An algebraic analysis approach to certain classes of nonlinear partial differential systems

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***Abstract*— Many partial differential systems appearing in mathematical physics, engineering sciences and mathematical biology are nonlinear. Unfortunately, the algebraic analysis approach, based on module theory, can only handle linear partial differential systems. This paper is a first step toward a generalization of this approach to certain classes of nonlinear partial differential systems. In particular, we show how constructive methods of differential algebra and algebraic analysis can be used to extend recent results on internal symmetries, conservation laws and the decomposition problem to these classes of nonlinear partial differential systems.**

I. Introduction

A linear partial differential (PD) system can generally be written as $R\,\eta = 0$, where $R \in D^{q\times p}$ is a $q \times p$ matrix with coefficients in a noncommutative polynomial ring D of PD operators, and η is a vector of unknown functions. More precisely, if $\mathcal{F}$ is a left D-module, then we can consider the *linear PD system* or *behaviour*:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

The *algebraic analysis* approach to linear PD systems (see, e.g., [3], [5], [9], [13]) is based on the fact that the linear PD system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$. Indeed, if $\hom_D(M, \mathcal{F})$ is the abelian group of left D-homomorphisms (i.e., left D-linear maps) from M to $\mathcal{F}$, then a remark due to Malgrange [13] asserts that the abelian group isomorphism

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$$

holds, which provides a dictionary between the systemic properties of $\ker_{\mathcal{F}}(R.)$ and the module properties of M and $\mathcal{F}$. Using homological algebra techniques [3], [5], [9], [13], [16], certain properties of M can be constructively checked [5], [7] using the recent development of noncommutative Gröbner/Janet bases. The corresponding algorithms were implemented in the OreModules [6], OreMorphisms [8], Janet [2] and `homalg` [1] packages.

Many PD systems studied in mathematical physics, engineering sciences and mathematical biology are nonlinear. Unfortunately, to our knowledge, due to its module-theoretic nature, algebraic analysis can only handle linear PD systems. In the literature, nonlinear PD systems are usually studied by means of linearization techniques, analytic approaches, differential algebra or Lie (pseudo)groups.

Using constructive methods of *differential algebra* [11], [15] and algebraic analysis, the purpose of this paper is to start the development of a new mathematical approach to certain classes of multidimensional (PD) nonlinear systems appearing in mathematical physics and engineering sciences (e.g., Burgers' flow, traffic flow, gas dynamics, shallow water equations, transonic flow, unsteady nonisentropic flow) [10]. More precisely, in this paper, we show how to extend some results developed in [5], [6], [7], [8] on homomorphisms, internal symmetries, conservation laws and decomposition problems to certain classes of nonlinear PD systems.

The paper is organized as follows. In Section II, we recall basic definitions and techniques of differential algebra. Section III shows how finitely presented left D-modules can be associated with certain nonlinear PD systems. Then, in Section IV, we first recall the characterization of a left D-homomorphism between two finitely presented left D-modules [7] and we then show how this can be used to compute internal symmetries of certain classes of nonlinear PD systems. Section V is concerned with the computation of conservation laws of nonlinear PD systems. Finally, in Section VI, we show that an approach similar to [7] can be used to decompose certain nonlinear PD systems.

II. Differential algebra

In this section, we recall basic definitions and constructions of *differential algebra* that will be used in what follows. For more details, see the standard books [11] and [15].

Definition 1: A *differential ring* $(A, \{\delta_1, \ldots, \delta_n\})$ is a ring A equipped with n *commuting derivations* $\delta_1, \ldots, \delta_n$, namely, for $i = 1, \ldots, n$, $\delta_i : A \longrightarrow A$ satisfies:

$$\forall\, a,\, b \in A,\ \forall\, j = 1, \ldots, n,\ \left\{\begin{array}{l} \delta_i(a+b) = \delta_i(a) + \delta_i(b), \\ \delta_i(a\,b) = \delta_i(a)\,b + a\,\delta_i(b), \\ \delta_i \circ \delta_j = \delta_j \circ \delta_i. \end{array}\right. \quad (1)$$

Moreover, if A is a field, then $(A, \{\delta_1, \ldots, \delta_n\})$ is called a *differential field*.

If $(A, \{\delta_1, \ldots, \delta_n\})$ is a differential ring, then we can define the *ring of constants* of A:

$$C(A) = \{a \in A \mid \forall\, i = 1, \ldots, n,\ \delta_i(a) = 0\}.$$

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In particular, we have:

$$\forall\, i = 1, \ldots, n, \quad \delta_i(1) = \delta_i\left(1^2\right) = 2\,\delta_i(1) \;\Rightarrow\; \delta_i(1) = 0. \tag{2}$$

In what follows, we shall assume that A contains a field k of characteristic 0, i.e., a field k isomorphic to $\mathbb{Q}$.

Definition 2: Let $(A, \{\delta_1, \ldots, \delta_n\})$ be a differential ring. An ideal I of A is a *differential ideal* of $(A, \{\delta_1, \ldots, \delta_n\})$ if it satisfies:

$$\forall\, i = 1, \ldots, n, \quad \forall\, a \in I, \quad \delta_i(a) \in I.$$

A differential ideal $\mathfrak{p}$ of $(A, \{\delta_1, \ldots, \delta_n\})$ is called *prime* if:

$$a\, b \in \mathfrak{p} \quad \Rightarrow \quad a \in \mathfrak{p} \quad \text{or} \quad b \in \mathfrak{p}.$$

If I is a differential ideal of a differential ring $(A, \{\delta_1, \ldots, \delta_n\})$, then we can define the quotient ring A/I formed by the residue classes $\kappa(a)$ of elements $a \in A$, and endowed with the following operations:

$$\forall\, a,\, b \in A, \quad \begin{cases} \kappa(a + b) \triangleq \kappa(a) + \kappa(b), \\ \kappa(a\, b) \triangleq \kappa(a)\, \kappa(b). \end{cases} \tag{3}$$

Then, A/I can be endowed with a differential ring structure by extending the action of the derivations δ_i's to A/I:

$$\forall\, i = 1, \ldots, n, \quad \forall\, a \in A, \quad \delta_i(\kappa(a)) \triangleq \kappa(\delta_i(a)). \tag{4}$$

Now, if $\mathfrak{p}$ is a prime differential ideal of $(A, \{\delta_1, \ldots, \delta_n\})$, then the quotient ring $B = A/\mathfrak{p}$ is an integral domain since:

$$\begin{aligned} \kappa(a)\, \kappa(b) = 0 \;\Leftrightarrow\; \kappa(a\, b) = 0 \;\Leftrightarrow\; a\, b \in \mathfrak{p} \\ \Leftrightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p} \;\Leftrightarrow\; \kappa(a) = 0 \text{ or } \kappa(b) = 0. \end{aligned}$$

Then, we can define the *quotient field* $Q(B)$ of B:

$$Q(B) = \left\{ q = n\, d^{-1} \mid d \in B \setminus \{0\},\; n \in B \right\}.$$

Using (2), $\delta_i(q)\, q^{-1} + q\, \delta_i(q^{-1}) = \delta_i(q\, q^{-1}) = \delta_i(1) = 0$, for all $q \in Q(B) \setminus \{0\}$, which yields:

$$\forall\, q \in Q(B) \setminus \{0\}, \quad \delta_i(q^{-1}) = -\delta_i(q)\, q^{-2}.$$

The field $Q(B)$ is then a differential field called the *differential quotient field* of B.

