the complex $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is exact if f is injective, g surjective and $\ker g = \operatorname{im} f$.

Let $M = D^{1\times p_0}/(D^{1\times p_1} R_1)$ be a left D-module finitely presented by $R_1 \in D^{p_1 \times p_0}$. Since D is noetherian, the left D-module $\ker_D(.R_1) = \{\lambda \in D^{1\times p_1} \mid \lambda R_1 = 0\}$ formed by left D-linear relations of the rows of R_1 is finitely generated [12]. Thus, there exists $R_2 \in D^{p_2 \times p_1}$ such that $\ker_D(.R_1) = D^{1\times p_2} R_2$. Then, we obtain the following exact sequence of left D-modules

$$D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \qquad (5)$$

where $(.R_i)(\lambda) := \lambda R_i$ for all $\lambda \in D^{1\times p_i}$ and for all $i \geq 1$. Repeating the above arguments with R_2 and so on, we get a *free resolution* of M [12], i.e., the exact sequence:

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0. \qquad (6)$$

A left D-module M admits different free resolutions [12].

Free resolutions can be computed for a commutative polynomial ring D over a *computable field* and for certain classes of noncommutative polynomial rings (see [2], [4], [10] and the references therein). For instance, the OreModules package [3] can handle such a computation for certain classes of *Ore algebras* of functional operators [2].

Let $\mathcal{F}$ be a left D-module. Since $R_{i+1} R_i = 0$, we can consider the following complex of abelian groups

$$\ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0, \qquad (7)$$

where $(R_i.)(\eta) := R_i \eta$ for all $\eta \in \mathcal{F}^{p_{i-1}}$ and for all $i \geq 1$. We can prove that the defects of exactness of (7) depend only on M and $\mathcal{F}$ up to isomorphism and not on the free resolution (6) of M. They are denoted by:

$$\begin{cases} \operatorname{ext}^0_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \operatorname{ext}^i_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/\operatorname{im}_{\mathcal{F}}(R_i.), \ \forall\, i \geq 1. \end{cases} \qquad (8)$$

If D is a commutative ring, then $\operatorname{ext}^i_D(M,\mathcal{F})$ has a D-module structure. If D is a noncommutative ring, then $\operatorname{ext}^i_D(M,\mathcal{F})$ generally only has an abelian group structure. If D is a k-algebra, where k is a field contained in the *center* $C(D) = \{d \in D \mid \forall\; e \in D : \;\; d\,e = e\,d\}$ of D, then $\operatorname{ext}^i_D(M,\mathcal{F})$ inherits a k-vector space structure.

Let us now study the inhomogeneous linear system

$$R_1\, \eta = \zeta, \qquad (9)$$

where ζ is a fixed element of $\mathcal{F}^{p_2}$. Since $R_2\, R_1 = 0$, a necessary condition for the existence of $\eta \in \mathcal{F}^{p_1}$ such that $R_1\, \eta = \zeta$ is $R_2\, \zeta = (R_2\, R_1)\, \eta = 0$. This condition is sufficient iff the residue class $\sigma_1(\zeta)$ of $\zeta \in \ker_{\mathcal{F}}(R_2.)$ in

$$\ker_{\mathcal{F}}(R_2.)/\operatorname{im}_{\mathcal{F}}(R_1.) \cong \operatorname{ext}^1_D(M,\mathcal{F}) \qquad (10)$$

is 0, where $\sigma_1 : \ker_{\mathcal{F}}(R_2.) \longrightarrow \ker_{\mathcal{F}}(R_2.)/\operatorname{im}_{\mathcal{F}}(R_1.)$ is the canonical projection. Indeed, $\sigma_1(\zeta) = 0$ is equivalent to $\zeta \in \operatorname{im}_{\mathcal{F}}(R_1.)$, i.e., $\zeta = R_1\, \eta$ for a certain $\eta \in \mathcal{F}^{p_0}$. Hence, the inhomogeneous linear system (9) is solvable iff $\sigma_1(\zeta) = 0$. The study of $\operatorname{ext}^i_D(M,\mathcal{F})$ is generally a difficult issue. In Section V, we shall explain how $\operatorname{ext}^i_D(M,\mathcal{F})$ can indirectly be studied by means of the computation of elements of $\operatorname{ext}^j_D(M,N)$, where N is another finitely presented left D-module (e.g., $N = M$).

Example 1: Let $D = \mathbb{Q}[\partial, \delta]$ be the commutative polynomial ring of differential time-delay operators

$$\partial\, \eta(t) = \dot{\eta}(t), \quad \delta\, \eta(t) = \eta(t-1),$$

$R_1 = (\partial \quad \delta - 1)^T$, $M = D/(D^{1\times 2} R_1) = D/(\partial, \delta - 1)$, where $(\partial, \delta - 1)$ is the ideal of D defined by ∂ and $\delta - 1$, and the D-module $\mathcal{F} = C^\infty(\mathbb{R})$. We can easily check that M admits the following free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_2 = (1 - \delta \quad \partial) \in D^{1\times 2}$. Then, the defects of exactness of the following complex

$$0 \longleftarrow \mathcal{F} \xleftarrow{R_2.} \mathcal{F}^2 \xleftarrow{R_1.} \mathcal{F} \longleftarrow 0$$

are defined by:

$$\begin{cases} \operatorname{ext}^0_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \operatorname{ext}^1_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_2.)/\operatorname{im}_{\mathcal{F}}(R_1.), \\ \operatorname{ext}^2_D(M,\mathcal{F}) \cong \mathcal{F}/\operatorname{im}_{\mathcal{F}}(R_2.). \end{cases}$$

We can easily check that $\operatorname{ext}^0_D(M,\mathcal{F}) = \mathbb{R}$ and $\operatorname{ext}^2_D(M,\mathcal{F}) = 0$ since for every $\vartheta \in \mathcal{F}$,

$$\zeta = \begin{pmatrix} 0 \\ \int_0^t \vartheta(t)\, dt + c \end{pmatrix} \in \mathcal{F}^2,$$

where c is any real constants, satisfies $\vartheta = R_2\,\zeta$. If c_1 and c_2 are two different real constants, then we clearly have $\zeta = (c_1 \quad c_2)^T \in \ker_{\mathcal{F}}(R_2.)$. If there exists $\eta \in \mathcal{F}$ satisfying $R_1\,\eta = \zeta$, i.e., $\dot{\eta}(t) = c_1$ and $\eta(t-1) - \eta(t) = c_2$, then, from the first equation, we get $\eta(t) = c_1\,t + c_3$, where $c_3 \in \mathbb{R}$, and thus $\eta(t-1) - \eta(t) - c_2 = c_1 - c_2$, which is not 0 since $c_1 \neq c_2$. Thus, the residue class $\sigma_1((c_1 \quad c_2)^T)$ of $(c_1 \quad c_2)^T$ is not 0, i.e., $\mathrm{ext}^1_D(M, \mathcal{F}) \neq 0$, and the inhomogeneous linear system $R_1\,\eta = \zeta$ is not solvable in $\mathcal{F} = C^\infty(\mathbb{R})$.

