Using effective homological algebra for factoring and decomposing linear functional systems

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Introduction: factorization and decomposition

- Let $L(\partial)$ be a scalar ordinary or partial differential operator.
- When is it possible to find $L_1(\partial)$ and $L_2(\partial)$ such that:

$$L(\partial) = L_2(\partial)\, L_1(\partial)?$$

- We note that $L_1(\partial)\, y = 0 \Rightarrow L(\partial)\, y = 0$.
- $L(\partial)\, y = 0$ is equivalent to the cascade integration:

$$L_1(\partial)\, y = z \quad \& \quad L_2(\partial)\, z = 0.$$

- When is the integration of $L(\partial)\, y = 0$ equivalent to:

$$L_2(\partial)\, z = 0 \quad \& \quad L_1(\partial)\, u = 0?$$

$$(L_1\, X + Y\, L_2 = 1 \Rightarrow L_1(X\, z) = z \Rightarrow y = u + X\, z)$$

Introduction: factorization and decomposition

- Let us consider the first order ordinary differential system:

$$\partial\, y = E(t)\, y, \quad E(t) \in k(t)^{p \times p}. \quad (\star)$$

- When does it exist an invertible change of variables

$$y = P(t)\, z,$$

such that

$$(\star) \quad \Leftrightarrow \quad \partial\, z = F(t)\, z,$$

where $F = -P^{-1}\,(\partial P - E\, P)$ is either of the form:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \quad \text{or} \quad F = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}?$$

Factorization: known cases

Square differential systems:

- Beke's algorithm (Beke1894, Schwarz89, Bronstein94, Tsarëv94...)
- Eigenring (Singer96, Giesbrecht98, Barkatou-Pflügel98, Barkatou01 - ideas in Jacobson37...)

Square (q-)difference systems (generalizations):

- Barkatou01, Bomboy01...

Square D-finite partial differential systems (connections):

- Li-Schwarz-Tsarëv03, Wu05...

Same cases in positive characteristic and modular approaches:

- van der Put95, C.03, Giesbrecht-Zhang03, C.-van Hoeij04,06, Barkatou-C.-Weil05...

General Setting

What about general linear functional systems?

• **Example** (Saint Venant equations): linearized model around the Riemann invariants (Dubois-Petit-Rouchon, ECC99):

$$\begin{cases} y_1(t-2\,h)+y_2(t)-2\,\dot{y}_3(t-h)=0, \\ y_1(t)+y_2(t-2\,h)-2\,\dot{y}_3(t-h)=0. \end{cases}$$

• Let $D=\mathbb{R}\left[\frac{d}{dt},\delta\right]$ and consider the system matrix:

$$R=\begin{pmatrix} \delta^2 & 1 & -2\,\delta\,\frac{d}{dt} \\ 1 & \delta^2 & -2\,\delta\,\frac{d}{dt} \end{pmatrix}\in D^{2\times 3}.$$

Question: $\exists\; U\in \mathrm{GL}_3(D),\; V\in \mathrm{GL}_2(D)$ such that:

$$V\,R\,U=\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \end{pmatrix},\;\; \alpha_1,\alpha_2,\alpha_3\in D?$$

Outline

• Type of systems: Partial differential/discrete/differential time-delay... linear systems (LFSs).

• General topic: Algebraic study of linear functional systems (LFSs) coming from mathematical physics, engineering sciences...

• Techniques: Module theory and homological algebra.

• Applications: Equivalences of systems, Galois symmetries, quadratic first integrals/conservation laws, decoupling problem...

• Implementation: package MORPHISMS based on OREMODULES:

`http://wwwb.math.rwth-aachen.de/OreModules`.

General methodology

1. A linear system is defined by means of a matrix R with entries in a ring D of functional operators:

$$R\, y = 0. \quad (\star)$$

2. We associate a finitely presented left D-module M with $(\star)$.
3. A dictionary exists between the properties of $(\star)$ and M.
4. Homological algebra allows us to check properties of M.
5. Effective algebra (non-commutative Gröbner/Janet bases) leads to constructive algorithms.
6. Implementation (Maple, Singular/Plural, Cocoa. . .).

I. Ore Module associated with a linear functional system

Consider a ring A, an automorphism σ of A and a σ-derivation δ:

$$\delta(a\,b) = \sigma(a)\,\delta(b) + \delta(a)\,b.$$

Definition: A non-commutative polynomial ring $D = A[\partial; \sigma, \delta]$ in

∂ is called skew if $\quad \forall\, a \in A, \quad \partial\, a = \sigma(a)\,\partial + \delta(a).$

Definition: Let us consider $A = k,\ k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$. The skew polynomial ring $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is called an Ore algebra if we have:

$$\begin{cases} \sigma_i\,\delta_j = \delta_j\,\sigma_i, & 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, & \delta_i(\partial_j) = 0, \quad j < i. \end{cases}$$

$\Rightarrow$ D is generally a non-commutative polynomial ring.

Examples of Ore algebras

- Partial differential operators: $A = k,\ k[x_1, \ldots, x_n],\ k(x_1, \ldots, x_n)$,

$$D = A\left[\partial_1; \mathrm{id}, \tfrac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \tfrac{\partial}{\partial x_n}\right],$$

$$P = \textstyle\sum_{0 \leq |\mu| \leq m} a_\mu(x)\, \partial^\mu \in D, \quad \partial^\mu = \partial_1^{\mu_1} \ldots \partial_n^{\mu_n}.$$

- Shift operators:

$$D = A[\partial; \sigma, 0], \quad A = k,\ k[n],\ k(n),$$

$$P = \textstyle\sum_{i=0}^m a_i(n)\, \partial^i \in D, \quad \sigma(a)(n) = a(n+1).$$

- Differential time-delay operators:

$$D = A\left[\partial_1; \mathrm{id}, \tfrac{d}{dt}\right] [\partial_2; \sigma, 0], \quad A = k,\ k[t],\ k(t),$$

$$P = \textstyle\sum_{0 \leq i+j \leq m} a_{ij}(t)\, \partial_1^i\, \partial_2^j \in D.$$

Exact sequences

• **Definition**: A sequence of D-morphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be exact at M if we have:

$$\ker g = \mathrm{im}\, f.$$

• **Example**: If $f : M \longrightarrow M'$ is a D-morphism, we then have the following exact sequences:

1. $0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{\rho} \mathrm{coim}\, f \triangleq M / \ker f \longrightarrow 0.$
2. $0 \longrightarrow \mathrm{im}\, f \xrightarrow{j} M' \xrightarrow{\kappa} \mathrm{coker}\, f \triangleq M' / \mathrm{im}\, f \longrightarrow 0.$
3. $0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{f} M' \xrightarrow{\kappa} \mathrm{coker}\, f \longrightarrow 0.$

A left D-module M associated with $R\,\eta = 0$

- Let D be an Ore algebra, $R \in D^{q\times p}$ and a left D-module $\mathcal{F}$.

