Factorization and decomposition of linear functional systems

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Introduction: factorization and decomposition

- Let $L(\partial)$ be a scalar ordinary or partial differential operator.
- When is it possible to find $L_1(\partial)$ and $L_2(\partial)$ such that:

$$L(\partial) = L_2(\partial)\, L_1(\partial)?$$

- We note that $L_1(\partial)\, y = 0 \Rightarrow L(\partial)\, y = 0$.
- $L(\partial)\, y = 0$ is equivalent to the cascade integration:

$$L_1(\partial)\, y = z \quad \& \quad L_2(\partial)\, z = 0.$$

- When is the integration of $L(\partial)\, y = 0$ equivalent to:

$$L_2(\partial)\, z = 0 \quad \& \quad L_1(\partial)\, u = 0?$$

$$(L_1\, X + Y\, L_2 = 1 \Rightarrow L_1(X\, z) = z \Rightarrow y = u + X\, z)$$

Introduction: factorization and decomposition

- Let us consider the first order ordinary differential system:

$$\partial\, y = E(t)\, y \quad (\star)$$

- When does it exist an invertible change of variables

$$y = P(t)\, z,$$

such that

$$(\star) \quad \Leftrightarrow \quad \partial\, z = F(t)\, z,$$

where $F = P^{-1}\,(E\,P - \partial P)$ is either of the form:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \quad \text{or} \quad F = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}?$$

- If $E(t) = E \in \mathbb{R}^{n \times n}$, then $F = P^{-1}\,E\,P$: Jordan form.

Factorization: known cases

Square differential systems:

- Beke's algorithm (Beke1894, Schwarz89, Bronstein94, Tsarëv94...)
- Eigenring (Singer96, Giesbrecht98, Barkatou-Pflügel98, Barkatou01 - ideas in Jacobson37...)

Square (q-)difference systems (generalizations):

- Barkatou01, Bomboy01...

Square D-finite partial differential systems (connections):

- Li-Schwarz-Tsarëv03, Wu05...

Square partial differential systems:

- Grigoriev-Schwarz05...

General linear functional systems?

- Linearized approximation of the steady two-dimensional rotational isentropic flow (Courant-Hilbert):

$$\begin{cases} u\,\rho \dfrac{\partial\,\omega}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial x} = 0, \\ u\,\rho\,\dfrac{\partial\,\lambda}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial y} = 0, \\ \rho\,\dfrac{\partial\,\omega}{\partial x} + \rho\,\dfrac{\partial\,\lambda}{\partial y} + u\,\dfrac{\partial \sigma}{\partial x} = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(u, \rho, c)[\partial_x, \partial_y]$ and the system matrix:

$$R = \begin{pmatrix} u\,\rho\,\partial_x & c^2\,\partial_x & 0 \\ 0 & c^2\,\partial_y & u\,\rho\,\partial_x \\ \rho\,\partial_x & u\,\partial_x & \rho\,\partial_y \end{pmatrix} \in D^{3\times 3}.$$

- Question: $\exists\ U \in \mathrm{GL}_3(D),\ V \in \mathrm{GL}_3(D)$ such that:

$$V\,R\,U = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3), \quad \alpha_1, \alpha_2, \alpha_3 \in D?$$

General linear functional systems?

- Model of a one-dimensional tank containing a fluid subjected to an horizontal move (Petit, Rouchon, IEEE TAC, 2002):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-2\,h) + \alpha\, \ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t-2\,h) - \dot{y}_2(t) + \alpha\, \ddot{y}_3(t-h) = 0, \end{cases} \qquad \alpha \in \mathbb{R}, \quad h \in \mathbb{R}_+.$$

- Let $D = \mathbb{R}\left[\frac{d}{dt}, \delta\right]$ and consider the system matrix:

$$R = \begin{pmatrix} \dfrac{d}{dt} & -\dfrac{d}{dt}\,\delta^2 & \alpha\,\dfrac{d^2}{dt^2}\,\delta \\ \dfrac{d}{dt}\,\delta^2 & -\dfrac{d}{dt} & \alpha\,\dfrac{d^2}{dt^2}\,\delta \end{pmatrix} \in D^{2\times 3}.$$

- Question: $\exists\ U \in \mathrm{GL}_3(D),\ V \in \mathrm{GL}_2(D)$ such that:

$$V\,R\,U = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \end{pmatrix},\ \alpha_1, \alpha_2, \alpha_3 \in D?$$

Factorization and decomposition problems

- Let D be an algebra of functional operators.
- Let $R \in D^{q \times p}$ be a matrix of functional operators.

Questions:

1. $\exists \, R_1 \in D^{r \times p}, \, R_2 \in D^{q \times r} : \quad R = R_2 \, R_1$?
2. $\exists \, W \in \mathrm{GL}_p(D), \, V \in \mathrm{GL}_q(D)$ s.t. $V \, R \, W = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$?
3. $\exists \, W \in \mathrm{GL}_p(D), \, V \in \mathrm{GL}_q(D)$ s.t. $V \, R \, W = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$?

Outline

• Type of systems: Partial differential/discrete/differential time-delay... linear systems (LFSs).