Within algebraic analysis [8] and *derived categories* [6], a linear partial differential (PD) system is defined as the collection of the $(n+1)$ k-vector spaces $\{\mathrm{ext}^i_D(M, \mathcal{F})\}_{i=0,\ldots,n}$ formed by the behaviour $\mathrm{ext}^0_D(M, \mathcal{F})$ and the obstructions $\mathrm{ext}^i_D(M, \mathcal{F})$'s of the solvability of the successive compatibility conditions induced by the linear PD system $R_1\,\eta = 0$, where $D = A\langle \partial_1, \ldots, \partial_n\rangle$ is a ring of PD operators in the ∂_i's with coefficients in a *differential ring* A containing a field $k \subseteq C(D)$, $R_1 \in D^{p_1 \times p_0}$ and $M = D^{1\times p_0}/(D^{1\times p_1}\,R_1)$.

A left D-module $\mathcal{F}$ (resp., M) is called *injective* (resp., *projective*) if $\mathrm{ext}^i_D(M, \mathcal{F}) = 0$ for all left D-modules M (resp., $\mathcal{F}$) and for all $i \geq 1$ [6], [12]. In these two cases, $\{\mathrm{ext}^i_D(M, \mathcal{F})\}_{i=0,\ldots,n}$ reduces to the behaviour $\mathrm{ext}^0_D(M, \mathcal{F})$. The purpose of this paper is to generalize results on *behaviour homomorphisms* [4] to this complete characterization $\{\mathrm{ext}^i_D(M, \mathcal{F})\}_{i=0,\ldots,n}$ of a linear system.

III. Characterization of $\mathrm{ext}^i_D(M, N)$

Let $R_1 \in D^{p_1\times p_0}$ and $S_1 \in D^{q_1 \times q_0}$ be two matrices and $M = D^{1\times p_0}/(D^{1\times p_1}\,R_1)$ and $N = D^{1\times q_0}/(D^{1\times q_1}\,S_1)$ two finitely presented left D-modules. Let us explicitly characterize the $\mathrm{ext}^i_D(M, N)$'s. If (6) is a free resolution of M, then (7) with $\mathcal{F} = N$ yields the following complex

$$\ldots \xleftarrow{R_3.} N^{p_2} \xleftarrow{R_2.} N^{p_1} \xleftarrow{R_1.} N^{p_0} \longleftarrow 0, \tag{11}$$

whose defects of exactness are:

$$\begin{cases} \mathrm{ext}^0_D(M,N) \cong \ker_N(R_1.) = \{\eta \in N^{p_0} \mid R_1\,\eta = 0\}, \\ \mathrm{ext}^i_D(M,N) \cong \ker_N(R_{i+1}.)/\mathrm{im}_N(R_i.), \ \forall\, i \geq 1. \end{cases}$$

See (8). Considering r direct copies of the finite presentation of N, $D^{1\times q_1} \xrightarrow{.S_1} D^{1\times q_0} \xrightarrow{\sigma} N \longrightarrow 0$, we get the exact sequence $D^{r\times q_1} \xrightarrow{.S_1} D^{r\times q_0} \xrightarrow{\mathrm{id}_r \otimes \sigma} N^r \longrightarrow 0$, where $(.S_1)(\Theta) = \Theta\, S_1$ for all $\Theta \in D^{r\times q_1}$ and:

$$\forall\, \Lambda \in D^{r \times q_0}, \quad (\mathrm{id}_r \otimes \sigma)(\Lambda) = (\sigma(\Lambda_{1\bullet}) \ \ldots \ \sigma(\Lambda_{r\bullet}))^T.$$

Let $k \geq 0$. Then, for all $P \in D^{p_k \times p_0}$, we have

$$\begin{aligned} & R_{k+1}((\mathrm{id}_{p_k} \otimes \sigma)(P)) \\ &= R_{k+1} \begin{pmatrix} \sigma(P_{1\bullet}) \\ \vdots \\ \sigma(P_{p_k \bullet}) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{p_k} (R_{k+1})_{1j}\, \sigma(P_{j\bullet}) \\ \vdots \\ \sum_{j=1}^{p_k} (R_{k+1})_{p_{k+1}j}\, \sigma(P_{j\bullet}) \end{pmatrix} \\ &= \begin{pmatrix} \sigma\left(\sum_{j=1}^{p_k} (R_{k+1})_{1j}\, P_{j\bullet}\right) \\ \vdots \\ \sigma\left(\sum_{j=1}^{p_k} (R_{k+1})_{p_{k+1}j}\, P_{j\bullet}\right) \end{pmatrix} \\ &= (\mathrm{id}_{p_{k+1}} \otimes \sigma)(R_{k+1}\, P), \end{aligned}$$

i.e., $(R_{k+1}.) \circ (\mathrm{id}_{p_k} \otimes \sigma) = (\mathrm{id}_{p_{k+1}} \otimes \sigma) \circ (R_{k+1}.)$. Thus, we obtain the following *commutative exact diagram*

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \uparrow & & \uparrow & & \uparrow \\ N^{p_2} & \xleftarrow{R_2.} & N^{p_1} & \xleftarrow{R_1.} & N^{p_0} \\ \uparrow \mathrm{id}_{p_2} \otimes \sigma & & \uparrow \mathrm{id}_{p_1} \otimes \sigma & & \uparrow \mathrm{id}_{p_0} \otimes \sigma \\ D^{p_2 \times q_0} & \xleftarrow{R_2.} & D^{p_1 \times q_0} & \xleftarrow{R_1.} & D^{p_0 \times q_0} \\ \uparrow .S_1 & & \uparrow .S_1 & & \uparrow .S_1 \\ D^{p_2 \times q_1} & \xleftarrow{R_2.} & D^{p_1 \times q_1} & \xleftarrow{R_1.} & D^{p_0 \times q_1}, \end{array} \tag{12}$$

i.e., every square commutes and the sequences are exact.