- Let us consider $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$.

- As in number theory or algebraic geometry, we associate with the system $\ker_{\mathcal{F}}(R.)$ the finitely presented left D-module:

$$M = D^{1\times p}/(D^{1\times q}\,R).$$

- Malgrange's remark: applying the functor $\hom_D(., \mathcal{F})$ to the finite free resolution (exact sequence)

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\,R & & & & \end{array}$$

we then obtain the exact sequence:

$$\begin{array}{ccccc} \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \xleftarrow{\pi^\star} & \hom_D(M, \mathcal{F}) \longleftarrow 0. \\ R\,\eta & \longleftarrow & \eta = (\eta_1, \ldots, \eta_p)^{\mathrm{T}} & & \end{array}$$

Example: Linearized Euler equations

• The linearized Euler equations for an incompressible fluid can be defined by the system matrix

$$R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} \in D^{4\times 4},$$

where $D = \mathbb{R}\left[\partial_1, \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2, \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3, \mathrm{id}, \frac{\partial}{\partial x_3}\right]\left[\partial_t, \mathrm{id}, \frac{\partial}{\partial t}\right]$.

• Let us consider the left D-module $\mathcal{F} = \mathcal{C}^{\infty}(\Omega)$ (Ω open convex subset of $\mathbb{R}^4$) and the D-module:

$$M = D^{1\times 4}/(D^{1\times 4}\, R).$$

The solutions of $R\, y = 0$ in $\mathcal{F}$ are in $1-1$ correspondence with the morphisms from M to $\mathcal{F}$, i.e., with the elements of:

$$\hom_D(M, \mathcal{F}).$$

II. Morphisms between Ore modules finitely presented by two matrices R and R' of functional operators

Morphims of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- Let us consider the finitely presented left D-modules:

$$M = D^{1 \times p}/(D^{1 \times q}\, R), \quad M' = D^{1 \times p'}/(D^{1 \times q'}\, R').$$

- We are interested in the abelian group $\hom_D(M, M')$ of D-morphisms from M to M':

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ & & & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. \end{array}$$

Morphims of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. \end{array}$$

$$\exists\, f : M \to M' \Longleftrightarrow \exists\, P \in D^{p\times p'},\ Q \in D^{q \times q'} \text{ such that:}$$

$$R\,P = Q\,R'.$$

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda\, P)$, for all $\lambda \in D^{1\times p}$.

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \sigma, \delta]$, $\quad E,\ F \in A^{p \times p},\ R = \partial\, I_p - E,\ R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \Longleftrightarrow \begin{cases} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E\, P - \sigma(P)\, F. \end{cases}$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E\, P).$$

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \sigma, \delta]$, $\quad E, F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
& & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \Longleftrightarrow \left\{ \begin{array}{l} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E\, P - \sigma(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E\, P).$$

- Differential case: $\delta = \frac{d}{dt}$, $\sigma = \mathrm{id}$:

$$\left\{ \begin{array}{l} \dot{P} = E\, P - P\, F, \\ F = -P^{-1}(\dot{P} - E\, P). \end{array} \right.$$

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \sigma, \delta]$, $\quad E,\, F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \Longleftrightarrow \left\{ \begin{array}{l} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E\, P - \sigma(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E\, P).$$

- Discrete case: $\delta = 0$, $\sigma(k) = k - 1$:

$$\left\{ \begin{array}{l} E(k)\, P(k) - P(k-1)\, F(k) = 0, \\ B = \sigma(P)^{-1}\, E\, P. \end{array} \right.$$

- Problem: Given $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, find $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the commutation relation $R\, P = Q\, R'$.

- If D is a commutative ring, then $\hom_D(M, M')$ is a D-module.

- The Kronecker product of $E \in D^{q \times p}$ and $F \in D^{r \times s}$ is:

$$E \otimes F = \begin{pmatrix} E_{11}\, F & \dots & E_{1p}\, F \\ \vdots & \vdots & \vdots \\ E_{q1}\, F & \dots & E_{qp}\, F \end{pmatrix} \in D^{(q\,r) \times (p\,s)}.$$

Lemma: If $U \in D^{a \times b}$, $V \in D^{b \times c}$ and $W \in D^{c \times d}$, then we have:

$$U\, V\, W = (V_1 \dots V_b)\, (U^T \otimes W).$$

$$R\, P\, I_{p'} = (P_1 \dots P_p)\, (R^T \otimes I_{p'}), \quad I_q\, Q\, R' = (Q_1 \dots Q_q)\, (I_q \otimes R').$$

We are reduced to compute $\ker_D \left(. \begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix} \right)$.

Computation of $\hom_D(M, M')$

- Problem: Given $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, find $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the commutation relation $R\,P = Q\,R'$.

- If D is a non-commutative ring, then $\hom_D(M, M')$ is an abelian group and generally an infinite-dimensional k-vector space.

$\Rightarrow$ find a k-basis of morphisms with given degrees in x_i and in ∂_j:

1. Take an ansatz for P with chosen degrees.
2. Compute $R\,P$ and a Gröbner basis G of the rows of R'.
3. Reduce the rows of $R\,P$ w.r.t. G.
4. Solve the system on the coefficients of the ansatz so that all the normal forms vanish.
5. Substitute the solutions in P and compute Q by means of a factorization.

Example: Bipendulum

- We consider the Ore algebra $D = \mathbb{R}(g, l)\left[\frac{d}{dt}\right]$.

- We consider the matrix of the bipendulum with $l = l_1 = l_2$:

$$R = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} & 0 & -\frac{g}{l} \\ 0 & \frac{d^2}{dt^2} + \frac{g}{l} & -\frac{g}{l} \end{pmatrix} \in D^{2 \times 3}.$$

- Let us consider the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$.