If $(k, \{\delta_1, \ldots, \delta_n\})$ is a differential field and $Y_1, \ldots, Y_p$ a set of indeterminates over k, then we can form the ring $k\{Y_1, \ldots, Y_p\}$ of *differential polynomials* in the Y_j's, namely, the commutative polynomial ring $k[\Delta\, Y]$ in the set of indeterminates $\Delta\, Y$ defined by

$$\Delta\, Y = \{Y_{j,\nu} \mid j = 1, \ldots, p,\; \nu = (\nu_1 \;\ldots\; \nu_n) \in \mathbb{N}^n\},$$

where $Y_{j,0} = Y_j$. An element of $k\{Y_1, \ldots, Y_p\}$ is a polynomial in a finite number of the indeterminates $Y_{j,\nu}$'s with coefficients in k. Now, if $\nu = (\nu_1 \;\ldots\; \nu_n) \in \mathbb{N}^n$ is a multi-index of length $|\nu| = \nu_1 + \cdots + \nu_n$ and $\nu + 1_i$ is the multi-index of length $|\nu| + 1$ obtained by adding 1 at the i^{th} position of ν, then $k\{Y_1, \ldots, Y_p\}$ can be endowed with a differential ring structure defined by the following derivations:

$$\forall\, i = 1, \ldots, n,\; d_i \triangleq \delta_i + \sum_{j = 1, \ldots, p,\; |\nu| \geq 0} Y_{j, \nu + 1_i}\, \frac{\partial}{\partial\, Y_{j,\nu}}. \tag{5}$$

The Y_j's are then called *differential indeterminates* over k.

Let $(k\{Y_1, \ldots, Y_p\}, \{d_1, \ldots, d_n\})$ be the ring of differential polynomials over the differential field $(k, \{\delta_1, \ldots, \delta_n\})$, where the d_i's are defined by (5), and $\mathfrak{p}$ a prime differential ideal of $(k\{Y_1, \ldots, Y_p\}, \{d_1, \ldots, d_n\})$. Let us consider the integral domain $A \triangleq k\{Y_1, \ldots, Y_p\}/\mathfrak{p}$ and y_j the residue class of Y_j in A, i.e., $y_j = \kappa(Y_j)$ for $j = 1, \ldots, p$. Using (4), $A = k\{y_1, \ldots, y_p\}$ inherits a differential ring structure:

$$\forall\, i = 1, \ldots, n, \quad \forall\, j = 1, \ldots, p, \quad d_i\, y_j = \kappa(d_i\, Y_j).$$

Since A is an integral domain, we denote by $k\langle y_1, \ldots, y_p\rangle$ its differential quotient field $Q(A)$.

Let us now introduce the concept of a *ring of partial differential (PD) operators* with coefficients in a differential ring $(A, \{d_1, \ldots, d_n\})$. Let $D = A\langle \partial_1, \ldots, \partial_n\rangle$ be the noncommutative polynomial ring of PD operators with coefficients in A, namely, the ring formed by elements of the form $\sum_{0 \leq |\nu| \leq r} a_\nu\, \partial^\nu$, where $\partial^\nu = \partial_1^{\nu_1} \cdots \partial_n^{\nu_n}$ and $a_\nu \in A$ satisfy the following relations:

$$\forall\, i, j = 1, \ldots, n, \quad \forall\, a \in A, \quad \begin{cases} \partial_i\, a = a\, \partial_i + d_i\, a, \\ \partial_i\, \partial_j = \partial_j\, \partial_i. \end{cases} \tag{6}$$

Remark 1: We point out that in (6), a has to be interpreted as the multiplication operator $b \longmapsto a\, b$. To simplify the notations, we shall simply denote the above multiplication operator (sometimes denoted by $a.$) and the element $a \in A$ by the same notation.

The ring A inherits a left D-module structure defined by:

$$\forall\, i = 1, \ldots, n, \quad \forall\, a \in A, \quad \partial_i\, a \triangleq d_i\, a. \tag{7}$$

In what follows, k will denote a differential field with the derivations $\{\delta_1, \ldots, \delta_n\}$ and we shall focus on differential rings A of the form $A = k\{Y_1, \ldots, Y_p\}/\mathfrak{p}$, where $\mathfrak{p}$ is a prime differential ideal of $k\{Y_1, \ldots, Y_p\}$.

Example 1: Let $(\mathbb{Q}, \{\delta_t, \delta_x\})$ be the differential field, where $\delta_t(q) = \delta_x(q) = 0$ for all $q \in \mathbb{Q}$. Moreover, let Y be an indeterminate over $\mathbb{Q}$ and let us consider the ring $\mathbb{Q}\{Y\}$ of differential polynomials in the differential indeterminate Y, namely, polynomials in a finite number of $d^\nu\, Y$, where $d^\nu = d_t^{\nu_1}\, d_x^{\nu_2}$ and d_t and d_x are defined by (5). For instance, we can easily check that $d_t(q) = 0$ for $q \in \mathbb{Q}$, $d_t\, Y = Y_{(1,0)}$ (also denoted by Y_t), $d_x\, Y = Y_{(0,1)}$ (also denoted by Y_x), $d_t(Y^2) = 2\, Y\, Y_t$, $d_t(Y_x) = Y_{(1,1)}$ (also denoted by $Y_{x,t}$),

$$\begin{aligned} d_t(Y^2\, Y_x) &= d_t(Y^2)\, Y_x + Y^2\, d_t(Y_x) \\ &= 2\, Y\, Y_t\, Y_x + Y^2\, Y_{x,t}. \end{aligned}$$

Consider the prime differential ideal $\mathfrak{p} = \{d_t\, Y + Y\, d_x\, Y\}$ of $\mathbb{Q}\{Y\}$ and $A = \mathbb{Q}\{Y\}/\mathfrak{p} = \mathbb{Q}\{y\}$, where $y = \kappa(Y)$. Then, A is a differential ring with the derivations $d_t\, y = \kappa(d_t\, Y)$ and $d_x\, y = \kappa(d_x\, Y)$. Since $\kappa(d_t\, Y + Y\, d_x\, Y) = 0$, using (3), we then get the so-called *Burgers' equation* [10]:

$$d_t\, y + y\, d_x\, y = 0.$$

Let $D = A\langle \partial_t, \partial_x \rangle$ be the noncommutative polynomial ring of PD operators in ∂_t and ∂_x with coefficients in A. An element of D has the form $\sum_{0 \leq |\nu| \leq r} a_\nu \, \partial^\nu$, where $a_\nu \in A$ and ∂^ν satisfy the commutation relations (6). For instance, we have $\partial_t \, y = y \, \partial_t + d_t \, y = y \, \partial_t - y \, d_x \, y$.