We use the commutative diagram (12) to characterize:

$$\begin{aligned} \ker_N(R_2.) = \{(\mathrm{id}_{p_1} \otimes \sigma)(A) \in N^{p_1} \mid A \in D^{p_1 \times q_0} : \\ R_2\left((\mathrm{id}_{p_1} \otimes \sigma)(A)\right) = 0\}, \\ \mathrm{im}_N(R_1.) = \{(\mathrm{id}_{p_1} \otimes \sigma)(A) \in N^{p_1} \mid A \in D^{p_1 \times q_0} : \\ \exists\, X \in D^{p_0 \times q_0},\ (\mathrm{id}_{p_1} \otimes \sigma)(A) = R_1\left((\mathrm{id}_{p_0} \otimes \sigma)(X)\right)\}. \end{aligned}$$

Since the columns of (12) are exact sequences, we get:

$$\begin{aligned} R_2((\mathrm{id}_{p_1} \otimes \sigma)(A)) &= (\mathrm{id}_{p_2} \otimes \sigma)(R_2\, A) = 0 \\ &\Leftrightarrow \ \exists\, B \in D^{p_2 \times q_1} : \ R_2\, A = B\, S_1. \end{aligned}$$

$$\begin{aligned} (\mathrm{id}_{p_1} \otimes \sigma)(A) &= R_1\left((\mathrm{id}_{p_0} \otimes \sigma)(X)\right) \\ &= (\mathrm{id}_{p_1} \otimes \sigma)(R_1\, X) \\ &\Leftrightarrow \ (\mathrm{id}_{p_1} \otimes \sigma)(A - R_1\, X) = 0 \\ &\Leftrightarrow \ \exists\, Y \in D^{p_1 \times q_1} : \ A = R_1\, B + Y\, S_1. \end{aligned}$$

Hence, we obtain:

$$\begin{aligned} \ker_N(R_2.) = \{(\mathrm{id}_{p_1} \otimes \sigma)(A) \in N^{p_1} \mid A \in D^{p_1 \times q_0} : \\ \exists\, B \in D^{p_2 \times q_1} : \ R_2\, A = B\, S_1\} \\ \mathrm{im}_N(R_1.) = \{(\mathrm{id}_{p_1} \otimes \sigma)(A) \in N^{p_1} \mid A \in D^{p_1 \times q_0} : \\ \exists\, X \in D^{p_0 \times q_0},\ \exists\, Y \in D^{p_1 \times q_1} : \ A = R_1\, X + Y\, S_1\} \\ = (R_1\, D^{p_0 \times q_0} + D^{p_1 \times q_1}\, S_1)/(D^{p_1 \times q_1}\, S_1). \end{aligned}$$

If we now introduce the following abelian groups

$$\begin{cases} \Omega := \{A \in D^{p_1 \times q_0} \mid \exists\, B \in D^{p_2 \times q_1} : \ R_2\, A = B\, S_1\}, \\ E := \Omega/(R_1\, D^{p_0 \times q_0} + D^{p_1 \times q_1}\, S_1), \end{cases} \tag{13}$$

then we have the following isomorphism of abelian groups

$$\begin{array}{rcl} \mathrm{ext}^1_D(M,N) \cong \ker_N(R_2.)/\mathrm{im}_N(R_1.) & \xrightarrow{\upsilon} & E, \\ \rho((\mathrm{id}_{p_1} \otimes \sigma)(A)) & \longmapsto & \varepsilon(A), \end{array} \tag{14}$$

where $\rho : \ker_N(R_2.) \longrightarrow \ker_N(R_2.)/\mathrm{im}_N(R_1.)$ (resp., $\varepsilon : \Omega \longrightarrow E$) is the canonical projection. Indeed, the *third isomorphism theorem* of module theory [12] yields:

$$\begin{gathered} \mathrm{ext}^1_D(M,N) \cong \ker_N(R_2.)/\mathrm{im}_N(R_1.) \\ = \\ (\Omega/(D^{p_1 \times q_1}\, S_1))/((R_1\, D^{p_0 \times q_0} + D^{p_1 \times q_1}\, S_1)/(D^{p_1 \times q_1}\, S_1)) \\ \cong E. \end{gathered}$$

For more details on $\mathrm{ext}^1_D(M,N)$, see [1], [7], [10], [11]. Note that $\ker_D(.R_1) = 0$, i.e., $R_2 = 0$, yields $\Omega = D^{p_1 \times q_0}$.

Example 2: Let us compute the D-module $\mathrm{ext}^1_D(M, M)$, where M is the D-module defined in Example 1. By (13) and (14), we have:

$$\begin{array}{l} \Omega = \{A \in D^2 \mid \exists\, B \in D^{1\times 2} :\ R_2\, A = B\, R_1\}, \\ \mathrm{ext}^1_D(M, M) \cong \Omega/(R_1\, D + D^{2\times 2}\, R_1). \end{array}$$

If $A \in \Omega$, then there exists $B \in D^{1\times 2}$ such that:

$$R_2\, A = B\, R_1 \quad \Leftrightarrow \quad (A^T \quad -B) \begin{pmatrix} R_2^T \\ R_1 \end{pmatrix} = 0.$$

Using *Gröbner basis techniques* (see, e.g., [2]), we get:

$$\ker_D(.(R_2 \quad R_1^T)^T) = D^{1\times 3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1-\delta & \partial \end{pmatrix}.$$

The D-module Ω is then generated by the matrices

$$A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., by A_1 and A_2, and thus $\{\varepsilon(A_1), \varepsilon(A_2)\}$ is a family of generators of the D-module $\mathrm{ext}^1_D(M, M)$.

