Polynomial Solutions and Annihilators of Ordinary Integro-Differential Operators ⋆

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Abstract: In this paper, we study algorithmic aspects of linear ordinary integro-differential operators with polynomial coefficients. Even though this algebra is not noetherian and has zero divisors, Bavula recently proved that it is coherent, which allows one to develop an algebraic systems theory. For an algorithmic approach to linear systems theory of integro-differential equations with boundary conditions, computing the kernel of matrices is a fundamental task. As a first step, we have to find annihilators, which is, in turn, related to polynomial solutions. We present an algorithmic approach for computing polynomial solutions and the index for a class of linear operators including integro-differential operators. A generating set for right annihilators can be constructed in terms of such polynomial solutions. For initial value problems, an involution of the algebra of integro-differential operators also allows us to compute left annihilators, which can be interpreted as compatibility conditions of integro-differential equations with boundary conditions. We illustrate our approach using an implementation in the computer algebra system Maple. Finally, system-theoretic interpretations of these results are given and illustrated on integro-differential equations.

1. INTRODUCTION

A standard RLC circuit is governed by the following linear *integro-differential (ID) equation*

$$L\,\frac{di(t)}{dt} + R\,i(t) + \frac{1}{C}\int_0^t i(s)\,ds = v(t), \tag{1}$$

where L is the inductor, R the resistor, C the capacitor, i the current, and v the voltage source. ID equations is a class of equations that naturally appear while modeling natural phenomena and they appear in many applications.

Using operator notation, (1) can be written as:

$$\left(L\,\partial + R + C^{-1}\textstyle\int\right)\, i(t) = v(t). \tag{2}$$

The integral operator is generally eliminated by differentiating once (1) to get the following linear ordinary differential (OD) equation:

$$L\,\frac{d^2 i(t)}{dt^2} + R\,\frac{di(t)}{dt} + \frac{1}{C}\,i(t) = \frac{dv(t)}{dt}. \tag{3}$$

If the current source v is constant, we find again the classical second order OD equation defining a RLC circuit. Equation (3) was obtained by pre-multiplying (2) by the differential operator ∂ and using the *fundamental theorem of analysis* stating that $\partial \int = \mathrm{id}$, i.e., $\int$ is a right inverse of ∂. We note that $\int$ is in general not a two-sided inverse since applying the operator $\int \partial$ to a function y, we get

$$\int_0^t \dot{y}(s)\,ds = y(t) - y(0),$$

which shows that $\int \partial = \mathrm{id} - \mathtt{E}$, where $\mathtt{E}$ denotes the evaluation at 0. Initial value problems of linear OD systems can be algebraically investigated using the *evaluation* $\mathtt{E}$.

Rings of *functional operators* (e.g., rings of OD operators, partial differential (PD) operators, differential time-delay operators, differential difference operators) were recently introduced in mathematical systems theory. Since many control linear systems can be defined by means of a matrix with entries in a *skew polynomial ring* or in an *Ore algebra* of functional operators (i.e., classes of univariate or multivariate noncommutative polynomial rings) [6], the classical *polynomial approach* to linear systems theory can be generalized yielding a *module-theoretic approach* to linear functional systems [9, 16, 17, 24]. Symbolic computation techniques (e.g., Gröbner basis techniques) and computer algebra systems can then be used to develop dedicated packages for algebraic systems theory [7, 15].

Algebras of ordinary ID operators have recently been studied within an algebraic approach in [1, 2, 3, 4] and within an algorithmic approach in [20, 21, 22]. The goal of the latter works is to provide an algebraic and algorithmic framework for studying *boundary value problems and Green's operators*.

Even though linear systems of ID equations play an important role in different domains and applications (e.g., PID controllers), it does not seem that they have been extensively studied by the mathematical systems community. For *boundary value systems*, we refer to [10, 11] and the references therein. The first purpose of this paper is to introduce concepts, techniques, and results developed in

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the above recents works. In particular, we emphasize that the algebraic structure of the ring of ID operators with polynomial coefficients

is much more involved (e.g., zero divisors, non noetherianity) than the one of the ring of OD operators with polynomial coefficients (the so-called *Weyl algebra*). The fundamental issue of computing left/right kernel of a matrix of ID operators has to be solved towards developing a system-theoretic approach to linear ID systems. The second goal of this paper is to study this problem for an ID operator $d \in \mathbb{I}$, by investigating the computation of zero divisors of d, which allows us to compute compatibility conditions of the inhomogeneous linear ID equation $d\,y = u$. Within a *representation approach*, we show that this problem is related to the computation of polynomial solutions of ID operators, a problem that is studied in detail in this paper.

2. ORDINARY INTEGRO-DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS

In what follows, let k be a field a field of characteristic zero (i.e., containing a subfield isomorphic to $\mathbb{Q}$). The k-algebra $A(k)$ of OD operators with coefficients in the polynomial ring $k[t]$ (*Weyl algebra*) can be defined in the following two ways (see, e.g., [8]):

(1) Let $k\langle X\rangle$ be the *free associative* k-algebra on $X = \{T, \Delta\}$ (i.e., the k-vector space with the basis formed by all words over X and the multiplication of basis elements defined by the concatenation). Then $A(k) = k\langle X\rangle/J$, where J is the two-sided ideal of $k\langle X\rangle$ generated by $\Psi := \Delta\,T - T\,\Delta - 1$, i.e., $J = k\langle X\rangle\,\Psi\,k\langle X\rangle$. If t (resp., ∂) is the residue class of T (resp., Δ) in $A(k) = k\langle t, \partial\rangle$, then using the relation $\partial\,t = t\,\partial + 1$, any element d of $A(k)$ can uniquely be written as a finite sum
$$d = \sum a_{ij}\,t^i\,\partial^j, \quad a_{ij} \in k,$$
which is called the *normal form* of $d \in A(k)$.

