# On the Stafford and Quillen-Suslin theorems and flat multidimensional linear systems


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# Monge problem (1784)

- Let $D$ be a ring of differential operators

$$(\text{e.g.,} \quad d_i = \partial/\partial x_i, \; D = k[x_1, \ldots, x_n][d_1, \ldots, d_n]).$$

- Let $\mathcal{F}$ be a functional space which satisfies:

$$\forall \, P_1, \, P_2 \in D, \, \forall \, y_1, \, y_2 \in \mathcal{F} : \; P_1 \, y_1 + P_2 \, y_2 \in \mathcal{F} \; \; (\text{e.g.,} \; \mathcal{F} = C^{\infty}(\mathbb{R}^n)).$$

Let us consider $R \in D^{q \times p}$ and the linear system of PDEs:

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R \, \eta = 0\}.$$

- Question: When does $Q \in D^{p \times m}$ exist such that:

$$\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.) \triangleq Q \, \mathcal{F}^m?$$

$\Rightarrow$ $Q$ is called a parametrization of $\ker_{\mathcal{F}}(R.)$.

# Example

- Example: $D = \mathbb{R}(t)\left[\frac{d}{dt}\right], \ \mathcal{F} = C^{\infty}(\mathbb{R})$,

$$R = \left(\frac{d^2}{dt^2} + \alpha(t)\,\frac{d}{dt} + 1, \ -\frac{d}{dt} - \alpha(t)\right) \in D^{1\times 2}.$$

$$\ddot{y}(t) + \alpha(t)\,\dot{y}(t) + y(t) - \dot{u}(t) - \alpha(t)\,u(t) = 0 \quad (\star)$$

$$\Leftrightarrow \left\{ \begin{array}{l} y(t) = \dot{\xi}(t) + \alpha(t)\,\xi(t), \\ u(t) = \ddot{\xi}(t) + \alpha(t)\,\dot{\xi}(t) + (\dot{\alpha}(t) + 1)\,\xi(t). \end{array} \right. \qquad (\star\star)$$

$(\star\star)$ is an injective parametrization of $(\star)$ as $\xi = -\dot{y} + u$.

- Example: $D = \mathbb{R}[d_1, d_2, d_3], \quad d_i = \partial/\partial x_i, \quad \mathcal{F} = C^{\infty}(\mathbb{R}^3)$,

$$\begin{array}{lcl} \mathrm{div}\vec{A} = 0 & \Leftrightarrow & \exists\, \vec{B} \in \mathcal{F}^3 : \ \vec{A} = \mathrm{curl}\,\vec{B}, \\ \mathrm{curl}\,\vec{B} = \vec{0} & \Leftrightarrow & \exists\, f \in \mathcal{F} : \ \vec{B} = \mathrm{grad}\, f. \end{array}$$

# Flatness: two pendula mounted on a car

- We consider two pendula mounted on a car:

$$\left\{\begin{array}{l} m_1\,L_1\,\ddot{w}_1(t)+m_2\,L_2\,\ddot{w}_2(t)-w_3(t)+(M+m_1+m_2)\,\ddot{w}_4(t)=0,\\ m_1\,L_1^2\,\ddot{w}_1(t)-m_1\,L_1\,g\,w_1(t)+m_1\,L_1\,\ddot{w}_4(t)=0,\\ m_2\,L_2^2\,\ddot{w}_2(t)-m_2\,L_2\,g\,w_2(t)+m_2\,L_2\,\ddot{w}_4(t)=0.\end{array}\right. \qquad (\star)$$

- $(\star)$ is parametrizable iff $L_1 \neq L_2$.

- If $L_1 \neq L_2$ then a parametrization of $(\star)$ is defined by:

$$\left\{\begin{array}{ll} w_1(t)= & -L_2\,\xi^{(4)}(t)+g\,\ddot{\xi}(t),\\ w_2(t)= & -L_1\,\xi^{(4)}(t)+g\,\ddot{\xi}(t),\\ w_3(t)= & L_1\,L_2\,M\,\xi^{(6)}(t)-(L_1\,m_2+L_2\,m_1+g\,(L_1+L_2)\,M)\,\xi^{(4)}(t)\\ & +g^2\,(m_1+m_2+M)\,\xi^{(2)(t)}\\ w_4(t)= & L_1\,L_2\,\xi^{(4)}(t)-g\,(L_1+L_2)\,\ddot{\xi}(t)+g^2\,\xi(t).\end{array}\right.$$

- The parametrization of $(\star)$ is injective as we have:

$$\xi(t)=\frac{1}{g^2\,(L_1-L_2)}\,(L_1^2\,w_1(t)-L_2^2\,w_2(t)+(L_1-L_2)\,w_4(t)). \quad (\star\star)$$

The system is called flat and $(\star\star)$ a flat output (Fliess & co.).

# Flatness: two pendula mounted on a car

- Patching problem $\Leftrightarrow$ controllability: $\quad T > 0$.

$w^p = (w_1^p, w_2^p, w_3^p, w_4^p)$ a past trajectory of $(\star)$ on $]-\infty, 0[$.