Similarly, the abelian group $\mathrm{ext}^i_D(M, N)$, $i \geq 1$, can be characterized. With the following abelian groups

$$\begin{array}{l} \Omega_i := \{A \in D^{p_i\times q_0} \mid \exists\, B \in D^{p_{i+1}\times q_1} :\ R_{i+1}\, A = B\, S_1\}, \\ E_i := \Omega_i/(R_i\, D^{p_{i-1}\times q_0} + D^{p_i\times q_1}\, S_1), \end{array}$$

we have the following $\mathbb{Z}$-isomorphism:

$$\begin{array}{rcl} \mathrm{ext}^i_D(M, N) \cong \ker_N(R_{i+1}.)/\mathrm{im}_N(R_i.) & \xrightarrow{\upsilon_i} & E_i, \\ \rho_i((\mathrm{id}_{p_i} \otimes \sigma)(A)) & \longmapsto & \varepsilon_i(A), \end{array} \tag{15}$$

where $\rho_i : \ker_N(R_{i+1}.) \longrightarrow \ker_N(R_{i+1}.)/\mathrm{im}_N(R_i.)$ (resp., $\varepsilon_i : \Omega_i \longrightarrow E_i$) is the canonical projection.

Let us now characterize $\mathrm{ext}^0_D(M, N) = \hom_D(M, N)$. By (4), we have $\hom_D(M, N) \cong \ker_N(R_1.)$. Similarly to what we have done for $\ker_N(R_2.)$, we obtain:

$$\begin{array}{c} \hom_D(M, N) \cong \\ \{P \in D^{p_0\times q_0} \mid \exists\, Q \in D^{p_1\times q_1} :\ R_1\, P = Q\, S_1\}/(D^{p_0\times q_0}\, S_1). \end{array} \tag{16}$$

Using (4), we get that $f \in \hom_D(M, N)$ is defined by

$$\forall\, \lambda \in D^{1\times p_0}, \quad f(\pi(\lambda)) = \sigma(\lambda\, P), \tag{17}$$

where the matrix $P \in D^{p_0\times q_0}$ satisfies $R_1\, P = Q\, S_1$ for a certain matrix $Q \in D^{p_1\times q_1}$. We note that (16) shows f can be defined by different matrices: $P' := P + Z\, S_1$, where $Z \in D^{p_0\times q_0}$ is any arbitrary matrices, also defines f, i.e., $f(\pi(\lambda)) = \sigma(\lambda\, P') = \sigma(\lambda\, P)$ for all $\lambda \in D^{1\times p_0}$.

It is interesting to compute $f \in \hom_D(M, N)$ because f induces the $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl} f^\star : \ker_{\mathcal{F}}(S_1.) & \longrightarrow & \ker_{\mathcal{F}}(R_1.) \\ \zeta & \longmapsto & \eta = P\, \zeta. \end{array} \tag{18}$$

Indeed, we have $R_1\, \eta = R_1\, (P\, \zeta) = Q\, (S_1\, \zeta) = 0$ for all $\zeta \in \ker_{\mathcal{F}}(S_1.)$. Hence, $f \in \hom_D(M, N)$ induces $f^\star \in \hom_{\mathbb{Z}}(\ker_{\mathcal{F}}(S_1.), \ker_{\mathcal{F}}(R_1.))$, i.e., maps $\mathcal{F}$-solutions of $S_1\, \zeta = 0$ to $\mathcal{F}$-solutions of $R_1\, \eta = 0$, or in other words, defines a *behaviour homomorphism* [4]. For instance, if $S_1 = R_1$, i.e., $N = M$, then $f \in \hom_D(M, M) := \mathrm{end}_D(M)$ induces an *internal symmetry* $f^\star$ of $\ker_{\mathcal{F}}(R_1.)$. For more details and applications, see [4], [5], [10].

Example 3: Let $D = \mathbb{Q}[\partial_t, \partial_x]$ be the commutative polynomial ring in the PD operators ∂_t and ∂_x with coefficients in $\mathbb{Q}$, the PD operator $R = \partial_t^2 - \partial_x^2 \in D$, $M = D/(D\, R)$ and $\mathcal{F} = C^\infty(\mathbb{R}^2)$. Using (16) and the commutativity of D, we obtain $\mathrm{end}_D(M) \cong M$. Hence, every $P \in D$ induces an internal symmetry of $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F} \mid \partial_t^2\, \eta - \partial_x^2\, \eta = 0\}$ defined by (18) with $S_1 = R_1 = R$. Now, if we consider the *Weyl algebra* $D = \mathbb{Q}[t, x]\langle \partial_t, \partial_x\rangle$ of PD operators in ∂_t and ∂_x with coefficients in $\mathbb{Q}[t, x]$, i.e., the noncommutative polynomial algebra formed by elements of the form $\sum_{0\leq|\nu|\leq r} a_\nu\, \partial^\nu$, where $a_\nu \in \mathbb{Q}[t, x]$, $\nu = (\nu_t, \nu_x) \in \mathbb{N}^2$ is a multi-index of length $|\nu| = \nu_t + \nu_x$, $\partial^\nu = \partial_t^{\nu_t}\, \partial_x^{\nu_x}$, where

$$\partial_t\, \partial_x = \partial_x\, \partial_t, \quad \partial_t\, t = t\, \partial_t + 1, \quad \partial_x\, x = x\, \partial_x + 1,$$

then (16) shows that $\mathrm{end}_D(M)$ is no longer a left or a right D-module. Using an algorithm developed in [4] and implemented in the OreMorphisms package [5], we get

$$P = a_0 + a_1\, \partial_t + a_2\, \partial_x + a_3\, (t\, \partial_t + x\, \partial_x) + a_4\, (x\, \partial_t + t\, \partial_x),$$

where $a_i \in \mathbb{Q}$ for $i = 0, \ldots, 4$, defines $f \in \mathrm{end}_D(M)$ since we have $R\, P = Q\, R$, where:

$$Q = a_0 + a_1\, \partial_t + a_2\, \partial_x + a_3\, (t\, \partial_t + x\, \partial_x) + a_4\, (x\, \partial_t + t\, \partial_x).$$

Now, a classical result due to d'Alembert shows that:

$$\ker_{\mathcal{F}}(R.) = \{\zeta(t, x) = \phi(t+x) + \psi(t-x) \mid \phi,\, \psi \in C^\infty(\mathbb{R})\}.$$