(2) Let $\mathrm{end}_k(k[t])$ be the k-algebra formed by all the k-endomorphisms of $k[t]$ (i.e., k-linear maps form $k[t]$ to $k[t]$). Then, $A(k)$ can also be be defined as the k-subalgebra of $\mathrm{end}_k(k[t])$ generated by the following three k-endomorphisms
$$\begin{cases} 1 \colon t^n \longmapsto t^n, \\ t \colon t^n \longmapsto t^{n+1}, \qquad \forall\, n \in \mathbb{N} \\ \partial \colon t^n \longmapsto n\,t^{n-1}, \end{cases}$$
defined on the basis $(t^n)_{n\in\mathbb{N}}$ of $k[t]$. In particular, they respectively correspond to the following operators
$$\begin{array}{lll} 1 \colon k[t] \longrightarrow k[t] & t \colon k[t] \longrightarrow k[t] & \partial \colon k[t] \longrightarrow k[t] \\ \quad p \longmapsto p, & \quad p \longmapsto t\,p, & \quad p \longmapsto \dot{p}, \end{array} \tag{4}$$
namely, the identity, the multiplication operator, and the derivation operator on the polynomial ring $k[t]$.

The first definition of $A(k)$ is by *generators* (T and Δ) and relations (Ψ). The second one is in terms of *representation theory.* We recall that $\partial\,t = t\,\partial + 1$ translates the following Leibniz rule in the operator language:
$$\partial(t\,y(t)) = t\,\partial\,y(t) + y(t) = (t\,\partial + 1)\,y(t).$$

Let us now introduce an important ring of ID operators.

Definition 1. The *k-algebra of ordinary ID operators with polynomial coefficients* $\mathbb{I}(k)$ is defined as the k-subalgebra of $\mathrm{end}_k(k[t])$ generated by 1, t, ∂ as in (4), and by
$$\begin{array}{rl} \int \colon k[t] & \longrightarrow k[t] \\ t^n & \longmapsto t^{n+1}/(n+1), \quad \forall\, n \in \mathbb{N}. \end{array}$$

The algebra $\mathbb{I}(k)$, simply be denoted by $\mathbb{I}$ in what follows, was studied in [1, 3] as a *generalized Weyl algebra.* See [20] for the construction of $\mathbb{I}$ as a factor algebra of a *skew polynomial ring.*

Note that the integral operator $\int$ corresponds to usual integral $p \in k[t] \longmapsto \int_0^t p(s)\,ds \in k[t]$. In the algebra $\mathbb{I}$, the *fundamental theorem* and a version of *integration by parts* can respectively be rewritten as:
$$\partial \int = 1, \quad \int\int = t\int - \int t.$$

Moreover, the *evaluation* at 0 can be defined as follows:
$$\mathtt{E} = 1 - \int\partial \colon p \in k[t] \longmapsto p(0) \in k. \tag{5}$$
The evaluation $\mathtt{E}$ can be used to study *initial value problems.*

Note that the operator $\mathtt{E}$ naturally induces the existence of *zero divisors* in $\mathbb{I}$ since, for instance, we have:
$$\partial\,\mathtt{E} = \mathtt{E}\int = \mathtt{E}\,t = 0.$$
The *left annihilator* of $d \in \mathbb{I}$, namely,
$$\mathrm{ann}_{\mathbb{I}}(.d) := \{e \in \mathbb{I} \mid e\,d = 0\},$$
can be interpreted as *compatibility conditions* of the inhomogeneous ID equation $d\,y(t) = u(t)$. Indeed, we have:
$$\forall\, e \in \mathrm{ann}_{\mathbb{I}}(.d), \quad e\,u(t) = e\,d\,y(t) = 0.$$
If d is not a zero divisor, then $d\,y = u$ does not admit compatibility condition of the form $e\,u = 0$, where $e \in \mathbb{I}$.

Example 2. Let us consider the following trivial example:
$$\int_0^t y(s)\,ds = u(t).$$
The compatibility condition $u(0) = 0$ corresponds to the left annihilator $\mathtt{E}$ of $\int$, i.e., $\mathtt{E}\int = 0$ in $\mathbb{I}$.

Let us consider the following inhomogeneous ID equation:
$$t^2\,\ddot{y}(t) - 2\,t\,\dot{y}(t) + (t+2)\,y(t) - (3\,t/5 + 2)\int_0^t y(s)\,ds + 3/5\int_0^t s\,y(s)\,ds = u(t). \tag{6}$$
The left annihilator of the following IO operator
$$d = t^2\,\partial^2 - 2\,t\,\partial + (t+2) - (3\,t/5 + 2)\int + 3/5\int t \in \mathbb{I} \tag{7}$$
yields the compatibility conditions of (6). See Example 22.

For the general construction of the algebra of ID operators $\mathcal{F}_\Phi[\partial, \int]$ defined over an ordinary ID algebra $\mathcal{F}$ and endowed with a set of *characters* (i.e., multiplicative linear functionals) Φ, we refer to [21, 22]. We note that the algebra $\mathbb{I}$ can be seen as a special case of this construction with $\mathcal{F} = (k[t], \partial, \int)$ and $\Phi = \{\mathtt{E}\}$. Hence, $\mathbb{I}$ can be defined as $k\langle t, \partial, \int\rangle = k\langle T, \Delta, I\rangle/J$, where J is the two sided ideal of the free algebra $k\langle T, \Delta, I\rangle$ generated by:
$$\Delta\,T - T\,\Delta - 1, \quad \Delta\,I - 1, \quad I\,I - T\,I + I\,T, \quad T - I\,\Delta\,T.$$
In particular, we have the following relations in $\mathbb{I}$
$$\partial\,t = t\,\partial + 1, \quad \partial\int = 1, \quad \int\int = t\int - \int t, \quad \mathtt{E}\,t = 0,$$
where $\mathtt{E} = 1 - \int\partial$.