III. An algebraic analysis approach to certain classes of nonlinear PD systems

Let us first recall basic ideas of the *algebraic analysis approach* to linear PD systems [3], [4], [9]. Let $(A, \{d_1, \ldots, d_n\})$ be a differential ring, $D = A\langle \partial_1, \ldots, \partial_n \rangle$ a noncommutative ring of PD operators, $R \in D^{q \times p}$ and $M = D^{1 \times p}/(D^{1 \times q} \, R)$ the left D-module *finitely presented* by R. Let us explicitly describe M in terms of generators and relations. Let $\pi : D^{1 \times p} \longrightarrow M$ be the left D-homomorphism (i.e., the left D-linear map) which sends $\lambda \in D^{1 \times p}$ to its residue class $\pi(\lambda)$ in the quotient left D-module M. Let $\{f_j\}_{j=1,\ldots,p}$ be the standard basis of $D^{1 \times p}$, i.e., f_j is the row vector of length p defined by 1 at the j^{th} position and 0 elsewhere, and $z_j = \pi(f_j)$ for $j = 1, \ldots, p$. Since every $m \in M$ has the form $m = \pi(\lambda)$ for a certain $\lambda \in D^{1 \times p}$, using the fact that π is a left D-homomorphism, we get

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j \, f_j\right) = \sum_{j=1}^{p} \lambda_j \, \pi(f_j) = \sum_{j=1}^{p} \lambda_j \, z_j,$$

which shows that $\{z_j\}_{j=1,\ldots,p}$ is a family of generators of M. This family of generators of M admits the relations

$$\begin{aligned} \textstyle\sum_{j=1}^{p} R_{kj} \, z_j &= \textstyle\sum_{j=1}^{p} R_{kj} \, \pi(f_j) \\ = \pi\left(\textstyle\sum_{j=1}^{p} R_{kj} \, f_j\right) &= \pi\left((R_{k1} \; \ldots \; R_{kp})\right) = 0, \end{aligned}$$

for all $k = 1, \ldots, q$, i.e., $R \, z = 0$, where $z = (z_1 \; \ldots \; z_p)^T$.

Theorem 1 ([13]): Let $D = A\langle \partial_1, \ldots, \partial_n \rangle$ be a noncommutative polynomial ring of PD operators in $\partial_1, \ldots, \partial_n$ with coefficients in a differential ring $(A, \{d_1, \ldots, d_n\})$, $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q} \, R)$ and $\mathcal{F}$ a left D-module. Then, the following isomorphism of abelian groups

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R \, \eta = 0\} \cong \hom_D(M, \mathcal{F})$$

holds, where $\hom_D(M, \mathcal{F})$ is the abelian group of the left D-homomorphisms from M to $\mathcal{F}$.

Theorem 1, first noticed by Malgrange, plays a fundamental role in the algebraic analysis approach to linear PD systems. Indeed, the *linear PD system* $\ker_{\mathcal{F}}(R.)$ (also called *behaviour*) can be intrinsically studied by means of the left D-modules M and $\mathcal{F}$. For more details, see [5], [7], [14].

The purpose of this paper is to use algebraic analysis to study some classes of nonlinear PD systems. More precisely, if $\mathfrak{p}$ is a prime differential ideal of the differential polynomial ring $k\{Y_1, \ldots, Y_p\}$, $A = k\{Y_1, \ldots, Y_p\}/\mathfrak{p} = k\{y_1, \ldots, y_p\}$ and $D = A\langle \partial_1, \ldots, \partial_n \rangle$, then we shall consider the class of nonlinear PD systems which can be written as:

$$R \, y = 0, \quad R \in D^{q \times p}, \quad y = (y_1 \; \ldots \; y_p)^T \in A^p. \tag{8}$$

Example 2: Let us consider again Burgers' equation:

$$\frac{\partial y}{\partial t} + y \, \frac{\partial y}{\partial x} = 0. \tag{9}$$

Considering the ring $D = A\langle \partial_t, \partial_x \rangle$ defined in Example 1. Burgers' equation can then be written as $R \, y = 0$, where:

$$R = (\partial_t + y \, \partial_x) \in D.$$

Example 3: Let us consider the 1-dimensional isentropic flow of an inviscid gas [10] described by:

$$\begin{cases} \partial_t \, u + u \, \partial_x \, u + c \, \partial_x \, \ell = 0, \\ \partial_t \, \ell + c \, \partial_x \, u + u \, \partial_x \, \ell = 0. \end{cases} \tag{10}$$

We shall assume that c is a polynomial function of u and ℓ.

Let $c \in \mathbb{Q}[U, L]$ and $\mathfrak{p}$ be the prime differential ideal

$$\mathfrak{p} = \{d_t \, U + U \, d_x \, U + c \, d_x \, L, \; d_t \, L + c \, d_x \, U + U \, d_x \, L\}$$

of the differential polynomial ring $\mathbb{Q}\{U, L\}$. Then, let us consider the differential ring $A = \mathbb{Q}\{U, L\}/\mathfrak{p} = \mathbb{Q}\{u, \ell\}$, and $D = A\langle \partial_t, \partial_x \rangle$ the ring of PD operators with coefficients in the commutative differential ring A. Then, we have:

$$(10) \quad \Leftrightarrow \quad \begin{pmatrix} \partial_t + u \, \partial_x & c \, \partial_x \\ c \, \partial_x & \partial_t + u \, \partial_x \end{pmatrix} \begin{pmatrix} u \\ \ell \end{pmatrix} = 0.$$

The main idea that we shall continuously used in what follows is to apply Theorem 1 to the case of a differential ring $A = k\{Y_1, \ldots, Y_p\}/\mathfrak{p} = k\{y_1, \ldots, y_p\}$, where $\mathfrak{p}$ is a prime differential ideal of the differential ring $k\{Y_1, \ldots, Y_p\}$, and to the left D-module $\mathcal{F} = A$ defined by (7). Since by construction, $y = (y_1 \; \ldots \; y_p)^T \in A^p$ satisfies the nonlinear PD system $R \, y = 0$, i.e., $y \in \ker_A(R.)$, the solution y corresponds to $f \in \hom_D(M, A)$ defined by:

$$\forall \; j = 1, \ldots, p, \quad f(z_j) = y_j. \tag{11}$$

Algebraic analysis techniques can then be used to study the finitely presented left D-module $M = D^{1 \times p}/(D^{1 \times q} \, R)$, and thus the nonlinear PD system $R \, y = 0$ by means of (11).

In the next sections, we shall show how to use algebraic analysis techniques developed in [7] to study certain problems concerning nonlinear PD systems.

IV. Homomorphisms and internal symmetries

Let us recall a characterization of a left D-homomorphism between two finitely presented left D-modules.

Proposition 1 ([7]): Let D be a ring of PD operators with coefficients in a differential ring A, $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$, $M = D^{1 \times p}/(D^{1 \times q} \, R)$ and $M' = D^{1 \times p'}/(D^{1 \times q'} \, R')$.

1) The existence of $f \in \hom_D(M, M')$ is equivalent to the existence of $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying

$$R \, P = Q \, R', \tag{12}$$

which is equivalent to the existence of the following *commutative exact diagram* [16]

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1 \times q'} & \xrightarrow{.R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0, \end{array}$$

where the left D-homomorphism $.R$ is defined by

$$\forall\, \mu \in D^{1\times q}, \quad (.R)(\mu) = \mu\, R,$$

(similarly for $.R'$, $.P$ and $.Q$) and $f \in \hom_D(M, M')$ is given by:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P). \tag{13}$$

2) Let $R_2' \in D^{q_2'\times q'}$ be such that $\ker_D(.R') = D^{1\times q_2'}\, R_2'$ and $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ two matrices satisfying $R\,P = Q\,R'$. Then, the matrices defined by

$$\begin{cases} \overline{P} = P + Z\,R', \\ \overline{Q} = Q + R\,Z + Z_2\,R_2', \end{cases}$$

where $Z \in D^{p\times q'}$ and $Z_2 \in D^{q\times q_2'}$ are two arbitrary matrices, satisfy $R\,\overline{P} = \overline{Q}\,R'$ and:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P) = \pi'(\lambda\, \overline{P}).$$

Using (12), we get the next corollary of Proposition 1.