Therefore, using (18), we obtain that

$$\begin{aligned} \eta = P\, \zeta = {} & a_0\, \phi(t + x) + a_0\, \psi(t - x) \\ & + (a_1 + a_2 + a_3\, (t + x) + a_4\, (x + t))\, \dot{\phi}(t + x) \\ & + (a_1 - a_2 + a_3\, (t - x) + a_4\, (x - t))\, \dot{\psi}(t - x) \end{aligned}$$

is a $\mathcal{F}$-solution of the wave equation. This result can be checked again by writing $\eta = \phi'(t + x) + \psi'(t - x)$, where:

$$\left\{ \begin{aligned} \phi'(t + x) = {} & a_0\, \phi(t + x) \\ & + (a_1 + a_2 + (a_3 + a_4)\, (t + x))\, \dot{\phi}(t + x), \\ \psi'(t - x) = {} & a_0\, \psi(t - x) \\ & + (a_1 - a_2 + (a_3 - a_4)\, (t - x))\, \dot{\psi}(t - x). \end{aligned} \right.$$

Finally, we note that P yields the vector fields ∂_t, ∂_x, $t\, \partial_t + x\, \partial_x$ and $x\, \partial_t + t\, \partial_x$, which are called *infinitesimal symmetries* of the wave equation $\partial_t^2\, \eta - \partial_x^2\, \eta = 0$ in the literature of Lie groups and symmetries of differential systems. Similarly, the infinitesimal symmetries of PD operators, which depend only on the independent variables, can be computed by following an algorithm developed in [4] and implemented in the OreMorphisms package [5]. The study of infinitesimal symmetries of PD operators will be developed in a forthcoming publication.

IV. The functor $\mathrm{ext}^i_D(\,\cdot\,,\mathcal{F})$

The purpose of this paper is to generalize the relations between (17) and (18) to the case of inhomogeneous linear systems. In Section III, we have shown that the *contravariant* $\mathrm{ext}^i_D(\,\cdot\,,\mathcal{F})$ associates an abelian group $\mathrm{ext}^i_D(M,\mathcal{F})$ to a finitely generated left D-module M. Let us now show that $\mathrm{ext}^i_D(\,\cdot\,,\mathcal{F})$ assigns a $\mathbb{Z}$-homomorphism $\mathrm{ext}^i_D(f,\mathcal{F})$ to $f \in \hom_D(M,N)$. If M and N are two finitely generated left D-modules and $f \in \hom_D(M,N)$, then considering free resolutions of M and N of the form (6), and using (16) and (17), there exist $P_0 \in D^{p_0\times q_0}$ and $P_1 \in D^{p_1\times q_1}$ such that $R_1\,P_0 = P_1\,S_1$. Now, since $\ker_D(.R_1) = D^{1\times p_2}\,R_2$, $R_2\,P_1\,S_1 = (R_2\,R_1)\,P_0 = 0$, i.e., $D^{1\times p_2}\,(R_2\,P_1) \subseteq \ker_D(.S_1) = D^{1\times q_2}\,S_2$, there exists a matrix $P_2 \in D^{p_2\times q_2}$ such that $R_2\,P_1 = P_2\,S_2$. Repeating the arguments, we get $P_i \in D^{p_i\times q_i}$ such that $R_i\,P_{i-1} = P_i\,S_i$ for all $i \geq 1$. Hence, we obtain the following commutative exact diagram

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{.R_2} & D^{1\times p_1} & \xrightarrow{.R_1} & D^{1\times p_0} & \xrightarrow{\pi} & M & \longrightarrow 0 \\
 & & \downarrow .P_1 & & \downarrow .P_0 & & \downarrow f & \\
\ldots & \xrightarrow{.S_2} & D^{1\times q_1} & \xrightarrow{.S_1} & D^{1\times q_0} & \xrightarrow{\sigma} & N & \longrightarrow 0.
\end{array}$$

which yields the following *chain complex*:

$$\begin{array}{ccccccccc}
\ldots & \xleftarrow{R_3.} & \mathcal{F}^{p_2} & \xleftarrow{R_2.} & \mathcal{F}^{p_1} & \xleftarrow{R_1.} & \mathcal{F}^{p_0} & \longleftarrow 0 \\
 & & \uparrow P_2. & & \uparrow P_1. & & \uparrow P_0. & \\
\ldots & \xleftarrow{S_3.} & \mathcal{F}^{q_2} & \xleftarrow{S_2.} & \mathcal{F}^{q_1} & \xleftarrow{S_1.} & \mathcal{F}^{q_0} & \longleftarrow 0.
\end{array}$$

If $\eta \in \ker_{\mathcal{F}}(S_2.)$ and $\zeta = P_1\,\eta$, then we have:

$$R_2\,\zeta = (R_2\,P_1)\,\eta = P_2\,(S_2\,\eta) = 0 \;\Rightarrow\; P_1\,\eta \in \ker_{\mathcal{F}}(R_2.).$$

Now, if $\theta \in \mathrm{im}_{\mathcal{F}}(S_1.)$, i.e., if there exists $\xi \in \mathcal{F}^{q_0}$ such that $\theta = S_1\,\xi$, then $\omega := P_1\,\theta$ satisfies:

$$\omega = (P_1\,S_1)\,\xi = R_1\,(P_0\,\xi) \in \mathrm{im}_{\mathcal{F}}(R_1.).$$

Thus, if κ_i and τ_i are the canonical projections, i.e.,

$$\kappa_i : \ker_{\mathcal{F}}(R_{i+1}.) \longrightarrow \ker_{\mathcal{F}}(R_{i+1}.)/\mathrm{im}_{\mathcal{F}}(R_i.) \cong \mathrm{ext}^i_D(M,\mathcal{F}),$$
$$\tau_i : \ker_{\mathcal{F}}(S_{i+1}.) \longrightarrow \ker_{\mathcal{F}}(S_{i+1}.)/\mathrm{im}_{\mathcal{F}}(S_i.) \cong \mathrm{ext}^i_D(N,\mathcal{F}),$$

then, up to isomorphism, we get the $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl} f_1 : \mathrm{ext}^1_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^1_D(M,\mathcal{F}) \\ \tau_1(\eta) & \longmapsto & \kappa_1(P_1\,\eta). \end{array} \tag{19}$$

f_1 is well-defined: if $\tau_1(\eta) = \tau_1(\eta')$, then $\eta' = \eta + \theta$ for a certain $\theta \in \mathrm{im}_{\mathcal{F}}(S_1.)$, which yields $\kappa_1(P_1\,\eta') = \kappa_1(P_1\,\eta)$ since $P_1\,\theta \in \mathrm{im}_{\mathcal{F}}(R_1.)$, i.e., $\kappa_1(P_1\,\theta) = 0$.