$w^f = (w_1^f, w_2^f, w_3^f, w_4^f)$ a future trajectory of $(\star)$ on $]T, +\infty[$.

$$\Rightarrow \exists \, w = (w_1, w_2, w_3, w_4) \in C^\infty(\mathbb{R})^4 \text{ trajectory of } (\star):$$

$$\begin{cases} w_{]-\infty,0[} = w^p, \\ w_{]T,+\infty[} = w^f. \end{cases}$$

- Using the flat output

$$\xi(t) = \frac{1}{g^2\,(L_1 - L_2)}\,(L_1^2\,w_1(t) - L_2^2\,w_2(t) + (L_1 - L_2)\,w_4(t))$$

and the parametrization, it is enough to find $\xi \in C^\infty(\mathbb{R})$ s.t.:

$$\xi_{]-\infty,0[} = \xi^p \quad \& \quad \xi_{]T,+\infty[} = \xi^f.$$

# Optimal control

- Let us minimize $\frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\,dt \quad (1)$ under:

$$\dot{x}(t) + x(t) - u(t) = 0, \quad x(0) = x_0. \quad (2)$$

- (2) is parametrized by $\left\{ \begin{array}{l} x(t) = \xi(t), \\ u(t) = \dot{\xi}(t) + \xi(t). \end{array} \right. \quad (3)$

- (1) & (3) $\Rightarrow$ min $\frac{1}{2}\int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\,dt$,

$$\Rightarrow \text{Euler-Lagrange equations} \quad \left\{ \begin{array}{l} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{array} \right.$$

$$\Rightarrow u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\,e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\,e^{-\sqrt{2}(t-T)}}\,x(t).$$

# Motion planning

- Flexible rod with a torque (Mounier 95):

$$\left\{\begin{array}{l} \sigma^2\,\frac{\partial^2 q(\tau,x)}{\partial\tau^2}-\frac{\partial^2 q(\tau,x)}{\partial x^2}=0,\\ \frac{\partial q}{\partial x}(\tau,0)=-u(\tau),\\ \frac{\partial q}{\partial x}(\tau,L)=-J\,\frac{\partial^2 q}{\partial\tau^2}(\tau,L),\\ y(\tau)=q(\tau,L). \end{array}\right. \qquad (\star)$$

- $q(\tau,x)=\phi(\tau+\sigma\,x)+\psi(\tau-\sigma\,x)$, $t=(\sigma/J)\,\tau$, $v=(2\,J/\sigma^2)\,u$,

$$(\star)\Rightarrow \ddot{y}(t+1)+\ddot{y}(t-1)+\dot{y}(t+1)-\dot{y}(t-1)=v(t)$$

$$\Leftrightarrow \left\{\begin{array}{l} y(t)=\xi(t-1),\\ v(t)=\ddot{\xi}(t)+\ddot{\xi}(t-2)+\dot{\xi}(t)-\dot{\xi}(t-2). \end{array}\right.$$

- If $y_r$ is a desired trajectory then $\xi_r(t)=y_r(t+1)$ and we obtain the open-loop control law:

$$v_r(t)=\ddot{y}_r(t+1)+\ddot{y}_r(t-1)+\dot{y}_r(t+1)-\dot{y}_r(t-1).$$

# Variational problems

- Let us extremize the electromagnetism Lagragian

$$\int \left( \frac{1}{2\,\mu_0} \; \| \vec{B} \|^2 - \frac{\epsilon_0}{2} \; \| \vec{E} \|^2 \right) \, dx_1 \, dx_2 \, dx_3 \, dt, \quad (1)$$

where $\vec{B}$ and $\vec{E}$ satisfy:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \, . \, \vec{B} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = - \vec{\nabla} \, V - \dfrac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right. \qquad (3)$$

- Substituting (3) in (1) and using Lorentz gauge

$$\vec{\nabla} . \vec{A} + \frac{1}{c^2} \, \frac{\partial V}{\partial t} = 0, \quad c^2 = 1/(\epsilon_0 \, \mu_0),$$

$$\Rightarrow \left\{ \begin{array}{l} \dfrac{1}{c^2} \, \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta \, \vec{A} = 0, \\ \dfrac{1}{c^2} \, \dfrac{\partial^2 V}{\partial t^2} - \Delta \, V = 0. \end{array} \right. \qquad \text{(electromagnetic waves).}$$

# Definitions

- Definition: 1. $M$ is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^r$.

2. $M$ is stably free if $\exists\, r, s \in \mathbb{Z}_+$ such that $M \oplus D^s \cong D^r$.

3. $M$ is projective if $\exists\, r \in \mathbb{Z}_+$ and a $D$-module $P$ such that:

$$M \oplus P \cong D^r.$$

4. $M$ is reflexive if $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall\, m \in M, \quad f \in \hom_D(M, D).$$

5. $M$ is torsion-free if:

$$t(M) = \{m \in M \,|\, \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

# Classification of modules

- Theorem:

1. We have the following implications:

$$\text{free} \Rightarrow \text{stably free} \Rightarrow \text{projective} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free}.$$

2. If $D$ is a principal domain (e.g., $\mathbb{Q}(t)\left[\frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{free}.$$

3. If $D$ is a hereditary ring (e.g., $\mathbb{Q}[t]\left[\frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{projective}.$$

4. If $D = k[d_1, \ldots, d_n]$, $k$ is a field of constants, then:

$$\text{projective} = \text{free} \quad \text{(Quillen-Suslin theorem)}.$$

# Involution & formal adjoint

- Let $\mathbb{Q} \subseteq k$ be a field and $D$ a Weyl algebra:

$$A_n(k) = k[x_1, \ldots, x_n][d_1, \ldots, d_n],$$
$$B_n(k) = k(x_1, \ldots, x_n)[d_1, \ldots, d_n].$$

- Let $\theta$ be the involution of $D$ defined by:

$$\theta(d_i) = -d_i, \quad \theta(x_i) = x_i, \quad \theta(a) = a, \quad \forall\, a \in k.$$

$$(\theta : D \longrightarrow D \ k\text{-linear map}, \quad \theta(P\,Q) = \theta(Q)\,\theta(P), \quad \theta^2 = \mathrm{id}).$$

- If $R \in D^{q \times p}$, then the formal adjoint of $R$ is defined by:

$$\theta(R) = (\theta(R_{ij}))^T.$$

- $M = D^{1 \times p}/(D^{1 \times q}\, R), \quad \widetilde{N} = D^{1 \times q}/(D^{1 \times p}\, \theta(R)).$
- If $\mathcal{F}$ is a left $D$-module, then $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$.