- We obtain that $\mathrm{end}_D(M)$ is defined by the matrices:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3\, g \\ \alpha_4 & \alpha_1 + \alpha_2 - \alpha_4 & \alpha_3\, g \\ 0 & 0 & \alpha_3\, \mathrm{D}^2\, l + \alpha_1 + \alpha_2 + \alpha_3\, g \end{pmatrix},$$

$$Q = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_1 + \alpha_2 - \alpha_4 \end{pmatrix}, \quad \forall\, \alpha_1, \ldots, \alpha_4 \in D.$$

Example: Bomboy's PhD, p. 80

q-dilatation case: $D = \mathbb{R}(q)(x)\,[H]$ where $H(f(x)) = f(q\,x)$ and:

$$R = \begin{pmatrix} H & -1 \\ -\dfrac{1-q^3\,x^2}{1-q\,x^2} & \dfrac{x\,(1-q^2)}{1-q\,x^2} + H \end{pmatrix} \in D^{2\times 2}.$$

• Searching for endomorphisms with degree 0 in H and 2 in x (both in numerator and denominator), we obtain

$$P = \begin{pmatrix} \dfrac{-a+b\,x\,q-b\,x+a\,q\,x^2}{c\,(-1+q\,x^2)} & \dfrac{b\,(-1+x^2)}{c\,(-1+q\,x^2)} \\ \dfrac{b\,(-1+q^2\,x^2)}{c\,(-1+q\,x^2)} & -\dfrac{a+b\,x\,q-b\,x-a\,q\,x^2}{c\,(-1+q\,x^2)} \end{pmatrix},$$

where, a, b, c are constants or $P = I_2$ (and corresponding Q).

Saint-Venant equations

- Let $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \sigma, 0]$ be the ring of differential time-delay operators and consider the matrix of the tank model:

$$R = \begin{pmatrix} \partial_2^2 & 1 & -2\,\partial_1\,\partial_2 \\ 1 & \partial_2^2 & -2\,\partial_1\,\partial_2 \end{pmatrix} \in D^{2\times 3}.$$

- The endomorphisms of $M = D^{1\times 3}/(D^{1\times 2}\,R)$ are defined by:

$$P_\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & 2\,\alpha_3\,\partial_1\,\partial_2 \\ \alpha_2 + 2\,\alpha_4\,\partial_1 + 2\,\alpha_5\,\partial_1\,\partial_2 & \alpha_1 - 2\,\alpha_4\,\partial_1 - 2\,\alpha_5\,\partial_1\,\partial_2 & 2\,\alpha_3\,\partial_1\,\partial_2 \\ \alpha_4\,\partial_2 + \alpha_5 & -\alpha_4\,\partial_2 - \alpha_5 & \alpha_1 + \alpha_2 + \alpha_3\,(\partial_2^2 + 1) \end{pmatrix},$$

$$Q_\alpha = \begin{pmatrix} \alpha_1 - 2\,\alpha_4\,\partial_1 & \alpha_2 + 2\,\alpha_4\,\partial_1 \\ \alpha_2 + 2\,\alpha_5\,\partial_1\,\partial_2 & \alpha_1 - 2\,\alpha_5\,\partial_1\,\partial_2 \end{pmatrix}, \quad \forall\, \alpha_1, \ldots, \alpha_5 \in D.$$

Euler-Tricomi equation

- Let us consider the Euler-Tricomi equation (transonic flow):

$$\partial_1^2 \, u(x_1, x_2) - x_1 \, \partial_2^2 \, u(x_1, x_2) = 0.$$

- Let $D = A_2(\mathbb{Q})$, $R = (\partial_1^2 - x_1 \, \partial_2^2) \in D$ and $M = D/(D\,R)$.
  - $\mathrm{end}_D(M)_{1,1}$ is defined by:

$$\begin{cases} P = a_1 + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1, \\ Q = (a_1 + 2 \, a_3) + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1, \end{cases}$$

  - $\mathrm{end}_D(M)_{2,0}$ is defined by $P = Q = a_1 + a_2 \, \partial_2 + a_3 \, \partial_2^2$.
  - $\mathrm{end}_D(M)_{2,1}$ is defined by:

$$\begin{cases} P = & a_1 + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1 \\ & + a_4 \, \partial_2^2 + \frac{3}{2} \, a_5 \, x_2 \, \partial_2^2 + a_5 \, x_1 \, \partial_1 \, \partial_2, \\ Q = & (a_1 + 2 \, a_3) + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1 \\ & + a_4 \, \partial_2^2 + a_5 \, x_1 \, \partial_1 \, \partial_2 + 2 \, a_5 \, \partial_2 + \frac{3}{2} \, a_5 \, x_2 \, \partial_2^2. \end{cases}$$

III. A few applications: Galois symmetries, quadratic first integrals of motion and conservation laws

Galois Symmetries

We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 & \\ \downarrow .Q & & \downarrow .P & & \downarrow f & & (\star) \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. & \end{array}$$

If $\mathcal{F}$ is a left D-module, by applying the functor $\hom_D(\cdot, \mathcal{F})$ to $(\star)$, we then obtain the following commutative exact diagram:

$$\begin{array}{ccccccc} 0 = Q\,(R'\,y) = R\,(P\,y) & \longleftarrow\!\shortmid & P\,y & & & & \\ \mathcal{F}^{q} & \xleftarrow{R.} & \mathcal{F}^{p} & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\ \uparrow Q. & & \uparrow P. & & \uparrow f^{\star} & & \\ \mathcal{F}^{q'} & \xleftarrow{R'.} & \mathcal{F}^{p'} & \longleftarrow & \ker_{\mathcal{F}}(R'.) & \longleftarrow & 0. \\ 0 = R'\,y & \longleftarrow\!\shortmid & y & & & & \end{array}$$

$\Rightarrow$ $f^{\star}$ sends $\ker_{\mathcal{F}}(R.')$ to $\ker_{\mathcal{F}}(R.)$ ($R' = R$: Galois symmetries).