• General topic: Algebraic study of linear functional systems (LFSs) coming from mathematical physics, engineering sciences...

• Techniques: Module theory and homological algebra.

• Applications: Equivalences of systems, Galois symmetries, quadratic first integrals/conservation laws, decoupling problem...

• Implementation: package MORPHISMS based on OREMODULES:

http://wwwb.math.rwth-aachen.de/OreModules.

I. Ore Module associated with a linear functional system

Ore algebras

Consider a ring A, an automorphism σ of A and a σ-derivation δ:

$$\delta(a\,b) = \sigma(a)\,\delta(b) + \delta(a)\,b.$$

Definition: A non-commutative polynomial ring $D = A[\partial; \sigma, \delta]$ in

∂ is called skew if $\forall\, a \in A, \quad \partial\, a = \sigma(a)\,\partial + \delta(a).$

Definition: Let us consider $A = k,\ k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$. The skew polynomial ring $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is called an Ore algebra if we have:

$$\begin{cases} \sigma_i\,\delta_j = \delta_j\,\sigma_i, & 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, & \delta_i(\partial_j) = 0, \quad j < i. \end{cases}$$

$\Rightarrow$ D is generally a non-commutative polynomial ring.

Examples of Ore algebras

- Partial differential operators: $A = k,\ k[x_1, \ldots, x_n],\ k(x_1, \ldots, x_n)$,

$$D = A\left[\partial_1; \mathrm{id}, \tfrac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \tfrac{\partial}{\partial x_n}\right],$$

$$P = \textstyle\sum_{0 \leq |\mu| \leq m} a_\mu(x)\, \partial^\mu \in D, \quad \partial^\mu = \partial_1^{\mu_1} \ldots \partial_n^{\mu_n}.$$

- Shift operators:

$$D = A[\partial; \sigma, 0], \quad A = k,\ k[n],\ k(n),$$

$$P = \textstyle\sum_{i=0}^m a_i(n)\, \partial^i \in D, \quad \sigma(a)(n) = a(n+1).$$

- Differential time-delay operators:

$$D = A\left[\partial_1; \mathrm{id}, \tfrac{d}{dt}\right] [\partial_2; \sigma, 0], \quad A = k,\ k[t],\ k(t),$$

$$P = \textstyle\sum_{0 \leq i+j \leq m} a_{ij}(t)\, \partial_1^i\, \partial_2^j \in D.$$

Exact sequences

• **Definition**: A sequence of D-morphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be exact at M if we have:

$$\ker g = \mathrm{im}\, f.$$

• **Example**: If $f : M \longrightarrow M'$ is a D-morphism, we then have the following exact sequences:

1. $0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{\rho} \mathrm{coim}\, f \triangleq M / \ker f \longrightarrow 0.$
2. $0 \longrightarrow \mathrm{im}\, f \xrightarrow{j} M' \xrightarrow{\kappa} \mathrm{coker}\, f \triangleq M' / \mathrm{im}\, f \longrightarrow 0.$
3. $0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{f} M' \xrightarrow{\kappa} \mathrm{coker}\, f \longrightarrow 0.$

A left D-module M associated with $R\,\eta = 0$

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D-module $\mathcal{F}$.

- Let us consider $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$.

- As in number theory or algebraic geometry, we associate with the system $\ker_{\mathcal{F}}(R.)$ the finitely presented left D-module:

$$M = D^{1 \times p}/(D^{1 \times q}\,R).$$

- Malgrange's remark: applying the functor $\hom_D(., \mathcal{F})$ to the finite free resolution (exact sequence)

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\,R & & & & \end{array}$$

we then obtain the exact sequence:

$$\begin{array}{ccccc} \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \xleftarrow{\pi^\star} & \hom_D(M, \mathcal{F}) \longleftarrow 0. \\ R\,\eta & \longleftarrow & \eta = (\eta_1, \ldots, \eta_p)^{\mathrm{T}} & & \end{array}$$

Example: Linearized Euler equations

• The linearized Euler equations for an incompressible fluid can be defined by the system matrix

$$R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} \in D^{4\times 4},$$

where $D = \mathbb{R}\left[\partial_1, \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2, \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3, \mathrm{id}, \frac{\partial}{\partial x_3}\right]\left[\partial_t, \mathrm{id}, \frac{\partial}{\partial t}\right]$.

• Let us consider the left D-module $\mathcal{F} = \mathcal{C}^\infty(\Omega)$ (Ω open convex subset of $\mathbb{R}^4$) and the D-module:

$$M = D^{1\times 4}/(D^{1\times 4}\, R).$$

The solutions of $R\, y = 0$ in $\mathcal{F}$ are in $1-1$ correspondence with the morphisms from M to $\mathcal{F}$, i.e., with the elements of:

$$\hom_D(M, \mathcal{F}).$$

II. Morphisms between Ore modules finitely presented by two matrices R and R' of functional operators

Morphims of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- Let us consider the finitely presented left D-modules:

$$M = D^{1 \times p}/(D^{1 \times q}\, R), \quad M' = D^{1 \times p'}/(D^{1 \times q'}\, R').$$

- We are interested in the abelian group $\hom_D(M, M')$ of D-morphisms from M to M':

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

Morphims of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$$\exists\, f : M \to M' \Longleftrightarrow \exists\, P \in D^{p \times p'},\ Q \in D^{q \times q'} \text{ such that:}$$

$$R\, P = Q\, R'.$$

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda\, P)$, for all $\lambda \in D^{1 \times p}$.