| **Module** $M$ | **Homological algebra** | $\mathcal{F}$ **injective cogenerator** |
|---|---|---|
| with torsion | $t(M) \cong \mathrm{ext}^1_D(\widetilde{N}, D)$ | $\emptyset$ |
| torsion-free | $\mathrm{ext}^1_D(\widetilde{N}, D) = 0$ | $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^{l_1}$ |
| reflexive | $\mathrm{ext}^i_D(\widetilde{N}, D) = 0$<br>$i = 1, 2$ | $\ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^{l_1}$<br>$\ker_{\mathcal{F}}(Q_1.) = Q_2\,\mathcal{F}^{l_2}$ |
| projective<br>=<br>stably free | $\mathrm{ext}^i_D(\widetilde{N}, D) = 0$<br>$1 \leq i \leq n$ | $\ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^{l_1}$<br>$\ker_{\mathcal{F}}(Q_1.) = Q_2\,\mathcal{F}^{l_2}$<br>$\ldots$<br>$\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n\,\mathcal{F}^{l_n}$ |
| free | ? | $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^{l}$<br>$\exists\ T :\ T\,Q = I$ |

# Computation bases

- $V = \{(x,\ y,\ z)^T \in k^3 \mid 2\,x + 3\,y + 5\,z = 0\}, \quad k = \mathbb{Q},\, \mathbb{R},\, \mathbb{C}.$

$$2\,x+3\,y+5\,z = 0 \Rightarrow x = -\frac{3}{2}\,y - \frac{5}{2}\,z \Rightarrow \left\{ \begin{array}{l} x = \frac{3}{2}\,y - \frac{5}{2}\,z, \\ y = y, \\ z = z, \end{array} \right. \quad \forall\, y,\, z \in k.$$

$$\Rightarrow V = k \left(-\frac{3}{2},\ 1,\ 0\right)^T + k \left(-\frac{5}{2},\ 0,\ 1\right)^T \quad \text{basis of } V.$$

- $M = \{(x,\ y,\ z)^T \in \mathbb{Z}^3 \mid 2\,x + 3\,y + 5\,z = 0\}.$

$$M = \mathbb{Z}\,(\alpha_1,\ \beta_1,\ \gamma_1)^T + \mathbb{Z}\,(\alpha_2,\ \beta_2,\ \gamma_2)^T \Leftrightarrow \left\{ \begin{array}{l} x = \alpha_1\,t_1 + \alpha_2\,t_2, \\ y = \beta_1\,t_1 + \beta_2\,t_2,\ \forall\, t_i \in \mathbb{Z}, \\ z = \gamma_1\,t_1 + \gamma_2\,t_2, \quad (\star) \end{array} \right.$$

$\Rightarrow \{(\alpha_i,\ \beta_i,\ \gamma_i)^T\}_{1 \leq i \leq 2}$ is a basis of $M$ iff $(\star)$ is injective, i.e.:

$$t_i = a_{i1}\,x + a_{i2}\,y + a_{i3}\,z, \quad a_{ij} \in \mathbb{Z}, \quad i = 1,\, 2.$$

# Stafford results

- Theorem: Let us consider $a_1, a_2, a_3 \in D$ and the left ideal:

$$I = D\,a_1 + D\,a_2 + D\,a_3.$$

$$\Rightarrow \exists\, \lambda, \mu \in D : \quad I = D\,(a_1 + \lambda\, a_3) + D\,(a_2 + \mu\, a_3).$$

- Two constructive proofs have been developed in:

⋆ A. Hillebrand, W. Schmale, "Towards an effective version of a theorem of Stafford", J. Symbolic Computation, 32 (2001), 699-716.

⋆ A. Leykin, "Algorithmic proofs of two theorems of Stafford", J. Symbolic Computation, 38 (2004), 15 35-1550.

- Implementation in the package Stafford of OreModules.
- Corollary: A stably free left $D$-module $M$ with $\mathrm{rank}_D(M) \geq 2$ is free, i.e., $M$ admits a finite basis over $D$.

# Elementary operations

• Definition: 1. The general linear group $\mathrm{GL}_m(D)$ is the group of invertible matrices with entries in $D$:

$$\mathrm{GL}_m(D) = \{\, U \in D^{m \times m} \mid \exists\, V \in D^{m \times m} : \; U\,V = V\,U = I_m \,\}.$$

2. The elementary group $\mathrm{EL}_m(D)$ is the subgroup of $\mathrm{GL}_m(D)$ generated by all matrices of the form

$$I_m + r\,E_{ij}, \quad r \in D, \quad i \neq j,$$

$E_{ij}$ is the matrix defined by 1 at the position $(i, j)$ and 0 else.

3. $a = (a_1, \ldots, a_m)^T \in D^m$ is called unimodular if:

$$\exists\, b = (b_1, \ldots, b_m) \in D^{1 \times m} : \quad b\,a = \sum_{i=1}^{m} b_i\,a_i = 1.$$

$\mathrm{U}_m(D)$ denotes the set of unimodular vectors of $D^m$.

• Theorem: Let $m \geq 3$ and $a = (a_1, \ldots, a_m)^T \in \mathrm{U}_m(D)$. Then, there exists $E \in \mathrm{EL}_m(D)$ which satisfies:

$$E\, a = (1,\ 0, \ldots, 0)^T.$$

• Using Stafford's result, there exist $\lambda,\, \mu \in D$ such that:

$$a' = (a_1 + \lambda\, a_m,\ a_2 + \mu\, a_m,\ a_3, \ldots, a_{m-1})^T \in \mathrm{U}_{m-1}(D).$$

• $a'_1 = a_1 + \lambda\, a_m, \quad a'_2 = a_2 + \mu\, a_m, \quad a'_i = a_i, \quad i \geq 3,$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \lambda \\ 0 & 1 & 0 & \ldots & 0 & \mu \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathrm{EL}_m(D).$$

Then, we have $E_1\, a = (a'_1,\ a'_2, \ldots, a'_{m-1},\ a_m)^T$.