Example: Linear elasticity

- Consider the Killing operator for the euclidian metric defined by:

$$R = \begin{pmatrix} \partial_1 & 0 \\ \partial_2/2 & \partial_1/2 \\ 0 & \partial_2 \end{pmatrix}.$$

- The system $R\, y = 0$ admits the following general solution:

$$y = \begin{pmatrix} c_1\, x_2 + c_2 \\ -c_1\, x_1 + c_3 \end{pmatrix}, \quad c_1,\, c_2,\, c_3 \in \mathbb{R}. \quad (\star)$$

- We find that $\mathrm{end}_D(D^{1\times 2}/(D^{1\times 3}\, R))$ is defined by:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2\, \partial_1 \\ 0 & 2\, \alpha_3\, \partial_1 + \alpha_1 \end{pmatrix}, \quad \alpha_1,\, \alpha_2,\, \alpha_3 \in D.$$

- Applying P to $(\star)$, we then get the new solution:

$$\overline{y} = P\, y = \begin{pmatrix} \alpha_1\, c_1\, x_2 + \alpha_1\, c_2 - \alpha_2\, c_1 \\ -\alpha_1\, c_1\, x_1 + \alpha_1\, c_3 - 2\, \alpha_3\, c_1 \end{pmatrix}, \text{ i.e., } R\, \overline{y} = 0.$$

Quadratic first integrals of motion

Let us consider a morphism f from $\widetilde{N}$ to M defined by:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p + E^T)} & D^{1\times p} & \xrightarrow{\pi} & \widetilde{N} & \longrightarrow & 0 \\ & & \downarrow .P & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - E)} & D^{1\times p} & \xrightarrow{\pi'} & M & \longrightarrow & 0. \end{array}$$

We then have:

$$\dot{P} + E^T\, P + P\, E = 0.$$

If $V(x) = x^T\, P\, x$, then $\dot{V}(x) = x^T\, (\dot{P} + E^T\, P + P\, E)\, x$ so that:

$$\dot{P} + E^T\, P + P\, E = 0 \Longleftrightarrow V(x) = x^T\, P\, x \ \text{ first integral.}$$

$\Rightarrow$ Morphisms from $\widetilde{N}$ to M give quadratic first integrals.

If E is a skew-symmetric matrix, i.e., $E = -E^T$, then we have:

$$(\partial\, I_p + E^T) = (\partial\, I_p - E), \quad \widetilde{N} = M, \quad \hom_D(\widetilde{N}, M) = \mathrm{end}_D(M).$$

Example: Landau & Lifchitz (p. 117)

- Consider $R = \partial\, I_4 - E$, where $E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & \alpha \end{pmatrix}$.

- We find that the morphisms from $\widetilde{N}$ to M are defined by

$$P = \begin{pmatrix} c_1\,\omega^4 & c_2\,\omega^2 & -\omega^2\,(c_1\,\alpha + c_2) & c_1\,\omega^2 \\ -c_2\,\omega^2 & c_1\,\omega^2 & -c_1\,\omega^2 + c_2\,\alpha & -c_2 \\ -\omega^2\,(c_1\,\alpha - c_2) & -c_1\,\omega^2 - c_2\,\alpha & c_1\,(\alpha^2 + \omega^2) & -c_1\,\alpha + c_2 \\ c_1\,\omega^2 & c_2 & -c_1\,\alpha - c_2 & c_1 \end{pmatrix},$$

which leads to the quadratic first integral $V(x) = x^T\, P\, x$:

$$\begin{aligned} V(x) \;&=\; c_1\,\omega^4\,x_1(t)^2 - 2\,x_1(t)\,\omega^2\,x_3(t)\,c_1\,\alpha + 2\,x_1(t)\,c_1\,\omega^2\,x_4(t) \\ &\quad + x_2(t)^2\,c_1\,\omega^2 - 2\,x_2(t)\,c_1\,x_3(t)\,\omega^2 + c_1\,x_3(t)^2\,\alpha^2 \\ &\quad + c_1\,x_3(t)^2\,\omega^2 - 2\,x_3(t)\,x_4(t)\,c_1\,\alpha + c_1\,x_4(t)^2. \end{aligned}$$

Formal adjoint

- Let $D = A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$ be the ring of differential operators with coefficients in A (e.g., $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$).

- The formal adjoint $\widetilde{R} \in D^{p\times q}$ of $R \in D^{q\times p}$ is defined by:

$$< \lambda, R\,\eta > = < \widetilde{R}\,\lambda, \eta > + \sum_{i=1}^{n} \partial_i\,\Phi_i(\lambda, \eta).$$

- The formal adjoint $\widetilde{R}$ can be defined by $\widetilde{R} = (\theta(R_{ij}))^T \in D^{p\times q}$, where $\theta : D \longrightarrow D$ is the involution defined by:

1. $\forall\ a \in A, \quad \theta(a) = a.$
2. $\theta(\partial_i) = -\partial_i, \quad i = 1, \ldots, n.$

Involution: $\theta^2 = \mathrm{id}_D, \quad \forall\ P_1,\ P_2 \in D: \quad \theta(P_1\,P_2) = \theta(P_2)\,\theta(P_1).$

Conservation laws

- Let us consider the left D-modules:

$$M = D^{1\times p}/(D^{1\times q}\, R), \quad \widetilde{N} = D^{1\times q}/(D^{1\times p}\, \widetilde{R}).$$

- Let $f : \widetilde{N} \longrightarrow M$ be a morphism defined by the matrices P and Q.
- Let $\mathcal{F}$ be a left D-module and the commutative exact diagram:

$$\begin{array}{ccccccc}
\mathcal{F}^p & \xleftarrow{\widetilde{R}.} & \mathcal{F}^q & \longleftarrow & \ker_{\mathcal{F}}(\widetilde{R}.) & \longleftarrow & 0\\
\uparrow Q. & & \uparrow P. & & \uparrow f^{\star} & & \\
\mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0.
\end{array}$$

- $\eta \in \mathcal{F}^p$ solution of $R\,\eta = 0 \Rightarrow \lambda = P\,\eta$ is a solution of $\widetilde{R}\,\lambda = 0$.

$$\Rightarrow < P\,\eta, R\,\eta > - < \widetilde{R}\,(P\,\eta), \eta >= \sum_{i=1}^{n} \partial_i\, \Phi_i(P\,\eta, \eta) = 0,$$

i.e., $\Phi = (\Phi_1(P\,\eta, \eta), \ldots, \Phi_n(P\,\eta, \eta))^T$ satisfies $\mathrm{div}\,\Phi = 0$.