More generally, we denote the evaluation at $\alpha \in k$ by
$$\mathtt{E}_\alpha \colon p \in k[t] \longmapsto p(\alpha) \in k.$$

The corresponding relations are

$$\forall\, \alpha,\, \beta \in k, \quad \mathtt{E}_\alpha\, t = \alpha \quad \text{and} \quad \mathtt{E}_\beta\, \mathtt{E}_\alpha = \mathtt{E}_\alpha.$$

In contrast to [1, 3], this last approach allows one to have more than one point evaluation, which is crucial for the study of *boundary value problems*.

Let $\Phi \subseteq k$. Identifying $\alpha \in \Phi$ with the evaluation $\mathtt{E}_\alpha$, we denote by $\mathbb{I}_\Phi$ the algebra of ID operators with polynomial coefficients endowed with the set of characters Φ. Then, every ID operator $d \in \mathbb{I}_\Phi$ can be uniquely written as a sum $d = d_1 + d_2 + d_3$, where $d_1 = \sum a_{ij}\, t^i\, \partial^j$ is an OD operator, $d_2 = \sum b_{ij}\, t^i \int t^j$ an *integral operator*, and

$$d_3 = \sum_{\alpha \in \Phi} \left(\sum f_{ij}\, t^i\, \mathtt{E}_\alpha\, \partial^j + \sum g_{ij}\, t^i\, \mathtt{E}_\alpha \int t^j \right) \qquad (8)$$

a *boundary operator*, where a_{ij}, b_{ij}, f_{ij}, and $g_{ij} \in k$, and d_1, d_2, and d_3 contain only finitely nonzero summands. See [21, 20] for details, in particular, for a Gröbner basis of the defining relations. For $\alpha = 0$, a boundary operator (8) is of the form $\sum c_{ij}\, t^i\, \mathtt{E}_0\, \partial^k$ since $\mathtt{E} \int = 0$.

In the following, we discuss some important algebraic properties of $\mathbb{I}$. First, since the integral operator $\int$ is a right but not a left inverse of the derivation ∂, it is known that the algebra $\mathbb{I}$ is necessarily non *noetherian* [12]. More explicitly, if $\int^i = \int \cdots \int$ denotes the product of i integral operators, one verifies that $e_{ij} = \int^i \mathtt{E}\, \partial^j$ satisfy

$$e_{ij}\, e_{lm} = \delta_{jl}\, e_{im}, \qquad (9)$$

where $\delta_{jl} = 1$ for $j = l$, and 0 otherwise; see [12] or [14, Ex. 21.26]. In particular, $\mathbb{I}$ contains infinitely many *orthogonal idempotents* e_{ii} for all $i \in \mathbb{N}$, i.e., $e_{ii}\, e_{jj} = \delta_{ij}$ for all $i,\, j \in \mathbb{N}$. If we introduce $e_k = e_{11} + \cdots + e_{kk} \in \mathbb{I}$ for $k \geq 1$, then using (9), we get $e_{ii} = e_{ii}\, e_k = e_k\, e_{ii}$ for $1 \leq i \leq k$, and the increasing sequence $\{I_k := \mathbb{I}\, e_k\}_{k \geq 1}$ (resp., $\{I_k := e_k\, \mathbb{I}\}_{k \geq 1}$) of principal left (resp., right) ideals of $\mathbb{I}$ is not stationary (see [12]), which proves that $\mathbb{I}$ is not a left (resp., a right) noetherian ring.

The following fundamental result was obtained by Bavula.

Theorem 3. ([1]). The ring $\mathbb{I}$ is *coherent*, i.e., for every $r \geq 1$, and for all $d_1, \ldots, d_r \in \mathbb{I}$, the left (resp., right) $\mathbb{I}$-module $S = \left\{(e_1, \ldots, e_r) \in \mathbb{I}^{1 \times r} \mid \sum_{i=1}^r e_i\, d_i = 0\right\}$ (resp., $S = \left\{(e_1, \ldots, e_r)^T \in \mathbb{I}^{r \times 1} \mid \sum_{i=1}^r d_i\, e_i = 0\right\}$) is finitely generated as a left (resp., right) $\mathbb{I}$-module.

Linear systems are usually described by means of finite matrices with entries in a certain ring D. As explained in [18], if D is a coherent ring, an algebraic systems theory can be developed as if D was a noetherian ring. Hence, Theorem 3 shows that an algebraic systems theory can be developed over $\mathbb{I}$. In particular, basic module-theoretic operations of *finitely presented* left/right $\mathbb{I}$-modules, namely, left/right $\mathbb{I}$-modules defined by matrices, are finitely presented, and thus, finitely generated. For more details, see, e.g., [14, 23]. It is shown in [4] that Theorem 3 cannot be generalized for more than one differential operator, i.e., for $\mathbb{I}_n$ and $n > 1$ (partial analogues).

Based on the normal forms for generalized Weyl algebras, it is shown in [3] that $\mathbb{I}$ admits the *involution* θ

$$\theta(\partial) = \textstyle\int, \quad \theta(\int) = \partial, \quad \theta(t) = t\, \partial^2 + \partial = (t\, \partial + 1)\, \partial, \quad (10)$$

i.e., an *anti-automorphism* of D of order two, namely, the k-linear map θ satisfies the following two properties:

$$\forall\, d,\, e \in D, \quad \theta(d\, e) = \theta(e)\, \theta(d), \quad \theta^2(d) = d.$$

An important consequence is that many algebraic properties of left $\mathbb{I}$-modules have a right analogue and conversely.

The computation of *syzygies*, namely, left/right kernel of a matrix with entries in $\mathbb{I}$ is a central task towards developing an algorithmic approach to linear systems of ID equations with boundary conditions based on module theory and homological algebra. See [6, 15, 19] and references therein. As a first step, we have to find left/right zero divisors of elements of $\mathbb{I}$. This problem leads, in turn, to computing polynomial solutions of ordinary ID equations with boundary conditions.