- $a' \in \mathrm{U}_{m-1}(D) \Rightarrow \exists\, b_1, \ldots, b_{m-1} \in D$ such that:

$$\sum_{i=1}^{m-1} b_i\, a'_i = 1 \Rightarrow \sum_{i=1}^{m-1} (a'_1 - 1 - a_m)\, b_i\, a'_i = (a'_1 - 1 - a_m).$$

- Let us define $a''_i = (a'_1 - 1 - a_m)\, b_i, \quad i \geq 1$, and:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ a''_1 & a''_2 & a''_3 & \ldots & a''_{m-1} & 1 \end{pmatrix} \in \mathrm{EL}_m(D).$$

Then, we have:

$$E_2\, (a'_1, \ldots, a'_{m-1}, a_m)^T = (a'_1, \ldots, a'_{m-1}, a'_1 - 1)^T.$$

• If we define by

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{EL}_m(D),$$

then we have:

$$E_3\,(a'_1, \dots, a'_{m-1}, a'_1 - 1)^T = (1,\; a'_2, \dots, a'_{m-1},\; a'_1 - 1)^T.$$

- Finally, if we denote by

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -a'_2 & 1 & 0 & \dots & 0 & 0 \\ -a'_3 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a'_{m-1} & 0 & 0 & \dots & 1 & 0 \\ -a'_1 + 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{EL}_m(D),$$

then we finally get:

$$E_4\,(1,\ a'_2, \dots, a'_{m-1},\ a'_1 - 1)^T = (1,\ 0, \dots,\ 0)^T.$$

- Hence, if we denote by $E = E_4\,E_3\,E_2\,E_1 \in \mathrm{EL}_m(D)$, then:

$$E\,(a_1, \dots, a_m)^T = (1,\ 0, \dots,\ 0)^T.$$

# Computation of basis

• Let $R \in D^{q \times p}$ be a matrix such that $p \geq q+2$ and which admits a right-inverse $S \in D^{p \times q}$.

$$0 \longrightarrow D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi_0} M \longrightarrow 0.$$

$\Rightarrow$ $M$ is a stably free left $D$-module with:

$$\mathrm{rank}_D(M) = p - q \geq 2.$$

• Compute the formal adjoint $\widetilde{R} = \theta(R) \in D^{p \times q}$:

$$0 \longleftarrow D^{1 \times q} \xleftarrow{.\widetilde{R}} D^{1 \times p} \longleftarrow \ker_D(.\widetilde{R}) \longleftarrow 0.$$

- Compute $\widetilde{E_1} \in \mathrm{EL}_p(D)$ such that:

$$\widetilde{E_1}\,\widetilde{R} = \begin{pmatrix} 1 & \star \\ 0 & \\ \vdots & \widetilde{R_2} \\ 0 & \end{pmatrix}, \quad \widetilde{R_2} \in D^{(p-1)\times(q-1)}.$$

- Compute $\widetilde{E_2} \in \mathrm{EL}_{p-1}(D)$ such that:

$$\widetilde{E_2}\,\widetilde{R_2} = \begin{pmatrix} 1 & \star \\ 0 & \\ \vdots & \widetilde{R_3} \\ 0 & \end{pmatrix}, \quad \widetilde{R_3} \in D^{(p-2)\times(q-2)}.$$

$$\widetilde{E_2}' = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{E_2} \end{pmatrix} \Rightarrow (\widetilde{E_2}'\,\widetilde{E_1})\,\widetilde{R} = \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ \vdots & 0 & \\ \vdots & \vdots & \widetilde{R_3} \\ 0 & 0 & \end{pmatrix}.$$

- By induction, we obtain $\widetilde{E} \in \mathrm{EL}_n(D)$ such that:

$$\widetilde{T} = \widetilde{E}\,\widetilde{R} = \begin{pmatrix} 1 & \star & \star & \star & \star \\ 0 & 1 & \star & \star & \star \\ 0 & 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- We easily check that we have:

$$\ker_D(.\widetilde{T}) = D^{1\times(p-q)}\,(0 \quad I_{p-q}).$$

• If we denote by $\widetilde{U} = (0 \quad I_{p-q}) \in D^{(p-q)\times p}$, then we obtain the commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longleftarrow & D^{1\times q} & \xleftarrow{.\widetilde{R}} & D^{1\times p} & \longleftarrow & \ker_D(.\widetilde{R}) \longleftarrow 0 \\
 & & & & \| & & \uparrow .\widetilde{E} & & \\
0 \longleftarrow & \operatorname{coker}_D(.\widetilde{T}) & & \xleftarrow{\kappa_0'} & D^{1\times q} & \xleftarrow{.\widetilde{T}} & D^{1\times p} & \xleftarrow{.\widetilde{U}} & D^{1\times(p-q)} \longleftarrow 0. \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

• In particular, we obtain that:

$$\ker_D(.\widetilde{R}) = D^{1\times(p-q)}\,(\tilde{U}\,\tilde{E}) \cong D^{1\times(p-q)}.$$

Therefore, we have the splitting exact sequence:

$$0 \longleftarrow D^{1\times q} \xleftarrow{.\widetilde{R}} D^{1\times p} \xleftarrow{.(\tilde{U}\,\tilde{E})} D^{1\times (p-q)} \longleftarrow 0.$$