Example: Laplacian operator

- Let us consider the Laplacian operator $\Delta\, y(x_1, x_2) = 0$, where:

$$\Delta = \partial_1^2 + \partial_2^2 \in D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right].$$

- The formal adjoint $\widetilde{R}$ of R is then defined by:

$$\lambda\,(\Delta\,\eta) - (\Delta\,\lambda)\,\eta = \partial_1\,(\lambda\,(\partial_1\,\eta) - (\partial_1\,\lambda)\,\eta) + \partial_2\,(\lambda\,(\partial_2\,\eta) - (\partial_2\,\lambda)\,\eta).$$

- $R = \Delta = \widetilde{R} \in D \Rightarrow \hom_D(\widetilde{N}, M) = \mathrm{end}_D(M) = D.$

- if $\mathcal{F}$ is a D-module (e.g., $C^\infty(\Omega)$), then we have:

$$\forall\, \alpha \in D, \forall\, \eta \in \ker_{\mathcal{F}}(\Delta .), \quad \lambda = \alpha\, y \in \ker_{\mathcal{F}}(\Delta .).$$

$$\Rightarrow \mathrm{div}\,\Phi = \partial_1\,\Phi_1 + \partial_2\,\Phi_2 = 0, \quad \Phi = \left(\begin{array}{c} (\alpha\, y)\,(\partial_1\, y) - y\,(\partial_1\,\alpha\, y) \\ (\alpha\, y)\,(\partial_2\, y) - y\,(\partial_2\,\alpha\, y) \end{array}\right).$$

IV. Factorization of linear functional systems

Kernel and factorization

$$
\begin{array}{ccccccc}
 & & \lambda & \longmapsto & y & & \\
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow f & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\
\exists\, \mu & \longmapsto & \mu\, R' = \lambda\, P & \longmapsto & 0 & &
\end{array}
$$

- $\ker_D \left(. \begin{pmatrix} P \\ R' \end{pmatrix} \right) = D^{1\times r} \, (S \quad -T)$

$$\Rightarrow \{\lambda \in D^{1\times p} \mid \lambda\, P \in D^{1\times q}\, R\} = D^{1\times r}\, S$$

$$\Rightarrow \ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R).$$

- $(D^{1\times q}\,(R \quad -Q)) \in \ker_D \left(. \begin{pmatrix} P \\ R' \end{pmatrix} \right) \Rightarrow (D^{1\times q}\, R) \subseteq (D^{1\times r}\, S).$

$$\exists\, V \in D^{q\times r} : \quad R = V\, S.$$

Kernel and factorization

We have the following commutative exact diagram:

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \ker f & & \\
 & & & & \downarrow i & & \\
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .V & & \| & & \downarrow \kappa & & \\
D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\pi'} & M/\ker f & \longrightarrow & 0. \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}
$$

Example: Linearized Euler equations

- Let $R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix}$ over $D = \mathbb{R}[\partial_1, \partial_2, \partial_3, \partial_t]$.
- Let us consider $f \in \mathrm{end}_D(M)$ defined by:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \partial_3^2 & -\partial_2\,\partial_3 & 0 \\ 0 & -\partial_2\,\partial_3 & \partial_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Computing $\ker_D \left(. \begin{pmatrix} P \\ R \end{pmatrix} \right)$ and factorizing R by S, we obtain:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & 0 \\ 0 & -\partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \partial_1 & 1 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & -1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix}.$$

Example: Linearized Euler equations

- We have $R = V\,S$ where:

$$\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} = \begin{pmatrix} \partial_1 & 1 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & -1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & 0 \\ 0 & -\partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The solutions of $S\,y = 0$ are particular solutions of $R\,y = 0$.

$\Rightarrow$ Integrating S, we obtain the following solutions of $R\,y = 0$:

$$y(x_1, x_2, x_3, t) = \begin{pmatrix} 0 \\ -\frac{\partial}{\partial x_3}\,\xi(x_1,\,x_2,\,x_3) \\ \frac{\partial}{\partial x_2}\,\xi(x_1,\,x_2,\,x_3) \\ 0 \end{pmatrix}, \quad \forall\,\xi \in C^{\infty}(\Omega).$$

Free modules & similarity transformations

- **Definition**: A left D-module M is free if there exists $l \in \mathbb{Z}_+$ s.t.:

$$M \cong D^{1 \times l}.$$

- Proposition: Let $P \in D^{p \times p}$. We have the equivalences:

1. $\ker_D(.P)$ and $\mathrm{coim}_D(.P)$ are free left D-modules of rank p and $p - m$.

2. There exists a unimodular matrix $U \in D^{p \times p}$, i.e., $U \in \mathrm{GL}_p(D)$, such that:

$$J \triangleq U\, P\, U^{-1} = \begin{pmatrix} 0 \\ J_2 \end{pmatrix}, \quad J_2 \in D^{(p-m) \times p}.$$

$$\Rightarrow U = (U_1^T \quad U_2^T)^T, \quad \begin{cases} \ker_D(.P) = D^{1 \times m}\, U_1 \\ \mathrm{coim}_D(.P) = \pi'(D^{1 \times (p-m)}\, U_2). \end{cases}$$

A useful proposition

• Proposition: Let $R \in D^{q \times p}$ and $P \in D^{p \times p}$, $Q \in D^{q \times q}$ be two matrices satisfying:

$$R\,P = Q\,R.$$

Let $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that

$$\begin{cases} P = U^{-1}\,J_P\,U, \\ Q = V^{-1}\,J_Q\,V, \end{cases}$$

for certain $J_P \in D^{p \times p}$ and $J_Q \in D^{q \times q}$.

Then, the matrix $\overline{R} = V\,R\,U^{-1}$ satisfies:

$$\overline{R}\,J_P = J_Q\,\overline{R}.$$

A commutative diagram

The following commutative diagram

$$
\begin{tikzcd}
 & D^{1\times q} \arrow[rr, ".(V\,R\,U^{-1})"] \arrow[dl, ".V"'] \arrow[dd, ".J_Q"] & & D^{1\times p} \arrow[dl, ".U"'] \arrow[dd, ".J_p"] \\
D^{1\times q} \arrow[rr, ".R"] \arrow[dd, ".Q"'] & & D^{1\times p} \arrow[dd, ".P"] & \\
 & D^{1\times q} \arrow[rr, ".(V\,R\,U^{-1})"] \arrow[dl, ".V"'] & & D^{1\times p} \arrow[dl, ".U"'] \\
D^{1\times q} \arrow[rr, ".R"] & & D^{1\times p} &
\end{tikzcd}
$$

implies $(V\,R\,U^{-1})\,J_p = J_Q\,(V\,R\,U^{-1})$.