Finally, in [1, 2, 3], various algebraic properties of $\mathbb{I}$ and important results are proven amongst them a classification of *simple modules*, an analogue of *Stafford's theorem*, and of the *first conjecture of Dixmier*.

3. FREDHOLM AND FINITE-RANK OPERATORS

Several properties of *Fredholm operators* can be studied in the purely algebraic setting of linear maps on infinite-dimensional vector spaces. In [1], such properties are used to investigate $\mathbb{I}$. It turns out that Fredholm operators are also very useful for an algorithmic approach to operator algebras. We review some algebraic properties of Fredholm operators in the following.

Definition 4. A k-linear map $f \colon V \longrightarrow W$ between two k-vector spaces is called *Fredholm* if it has finite dimensional kernel and cokernel, where $\operatorname{coker} f = W / \operatorname{im} f$. The *index* of a Fredholm operator f is defined by:

$$\operatorname{ind}_k f = \dim_k(\ker f) - \dim_k(\operatorname{coker} f).$$

We have the *long exact sequence* of k-vector spaces ([23])

$$0 \longrightarrow \ker f \xrightarrow{\ i\ } V \xrightarrow{\ f\ } W \xrightarrow{\ p\ } \operatorname{coker} f \longrightarrow 0,$$

i.e., i is injective, $\ker f = \operatorname{im} i$, $\ker p = \operatorname{im} f$, and p is surjective. Then, $\dim_k(\operatorname{coker} f)$ gives the number of independent k-linear compatibility conditions $g(w) = 0$ on w for the solvability of the inhomogeneous linear system $f(v) = w$ (e.g., f is surjective iff $\operatorname{coker} f = 0$), while $\dim_k(\ker f)$ measures the degrees of freedom in a solution ($v + u$ is solution for all $u \in \ker f$).

Example 5. Viewing the basic operators 1, t, ∂, $\int \in \mathbb{I}$ as k-linear maps on $k[t]$, we get:

$$\begin{array}{ll} \ker 1 = \ker t = \ker \int = 0, & \ker \partial = k, \\ \operatorname{im} 1 = \operatorname{im} \partial = k[t], & \operatorname{im} t = \operatorname{im} \int = k[t]\, t. \end{array}$$

Hence, they are also Fredholm with index:

$$\operatorname{ind}_k 1 = 0, \quad \operatorname{ind}_k t = \operatorname{ind}_k \textstyle\int = -1, \quad \operatorname{ind}_k \partial = 1.$$

If V and W are two finite-dimensional k-vector spaces, then $\dim_k(\operatorname{coker} f) = \dim_k(W) - \dim_k(\operatorname{im} f)$ and the rank-nullity theorem yields $\dim_k V = \dim_k(\operatorname{im} f) + \dim_k(\ker f)$,

$$\operatorname{ind}_k f = \dim_k V - \dim_k W, \qquad (11)$$

i.e., $\operatorname{ind}_k f$ depends only on the dimensions of V and W.

We also recall the index formula for Fredholm operators.

Proposition 6. Let $V' \xrightarrow{\ f\ } V \xrightarrow{\ g\ } V''$ be k-linear maps between k-vector spaces. If two of the maps f, g, and $g \circ f$ are Fredholm, then so is the third, and:

$$\operatorname{ind}_k(g \circ f) = \operatorname{ind}_k g + \operatorname{ind}_k f.$$

Definition 7. A k-linear map between two k-vector spaces is called *finite-rank* if its image is finite-dimensional.

Example 8. Let us consider $\mathtt{E} = 1 - \int\partial \in \mathbb{I} \subset \mathrm{end}_k(k[x])$. It has an infinite-dimensional kernel $\ker_k \mathtt{E} = k[t]\, t$, but its image $\mathrm{im}_k \mathtt{E} = k$ is one-dimensional. More generally, every boundary operator $d_3 \in \mathbb{I}_\Phi$ is obviously of finite rank since its image is contained in the k-vector space of polynomials with degree less than or equal n, where n is the maximal index i with a nonzero coefficient f_{ij} or g_{ij} in (8).

Clearly, composing a finite-rank map with a linear map from either side gives again finite-rank map and Proposition 6 shows that the composition of two Fredholm operators is a Fredholm operator.

Proposition 9. Let V be a k-vector space and A a k-subalgebra of $\mathrm{end}_k(V)$. Then, $\mathcal{F}_A = \{a \in A \mid a \text{ Fredholm}\}$ forms a monoid and $\mathcal{C}_A = \{c \in A \mid c \text{ finite-rank}\}$ is a two-sided ideal of A.

In particular, the boundary operators $(\Phi) \subset \mathbb{I}_\Phi$ form a two-sided subideal of $\mathcal{C}_{\mathrm{end}_k(k[t])}$ generated by the evaluations $\mathtt{E}_\alpha \in \Phi$, and all other ID operators $\mathbb{I}_\Phi \setminus (\Phi)$ are Fredholm as we shall see in Proposition 15. More generally, Bavula has introduced in [1] the notion of *(strong) compact-Fredholm alternative* for an arbitrary k-algebra A.

4. POLYNOMIAL SOLUTIONS OF RATIONAL INDICIAL MAPS AND POLYNOMIAL INDEX

Computing polynomial solutions of linear systems of OD is well-studied in symbolic computation since it appears as a subproblem of many important algorithms. See [5] and the references therein. In this section, we discuss an algebraic setting and an algorithmic approach for the computation of polynomial solutions (kernel), cokernel, and the "polynomial" index for a general class of linear operators including ID operators.

For computing the kernel and cokernel of a k-linear map $L \colon V \longrightarrow V'$ on infinite-dimensional k-vector spaces V and V', we can use the following simple consequence of the snake lemma.