- By duality, we obtain the splitting exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.(E\;U)} D^{1\times (p-q)} \longrightarrow 0,$$

where $E = \theta(\widetilde{E})$, which shows that:

$$M = D^{1\times p}\,(E \;\; U) \cong D^{1\times (p-q)}.$$

- A basis of $M$ is then defined by taking the last $p-q$ columns of the matrix $E = \theta(\widetilde{E})$ and:

$$T = R\,E = (I_q \quad 0) \Leftrightarrow E^{-1} = \begin{pmatrix} R \\ \star \end{pmatrix} \in \mathrm{GL}_p(D).$$

# Example

- Let us consider the linear control system:

$$\begin{cases} \dot{x}_1(t) - t\, u_1(t) = 0, \\ \dot{x}_2(t) - u_2(t) = 0. \end{cases} \quad (\star) \quad \Rightarrow R = \begin{pmatrix} \frac{d}{dt} & 0 & -t & 0 \\ 0 & \frac{d}{dt} & 0 & -1 \end{pmatrix}.$$

- We have the injective parametrization of $(\star)$.

$$\begin{cases} x_1(t) = \xi_1(t), \\ x_2(t) = \xi_2(t), \\ u_1(t) = \dfrac{1}{t}\, \dot{\xi}_1(t), \\ u_2(t) = \dot{\xi}_2(t). \end{cases} \quad (\star\star)$$

- The parametrization $(\star\star)$ is singular at $t = 0$.
- $M = B_1(\mathbb{Q})^{1\times 4}/(B_1(\mathbb{Q})^{1\times 2}\, R)$ is free with basis $\{x_1, x_2\}$.

- We define the left $D = \mathbb{Q}[t]\left[\frac{d}{dt}\right]$-module $P = D^{1\times 4}/(D^{1\times 2}\, R)$.
- $P$ is a stably free $D$-module as $R$ admits the right-inverse:

$$S = \begin{pmatrix} 0 & 0 & 0 & -1 \\ t & 0 & \frac{d}{dt} & 0 \end{pmatrix}^T.$$

- Using a constructive algorithm developed in previous publications, we can obtain the parametrization of $(\star)$:

$$\left\{ \begin{array}{l} x_1(t) = -t^2\,\xi_1(t) + t\,\dot{\xi}_2(t) - \xi_2(t), \\ x_2(t) = -\xi_3(t), \\ u_1(t) = -t\,\dot{\xi}_1(t) - 2\,\xi_1(t) + \ddot{\xi}_2(t), \\ u_2(t) = -\dot{\xi}_3(t). \end{array} \right. \qquad (\star\star\star)$$

$(\star\star\star)$ is clearly non-injective as $\mathrm{rank}_D(P) = 2$.

- $P$ is a stably free left $D$-module of $\mathrm{rank}_D(P) = 2$, i.e., free.

- The formal adjoint of $R$ is $\widetilde{R} = \begin{pmatrix} 0 & -\frac{d}{dt} & 0 & -1 \\ -\frac{d}{dt} & 0 & -t & 0 \end{pmatrix}^T$.
- $D\,0 + D\left(-\frac{d}{dt}\right) + D\,1 = D\,(0+1) + D\left(-\frac{d}{dt} + 0 \times 1\right)$.

Taking $a_1 = 1$ and $a_2 = 0$, we define the elementary matrices:

$$\widetilde{E_1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{E_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$\widetilde{E_3} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{E_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{d}{dt} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

- Defining $\widetilde{E} = \widetilde{E_4}\,\widetilde{E_3}\,\widetilde{E_2}\,\widetilde{E_1}$, we get:

$$\widetilde{E}\,\widetilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -t & -\frac{d}{dt} \end{pmatrix}^T.$$

- We have $D\,0 + D\,(-t) + D\,\left(-\frac{d}{dt}\right) = D\,\left(0 - \frac{d}{dt}\right) + D\,(-t)$.

Taking $b_1 = 1$ and $b_2 = 0$, we define the elementary matrices:

$$\widetilde{F_1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{F_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & \frac{d}{dt} & 1 \end{pmatrix},$$

$$\widetilde{F_3} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{F_4} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{d}{dt} + 1 & 0 & 1 \end{pmatrix},$$

- If we define $\widetilde{F} = \widetilde{F_4}\,\widetilde{F_3}\,\widetilde{F_2}\,\widetilde{F_1}$ and $\widetilde{G} = \mathrm{diag}(1, \widetilde{F})$, we then get:

$$(\widetilde{G}\,\widetilde{E})\,\widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

• Taking the last two columns of the formal adjoint of $\widetilde{G}\,\widetilde{E}$, we obtain the matrix defining a parametrization of $(\star)$:

$$Q = \begin{pmatrix} t^2 & -t\,\frac{d}{dt} + 1 \\ t\,(t+1) & -(t+1)\,\frac{d}{dt} + 1 \\ t\,\frac{d}{dt} + 2 & -\frac{d^2}{dt^2} \\ t\,(t+1)\,\frac{d}{dt} + 2\,t + 1 & -(t+1)\,\frac{d^2}{dt^2} \end{pmatrix}$$

• $Q$ is an injective parametrization of $(\star)$ as

$$T = \begin{pmatrix} 0 & 0 & t+1 & -1 \\ t+1 & -t & 0 & 0 \end{pmatrix}$$

is a left-inverse of $Q$, i.e., $T\,Q = I_2$.