Let us show that f_1 depends only on $f \in \hom_D(M,N)$ and not on a particular choice of P_0 and P_1 satisfying $R_1\,P_0 = P_1\,S_1$. If $P_0' := P_0 + Z_0\,S_1$, where $Z_0 \in D^{p_0\times q_1}$, then, in Section III, we proved that $f(\pi(\lambda)) = \sigma(\lambda\,P_0')$ for all $\lambda \in D^{1\times p_0}$. Now, we have

$$\begin{aligned} R_1\,P_0' &= R_1\,(P_0 + Z_0\,S_1) = (P_1 + R_1\,Z_0)\,S_1 \\ &= (P_1 + R_1\,Z_0 + Z_1\,R_2)\,S_1, \quad \forall\, Z_1 \in D^{p_1\times q_2}, \end{aligned}$$

i.e., $R_1\,P_0' = P_1'\,S_1$, where $P_1' := P_1 + R_1\,Z_0 + Z_1\,S_2$. Then, for $\eta \in \ker_{\mathcal{F}}(S_2.)$, we obtain

$$\begin{aligned} \kappa_1(P_1'\,\eta) &= \kappa_1((P_1 + R_1\,Z_0 + Z_1\,S_2)\,\eta) \\ &= \kappa_1(P_1\,\eta) + \kappa_1(R_1\,(Z_0\,\eta)) = \kappa_1(P_1\eta), \end{aligned}$$

which shows that f_1 depends only f, and thus f_1 can be denoted by $\mathrm{ext}^1_D(f,\mathcal{F})$. We get the following map:

$$\begin{array}{rcl} \hom_D(M,N) \times \mathrm{ext}^1_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^1_D(M,\mathcal{F}) \\ (f,\,\tau_1(\eta)) & \longmapsto & \kappa_1(P_1\,\eta). \end{array}$$

Similarly, for $i \in \mathbb{N}$, we have

$$\begin{array}{rcl} \mathrm{ext}^i_D(f,\mathcal{F}) : \mathrm{ext}^i_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^i_D(M,\mathcal{F}) \\ \tau_i(\eta) & \longmapsto & \kappa_i(P_i\,\eta), \end{array}$$

and we obtain the following map:

$$\begin{array}{rcl} \mathrm{ext}^0_D(M,N) \times \mathrm{ext}^i_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^i_D(M,\mathcal{F}) \\ (f,\,\tau_i(\eta)) & \longmapsto & \kappa_i(P_i\,\eta). \end{array} \tag{20}$$

We can check that (20) is a $\mathbb{Z}$-bilinear map, a fact which yields the following $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl} \mathrm{ext}^0_D(M,N) \otimes_{\mathbb{Z}} \mathrm{ext}^i_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^i_D(M,\mathcal{F}) \\ f \otimes \tau_i(\eta) & \longmapsto & \kappa_i(P_i\,\eta). \end{array}$$

If $\xi \in \mathcal{F}^{p_{i-1}}$ is a solution of $S_i\,\xi = \eta$ for a fixed right member $\eta \in \ker_{\mathcal{F}}(S_{i+1}.)$, then $\psi := P_{i-1}\,\xi$ is a solution of $R_i\,\psi = P_i\,\eta$. Hence, if $\zeta = P_i\,\eta$ for a certain $\eta \in \mathcal{F}^{q_i}$, then a particular solution of $S_i\,\xi = \eta$ yields a particular solution of the inhomogeneous linear system $R_i\,\psi = \zeta$.

V. Yoneda product

A. $\mathrm{ext}^1_D(M,N) \otimes_{\mathbb{Z}} \mathrm{ext}^0_D(N,\mathcal{F}) \longrightarrow \mathrm{ext}^1_D(M,\mathcal{F})$

With the notations of Section III, if we consider $A \in \Omega$ and $\eta \in \ker_{\mathcal{F}}(S_1.)$, then we have

$$R_2\,(A\,\eta) = B\,(S_1\,\eta) = 0,$$

which shows that $A \in \Omega$ induces the $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl} A. : \ker_{\mathcal{F}}(S_1.) & \longrightarrow & \ker_{\mathcal{F}}(R_2.) \\ \eta & \longmapsto & A\,\eta. \end{array} \tag{21}$$

If $Z \in (R_1\,D^{p_0\times q_0} + D^{p_1\times q_1}\,S_1)$, i.e., $Z = R_1\,U + V\,S_1$, where $U \in D^{p_0\times q_0}$ and $V \in D^{p_1\times q_1}$, and if $\eta \in \ker_{\mathcal{F}}(S_1.)$, then $Z\,\eta = R_1\,U\,\eta + V\,(S_1\,\eta) = R_1\,(U\,\eta) \in \mathrm{im}_{\mathcal{F}}(R_1.)$, which shows that Z induces the $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl} Z. : \ker_{\mathcal{F}}(S_1.) & \longrightarrow & \mathrm{im}_{\mathcal{F}}(R_1.) \\ \eta & \longmapsto & Z\,\eta. \end{array}$$