Lemma 10. Let $L \colon V \longrightarrow V'$ be a k-linear map and $U \subseteq V$, $U' \subseteq V'$ k-subspaces such that $L(U) \subseteq U'$. Let $L' = L_{|U} \colon U \longrightarrow U'$ and $\overline{L} \colon V/U \longrightarrow V'/U'$ the induced k-linear map defined by $\overline{L}(\pi(v)) = \pi'(L(v))$ for all $v \in V$, where $\pi \colon V \longrightarrow V/U$ (resp., $\pi' \colon V' \longrightarrow V'/U'$) is the canonical projection onto V/U (resp., V'/U'). Then, we have the following commutative exact diagram:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d, "L'"] & V \arrow[r, "\pi"] \arrow[d, "L"] & V/U \arrow[r] \arrow[d, "\overline{L}"] & 0 \\ 0 \arrow[r] & U' \arrow[r] & V' \arrow[r, "\pi'"] & V'/U' \arrow[r] & 0. \end{tikzcd} \tag{12}$$

If $\overline{L}$ is an isomorphism, i.e., $V/U \cong V'/U'$, then:

$$\ker L' = \ker L, \quad \mathrm{coker}\, L' \cong \mathrm{coker}\, L.$$

Moreover, if U and U' are two finite-dimensional k-vector spaces, then L is Fredholm and $\mathrm{ind}_k\, L = \dim_k U - \dim_k U'$.

Proof. Since $\overline{L}$ is an isomorphism, applying the standard the snake lemma (see, e.g., [23]) to (12), we obtain the following long exact sequence of k-vector spaces

$$0 \longrightarrow \ker L' \longrightarrow \ker L \longrightarrow 0 \longrightarrow \mathrm{coker}\, L' \longrightarrow \mathrm{coker}\, L \longrightarrow 0,$$

and the statements about the kernel and cokernel follow. If U and U' are two finite-dimensional k-vector spaces, then so are $\ker L' = \ker L$ and $\mathrm{coker}\, L' \cong \mathrm{coker}\, L$ and $\mathrm{ind}_k\, L = \mathrm{ind}_k\, L' = \dim_k U - \dim_k U'$ by (11).

From an algorithmic point of view, we want to find finite-dimensional k-subspaces U and U', and an algorithmic criterion for $\overline{L}$ being an isomorphism on the remaining infinite-dimensional parts V/U and V'/U'.

The cokernel of a k-linear map $f \colon V \longrightarrow W$ between two finite-dimensional k-vector spaces V and W can be characterized as follows. Choosing bases of V and W, there exists a matrix $C \in k^{m\times n}$ such that $f(v) = C\, v$ for all $v \in V \cong k^n$. Computing a basis of the finite-dimensional k-vector space $\ker C^T$ and stacking the elements of this basis into a matrix $D \in k^{m\times p}$, we get $\ker C^T = \mathrm{im}\, D$. Then, $\mathrm{coker}\, f \cong \mathrm{im}\, D^T$ and, more precisely, if $\pi \colon W \longrightarrow \mathrm{coker}\, f$ is the canonical projection onto $\mathrm{coker}\, f$, then the k-linear map $\sigma \colon \mathrm{coker}\, f \longrightarrow \mathrm{im}\, D$ defined by $\sigma(\pi(w)) = D\, w$ for all $w \in W$, is an isomorphism of k-vector spaces.

Let us study when the k-linear map $\overline{L} \colon V/U \longrightarrow V'/U'$ is an isomorphism. In what follows, we shall focus on the polynomial case, namely, $V = V' = k[t]$. To do that, let us introduce the degree filtration of $k[t]$, namely,

$$k[t] = \bigcup_{i\in\mathbb{N}} k[t]_{\le i}, \quad k[t]_{\le i} = \bigoplus_{j=0}^{i} k\, t^j,$$

defined by the finite-dimensional k-vector spaces $k[t]_{\le i}$ formed by the polynomials of $k[t]$ of degree less than or equal to i (we set $k[t]_{\le -1} = 0$). Note that this filtration is induced by any basis $\{p_i\}_{i\in\mathbb{N}}$ of $k[t]$ with $\deg p_i = i$ for all $i \in \mathbb{N}$. We recall that the multiplication operator, derivation, and integral operator are defined by (4), and we can check that:

$$\begin{cases} (t^i\, \partial^j)(t^n) = \dfrac{n!}{(n-j)!}\, t^{n-j+i}, \\ (t^i \int t^j)(t^n) = \dfrac{1}{n+j+1}\, t^{i+j+n+1}. \end{cases}$$

Definition 11. A k-linear map $L \colon k[t] \longrightarrow k[t]$ is called *rational indicial* with *rational symbol* $\mathrm{rsym}(L) = (s,\, q)$ if there exist a nonzero rational function $q \in k(n)$, $c_n \in k^*$, and $M \in \mathbb{N}$ such that:

$\forall\, n \ge M \ge -s,\ L(t^n) = c_n\, q(n)\, t^{n+s}$+lower degree terms.

Example 12. The rational symbols of (4) are:

$$\mathrm{rsym}(1) = (0,\, 1), \qquad \mathrm{rsym}(t) = (1,\, 1),$$
$$\mathrm{rsym}(\partial) = (-1,\, n), \quad \mathrm{rsym}(\textstyle\int) = \left(1,\, \dfrac{1}{(n+1)}\right).$$

Operators such as shift, dilation, convolution operators on $k[t]$ are also rational indicial. The sum of a rational indicial map and a finite-rank map is also rational indicial with the same symbol, as one sees, by choosing the bound M large enough, e.g., for $1 + t^3\mathtt{E}_0$, one can take $M = 4$.

Let us now state a result for the computation of the kernel and cokernel of rational indicial maps (compare with Lemma 6.5. of [1]).