Corollary 1 ([7]): With the notations of Proposition 1, if $\mathcal{F}$ is a left D-module, then the abelian group homomorphism

$$\begin{array}{rcl} P. : \mathcal{F}^{p'} & \longrightarrow & \mathcal{F}^{p} \\ \zeta & \longmapsto & P\,\zeta, \end{array}$$

sends elements of $\ker_{\mathcal{F}}(R'.)$ to elements of $\ker_{\mathcal{F}}(R.)$, i.e., $\mathcal{F}$-solutions of the linear PD system $R'\,\zeta = 0$ to $\mathcal{F}$-solutions of the linear PD system $R\,\eta = 0$. If $R' = R$, then $P.$ is called an *internal symmetry* of $\ker_{\mathcal{F}}(R.)$.

Applying Corollary 1 to the class of nonlinear PD systems considered, we get the following new result.

Theorem 2: Let $\mathfrak{p}$ be a prime differential ideal of $k\{Y_1', \ldots, Y_{p'}'\}$, $A = k\{Y_1', \ldots, Y_{p'}'\}/\mathfrak{p} = k\{y_1', \ldots, y_{p'}'\}$, $D = A\langle \partial_1, \ldots, \partial_n\rangle$ and $R' \in D^{q'\times p'}$ such that:

$$R'\,y' = 0, \quad y' = (y_1' \;\ldots\; y_{p'}')^T \in A^{p'}.$$

Moreover, let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$, $M' = D^{1\times p'}/(D^{1\times q'}\,R')$ and $f \in \hom_D(M, M')$ be defined by (13) for a certain matrix $P \in D^{p\times p'}$. Then, $y = P\,y' \in A^p$ satisfies $R\,y = 0$, i.e., $y = P\,y' \in \ker_A(R.)$

Example 4: Let us consider again Burgers' equation (9). Considering the ring $D = A\langle \partial_t, \partial_x\rangle$ of PD operators with coefficients in the differential ring

$$A = \mathbb{Q}\{Y\}/\{d_t\,Y + Y\,d_x Y\} = \mathbb{Q}\{y\},$$

Burgers' equation can then be written as $R\,y = 0$, where $R = \partial_t + y\,\partial_x = \partial_t - E$ and $E = -y\,\partial_x \in D' \triangleq A\langle \partial_x\rangle$. Let $M = D/(D\,R)$ be the left D-module finitely presented by R. By 1 of Proposition 1, $f \in \mathrm{end}_D(M) \triangleq \hom_D(M, M)$ is defined by $f(\pi(\lambda)) = \pi(\lambda\,P)$, for all $\lambda \in D$, where $\pi : D \longrightarrow M$ is the canonical projection and $P \in D$ is such that $R\,P = Q\,R$ for a certain $Q \in D$. Since $P \in D$ is defined up to a homotopy equivalence (see 2 of Proposition 1) and R has order 1 in ∂_t, we can assume without loss of generality that $P \in D'$. The relation $R\,P = Q\,R$ then yields

$$(\partial_t - E)\,P = Q\,(\partial_t - E) \Leftrightarrow P\,\partial_t + \frac{\partial P}{\partial t} - E\,P = Q\,\partial_t - Q\,E,$$

which implies $P = Q$ and $\frac{\partial P}{\partial t} = E\,P - P\,E$. We then get

$$\frac{\partial P}{\partial t} = -y\,\partial_x\,P + P\,y\,\partial_x = -y\,P\,\partial_x - y\,\frac{\partial P}{\partial x} + P\,y\,\partial_x,$$

and $f \in \mathrm{end}_D(M)$ is defined by $f(\pi(\lambda)) = \pi(\lambda\,P)$ for all $\lambda \in D$, where $P \in D'$ satisfies:

$$\frac{\partial P}{\partial t} = (P\,y - y\,P)\,\partial_x - y\,\frac{\partial P}{\partial x}. \tag{14}$$

For instance, since y satisfies Burgers' equation, $P = y \in A$ is a solution of (14). Hence, if $\mathcal{F}$ is a left D-module, then we obtain the following abelian group homomorphism:

$$\begin{array}{rcl} y. : \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \eta & \longmapsto & y\,\eta. \end{array}$$

Now, if $\mathcal{F} = A$ and $\eta = y \in \ker_A(R.)$, then $y^2 \in \ker_A(R.)$. Considering $\eta = y^2$, then we get $y^3 \in \ker_A(R.)$ and so on. Therefore, for all $n \in \mathbb{N}$, $y^n \in \ker_A(R.)$, i.e.:

$$\forall\, n \in \mathbb{N}, \quad \frac{\partial y^n}{\partial t} + y\,\frac{\partial y^n}{\partial x} = 0.$$

Hence, endomorphisms of the left D-module $M = D/(D\,R)$ naturally induce internal symmetries of Burgers' equation.

Example 5: Let us consider the prime differential ideal $\mathfrak{p} = \left\{d_t\,U - 6\,U\,d_t\,U + d_x^3\,U\right\}$ of $\mathbb{Q}\{U\}$, the differential ring $A = \mathbb{Q}\{U\}/\mathfrak{p} = \mathbb{Q}\{u\}$ defining the *Korteweg-de Vries (KdV) equation*:

$$\frac{\partial u}{\partial t} - 6\,u\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^3 u}{\partial x^3} = 0. \tag{15}$$

Let us consider the rings of PD operators $D' = A\langle \partial_x\rangle$ and $D = D'\langle \partial_t\rangle$, the two PD operators

$$\begin{cases} E = -4\,\partial_x^3 + 6\,u\,\partial_x + 3\left(\dfrac{\partial\,u}{\partial\,x}\right) \in D', \\ R = \partial_t - E \in D, \end{cases}$$

and the finitely presented left D-module $M = D/(D\,R)$. According to Proposition 1, $f \in \mathrm{end}_D(M)$ is defined by $f(\pi(\lambda)) = \pi(\lambda\,P)$, where $P \in D'$ satisfies $R\,P = Q\,R$. Using 2 of Lemma 1 and the fact that R has order 1 in ∂_t, we can suppose without loss of generality that $P \in D'$. Moreover, proceeding as in Example 4, we get $Q = P$ and:

$$R\,P - P\,R = \frac{\partial P}{\partial t} - E\,P + P\,E = 0.$$

If we consider the *Schrödinger operator* $P = -\partial_x^2 + u$ with the potential u, then, after tedious computations, we can check that $R\,P - P\,R = 0$. Hence, if u satisfies the KdV equation (15), then the Schrödinger operator P defines a left D-endomorphism of the left D-module M. The pair (E, P) is called a *Lax pair* [12] and it plays an important role in the study of integrable evolution equations. Within the inverse scattering theory, an important result asserts that the smooth one-parameter family of OD operators

$$t \longmapsto -\partial_x^2 + u(x,t)$$

defines an *isospectral flow* on the solutions of $\partial_t\,\eta = E\,\eta$, namely, if $\psi(x)$ is an eigenvector of the OD operator

$-\partial_x^2 + u(x,0)$ with eigenvalue λ, then the solution $\eta(x,t)$ of the equation $\partial_t\,\eta(x,t) = E\,\eta(x,t)$ with the initial value $\eta(x,0) = \psi(x)$ is an eigenvector of the OD operator $-\partial_x^2+u(x,t)$ with the same eigenvalue λ. This result directly follows from the integrability condition $\partial_t P = E\,P - P\,E$, i.e., from the KdV equation. Based on this result, we can prove that the KdV equation is *completely integrable* [12].