Now, if $\varepsilon(A') = \varepsilon(A)$, then we have $A' = A + Z$, where $Z \in (R_1\,D^{p_0\times q_0} + D^{p_1\times q_1}\,S_1)$, and using $\mathrm{im}_{\mathcal{F}}(R_1.) \subseteq \ker_{\mathcal{F}}(R_2.)$, we obtain $A'\,\eta = A\,\eta + Z\,\eta \in \ker_{\mathcal{F}}(R_2.)$ for all $\eta \in \ker_{\mathcal{F}}(S_1.)$, and thus $\sigma_1(A'\,\eta) = \sigma_1(A\,\eta)$, where $\sigma_1 : \ker_{\mathcal{F}}(R_2.) \longrightarrow \ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R.) \cong \mathrm{ext}^1_D(M,\mathcal{F})$ is the canonical projection, since $Z\,\eta \in \mathrm{im}_{\mathcal{F}}(R_1.)$. Thus, we get the following $\mathbb{Z}$-homomorphism

$$\begin{array}{rcl} \varepsilon(A) : \ker_{\mathcal{F}}(S_1.) & \longrightarrow & \ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R_1.) \\ \eta & \longmapsto & \sigma_1(A\,\eta), \end{array} \tag{22}$$

and then, up to isomorphism, the $\mathbb{Z}$-bilinear map

$$\begin{array}{rcl}\mathrm{ext}^1_D(M,N)\times \mathrm{ext}^0_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^1_D(M,\mathcal{F})\\ (\varepsilon(A),\,\eta) & \longmapsto & \sigma_1(A\,\eta),\end{array}$$

which finally yields the following $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl}\mathrm{ext}^1_D(M,N)\otimes_{\mathbb{Z}} \mathrm{ext}^0_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^1_D(M,\mathcal{F})\\ \varepsilon(A)\otimes\eta & \longmapsto & \sigma_1(A\,\eta).\end{array}$$

Example 4: Let us consider again Example 1. Using Example 2, (22) yields:

$$\begin{array}{rcl}\varepsilon(A_1):\ker_{\mathcal{F}}(R_1.)=\mathbb{R} & \longrightarrow & \mathrm{ext}^1_D(M,\mathcal{F})\\ \eta=c & \longmapsto & \sigma(A_1\,\eta)=\sigma_1((c\quad 0)^T),\\ \varepsilon(A_2):\ker_{\mathcal{F}}(R_1.)=\mathbb{R} & \longrightarrow & \mathrm{ext}^1_D(M,\mathcal{F})\\ \eta=c & \longmapsto & \sigma(A_2\,\eta)=\sigma_1((0\quad c)^T).\end{array}$$

B. $\mathrm{ext}^i_D(M,N)\otimes_{\mathbb{Z}}\mathrm{ext}^j_D(N,\mathcal{F})\longrightarrow \mathrm{ext}^{i+j}_D(M,\mathcal{F})$

With the notations of Section III, an element $P_1\in\Omega$, i.e., satisfying $R_2\,P_1=P_2\,S_1$ for a certain $P_2\in D^{p_2\times q_1}$, induces the commutative exact diagram (24) for $i=1$. Dualizing (24) for $i=1$, we obtain the chain complex:

$$\begin{array}{ccccc}\mathcal{F}^{p_3} & \xleftarrow{R_3.} & \mathcal{F}^{p_2} & \xleftarrow{R_2.} & \mathcal{F}^{p_1}\\ \uparrow P_3. & & \uparrow P_2. & & \uparrow P_1.\\ \mathcal{F}^{q_2} & \xleftarrow{S_2.} & \mathcal{F}^{q_1} & \xleftarrow{S_1.} & \mathcal{F}^{q_0}.\end{array}$$

Up to isomorphism, we get the $\mathbb{Z}$-group homomorphism

$$\begin{array}{rcl}\gamma_1:\ker_{\mathcal{F}}(S_2.)/\mathrm{im}_{\mathcal{F}}(S_1.) & \longrightarrow & \ker_{\mathcal{F}}(R_3.)/\mathrm{im}_{\mathcal{F}}(R_2.)\\ \varpi_1(\zeta) & \longmapsto & \sigma_2(P_2\,\zeta),\end{array}$$

where ϖ_1 and σ_2 are the canonical projections:

$$\varpi_1:\ker_{\mathcal{F}}(S_2.)\longrightarrow \ker_{\mathcal{F}}(S_2.)/\mathrm{im}_{\mathcal{F}}(S_1.)\cong \mathrm{ext}^1_D(N,\mathcal{F}),$$
$$\sigma_2:\ker_{\mathcal{F}}(R_3.)\longrightarrow \ker_{\mathcal{F}}(R_3.)/\mathrm{im}_{\mathcal{F}}(R_2.)\cong \mathrm{ext}^2_D(M,\mathcal{F}).$$

Let us now prove that γ_1 depends only on $\varepsilon(P_1)$. Let $P_1'\in\Omega$ be such that $\varepsilon(P_1')=\varepsilon(P_1)$, i.e., $P_1'=P_1+Z$, where $Z=R_1\,U+V\,S_1$, $U\in D^{p_0\times q_0}$ and $V\in D^{p_1\times q_1}$. Then, using $R_2\,R_1=0$ and $R_2\,P_1=P_2\,S_1$, we get:

$$\begin{aligned}R_2\,P_1' &= R_2\,(P_1+R_1\,U+V\,S_1)=R_2\,P_1+R_2\,V\,S_1\\ &= P_2\,S_1+R_2\,V\,S_1=(P_2+R_2\,V)\,S_1.\end{aligned}$$

Hence, for every $W\in D^{p_2\times q_2}$, $P_2':=P_2+R_2\,V+W\,S_2$ satisfies $R_2\,P_1'=P_2'\,S_1$. Then, for every $\zeta\in\ker_{\mathcal{F}}(S_2.)$,

$$\sigma_2(P_2'\,\zeta)=\sigma_2(P_2\,\zeta)+\sigma_2(R_2\,V\,\zeta)+\sigma_2(W\,S_2\,\zeta)=\sigma_2(P_2\,\zeta'),$$

since $R_2\,(V\,\zeta)\in\mathrm{im}_{\mathcal{F}}(R_2.)$, and thus $\sigma_2(R_2\,V\,\zeta)=0$, which proves that the $\mathbb{Z}$-homomorphism γ_1 depends only on $\varepsilon(P_1)$. Then, we have the following $\mathbb{Z}$-bilinear map

$$\begin{array}{rcl}\mathrm{ext}^1_D(M,N)\times \mathrm{ext}^1_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^2_D(M,\mathcal{F})\\ (\varepsilon(P_1),\,\varpi_1(\zeta)) & \longmapsto & \sigma_2(P_2\,\zeta),\end{array}$$

where $P_2\in D^{p_2\times q_2}$ is a matrix satisfying $R_2\,P_1=P_2\,S_1$, which finally yields the following $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl}\mathrm{ext}^1_D(M,N)\otimes_{\mathbb{Z}} \mathrm{ext}^1_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^2_D(M,\mathcal{F})\\ \varepsilon(P_1)\otimes\varpi_1(\zeta) & \longmapsto & \sigma_2(P_2\,\zeta).\end{array}\tag{23}$$