Proposition 13. Let $L \colon k[t] \longrightarrow k[t]$ be a k-linear map. Let

$$-1 \le N,\ -(N+1) \le s, \quad U = k[t]_{\le N}, \quad U' = k[t]_{\le N+s}$$

be such that $L(U) \subseteq U'$. Let $L' = L_{|U} \colon U \longrightarrow U'$ be the induced map. If $\deg L(t^n) = n + s$ for all $n \ge N + 1$, then:

$$\ker L' = \ker L, \quad \operatorname{coker} L' \cong \operatorname{coker} L.$$

Moreover, L is a Fredholm operator with $\operatorname{ind}_k L = -s$.

Proof. Let $V = V' = k[t]$ and $\pi : V \longrightarrow V/U$ (resp., $\pi' : V' \longrightarrow V'/U'$) be the canonical projection onto V/U (resp., V'/U'). Then, $\overline{L}(\pi(t^n)) = \pi'(L(t^n))$ for all $n \in N$.

Now, the condition on the degree of the image $L(t^n)$ for $n \geq N$ shows that $\overline{L}$ maps the basis $\{\pi(t^n)\}_{n\geq N}$ of V/U to a basis of V'/U', and thus, defines an isomorphism. The result then follows from Lemma 10 after noting that:

$$\dim_k U - \dim_k U' = N + 1 - (N + 1 + s) = -s.$$

Given a rational indicial operator with rational symbol (s, q) and bound M, we obtain a bound N for Proposition 13 by computing the largest nonnegative integer root l of q and taking $N = \max(l, M)$. Hence computing the kernel and cokernel of $L\colon k[t] \longrightarrow k[t]$ reduces to the same problem for the k-linear map $L' = L_{|U} : U \longrightarrow U'$ between two finite-dimensional k-vector spaces, which can be solved using basic linear algebra techniques. We have implemented in Maple the computation of kernel and cokernel of rational indicial maps.

Corollary 14. A rational indicial operator with rational symbol (s, q) is Fredholm with index $-s$ and its kernel and cokernel can be effectively computed.

We can explicitly compute the rational symbol (s, q) for $d \notin (\Phi)$ from its normal form. The following proposition is a purely algebraic version of an *index theorem* (compare with [1, Proposition 6.1]).

Proposition 15. Let $d = \sum a_{ij}\, t^i\, \partial^j + \sum b_{ij}\, t^i \int t^j + d_3 \in \mathbb{I}_\Phi$ be an ID operator, where $d_3 \in (\Phi)$, such that $d \notin (\Phi)$. Then, the k-linear map

$$\begin{aligned} L_d\colon k[t] &\longrightarrow k[t], \\ p &\longmapsto d(p), \end{aligned} \tag{13}$$

is rational indicial with rational symbol

$s = -\operatorname{ind}_k d = \max(\{i-j \mid a_{ij} \neq 0\} \cup \{i+j+1 \mid b_{ij} \neq 0\})$,

and $q(n) = \sum_{i-j=s} a_{ij}\, \frac{n!}{(n-j)!} + \sum_{i+j+1=s} b_{ij}\, \frac{1}{n+j+1}$.

5. POLYNOMIAL SOLUTION AND ANNIHILATORS

In the proof of Theorem 3, the fact that the left and right annihilators are finitely generated $\mathbb{I}$-modules is used, for which a non-constructive argument is given in [1].

Theorem 16. ([1]). If $d \in \mathbb{I}$, then the left (resp., right) annihilator $\operatorname{ann}_{\mathbb{I}}(.d)$ (resp., $\operatorname{ann}_{\mathbb{I}}(d.) := \{e \in \mathbb{I} \mid d\,e = 0\}$) of d is a finitely generated left (resp., right) $\mathbb{I}$-module.

We generalize this result to the right annihilator of a Fredholm operator $d \in \mathbb{I}_\Phi$ with more than one evaluation using a constructive approach. It is based on the fact that we can identify (as for the Weyl algebra and $\mathbb{I}$) an integro-differential operator d with the corresponding linear map L_d on the polynomial ring $k[t]$.

Theorem 17. The k-algebra homomorphism

$$\begin{aligned} \chi\colon \mathbb{I}_\Phi &\longrightarrow \operatorname{end}_k(k[t]) \\ d &\longmapsto L_d, \end{aligned}$$

is a *faithful representation* of $\mathbb{I}_\Phi$, i.e., χ is injective.

For a proof that χ is injective, we first observe that for $d \notin (\Phi)$, the k-linear map L_d is obviously nonzero by Proposition 15. So, let $d \in (\Phi)$ be a boundary operator. By (8), d is a finite $k[t]$-linear combination of terms of the form $\mathtt{E}_\alpha\, \partial^i$ and $\mathtt{E}_\alpha \int t^i$, where $\alpha \in \Phi$, namely

$$d = \sum_{\alpha\in\Phi} \left(\sum_{i=0}^{l} p_{\alpha,i}\, \mathtt{E}_\alpha\, \partial^i + \sum_{i=0}^{m} q_{\alpha,i}\, \mathtt{E}_\alpha \textstyle\int t^i \right), \tag{14}$$

where $p_{\alpha,i},\ q_{\alpha,i} \in k[t]$.

Lemma 18. The k-linear functionals $\mathtt{E}_\alpha\, \partial^i$ and $\mathtt{E}_\alpha \int t^i$ on $k[t]$ for $i \in \mathbb{N}$ and $\alpha \in k$ are k-linearly independent.