V. Conservation laws

In this section, we shall show how the techniques previously developed can be used to compute conservation laws for the class of nonlinear PD systems considered.

Let D be a ring of PD operators with coefficients in a differential ring A and $R \in D^{q\times p}$. We first recall the notion of *formal adjoint* of the matrix R.

Definition 3 ([16]): An *involution* θ of a ring D is an anti-automorphism of D of order two, namely:

$$\forall\, d_1,\, d_2 \in D, \quad \begin{cases} \theta(d_1+d_2) = \theta(d_1)+\theta(d_2), \\ \theta(d_1\,d_2) = \theta(d_2)\,\theta(d_1), \\ \theta\circ\theta = \mathrm{id}_D. \end{cases}$$

Example 6: The ring $D = A\langle\partial_1,\ldots,\partial_n\rangle$ of PD operators with coefficients in a differential ring A admits the involution θ defined by:

$$\forall\, a\in A, \quad \theta(a) = a, \quad \theta(\partial_i) = -\partial_i, \quad i=1,\ldots,n.$$

Definition 4 ([5]): Let D be ring admitting an involution θ and $R\in D^{q\times p}$. The *formal adjoint* $\widetilde{R}$ of R is defined by:

$$\widetilde{R} \triangleq (\theta(R_{ij}))^T_{i=1,\ldots,q,\, j=1,\ldots,p} \in D^{p\times q}.$$

If $M = D^{1\times p}/(D^{1\times q}\,R)$ is the left D-module finitely presented by $R\in D^{q\times p}$, then the *adjoint module* of M is the left D-module defined by $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$.

Example 7: Let us consider the *Euler equations* for an incompressible fluid defined by

$$\begin{cases} \rho\,(\partial_t\,\vec{u} + (\vec{u}\,.\,\vec{\nabla})\,\vec{u}) + \vec{\nabla}\,p = \vec{0}, \\ \vec{\nabla}\,.\,\vec{u} = 0, \end{cases} \tag{16}$$

where $\vec{u} = (u_1 \quad u_2 \quad u_3)^T$ is the fluid velocity vector, p the pressure and ρ the constant fluid density. Let us consider the following prime differential ideal

$$\mathfrak{p} = \Big\{\rho\,d_t\,U_i + \rho\,\textstyle\sum_{j=1}^3 U_j\,d_{x_j}\,U_i + d_{x_i}\,P, i=1,\ldots,3, \\ \textstyle\sum_{j=1}^3 d_{x_j}\,U_j\Big\},$$

of the differential polynomial ring $\mathbb{Q}(\rho)\{U_1,U_2,U_3,P\}$ defined by the four differential polynomials defining (16), the differential polynomial ring $A = \mathbb{Q}(\rho)\{U_1,U_2,U_3,P\}/\mathfrak{p} = \mathbb{Q}(\rho)\{u_1,u_2,u_3,p\}$, the ring $D = \mathbb{Q}(\rho)\{u_1,u_2,u_3,p\}\langle\partial_t,\partial_{x_1},\partial_{x_2},\partial_{x_3}\rangle$ of PD operators in ∂_t and ∂_{x_i} for $i=1,\,2,\,3$ with coefficients in the differential ring A. Then, (16) can be rewritten as

$$\begin{pmatrix} d & 0 & 0 & \partial_{x_1} \\ 0 & d & 0 & \partial_{x_3} \\ 0 & 0 & d & \partial_{x_3} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{pmatrix} = 0, \tag{17}$$

where $d = \rho\,(\partial_t + \sum_{j=1}^3 u_j\,\partial_{x_j}) = \rho\,(\partial_t + \vec{u}\,.\,\vec{\nabla}) \in D$, i.e., $R\,y=0$, where $y = (u_1 \quad u_2 \quad u_3 \quad p)^T$ and $R\in D^{4\times 4}$ is the matrix appearing in the left-hand side of (17). Using

$$\forall\, j=1,\,2,\,3, \quad \partial_{x_j}\,u_j = u_j\,\partial_{x_j} + d_{x_j}\,u_j, \quad \sum_{k=1}^3 d_{x_k}\,u_k = 0,$$

we can check that the formal adjoint $\widetilde{R}$ of R for the involution θ defined in Example 6 is

$$\widetilde{R} = \begin{pmatrix} -d & 0 & 0 & -\partial_{x_1} \\ 0 & -d & 0 & -\partial_{x_3} \\ 0 & 0 & -d & -\partial_{x_3} \\ -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} & 0 \end{pmatrix},$$

so that $\widetilde{R} = -R$, i.e., R is a skew-adjoint matrix.

The formal adjoint $\widetilde{R}\in D^{p\times q}$ of a matrix $R\in D^{q\times p}$, where $D = A\langle\partial_1,\ldots,\partial_n\rangle$ is a ring of PD operators with coefficients in a differential ring A, can also be obtained by contracting the column vector $R\,\eta$ by a row vector λ^T and integrating the result by parts to get the following relation

$$\lambda^T\,(R\,\eta) = \eta^T(\widetilde{R}\,\lambda) + \mathrm{div}\,\Phi, \tag{18}$$

where $\Phi = (\Phi_1(\lambda,\eta),\ldots,\Phi_n(\lambda,\eta))^T$ is the vector containing the boundary terms Φ_i (bilinear forms in λ and η) of the integration by parts (see references in [14]) and:

$$\mathrm{div}\,\Phi \triangleq \sum_{i=1}^n \partial_i\,\Phi_i.$$

From (18) and Corollary 1, we get the following result.

Theorem 3: Let $\mathfrak{p}$ be a prime differential ideal of $k\{Y_1,\ldots,Y_p\}$, $A = k\{Y_1,\ldots,Y_p\}/\mathfrak{p} = k\{y_1,\ldots,y_p\}$, $D = A\langle\partial_1,\ldots,\partial_n\rangle$ and $R\in D^{q\times p}$ such that:

$$R\,y = 0, \quad y = (y_1 \ \ldots \ y_p)^T \in A^p.$$

Moreover, let $M = D^{1\times p}/(D^{1\times q}\,R)$ be the left D-module finitely presented by R, $\widetilde{R}$ the formal adjoint of the matrix R, $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$ the adjoint module of M and Φ defined by (18). Let $f : \widetilde{N} \longrightarrow M$ be a left D-homomorphism defined by two matrices $P\in D^{q\times p}$ and $Q\in D^{p\times q}$ such that $\widetilde{R}\,P = Q\,R$. Then, the nonlinear PD system $R\,y=0$ admits the following *conservation law*:

$$\mathrm{div}\,\Phi(P\,y,y) = \sum_{i=1}^n \partial_i\,\Phi_i(P\,y,y) = 0. \tag{19}$$