Block triangular decomposition

- Theorem: Let $R \in D^{q \times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying $R\,P = Q\,R$.

If the left D-modules

$$\ker_D(.P), \quad \mathrm{coim}_D(.P), \quad \ker_D(.Q), \quad \mathrm{coim}_D(.Q)$$

are free of rank m, $p-m$, l, $q-l$, then there exist two matrices $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$ and $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$ such that

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ V_2\,R\,W_1 & V_2\,R\,W_2 \end{pmatrix} \in D^{q\times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$, $W_2 \in D^{p\times (p-m)}$ and:

$$U_1 \in D^{m\times p}, \quad U_2 \in D^{(p-m)\times p}, \quad V_1 \in D^{l\times q}, \quad V_2 \in D^{(q-l)\times q}.$$

Example: OD system

- Let $D = k[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $M = D^{1\times 4}/(D^{1\times 4}\, R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}.$$

- An endomorphism f of M is defined by:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

- We can prove that the left D-modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$ are free of rank 2.

Example: OD system

- We obtain:

$$\left\{ \begin{array}{ll} U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t-1 & -t \end{pmatrix}, & V_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \end{array} \right.$$

$$\Rightarrow U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t-1 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

we then obtain that R is equivalent to:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} -\partial & 1 & 0 & 0 \\ t\,\partial - t & -\partial - t & 0 & 0 \\ \partial + t & \partial - 1 & \partial & -t \\ -\partial & 1 & 0 & \partial \end{pmatrix}.$$

Example: Saint-Venant equations

- We consider $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \sigma, 0]$ and:

$$R = \begin{pmatrix} \partial_2^2 & 1 & -2\,\partial_1\,\partial_2 \\ 1 & \partial_2^2 & -2\,\partial_1\,\partial_2 \end{pmatrix}.$$

- A endomorphism f of M is defined by:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 2\,\partial_1\,\partial_2 & -2\,\partial_1\,\partial_2 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 2\,\partial_1\,\partial_2 & -2\,\partial_1\,\partial_2 \end{pmatrix}.$$

- We can check that $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q))$ are free D-modules of rank respectively 2, 1, 1, 1.

$$\Rightarrow \begin{cases} U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\,\partial_1\,\partial_2 \end{pmatrix}, & U_2 = (0 \quad 0 \quad 1), \\ V_1 = (1 \quad 0), & V_2 = (0 \quad 1). \end{cases}$$

Example: Saint-Venant equations

- If we denote by

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\,\partial_1\,\partial_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we then obtain

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\,\partial_1\,\partial_2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D),$$

and the matrix R is equivalent to:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial_1^2 & -1 & 0 \\ 1 & -\partial_1^2 & 2\,\partial_1\,\partial_2\,(\partial_1^2 - 1) \end{pmatrix}.$$

Ker f, im f, coim f and coker f

• Proposition: Let $M = D^{1\times p}/(D^{1\times q}\, R)$, $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ and $f : M \longrightarrow M'$ be a morphism defined by $R\, P = Q\, R'$.

Let us consider the matrices $S \in D^{r\times p}$, $T \in D^{r\times q'}$, $U \in D^{s\times r}$ and $V \in D^{q\times r}$ satisfying $R = V\, S$, $\ker_D(.S) = D^{1\times s}\, U$ and:

$$\ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = D^{1\times r}\,(S \quad -T).$$

Then, we have:

- $\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R) \cong D^{1\times l}/\left(D^{1\times(q+s)}\begin{pmatrix} U \\ V \end{pmatrix}\right)$,
- $\mathrm{coim}\, f \triangleq M/\ker f = D^{1\times p}/(D^{1\times r}\, S)$,
- $\mathrm{im}\, f = D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}/(D^{1\times q}\, R) \cong D^{1\times p}/(D^{1\times r}\, S)$,
- $\mathrm{coker}\, f \triangleq M'/\mathrm{im}\, f = D^{1\times p}/\left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right)$.

Equivalence of systems

- Corollary: Let us consider $f \in \hom_D(M, M')$. Then, we have:

1. f is injective iff one of the assertions holds:
   - There exists $L \in D^{r \times q}$ such that $S = L\, R$.
   - $\begin{pmatrix} U \\ V \end{pmatrix}$ admits a left-inverse.
2. f is surjective iff $\begin{pmatrix} P \\ R' \end{pmatrix}$ admits a left-inverse.
3. f is an isomorphism, i.e., $M \cong M'$, iff 1 and 2 are satisfied.

Pommaret's example

- Equivalence of the systems defined by the following R and R'?

$$R = \begin{pmatrix} \partial_1^2\,\partial_2^2 - 1 & -\partial_1\,\partial_2^3 - \partial_2^2 \\ \partial_1^3\,\partial_2 - \partial_1^2 & -\partial_1^2\,\partial_2^2 \end{pmatrix}, \quad R' = (\partial_1\,\partial_2 - 1 \quad -\partial_2^2).$$

- We find a morphism given by $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 + \partial_1\,\partial_2 \\ \partial_1^2 \end{pmatrix}$.

- $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 + \partial_1\,\partial_2 \\ \partial_1^2 \end{pmatrix}$ admits the left-inverse $(1 - \partial_1\,\partial_2 \quad \partial_2^2)$.

- $\begin{pmatrix} P \\ R' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1\,\partial_2 - 1 & -\partial_2^2 \end{pmatrix}$ admits the left-inverse $(I_2 \quad 0)$.

$$\Rightarrow M = D^{1\times 2}/(D^{1\times 2}\,R) \cong M' = D^{1\times 2}/(D\,R').$$

V. Decomposition of linear functional systems

Projectors of $\mathrm{end}_D(M)$

• Lemma: An endomorphism f of $M = D^{1\times p}/(D^{1\times q}\, R)$, defined by the matrices P and Q, is a projector, i.e., $f^2 = f$, iff there exist $Z \in D^{p\times q}$ and $Z' \in D^{q\times t}$ such that

$$\begin{cases} P^2 = P + Z\,R, \\ Q^2 = Q + R\,Z + Z'\,R_2, \end{cases}$$

where $R_2 \in D^{t\times q}$ satisfies $\ker_D(.R) = D^{1\times t}\, R_2$.