• Equivalently, time-varying linear control system

$$\begin{cases} \dot{x}_1(t) - t\, u_1(t) = 0, \\ \dot{x}_2(t) - u_2(t) = 0, \end{cases}$$

is injectively parametrized by

$$(\star) \Leftrightarrow \begin{cases} x_1(t) = t^2\, \xi_1(t) - t\, \dot{\xi}_2(t) + \xi_2(t), \\ x_2(t) = t\,(t+1)\, \xi_1(t) - (t+1)\, \dot{\xi}_2(t) + \xi_2(t), \\ u_1(t) = t\, \dot{\xi}_1(t) + 2\, \xi_1(t) - \ddot{\xi}_2(t), \\ u_2(t) = t\,(t+1)\, \dot{\xi}_1(t) + (2\,t+1)\, \xi_1(t) - (t+1)\, \ddot{\xi}_2(t) \end{cases}$$

and $\{\xi_1, \xi_2\}$ is a basis of the free left $D$-module $P$ as:

$$\begin{cases} \xi_1(t) = (t+1)\, u_1(t) - u_2(t), \\ \xi_2(t) = (t+1)\, x_1(t) - t\, x_2(t). \end{cases}$$

# Controllability v.s. Flatness

- Theorem: A controllable linear ordinary differential system with polynomial coefficients and at least two inputs is flat.

- Question: Can we extend the result to the case of analytic coefficients (i.e., coefficients ring $C\{t\}$)?

- The question is equivalent to know if the ring $D = C\{t\}\left[\frac{d}{dt}\right]$ is a strongly simple, namely, for all $a_1,\, a_2,\, a_3 \in D$:

$$\exists\, \lambda,\, \mu \in D: \quad D\, a_1 + D\, a_2 + D\, a_3 = D\,(a_1 + \lambda\, a_3) + D\,(a_2 + \mu\, a_3).$$

# Sontag's example

- We consider the time-varying ordinary differential system:

$$\dot{x}(t) - t\,u(t) = 0.$$

- The system is controllable in a neighborhood of $t = 0$ as:

$$\mathrm{rank}_{\mathbb{R}}(B(t) = t,\ \dot{B}(t) - A(t)\,B(t) = 1)(0) = 1.$$

- See Sontag's textbook for more details.

- Let $D = A_1(\mathbb{Q})$, $R = \left(\frac{d}{dt} \quad -t\right) \in D^{1\times 2}$ and $M = D^{1\times 2}/(D\,R)$:

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{\pi} M \longrightarrow 0.$$

- The matrix $R$ admits a right-inverse $S = \left(t \quad \frac{d}{dt}\right)^T$, i.e., $R\,S = 1$

$$\Rightarrow M \text{ is a stably free left } D\text{-module of rank } 1.$$

# Sontag's example

- The left $D$-module $M$ admits the minimal parametrization

$$Q = \begin{pmatrix} t^2 \\ t\,\frac{d}{dt} + 2 \end{pmatrix} \in D^2,$$

i.e., we have the following exact sequence

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{.Q} D \longrightarrow L \longrightarrow 0,$$

where $L$ is a torsion left $D$-module.

- We obtain $M \cong D^{1\times 2}\,Q = D\,t^2 + D\,\left(t\,\frac{d}{dt} + 2\right)$

$$\Rightarrow M \text{ is free iff } I = D\,t^2 + D\,\left(t\,\frac{d}{dt} + 2\right) \text{ is principal.}$$

# Sontag's example: Coutinho's trick

- Definition: 1. We define by $L(P) = a_m$ the leading term of

$$P = \sum_{i=0}^{m} a_i(t) \, \frac{d^i}{dt^i} \in D, \quad a_m \neq 0.$$

2. The order $\mathrm{ord}(P)$ of $P$ is $m$.

3. We define the family of ideals of $\mathbb{Q}[t]$ defined by:

$$I_m = \{L(P) \in \mathbb{Q}[t] \mid P \in D, \; \mathrm{ord}(P) = m\} \, \cup \{0\}.$$

- Let us suppose that $I$ is principal, i.e., $I = D\,P$, $\mathrm{ord}(P) = r$.

$$\left\{ \begin{array}{l} L(t\,P) = t\,L(P), \\ L\left(\frac{d}{dt}\,P\right) = L(P), \end{array} \right. \quad \Rightarrow I_{r+s} = I_r, \quad \forall\, s \geq 0.$$

- The system $\dot{x}(t) - t\,u(t) = 0$ is not flat as we have:

$$I = D\,t^2 + D\left(t\,\frac{d}{dt} + 2\right) \Rightarrow I_0 = (t^2) \subsetneq I_1 = (t^2, t) = (t).$$

# Example

- Let $D = A_3(\mathbb{Q})$ and $R = -(d_1 - x_3, \quad d_2, \quad d_3)$.

- We define the left $D$-module $M = D^{1\times 3}/(D\,R)$ which corresponds to the equation:

$$d_1\,y_1(x) + d_2\,y_2(x) + d_3\,y_3(x) - x_3\,y_1(x) = 0. \quad (\star)$$

- Does $(\star)$ admit an injective parametrization?

- $S = (-d_3, \quad 0, \quad d_1 - x_3)^T$ satisfies $R\,S = 1$, and thus, $M$ is stably free of rank 3-1=2, i.e., free.

- The formal adjoint $\widetilde{R}$ of $R$ is defined by

$$\widetilde{R} = (d_1 + x_3, \quad d_2, \quad d_3)^T$$

is then unimodular as we have $\widetilde{S}\,\widetilde{R} = 1$.

- An effective version of Stafford's result gives

$$D\,(d_1 + x_3) + D\,d_2 + D\,d_3 = D\,(d_1 + x_3) + D\,(d_2 + d_3),$$

$$\text{as} \left\{ \begin{array}{l} d_2 = (d_2\,(d_2 + d_3))\,P_1 - (d_2\,(d_1 + x_3))\,P_2, \\ d_3 = (d_3\,(d_2 + d_3))\,P_1 - (d_3\,(d_1 + x_3))\,P_2, \end{array} \right.$$

where $P_1 = d_1 + x_3$ and $P_2 = d_2 + d_3$.