More generally, we get the following $\mathbb{Z}$-homomorphism

$$\begin{array}{rcl}\mathrm{ext}^i_D(M,N)\otimes_{\mathbb{Z}} \mathrm{ext}^j_D(N,\mathcal{F}) & \longrightarrow & \mathrm{ext}^{i+j}_D(M,\mathcal{F})\\ \varepsilon_i(P_i)\otimes\varpi_j(\zeta) & \longmapsto & \sigma_{i+j}(P_{i+j}\,\zeta),\end{array}$$

called the *Yoneda product* [1], [6], where $P_i\in D^{p_i\times q_0}$ satisfies $R_{i+1}\,P_i=P_{i+1}\,S_1$ for a certain matrix $P_{i+1}\in D^{p_{i+1}\times q_1}$, i.e., which induces

$$\begin{array}{rcl}g_i:\mathrm{im}_D(.R_i)=D^{1\times p_i}\,R_i & \longrightarrow & N\\ \lambda\,R_i & \longmapsto & \sigma(\lambda\,P_i),\end{array}$$

and thus yields the following commutative exact diagram

$$\begin{array}{ccccccccc}\ldots & \xrightarrow{.R_{i+2}} & D^{1\times p_{i+1}} & \xrightarrow{.R_{i+1}} & D^{1\times p_i} & \xrightarrow{.R_i} & \mathrm{im}_D(.R_i) & \longrightarrow & 0\\ & & \downarrow .P_{i+1} & & \downarrow .P_i & & \downarrow g_i & & \\ \ldots & \xrightarrow{.S_2} & D^{1\times q_1} & \xrightarrow{.S_1} & D^{1\times q_0} & \xrightarrow{\sigma} & N & \longrightarrow & 0,\end{array}\tag{24}$$

where $P_{i+j}\in D^{p_{i+j}\times q_j}$ satisfies $R_{i+j}\,P_{i+j-1}=P_{i+j}\,S_j$ for $j\geq 1$. The above commutative exact diagram yields the following chain complex

$$\begin{array}{ccccc}\mathcal{F}^{p_{i+j+1}} & \xleftarrow{R_{i+j+1}.} & \mathcal{F}^{p_{i+j}} & \xleftarrow{R_{i+j}.} & \mathcal{F}^{p_{i+j-1}}\\ \uparrow P_{i+j+1}. & & \uparrow P_{i+j}. & & \uparrow P_{i+j-1}.\\ \mathcal{F}^{q_{j+1}} & \xleftarrow{S_{j+1}.} & \mathcal{F}^{q_j} & \xleftarrow{S_j.} & \mathcal{F}^{q_{j-1}},\end{array}$$

and thus the following $\mathbb{Z}$-homomorphism:

$$\begin{array}{rcl}\gamma_i:\ker_{\mathcal{F}}(S_{j+1}.)/\mathrm{im}_{\mathcal{F}}(S_j.) & \longrightarrow & \ker_{\mathcal{F}}(R_{i+j+1}.)/\mathrm{im}_{\mathcal{F}}(R_{i+j}.)\\ \varpi_j(\zeta) & \longmapsto & \sigma_{i+j}(P_{i+j}\,\zeta).\end{array}$$

Finally, $\varepsilon(P_i')=\varepsilon(P_i)$ yields $P_i'=P_i+Z$ for a certain $Z=R_i\,U+V\,S_1$, where $U\in D^{p_{i-1}\times q_0}$ and $V\in D^{p_i\times q_1}$, which induces a *homotopy* of g_i [4], [12], and thus γ_i is a $\mathbb{Z}$-homomorphism which depends only on $\varepsilon(P_i)$.

References

[1] M. Barakat, B. Bremer, "Higher extension modules and the Yoneda product", http://arxiv.org/abs/0802.3179, submitted for publication.

[2] F. Chyzak, A. Quadrat, D. Robertz, "Effective algorithms for parametrizing linear control systems over Ore algebras", *Appl. Algebra Engrg. Comm. Comput.*, 16 (2005), 319-376.

[3] F. Chyzak, A. Quadrat, D. Robertz, "OREMODULES: A symbolic package for the study of multidimensional linear systems", *Applications of Time-Delay Systems*, LNCIS 352, Springer, 2007, 233-264, OREMODULES project, http://wwwb.math.rwth-aachen.de/OreModules.

[4] T. Cluzeau, A. Quadrat, "Factoring and decomposing a class of linear functional systems", *Linear Algebra Appl.*, 428 (2008), 324-381.

[5] T. Cluzeau, A. Quadrat, "OREMORPHISMS: A homological algebraic package for factoring and decomposing linear functional systems", *Topics in Time-Delay Systems: Analysis, Algorithms and Control*, LNCIS 388, Springer, 2009, 179-196, OREMORPHISMS project, http://pages.saclay.inria.fr/alban.quadrat//OreMorphisms/index.html.

[6] S. I. Gelfand, Y. I. Manin, *Methods of Homological Algebra*, Springer, 2003.

[7] V. Lomadze, E. Zerz, "Control and interconnection revisited: The linear multidimensional case", Proceedings of *nDS*, Poland, 2000.

[8] M. Kashiwara, T. Kawai, T. Kimura, *Foundations of algebraic analysis*, Princeton Mathematical Series 37, 1986.

[9] U. Oberst, "Multidimensional constant linear systems", *Acta Appl. Math.*, 20 (1990), 1-175.

[10] A. Quadrat, "An introduction to constructive algebraic analysis and its applications", *Les cours du CIRM*, 1 (2010), 281-471, http://hal.archives-ouvertes.fr/inria-00506104/fr/.

[11] A. Quadrat, D. Robertz, "Baer's extension problem for multidimensional linear systems", Proceedings of the *MTNS 2008*, USA.

[12] J. J. Rotman, *An Introduction to Homological Algebra*, Springer, 2009.