Proof. This can be seen by evaluating $\mathtt{E}_\alpha\, \partial^i$ and $\mathtt{E}_\alpha \int t^i$ on sufficiently many polynomials of the form $(t - c)^n$ for $c \in k$ and $n \in \mathbb{N}$ since

(1) Evaluating $\mathtt{E}_{\alpha_1}, \ldots, \mathtt{E}_{\alpha_m}$ for distinct $\alpha_1, \ldots, \alpha_m \in k$ on $1, t, \ldots, t^{m-1}$ gives a *Vandermonde* matrix.
(2) Evaluating the functionals $\mathtt{E}_\alpha\, \partial, \mathtt{E}_\alpha\, \partial^2, \ldots, \mathtt{E}_\alpha\, \partial^m$ at $(t-c), (t-c)^2, \ldots, (t-c)^m$, for arbitrary $c \in k$, gives a upper triangular matrix with diagonal $1, 2!, \ldots, m!$.
(3) Evaluating the functionals $\mathtt{E}_\alpha \int, \mathtt{E}_\alpha \int t, \ldots, \mathtt{E}_\alpha \int t^{m-1}$, for $\alpha \neq 0$, at $1, (t-c), (t-c)^2, \ldots, (t-c)^{m-1}$ gives matrices A_m with entries $\int_0^\alpha x^j (x-c)^n\, dx$. For $\alpha = 1$ and $c = 0$, this is a *Hilbert matrix* H_m of order m. It is well-known that Hilbert matrices and all its submatrices are invertible. One can verify that $\det A_m$ is independent of c and is a nonzero multiple of $\det H_m$.

We can therefore apply the following lemma for linear functionals on arbitrary vector spaces to describe the image of a finite-rank operator L_d for a $d \in (\Phi)$ in terms of its normal form (14).

Lemma 19. Let V be a k-vector space and $\lambda_1, \ldots, \lambda_n \in V^*$ k-linear functionals. Then, the λ_i are k-linearly independent iff there exist $v_1, \ldots, v_n \in V$ such that:

$$\forall\, i, j = 1, \ldots, n, \quad \lambda_i(v_j) = \delta_{ij}.$$

Proposition 20. Let $d \in (\Phi)$ as in (14). Then, we have:

$$\operatorname{im} L_d = \sum_{\alpha\in\Phi} \sum_{i=0}^{l} k\, p_{\alpha,i} + \sum_{\alpha\in\Phi} \sum_{i=0}^{m} k\, q_{\alpha,i}.$$

Proof. The inclusion $\subseteq$ is obvious since $\mathtt{E}_\alpha\, \partial^i$ and $\mathtt{E}_\alpha \int t^i$ are functionals. Let $\mathtt{E}_\alpha\, \partial^i$ or $\mathtt{E}_\alpha \int t^i$ be a linear functional corresponding to a nonzero summand in (14). By Lemma 19 with $V = k[t]$, there exists a polynomial $p \in k[t]$ such that $(\mathtt{E}_\alpha\, \partial^i)(p) = 1$ (resp., $(\mathtt{E}_\alpha \int t^i)(p) = 1$) and $(\mathtt{E}_\beta\, \partial^j)(p) = 0$ (resp., $(\mathtt{E}_\beta \int t^j)(p) = 0$) for all other functionals corresponding to nonzero summands of (14). Then, we get $L_d(p) = d(p) = p_{\alpha,i}$ or $L_d(p) = d(p) = q_{\alpha,i}$, which proves the reverse inclusion.

In particular, by Proposition 20, we know that $L_d = 0$ implies $d = 0$ also for $d \in (\Phi)$. Hence χ is injective and Theorem 17 is proved.

To characterize $\operatorname{ann}_{\mathbb{I}_\Phi}(d.)$, we use the equivalences

$$d\,e = 0 \quad \Leftrightarrow\ L_{d\,e} = L_d \circ L_e = 0\ \Leftrightarrow\ \operatorname{im} L_e \subseteq \ker L_d. \tag{15}$$

If d is Fredholm, i.e., $d \in \mathbb{I} \setminus (\Phi)$, then $\ker L_d$ is a finite-dimensional k-vector space, and thus, e has to be finite-rank. Thus, we have to compute polynomial solutions of the Fredholm operator d, i.e., $\ker L_d$, and then find generators for all the e's satisfying $\operatorname{im} L_e \subseteq \ker L_d$.

Theorem 21. Let $\Phi \subset k$. Let $d \in \mathbb{I}_\Phi$ be Fredholm with $\ker L_d = \sum_{i=1}^n k\, r_i$, where $r_i \in k[t]$. Then, we have:

$$\mathrm{ann}_{\mathbb{I}}(d.) = \sum_{\alpha\in\Phi} \sum_{i=1}^{n} (r_i\, \mathtt{E}_\alpha)\, \mathbb{I}_\Phi.$$

If Φ is finite (i.e., only finitely many evaluations points), then $\mathrm{ann}_{\mathbb{I}}(d.)$ is a finitely generated right $\mathbb{I}$-module.

Proof. Since $\mathrm{im}\, L_{r_i \mathtt{E}_\alpha} = k\, r_i \subseteq \ker L_d$, the inclusion $\supseteq$ follows by (15). Conversely, let $e \in \mathbb{I}_\Phi$ as in (14) with $d\,e = 0$. Then, by (15) and Proposition 20, we have

$$\mathrm{im}\, L_e = \sum_{\alpha\in\Phi}\sum_{i=0}^{l} k\, p_{\alpha,i} + \sum_{\alpha\in\Phi}\sum_{i=0}^{m} k\, q_{\alpha,i} \subseteq \ker L_d = \sum_{i=1}^{n} k\, r_i.$$

Hence, every nonzero $p_{\alpha,i}$ and $q_{\alpha,i}$ can be written as a k-linear combination of the r_i's. The reverse inclusion then follows by post-multiplying the generators $r_i\,\mathtt{E}_\alpha$ with suitable ∂^i or $\int t^i$.

The computation of the left annihilator $\mathrm{ann}_{\mathbb{I}}(.d)$ (e.g., for initial value problems) can be solved by computing the right annihilator $\mathrm{ann}_{\mathbb{I}}(\theta(d).)$, where θ is defined by (10), and then apply θ to each generator of $\mathrm{ann}_{\mathbb{I}}(\theta(d).)$.

All necessary steps for computing right and left annihilators have been implemented based on the Maple package *IntDiffOp* [13] for ID operators and boundary problems.