Example 8: Let us consider again the Euler equations for an incompressible fluid defined in Example 7. We know that R is a skew-adjoint matrix, i.e., $\widetilde{R} = -R$. If $\lambda = (\vec{v}^T \quad \lambda_4)^T$ and $\eta = (\vec{w}^T \quad \eta_4)^T$, then (18) can be written as

$$\lambda^T(R\,\eta) = \eta^T(\widetilde{R}\,\lambda) + \partial_t\,\rho\,(\vec{v}\,.\,\vec{w}) + \vec{\nabla}.(\rho\,(\vec{v}\,.\,\vec{w})\,\vec{u} + \eta_4\,\vec{v} + \lambda_4\,\vec{w}),$$

which yields:

$$\lambda^T(R\,\eta) = -\eta^T\,R\,\lambda + \partial_t\,\rho\,(\vec{v}\,.\,\vec{w}) + \vec{\nabla}.(\rho\,(\vec{v}\,.\,\vec{w})\,\vec{u} + \eta_4\,\vec{v} + \lambda_4\,\vec{w}). \tag{20}$$

By construction, $y = (u_1 \quad u_2 \quad u_3 \quad p)^T \in A^4$ satisfies $R\,y = 0$. Since R is a skew-adjoint matrix, we can take $P = I_4$ in Theorem 3 and $\lambda = P\,y = (u_1 \quad u_2 \quad u_3 \quad p)^T$. We then obtain the following cubic conservation law of (16):

$$\partial_t \left(\frac{\rho}{2} \parallel \vec{u} \parallel^2\right) + \vec{\nabla} \,.\, \left(\frac{\rho}{2} \parallel \vec{u} \parallel^2 \; \vec{u} + p\,\vec{u}\right) = 0. \qquad (21)$$

Let us show how to obtain more conservation laws of (16). We first note that:

$$\widetilde{N} = D^{1\times 4}/(D^{1\times 4}\,\widetilde{R}) = D^{1\times 4}/(D^{1\times 4}\,R) = M.$$

Hence, $f \in \hom_D(\widetilde{N}, M) = \mathrm{end}_D(M)$ is defined by two matrices $P, Q \in D^{4\times 4}$ satisfying $\widetilde{R}\,P = Q\,R$, i.e., $R\,(-P) = Q\,R$. If $y = (u_1 \quad u_2 \quad u_3 \quad p)^T \in \ker_A(R.)$, then $\lambda = P\,y \in \ker_A(\widetilde{R}.) = \ker_A(R.)$. Therefore, if we write $\lambda = P\,y = (\vec{v}^T \quad \lambda_4)^T$, then (20) yields

$$\partial_t\,(\vec{v}\,.\,\vec{u}) + \vec{\nabla}\,.\,(\rho\,(\vec{v}\,.\,\vec{u})\,\vec{u} + p\,\vec{v} + \lambda_4\,\vec{u}) = 0,$$

and shows that

$$\Phi = \begin{pmatrix} \vec{v}\,.\,\vec{u} \\ \rho\,(\vec{v}\,.\,\vec{u})\,\vec{u} + p\,\vec{v} + \lambda_4\,\vec{u} \end{pmatrix}$$

is a conservation law of the nonlinear PD system (16).

These results show that the endomorphism ring $\mathrm{end}_D(M)$ of the left D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$ contains important physical information on the Euler equations (16), and thus it should be more investigated in the future. A similar study should also be done for different classical nonlinear PD systems encountered in mathematical physics and engineering sciences such as, for instance, in magnetohydrodynamics.

Finally, once the Φ_i's defined by (18) are known, we point out that the degrees in the y_j's of the conservation law (19) depends only on the degrees of the entries in the y_j's of the matrix $P \in D^{q\times p}$ defining $f \in \hom_D(\widetilde{N}, M)$. Consequently, if we are only interested in conservation laws of certain degrees in the y_j's, then we have to compute left D-homomorphisms $f : \widetilde{N} \longrightarrow M$ defined by a matrix $P \in D^{q\times p}$ of appropriate degrees in the y_j's.

VI. Decomposition problems

Let $\mathrm{GL}_r(D)$ be the *general linear group of degree* r, i.e.:

$$\mathrm{GL}_r(D) = \{U \in D^{r\times r} \mid \exists\, V \in D^{r\times r} : \; U\,V = V\,U = I_r\}.$$

Moreover, let A be a differential ring, $D = A\langle \partial_1, \ldots, \partial_n\rangle$ the ring of PD operators in $\partial_1, \ldots, \partial_n$ with coefficients in A, $R \in D^{q\times p}$ and $\mathcal{F}$ a left D-module. Then, the *decomposition problem* of the linear PD system $\ker_{\mathcal{F}}(R.)$ aims at determining, when they exist, two matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that $\overline{R} \triangleq V\,R\,W$ is a block-diagonal matrix [7]. As explained in [7], the decomposition problem for a linear PD system $\ker_{\mathcal{F}}(R.)$ is related to the possibility of decomposing the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ into a direct sum of submodules:

$$M = M_1 \oplus M_2.$$

This can be achieved by computing *idempotents* of the endomorphism ring $\mathrm{end}_D(M)$ of the left D-module M, namely $e \in \mathrm{end}_D(M)$ satisfying:

$$e^2 = e.$$

If M_1 (resp., M_2) is finitely presented by the matrix R_1 (resp., R_2), then $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(R_1.) \oplus \ker_{\mathcal{F}}(R_2.)$ [7].

In this section, we study the decomposition problem of a nonlinear PD system defined by $R\,y = 0$, where $R \in D^{q\times p}$, $D = A\langle \partial_1, \ldots, \partial_n\rangle$, $A = k\{Y_1, \ldots, Y_p\}/\mathfrak{p} = k\{y_1, \ldots, y_p\}$ and $y = (y_1 \; \ldots \; y_p)^T \in A^p$. If there exist two matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that

$$\overline{R} \triangleq V\,R\,W = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

where $R_1 \in D^{l\times m}$ and $R_2 \in D^{(q-l)\times(p-m)}$, then

$$\overline{y} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} = W^{-1}\,y \in \ker_A(\overline{R}.),$$

where $\overline{y}_1 \in A^m$ (resp., $\overline{y}_2 \in A^{(p-m)}$), i.e.:

$$R_1\,\overline{y}_1 = 0, \quad R_2\,\overline{y}_2 = 0. \qquad (22)$$

The fact that (22) is an uncoupled nonlinear PD system can be used to determine interesting information on $\overline{y}$, and thus on the solution y of the original nonlinear PD system $R\,y = 0$ by the invertible transformation:

$$y = W\,\overline{y}.$$

We first recall lemmas before stating the new result.