• Some projectors of $\mathrm{end}_D(M)$ can be computed when a family of endomorphisms of M is known.

• **Example:** $D = A_1(\mathbb{Q})$, $R = (\partial^2 \quad -t\,\partial - 1)$, $M = D^{1\times 2}/(D\,R)$.

$$P = \begin{pmatrix} -(t+a)\,\partial + 1 & t^2 + a\,t \\ 0 & 1 \end{pmatrix}, \quad P^2 = P + \begin{pmatrix} (t+a)^2 \\ 0 \end{pmatrix} R.$$

- Particular case: $(R_2 = 0$ and $P^2 = P) \Longrightarrow Q^2 = Q$.

- Lemma: Let us suppose that $R_2 = 0$ and $P^2 = P + Z\,R$. If there exists a solution $\Lambda \in D^{p\times q}$ of the Riccati equation

$$\Lambda\, R\, \Lambda + (P - I_p)\,\Lambda + \Lambda\, Q + Z = 0, \quad (\star)$$

then the matrices $\overline{P} = P + \Lambda\, R$ and $\overline{Q} = Q + R\,\Lambda$ satisfy:

$$R\,\overline{P} = \overline{Q}\,R, \quad \overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}.$$

- **Example:** $\Lambda = (a\,t \quad a\,\partial - 1)^T$ is a solution of $(\star)$

$$\Rightarrow \overline{P} = \begin{pmatrix} a\,t\,\partial^2 - (t+a)\,\partial + 1 & t^2\,(1 - a\,\partial) \\ (a\,\partial - 1)\,\partial^2 & -a\,t\,\partial^2 + (t - 2\,a)\,\partial + 2 \end{pmatrix}, \overline{Q} = 0,$$

then satisfy $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$.

Projectors of $\mathrm{end}_D(M)$

- Proposition: f is a projector of $\mathrm{end}_D(M)$, i.e., $f^2 = f$, iff there exists a matrix $X \in D^{p\times s}$ such that $P = I_p - X\,S$ and we have the following commutative exact diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \ker f \arrow[d, "i"] & \\
 & D^{1\times q} \arrow[r, ".R"] \arrow[d, shift left, ".V"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, shift left, ".I_p"] & M \arrow[r] \arrow[d, shift left, "\kappa"] & 0 \\
D^{1\times s} \arrow[r, ".U"] & D^{1\times r} \arrow[r, ".S"] \arrow[u, shift left, ".T"] & D^{1\times p} \arrow[r, "\pi'"] \arrow[u, shift left, ".P"] \arrow[l, shift left=2, ".X"] & M/\ker f \arrow[r] \arrow[u, shift left, "f"] \arrow[d] & 0. \\
 & & & 0 &
\end{tikzcd}$$

$$\Rightarrow M \cong \ker f \oplus \mathrm{im}\, f \quad \& \quad S - S\,X\,S = T\,R. \quad (\star)$$

- Corollary: If $\ker_D(.S) = 0$, then $R = V\,S$ satisfies:

$$S\,X - T\,V = I_r.$$

• Corollary: Let us suppose that $\mathcal{F}$ is an injective left D-module. Then, we have the following commutative exact diagram:

$$
\begin{tikzcd}
 & V\,z = 0 = R\,y & y \arrow[l, mapsto] & & \\
 & \mathcal{F}^q & \mathcal{F}^p \arrow[l, "R."'] & \ker_{\mathcal{F}}(R.) \arrow[l] & 0 \arrow[l] \\
\mathcal{F}^s & \mathcal{F}^r \arrow[l, "U."'] \arrow[u, "V."'] \arrow[r, shift right=2, "X."'] & \mathcal{F}^p \arrow[l, "S."'] \arrow[u, equal] & \ker_{\mathcal{F}}(S.) \arrow[l] \arrow[u, "f^\star"'] & 0. \arrow[l] \\
0 = U\,z & z = S\,y \arrow[l, mapsto] & y \arrow[l, mapsto] & &
\end{tikzcd}
$$

Moreover, we have: $\quad R\,y = 0 \quad \Leftrightarrow \quad \begin{pmatrix} U \\ V \end{pmatrix} z = 0, \quad S\,y = z.$

General solution: $y = u + X\,z$ where $S\,u = 0$ and $\begin{pmatrix} U \\ V \end{pmatrix} z = 0.$

Example: OD system

- Let $D = k[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $M = D^{1\times 4}/(D^{1\times 4}\, R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}.$$

- We obtain the following idempotent:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in k^{4\times 4} : \quad P^2 = P.$$

- We obtain the factorization $R = V\, S$, where:

$$S = \begin{pmatrix} \partial & -t & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & t & \partial \\ 1 & t & \partial & -1 \\ 1 & 0 & \partial + t & \partial - 1 \\ 1 & 1 & t & \partial \end{pmatrix}.$$

Example

- Using the fact that we must have $I_p - P = X\,S$, we then obtain:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R\,y = 0 \Leftrightarrow y = u + X\,z: \quad V\,z = 0, \quad S\,u = 0.$$

- The solution of $S\,u = 0$ is defined by:

$$u_1 = \frac{1}{2}\,C_1\,t^2 + C_2, \quad u_2 = C_1, \quad u_3 = 0, \quad u_4 = 0.$$

- The solution of $V\,z = 0$ is defined by: $z_1 = 0$, $z_2 = 0$ and

$$z_3(t) = C_3\,\mathrm{Ai}(t) + C_4\,\mathrm{Bi}(t), \quad z_4(t) = C_3\,\partial\,\mathrm{Ai}(t) + C_4\,\partial\,\mathrm{Bi}(t).$$

- The general solution of $R\,y = 0$ is then given by:

$$y = u + X\,z = \left(\frac{1}{2}\,C_1\,t^2 + C_2 \quad C_1 \quad z_3(t) \quad z_4(t)\right)^T.$$

- **Definition:** A left D-module M is projective if there exists a left D-module N and $l \in \mathbb{Z}_+$ such that $M \oplus N \cong D^{1\times l}$.