- Taking $\lambda = 0$ and $\mu = 1$, we can then define:

$$\widetilde{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Q_1 & Q_2 & 1 \end{pmatrix},$$

$$\text{where} \left\{ \begin{array}{l} Q_1 = (d_1 + x_3 - 1 - d_3)\,(d_2 + d_3), \\ Q_2 = -(d_1 + x_3 - 1 - d_3)\,(d_1 + x_3), \end{array} \right.$$

$$\widetilde{E_3} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{E_4} = \begin{pmatrix} 1 & 0 & 0 \\ -(d_2 + d_3) & 0 & 1 \\ -(d_1 + x_3 - 1) & 0 & 1 \end{pmatrix}.$$

- Defining $\widetilde{E} = \widetilde{E_4}\,\widetilde{E_3}\,\widetilde{E_2}\,\widetilde{E_1}$, we finally get:

$$\widetilde{E}\,(d_1 + x_3, \quad d_2, \quad d_3)^T = (1, \quad 0, \quad 0)^T.$$

- Taking the last two columns of $\theta(\tilde{E})$, we obtain:

$$\begin{cases} y_1(x) = (1 - L_1)\,(d_2 + d_3)\,\xi_1(x) + ((1 - L_1)\,(d_1 - x_3) + 1)\,\xi_2(x), \\ y_2(x) = (-L_2\,(d_2 + d_3) + 1)\,\xi_1(x) - L_2\,(d_1 - x_3)\,\xi_2(x), \\ y_3(x) = (-(1 + L_2)\,(d_2 + d_3) + 1)\,\xi_1(x) - (1 + L_2)\,(d_1 - x_3)\,\xi_2(x), \end{cases}$$

$$\text{where } \begin{cases} L_1 = (d_2 + d_3)\,(d_1 - d_3 - x_3 + 1), \\ L_2 = -(d_1 + x_3)\,(d_1 - d_3 - x_3 + 1). \end{cases}$$

$$\begin{cases} y_1(x) = (1 - L_1)\,(d_2 + d_3)\,\xi_1(x) + ((1 - L_1)\,(d_1 - x_3) + 1)\,\xi_2(x), \\ y_2(x) = (-L_2\,(d_2 + d_3) + 1)\,\xi_1(x) - L_2\,(d_1 - x_3)\,\xi_2(x), \\ y_3(x) = (-(1 + L_2)\,(d_2 + d_3) + 1)\,\xi_1(x) - (1 + L_2)\,(d_1 - x_3)\,\xi_2(x), \end{cases}$$

is an injective parametrization of the system

$$d_1\,y_1(x) + d_2\,y_2(x) + d_3\,y_3(x) - x_3\,y_1(x) = 0, \quad (\star)$$

as we have:

$$\begin{cases} \xi_1(x) = & (-d_1^2 + d_1\,d_3 - x_3\,d_3 + (2\,x_3 - 1)\,d_1 - x_3^2 + x_3 + 1)\,y_2(x) \\ & +(d_1^2 - d_1\,d_3 + x_3\,d_3 - (2\,x_3 - 1)\,d_1 + x_3^2 - x_3)\,y_3(x), \\ \xi_2(x) = & y_1(x) + (-d_3^2 + d_1\,d_2 - d_2\,d_3 + d_1\,d_3 + d_2 - (x_3 - 1)\,d_3 - x_3 - 2)\,y_2(x) \\ & +(d_3^2 - d_1\,d_2 + d_2\,d_3 - d_1\,d_3 + (x_3 - 1)\,d_3 + (x_3 - 1)\,d_2 + 2)\,y_3(x). \end{cases}$$

- $\{\xi_1, \xi_2\}$ is a basis of the left $D$-module associated with $(\star)$.

# Minimal free resolutions

- Theorem: Let us consider a finite free resolution of $M$:

$$0 \longrightarrow D^{1\times p_m} \xrightarrow{.R_m} D^{1\times p_{m-1}} \xrightarrow{.R_{m-1}} \dots \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0.$$

1. If $m \geq 3$ and there exists $S_m \in D^{p_{m-1}\times p_m}$ such that $R_m\, S_m = I_{p_m}$, then we have the finite free resolution of $M$ (1):

$$0 \longrightarrow D^{1\times p_{m-1}} \xrightarrow{.T_{m-1}} D^{1\times (p_{m-2}+p_m)} \xrightarrow{.T_{m-2}} D^{1\times p_{m-3}} \xrightarrow{.R_{m-3}} \dots \xrightarrow{\pi} M \longrightarrow 0,$$

where $T_{m-1} = (R_{m-1} \quad S_m), \quad T_{m-2} = \begin{pmatrix} R_{m-2} \\ 0 \end{pmatrix}.$

2. If $m = 2$ and there exists $S_2 \in D^{p_1\times p_2}$ such that $R_2\, S_2 = I_{p_2}$, then we have the finite free resolution

$$0 \longrightarrow D^{1\times p_1} \xrightarrow{.T_1} D^{1\times (p_0+p_2)} \xrightarrow{\tau_0} M \longrightarrow 0, \quad (2)$$

where $T_1 = (R_1 \quad S_2), \quad \tau_0 = \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix}.$

# Example

- $\dot{\delta}$ satisfies the system $\quad t^2\, y(t) = 0, \quad t\, \dot{y}(t) + 2\, y(t) = 0.$
- Let us consider $D = A_1(\mathbb{Q})$ and the left $D$-module:

$$M = D / \left( D\, t^2 + D\, \left( t\, \frac{d}{dt} + 2 \right) \right).$$

- $M$ admits the following finite free resolution of $M$:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1 \times 2} D \xrightarrow{.R_1} D \xrightarrow{\delta_0} M \longrightarrow 0,$$

$$R_1 = \begin{pmatrix} t^2 & t\, \frac{d}{dt} + 2 \end{pmatrix}^T, \quad R_2 = \begin{pmatrix} \frac{d}{dt} & -t \end{pmatrix}.$$