Lemma 1 ([7]): Let us consider the beginning of a *finite free resolution* of a left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ [16], namely an exact sequence of the form

$$D^{1\times q_2} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

and a left D-endomorphism $f : M \longrightarrow M$ defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying $R\,P = Q\,R$. Then, f is an idempotent of $\mathrm{end}_D(M)$, i.e., $f^2 = f$, if and only if there exists a matrix $Z \in D^{p\times q}$ satisfying:

$$P^2 = P + Z\,R. \qquad (23)$$

Then, there exists a matrix $Z' \in D^{q\times q_2}$ such that:

$$Q^2 = Q + R\,Z + Z'\,R_2. \qquad (24)$$

If $R \in D^{q\times p}$ has *full row rank*, i.e., $R_2 = 0$, we then have:

$$Q^2 = Q + R\,Z. \qquad (25)$$

Lemma 2 ([7]): Let $P \in D^{p\times p}$ be an idempotent, i.e., $P^2 = P$. The following assertions are equivalent:

1) The left D-modules $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are free of rank respectively m and $p - m$.
2) There exist $U \in \mathrm{GL}_p(D)$ and $J_P \in D^{p\times p}$ of the form

$$J_P = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix},$$

which satisfy the relation:

$$U\,P = J_P\,U. \tag{26}$$

The matrix U has then the form

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{27}$$

where the matrices $U_1 \in D^{m\times p}$ and $U_2 \in D^{(p-m)\times p}$ have full row ranks and satisfy:

$$\begin{cases} \ker_D(.P) = D^{1\times m}\,U_1, \\ \mathrm{im}_D(.P) = D^{1\times (p-m)}\,U_2. \end{cases} \tag{28}$$

In particular, we have the relations $U_1\,P = 0$ and $U_2\,P = U_2$.

Lemma 3 ([7]): Let us consider the following matrices

$$\begin{cases} J_P = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix} \in D^{p\times p}, \\ J_Q = \begin{pmatrix} 0 & 0 \\ 0 & I_{q-l} \end{pmatrix} \in D^{q\times q}, \end{cases} \tag{29}$$

where $1 \leq m \leq p$ and $1 \leq l \leq q$, and a matrix $\overline{R} \in D^{q\times p}$ satisfying the following relation:

$$\overline{R}\,J_P = J_Q\,\overline{R}. \tag{30}$$

Then, $\overline{R}_1 \in D^{l\times m}$ and $\overline{R}_2 \in D^{(q-l)\times(p-m)}$ exist such that:

$$\overline{R} = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & \overline{R}_2 \end{pmatrix}. \tag{31}$$

Lemma 4 ([7]): Let us suppose that matrices $R \in D^{q\times p}$, $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfy $R\,P = Q\,R$. Moreover, let us assume that there exist $U \in \mathrm{GL}_p(D)$, $V \in \mathrm{GL}_q(D)$, $J_P \in D^{p\times p}$ and $J_Q \in D^{q\times q}$ such that:

$$\begin{cases} U\,P = J_P\,U, \\ V\,Q = J_Q\,V. \end{cases} \tag{32}$$

Then, we have:

$$(V\,R\,U^{-1})\,J_P = J_Q\,(V\,R\,U^{-1}). \tag{33}$$

We can now state the main result of this section.

Theorem 4: Let $\mathfrak{p}$ be a prime differential ideal of $k\{Y_1,\ldots,Y_p\}$, $A = k\{Y_1,\ldots,Y_p\}/\mathfrak{p} = k\{y_1,\ldots,y_p\}$, $D = A\langle\partial_1,\ldots,\partial_n\rangle$ and $R \in D^{q\times p}$ such that:

$$R\,y = 0, \quad y = (y_1 \ \ldots \ y_p)^T \in A^p.$$

Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be the left D-module finitely presented by R and $f \in \mathrm{end}_D(M)$ an idempotent endomorphism defined by $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

If the left D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q)$ are free of rank respectively m, $p-m$, l and $q-l$, where $1 \leq m \leq p$ and $1 \leq l \leq q$, then we have:

1) There exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ satisfying

$$\begin{cases} P = U^{-1}\,J_P\,U, \\ Q = V^{-1}\,J_Q\,V, \end{cases}$$

where J_P and J_Q are the matrices defined by (29). In particular, the matrices U and V are defined by

$$\begin{cases} U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad U_1 \in D^{m\times p}, \quad U_2 \in D^{(p-m)\times p}, \\ V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad V_1 \in D^{l\times q}, \quad V_2 \in D^{(q-l)\times q}, \end{cases}$$

where the full row rank matrices U_1, U_2, V_1 and V_2 are respectively bases of the free left D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$, i.e.:

$$\begin{cases} \ker_D(.P) = D^{1\times m}\,U_1, \\ \mathrm{im}_D(.P) = D^{1\times(p-m)}\,U_2, \\ \ker_D(.Q) = D^{1\times l}\,V_1, \\ \mathrm{im}_D(.Q) = D^{1\times(q-l)}\,V_2. \end{cases}$$

2) The matrix R is equivalent to $\overline{R} = V\,R\,U^{-1}$.

3) If we denote by $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$, $W_2 \in D^{p\times(p-m)}$, we then have:

$$\overline{R} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix} \in D^{q\times p}. \tag{34}$$

Proof: The proof is analogous to that given in [7] in the case of linear systems. We repeat it here for completeness.

1. The result directly follows from 2 of Lemma 2.

2. The fact that $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ yields:

$$R = V^{-1}\,\overline{R}\,U.$$

3. From Lemma 4, the matrix $\overline{R} = V\,R\,U^{-1}$ satisfies the relation (30). Then, applying Lemma 3 to $\overline{R}$, we obtain that $\overline{R}$ has the block-diagonal form (31), where $\overline{R}_1 \in D^{l\times m}$ and $\overline{R}_2 \in D^{(q-l)\times(p-m)}$. Finally, we have

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & V_1\,R\,W_2 \\ V_2\,R\,W_1 & V_2\,R\,W_2 \end{pmatrix} \in D^{q\times p},$$

where $V_1\,R\,W_1 \in D^{l\times m}$, $V_2\,R\,W_1 \in D^{(p-l)\times m}$ and $V_1\,R\,W_2 \in D^{l\times(p-m)}$, $V_2\,R\,W_2 \in D^{(p-l)\times(p-m)}$. ■

Example 9: Let us consider again the 1-dimensional isentropic flow of an inviscid gas given by (10). In Example 3, we saw that (10) can be written as follows:

$$R \begin{pmatrix} u \\ \ell \end{pmatrix} = 0, \quad R = \begin{pmatrix} \partial_t + u\,\partial_x & c\,\partial_x \\ c\,\partial_x & \partial_t + u\,\partial_x \end{pmatrix}.$$

Let us now introduce the following matrices

$$F = \begin{pmatrix} u & c \\ c & u \end{pmatrix} \in A^{2\times 2},$$

$$E = -F\,\partial_x \in A^{2\times 2}, \quad R = \partial_t\,I_2 - E \in D^{2\times 2},$$

$M = D^{1\times 2}/(D^{1\times 2}\,R)$ and let $\pi : D^{1\times 2} \longrightarrow M$ be the canonical projection onto M.

Now, $f \in \mathrm{end}_D(M)$ is defined by $f(\pi(\lambda)) = \pi(\lambda\,P)$, where the matrix $P \in D^{2\times 2}$ satisfies $R\,P = Q\,R$ for a certain $Q \in D^{2\times 2}$. If $y = (u \quad \ell)^T \in A^2$, then

$$\begin{array}{rcl} P. : \ker_A(R.) & \longrightarrow & \ker_A(R.) \\ y & \longmapsto & \overline{y} = P\,y, \end{array} \tag{35}$$

induces the nonlinear transformation (e.g., quadratic, cubic)

$$\left(\begin{array}{c} \overline{u} \\ \overline{\ell} \end{array}\right) = P(\partial_t, \partial_x, u, \ell, d_x\, u, d_x\, \ell, d_x^2\, u, d_x^2\, \ell \ldots) \left(\begin{array}{c} u \\ \ell \end{array}\right),$$

where $R\,(\overline{u} \quad \overline{\ell})^T = 0$.