Example 22. Let us compute the compatibility conditions of (6). Note $\mathrm{rsym}(\theta(d)) = (0, n^2 - 3\,n + 2)$, where $\theta(d) = (t^2 + t - 3/5)\,\partial^2 - (2\,t+1)\,\partial + 2$. The largest nonnegative integer root of q is 2. With this bound N for Proposition 13, we get $\ker L_{\theta(d)} = k\,(t^2+3/5) + k\,(t+1/2)$. By Theorem 21, $\mathrm{ann}_{\mathbb{I}}(\theta(d).) = ((t^2 + 3/5)\,\mathtt{E})\,\mathbb{I} + ((t+1/2)\,\mathtt{E})\,\mathbb{I}$. Computing the involution of these generators yield the left annihilator $\mathrm{ann}_{\mathbb{I}}(.d) = \mathbb{I}\,(2\,\mathtt{E}\,\partial^2 + 3/5\,\mathtt{E}) + \mathbb{I}\,(\mathtt{E}\,\partial + 1/2\,\mathtt{E})$ for (7), which correspond to the compatibility conditions:

$$2\,\ddot{u}(0) + 3/5\,u(0) = 0, \quad \dot{u}(0) + 1/2\,u(0) = 0.$$

REFERENCES

[1] V. V. Bavula. The algebra of integro-differential operators on an affine line and its modules. *J. Pure Appl. Algebra* 217(3):495–529, 2013.

[2] V. V. Bavula. An analogue of the Conjecture of Dixmier is true for the algebra of polynomial integro-differential operators. *J. Algebra* 372:237–250, 2012.

[3] V. V. Bavula. The algebra of integro-differential operators on a polynomial algebra. *J. Lond. Math. Soc. (2)*, 83(2):517–543, 2011.

[4] V. V. Bavula. The algebra of polynomial integro-differential operators is a holonomic bimodule over the subalgebra of polynomial differential operators. 2011. `http://arxiv.org/abs/arXiv:1011.3009`.

[5] A. Bostan, T. Cluzeau, and B. Salvy. Fast algorithms for polynomial solutions of linear differential equations. In *Proceedings of ISSAC 2005*, pages 45–52. ACM, 2005.

[6] F. Chyzak, A. Quadrat, and D. Robertz. Effective algorithms for parametrizing linear control systems over Ore algebras. *Appl. Algebra Engrg. Comm. Comput.*, 16(5):319–376, 2005.

[7] F. Chyzak, A. Quadrat, and D. Robertz. OreModules: A symbolic package for the study of multidimensional linear systems. In *Applications of time delay systems*, volume 352 of *LNCIS*, pages 233–264. Springer, Berlin, 2007.

[8] S. C. Coutinho. *A primer of algebraic D-modules*, volume 33. Cambridge University Press, 1995.

[9] M. Fliess. Some basic structural properties of generalized linear systems. *Systems Control Lett.*, 15(5): 391–396, 1990.

[10] I. Gohberg and M. A. Kaashoek. Time varying linear systems with boundary conditions and integral operators. I. The transfer operator and its properties. *Integral Equations Operator Theory*, 7(3):325–391, 1984.

[11] I. Gohberg, M. A. Kaashoek, and L. Lerer. Minimality and irreducibility of time-invariant linear boundary value systems. *Internat. J. Control*, 44(2):363–379, 1986.

[12] N. Jacobson. Some remarks on one-sided inverses. *Proc. Amer. Math. Soc.*, 1:352–355, 1950.

[13] A. Korporal, G. Regensburger, and M. Rosenkranz. Regular and singular boundary problems in Maple. In *Proceedings of CASC 2011*, volume 6885 of *LNCS*, pages 280–293, 2011. Springer, `http://www.risc.jku.at/people/akorpora/index.html`.

[14] T. Y. Lam. *A First Course in Noncommutative Rings.* Springer-Verlag, New York, 1991.

[15] V. Levandovskyy and E. Zerz. Algebraic systems theory and computer algebraic methods for some classes of linear control systems. In *Proceedings of MTNS 2006*, Kyoto, Japan, 2006.

[16] U. Oberst. Multidimensional constant linear systems. *Acta Appl. Math.*, 20(1-2):1–175, 1990.

[17] J.-F. Pommaret and A. Quadrat. A functorial approach to the behaviour of multidimensional control systems. *Int. J. Appl. Math. Comput. Sci.*, 13(1):7–13, 2003.

[18] A. Quadrat. The fractional representation approach to synthesis problems: An algebraic analysis viewpoint. I. (Weakly) doubly coprime factorizations. *SIAM J. Control Optim.*, 42(1):266–299, 2003.

[19] A. Quadrat. An introduction to constructive algebraic analysis and its applications. Inria Research Report n. 7354, 2010, `http://hal.archives-ouvertes.fr/inria-00506104/fr/`.

[20] G. Regensburger, M. Rosenkranz, and J. Middeke. A skew polynomial approach to integro-differential operators. In *Proceedings of ISSAC 2009*, pages 287–294, New York, NY, USA, 2009. ACM.

[21] M. Rosenkranz and G. Regensburger. Solving and factoring boundary problems for linear ordinary differential equations in differential algebras. *J. Symbolic Comput.*, 43(8):515–544, 2008.

[22] M. Rosenkranz, G. Regensburger, L. Tec, and B. Buchberger. Symbolic analysis for boundary problems: From rewriting to parametrized Gröbner bases. In *Numerical and Symbolic Scientific Computing: Progress and Prospects*, pages 273–331. SpringerWienNew York, Vienna, 2012.

[23] J. Rotman. *An Introduction to Homological Algebra.* Springer, New York, second edition, 2009.

[24] J. Wood. Modules and behaviours in nD systems theory. *Multidim. Syst. Signal Process.*, 11(1/2):11–48, 2000.