- $S_2 = \begin{pmatrix} t & \frac{d}{dt} \end{pmatrix}^T$ is a right-inverse of $R_2$, and thus, we have:

$$0 \longrightarrow D^{1 \times 2} \xrightarrow{.T_1} D^{1 \times 2} \xrightarrow{\tau_0} M \longrightarrow 0, \quad T_1 = \begin{pmatrix} t^2 & t \\ t\, \frac{d}{dt} + 2 & \frac{d}{dt} \end{pmatrix}.$$

# Example

- Let us consider $D = A_3(\mathbb{Q})$ and the matrix:

$$R_1 = \begin{pmatrix} \frac{1}{2}\,x_2\,d_1 & x_2\,d_2 + 1 & x_2\,d_3 + \frac{1}{2}\,d_1 \\ -\frac{1}{2}\,x_2\,d_2 - \frac{3}{2} & 0 & \frac{1}{2}\,d_2 \\ -d_1 - \frac{1}{2}\,x_2\,d_3 & -d_2 & -\frac{1}{2}\,d_3 \end{pmatrix}.$$

- The left $D$-module $M = D^{1\times 3}/(D^{1\times 3}\,R_1)$ admits the finite free resolution where $R_2 = (d_2 \quad -(d_1 + x_3\,d_3) \quad x_2\,d_2 + 2)$:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R_1} D^{1\times 3} \xrightarrow{\delta_0} M \longrightarrow 0.$$

- $S_2 = (-x_2 \quad 0 \quad 1)^T$ is a right-inverse of $R_2$ and we have:

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.T_1} D^{1\times 4} \xrightarrow{\tau_0} M \longrightarrow 0,$$

$$T_1 = \begin{pmatrix} \frac{1}{2}\,x_2\,d_1 & x_2\,d_2 + 1 & x_2\,d_3 + \frac{1}{2}\,d_1 & -x_2 \\ -\frac{1}{2}\,x_2\,d_2 - \frac{3}{2} & 0 & \frac{1}{2}\,d_2 & 0 \\ -d_1 - \frac{1}{2}\,x_2\,d_3 & -d_2 & -\frac{1}{2}\,d_3 & 1 \end{pmatrix}.$$

# Projective dimension

- Definition: A projective resolution of a left $D$-module $M$ is an exact sequence of the form

$$0 \longrightarrow P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0, \quad (\star)$$

where the $P_i$ are projective left $D$-modules.

- Definition: We call left projective dimension of a left $D$-module $M$, denoted by $\mathrm{lpd}_D(M)$, the smallest $n$ such that there exists a projective resolution of the form $(\star)$.

- Proposition: $\mathrm{lpd}_D(M) = n$ iff there exists a finite projective resolution $(\star)$ of $M$, where $\delta_n$ is nonsplit, i.e., there exists no $D$-morphism $\tau_n : P_{n-1} \longrightarrow P_n$ s.t. $\tau_n \circ \delta_n = id_{P_n}$ $(P_{-1} = M)$.

# Computation of the projective dimension

- Algorithm: 1. Compute a finite free resolution of $M$.

2. Set $j = m$ and $T_j = R_m$.

3. Check if $R_j$ admits a right-inverse $S_j$ over $D$.

$\Rightarrow$ If not, then exit and $\mathrm{lpd}_D(M) = j$.

$\Rightarrow$ If yes and:

(a) If $j = 1$, then exit with $\mathrm{lpd}_D(M) = 0$.
(b) If $j = 2$, then compute (2) and return to 3 with $j \leftarrow j - 1$.
(c) If $j \geq 3$, then compute (1) and return to 3 with $j \leftarrow j - 1$.

- Example: The left $A_1(\mathbb{Q})$-module $M$ associated with the annihilator of $\dot{\delta}$ satisfies $\mathrm{lpd}_D(M) = 1$.

- Example: The left $A_3(\mathbb{Q})$-module $M$ associated with the contact transformations satisfies $\mathrm{lpd}_D(M) = 1$.

# OreModules

- OreModules is a tool-box developed in *Maple*.
- OreModules uses *Ore_algebra* developed by F. Chyzak.
- OreModules handles linear systems of ODEs, PDEs, discrete equations, differential time-delay equations...
- OreModules computes:

1. free resolutions, $\mathrm{ext}^i_D(\cdot, D)$, projective dim., Hilbert series,
2. torsion elements, autonomous elements,
3. parametrizations of under-determined systems,
4. left-/right-/generalized inverses,
5. bases, flat outputs, $\pi$-polynomials,
6. first integrals of motion, Euler-Lagrange equations...

# Conclusion

• We obtain constructive algorithms for checking stably freeness and computing the bases over the Weyl algebras.

• These results seem to us much simpler than the ones of:

J. Gago-Vargas, "Bases for projective modules in $A_n(k)$", J. Symbolic Computation 36 (2003), 845-853.

• These results as well as the Stafford result on the number of generators of left ideal over $D = A_n(\mathbb{Q})$ or $B_n(\mathbb{Q})$ have been implemented in STAFFORD:

`http://wwwb.math.rwth-aachen.de/OreModules.`

• Algorithms for the computation of projective dimensions and minimal free resolutions have also been implemented.

# Conclusion

- We recall what E. Goursat said about the Monge problem:

> "Ces résultats sont encore bien particuliers. J'espère qu'ils pourront contribuer à appeler l'attention de quelques jeunes mathématiciens sur un sujet difficile et bien peu étudié. (1930)"


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