In what follows, we shall simply study *linear transformations*, namely, $P \in D'^{2\times 2}$, where $D' = \mathbb{Q}\langle \partial_t, \partial_x\rangle$ is the commutative polynomial ring in ∂_t and ∂_x with rational constant coefficients. Clearly, we have $D' \subset D$ and the left D-module M is a left D'-module.

Since R has degree 1 in ∂_t, 2 of Proposition 1 shows that we can assume without loss of generality that we have $Q = P \in \mathbb{Q}\langle \partial_x\rangle$. Then, the relation $R\,P = P\,R$ yields

$$E\,P - P\,E = 0 \;\Leftrightarrow\; F\,\partial_x\,P = P\,F\,\partial_x \;\Leftrightarrow\; F\,P = P\,F,$$

since the entries of P belong to D' (i.e., $\frac{\partial P}{\partial x} = 0$). Let

$$P = \left(\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array}\right),$$

where the $P_{ij} \in D'$, then $F\,P = P\,F$ is equivalent to:

$$\left\{\begin{array}{l} u\,P_{11} + c\,P_{21} = P_{11}\,u + P_{12}\,c, \\ u\,P_{12} + c\,P_{22} = P_{11}\,c + P_{12}\,u, \\ c\,P_{11} + u\,P_{21} = P_{21}\,u + P_{22}\,c, \\ c\,P_{12} + u\,P_{22} = P_{21}\,c + P_{22}\,u. \end{array}\right. \tag{36}$$

From now, let us suppose that P_{ij}'s belong to $\mathbb{Q}$. Then, (36) yields $P_{21} = P_{12}$ and $P_{22} = P_{11}$. Hence, $P^2 = P$ yields:

$$\left\{\begin{array}{l} P_{11}^2 + P_{12}^2 - P_{11} = 0, \\ (2\,P_{11} - 1)\,P_{12} = 0. \end{array}\right. \tag{37}$$

Using the second and then the first equation of (37), we get

1) $P_{11} = 1/2$ and $P_{12} = \pm 1/2$.
2) $P_{12} = 0$ and $P_{11} = 0$ or 1,

which shows that the matrices

$$P_1 = \frac{1}{2}\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right), \quad P_2 = \frac{1}{2}\left(\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right),$$

define two idempotents f_i of $\mathrm{end}_D(M)$. Let us consider f_1. The matrices U and V defined in Theorem 4 can easily be computed:

$$U = V = \left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right) \in \mathrm{GL}_2(D),\; U^{-1} = \frac{1}{2}\left(\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right).$$

We then obtain

$$\overline{R} = U\,R\,U^{-1} = \left(\begin{array}{cc} \partial_t + (u - c)\,\partial_x & 0 \\ 0 & \partial_t + (u + c)\,\partial_x \end{array}\right),$$

which shows that:

$$\left\{\begin{array}{l} (\partial_t + (u - c)\,\partial_x)\,\overline{u} = 0, \\ (\partial_t + (u + c)\,\partial_x)\,\overline{\ell} = 0, \end{array}\right. \quad \text{where} \quad \left\{\begin{array}{l} \overline{u} = u - \ell, \\ \overline{\ell} = u + \ell. \end{array}\right.$$

We then find the so-called *Riemann invariants* $\overline{u}$ and $\overline{\ell}$ of (10) (i.e., $\overline{u}$ and $\overline{\ell}$ are conserved along the *characteristics* of (10)) [10]. The fact that the decomposition method seems to give a new constructive way for computing of certain Riemann invariants for nonlinear PD systems will be further investigated in a future publication.

VII. Conlusion and future work

This paper is a first step toward a new approach mixing differential algebra techniques and algebraic analysis methods for studying certain classes of nonlinear PD systems. The results developed in this paper and the different explicit examples handled here show the relevance of the extension of the results developed in [5], [7], [8] to certain classes of nonlinear PD systems appearing in mathematical physics by considering the category of finitely presented left modules over a ring $D = A\langle \partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficients in a differential ring A that can be written $A = k\{Y_1, \ldots, Y_p\}/\mathfrak{p}$, where $\mathfrak{p}$ is a prime differential ideal of $k\{Y_1, \ldots, Y_p\}$ defining the polynomial PD system. This approach can also be used to study the generic linearization of polynomial PD systems using the *Kähler differentials* [11].

Based on the Janet package [2], an implementation of a Maple package called JanetMorphisms, extending the OreMorphisms package [8] to the class of nonlinear PD systems considered in this paper, is in development in collaboration with D. Robertz.

References

[1] M. Barakat and D. Robertz. *homalg*: A meta-package for homological algebra. *J. Algebra Appl.* 7 (2008), 299-317, http://wwwb.math.rwth-aachen.de/homalg/.
[2] Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, D. Robertz, "The MAPLE Package "Janet": I. Polynomial Systems", *Proceedings of Computer Algebra in Scientific Computing CASC 2003*, 31-40, http://wwwb.math.rwth-aachen.de/Janet.
[3] J. E. Björk, *Rings of Differential Operators*, North Holland, 1979.
[4] J. E. Björk, *Analytic $\mathcal{D}$-modules and Applications*, Kluwer, 1993.
[5] F. Chyzak, A. Quadrat, D. Robertz, "Effective algorithms for parametrizing linear control systems over Ore algebras", *Appl. Algebra Engrg. Comm. Comput.*, 16 (2005), 319-376.
[6] F. Chyzak, A. Quadrat, D. Robertz, "OreModules: A symbolic package for the study of multidimensional linear systems", LNCIS 352, Springer, 2007, 233-264, OreModules project, http://wwwb.math.rwth-aachen.de/OreModules.
[7] T. Cluzeau, A. Quadrat, "Factoring and decomposing a class of linear functional systems", *Linear Algebra Appl.*, 428 (2008), 324-381.
[8] T. Cluzeau, A. Quadrat, "OreMorphisms: A homological algebraic package for factoring and decomposing linear functional systems", *LNCIS* 388, Springer, 2009, 179-196, http://www-sop.inria.fr/members/Alban.Quadrat/OreMorphisms/index.html.
[9] M. Kashiwara, *Algebraic Study of Systems of Partial Differential Equations*, Master Thesis, Tokyo Univ. 1970, Mémoires de la Société Mathématique de France 63 (1995) (English translation).
[10] J. Kevorkian, *Partial Differential Equations: Analytical Solution Techniques*, Texts in Applied Mathematics 35, Springer, 2000.
[11] E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, Pure and Applied Mathematics 54, 1973.
[12] P. Lax, "Integrals of nonlinear equations of evolution and solitary waves", *Comm. Pure Appl. Math.*, 21 (1968), 467-490.
[13] B. Malgrange, "Systèmes différentiels à coefficients constants", *Séminaire Bourbaki*, 1962/63, 1-11.
[14] J.-F. Pommaret, A. Quadrat, "Algebraic analysis of linear multidimensional control systems", *IMA J. Control Inform.*, 16 (1999), 275-297.
[15] J. F. Ritt, *Differential Algebra*, American Mathematical Society, Colloquium publications XXXIII, 1950, reprinted by Dover, 1966.
[16] J. J. Rotman, *An Introduction to Homological Algebra*, Springer, 2nd edition, 2009.