- Lemma: If $P \in D^{p\times p}$ is an idempotent, then:
  - $\ker_D(.P)$ and $\mathrm{im}_D\,(.P)$ are projective left D-modules of rank m and $p-m$.
  - $\mathrm{im}_D(.P) = \ker_D(.(I_p - P))$.

- Proposition: Let $P \in D^{p\times p}$ be an idempotent. $1 \Leftrightarrow 2$:
  1. $\ker_D(.P)$ and $\mathrm{im}_D\,(.P)$ are free modules of rank m and $p-m$.
  2. $\exists\; U \in \mathrm{GL}_p(D)$ satisfying $U\,P\,U^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix}$

$$\Rightarrow U = (U_1^T \quad U_2^T)^T, \quad \left\{ \begin{array}{l} \ker_D(.P) = D^{1\times m}\, U_1, \\ \mathrm{im}_D(.P) = D^{1\times (p-m)}\, U_2. \end{array} \right.$$

Block diagonal decomposition

- Theorem: Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q} R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying:

$$P^2 = P, \quad Q^2 = Q.$$

If the left D-modules

$$\ker_D(.P), \quad \mathrm{im}_D(.P), \quad \ker_D(.Q), \quad \mathrm{im}_D(.Q)$$

are free of rank m, $p - m$, l, $q - l$, then there exist two matrices $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$ and $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$ such that

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Example: OD system

- Let us consider the matrix again:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix}.$$

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

i.e., P and Q satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

- Computing bases of the left D-modules

$$\ker_D(.P), \quad \mathrm{im}_D(.P), \quad \ker_D(.P), \quad \mathrm{im}_D(.Q),$$

we obtain the unimodular matrices:

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -t & -1 & 1 & t \\ t+1 & 1 & -1 & -t \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

- R is then equivalent to the following block diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial & -1 & 0 & 0 \\ t & \partial & 0 & 0 \\ 0 & 0 & \partial & -t \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

Example: Saint-Venant equations

- We consider $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \sigma, 0]$ and:

$$R = \left(\begin{array}{ccc} \partial_2^2 & 1 & -2\,\partial_1\,\partial_2 \\ 1 & \partial_2^2 & -2\,\partial_1\,\partial_2 \end{array}\right).$$

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents

$$P = \left(\begin{array}{ccc} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad Q = \left(\begin{array}{cc} 1/2 & 1/2 \\ 1/2 & 1/2 \end{array}\right),$$

i.e., P and Q satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

Example: Saint-Venant equations

$$\begin{cases} U_1 = \ker_D(.P) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \\ U_2 = \mathrm{im}_D(.P) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ V_1 = \ker_D(.Q) = \begin{pmatrix} 1 & -1 \end{pmatrix}, \\ V_2 = \mathrm{im}_D(.Q) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \end{cases}$$

and we obtain the following two unimodular matrices:

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

- We easily check that we have the following block diagonal matrix:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \partial_2^2 - 1 & 0 & 0 \\ 0 & 1 + \partial_2^2 & -4\, \partial_1\, \partial_2 \end{pmatrix}.$$

Corollary

- Corollary: Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $f \in \mathrm{end}_D(M)$ be defined by P and Q and satisfying $P^2 = P$ and $Q^2 = Q$. Let us suppose that one of the conditions holds:

1. $D = A[\partial; \sigma, \delta]$, where A is a field and σ is injective,
2. $D = k[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ is a commutative Ore algebra,
3. $D = A[\partial_1; \mathrm{id}, \delta_1] \ldots [\partial_n; \mathrm{id}, \delta_n]$, where $A = k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$ and k is a field of characteristic 0, and:

$$\mathrm{rank}_D(\ker_D(.P)) \geq 2, \quad \mathrm{rank}_D(\mathrm{im}_D\,(.P)) \geq 2,$$
$$\mathrm{rank}_D(\ker_D(.Q)) \geq 2, \quad \mathrm{rank}_D(\mathrm{im}_D\,(.Q)) \geq 2.$$

Then, there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that $\overline{R} = V\, R\, U^{-1}$ is a block diagonal matrix.

Example: Flexible rod

- Let us consider the flexible rod (Mounier 95):

$$R = \begin{pmatrix} \partial_1 & -\partial_1\,\partial_2 & -1 \\ 2\,\partial_1\,\partial_2 & -\partial_1\,\partial_2^2 - \partial_1 & 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1+\partial_2^2 & -\frac{1}{2}\,\partial_2^2\,(1+\partial_2) & 0 \\ 2\,\partial_2 & -\partial_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -\frac{1}{2}\,\partial_2 \\ 0 & 0 \end{pmatrix},$$

$$\Rightarrow U = \begin{pmatrix} -2\,\partial_2 & \partial_2^2+1 & 0 \\ -2 & \partial_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\partial_2 \end{pmatrix},$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1\,(\partial_2^2-1) & -2 \end{pmatrix}.$$

V. Implementation: the Maple MORPHISMS package

The MORPHISMS package

- The algorithms have been implemented in a Maple package called MORPHISMS based on the library OREMODULES developed by Chyzak, Q. and Robertz:

http://wwwb.math.rwth-aachen.de/OreModules

- List of functions:
  - Morphisms, MorphismsConst, MorphismsRat, MorphimsRat1.
  - Projectors, ProjectorsConst, ProjectorsRat, Idempotents.
  - KerMorphism, ImMorphism, CokerMorphism, CoimMorphism.
  - TestSurj, TestInj, TestBij.
  - QuadraticFirstIntegralConst...

- It will be soon available with a library of examples

Conclusion

- Contributions:
  - We use constructive homological algebra to provide algorithms for studying general LFSs (e.g., factoring or decomposing).
  - We apply the obtained results in control theory.
- Work in progress:

Using morphism computations for factoring and decomposing general linear functional systems, in the proceedings of the *Mathematical Theory of Networks and Systems* (MTNS), Kyoto (Japan), 2006, rapport INRIA.

- Open questions:
  - Bounds in the general case.
  - Criteria for choosing the right P.
  - Existence of a solution to the Riccati equation.
  - Formulas for connections. . .