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \sigma, \delta]$, $E,\ F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \Longleftrightarrow \left\{ \begin{array}{l} \sigma(P) = Q \in A^{p\times p}, \\ \delta(P) = E\, P - \sigma(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E\, P).$$

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \sigma, \delta]$, $E, F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \Longleftrightarrow \left\{ \begin{array}{l} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E\, P - \sigma(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E\, P).$$

- Differential case: $\delta = \frac{d}{dt}$, $\sigma = \mathrm{id}$:

$$\left\{ \begin{array}{l} \dot{P} = E\, P - P\, F, \\ F = -P^{-1}(\dot{P} - E\, P). \end{array} \right.$$

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \sigma, \delta]$, $E, F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \Longleftrightarrow \left\{ \begin{array}{l} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E\, P - \sigma(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E\, P).$$

- Discrete case: $\delta = 0$, $\sigma(k) = k - 1$:

$$\left\{ \begin{array}{l} E(k)\, P(k) - P(k-1)\, F(k) = 0, \\ F = \sigma(P)^{-1}\, E\, P. \end{array} \right.$$

Eigenring: example

- Let us consider the system $\dot{y}(t) = E(t)\, y(t)$, where:

$$E(t) = \begin{pmatrix} t\,(2\,t+1) & -2\,t^3 - 2\,t^2 + 1 \\ 2\,t & -t\,(2\,t+1) \end{pmatrix}.$$

- The eigenring of the system $\partial y(t) = E(t)\, y(t)$ is:

$$\mathcal{E} = \{P \in \mathbb{Q}(t)^{2\times 2} \mid \dot{P}(t) = E(t)\, P(t) - P(t)\, E(t)\}.$$

- Computing the rational solutions of $\dot{P} = [E, P]$, we then get:

$$\mathcal{E} = \left\{ P = \begin{pmatrix} a_1 - a_2\,(t+1) & a_2\,t\,(t+1) \\ -a_2 & a_2\,t + a_1 \end{pmatrix} \mid a_1,\ a_2 \in \mathbb{Q} \right\}.$$

- P is isospectral as (E, P) is a Lax pair:

$$\det(P - \lambda\, I_2) = (\lambda - a_1)\,(\lambda - a_1 + a_2).$$

Eigenring: example

- Computing a Jordan form of P, we obtain

$$J = V^{-1}\, P\, V = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 - a_2 \end{pmatrix},$$

where:

$$V = \begin{pmatrix} -t & 1+t \\ -1 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -(t+1) \\ 1 & -t \end{pmatrix}.$$

- Let us denote by $z = V^{-1}\, y = (y_1 - (t+1)\, y_2 \quad y_1 - t\, y_2)^T$:

$$\dot{y}(t) = E(t)\, y(t) \quad \Leftrightarrow \quad \dot{z}(t) = \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} z(t).$$

$$\Rightarrow \begin{cases} z_1(t) = C_1\, e^{-t^2/2}, \\ z_2(t) = C_2\, e^{t^2/2}, \end{cases} \Rightarrow \begin{cases} y_1(t) = -C_1\, t\, e^{-t^2/2} + C_2\, (t+1)\, e^{t^2/2}, \\ y_2(t) = -C_1\, e^{-t^2/2} + C_2\, e^{t^2/2}. \end{cases}$$

Integrable connections

- Let us consider n matrices $A_i \in A^{p \times p}$ and the connexion:

$$\left\{ \begin{array}{c} \partial_1 \, y(x) - A_1(x) \, y(x) = 0, \\ \dots \\ \partial_n \, y(x) - A_n(x) \, y(x) = 0. \end{array} \right.$$

- $\nabla_i = \partial_i \, I_p - A_i \in D^{p \times p} = A[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]^{p \times p}$.
- The connexion is integrable if the integrability conditions hold:

$$[\nabla_i, \nabla_j] \triangleq \nabla_i \, \nabla_j - \nabla_j \, \nabla_i = \delta_j(A_i) - \delta_i(A_j) + \sigma_j(A_i) \, A_j - \sigma_i(A_j) \, A_i = 0.$$

- Let us consider the matrix of functional operators

$$R = ((\partial_1 \, I_p - A_1)^T \cdots (\partial_n \, I_p - A_n)^T)^T \in D^{n \, p \times p},$$

and the left D-module $M = D^{1 \times p} / (D^{1 \times n \, p} \, R)$.

- $f \in \mathrm{end}_D(M)$ is defined by the pair $(P, \; Q)$, where:

$$\delta_i(P) + \sigma_i(P) \, A_i - A_i \, P = 0, \quad i = 1, \dots, n, \quad Q = \mathrm{diag}(\sigma_i(P)).$$

Example: Linear elasticity

- The deformation symmetric tensor $\epsilon = (\epsilon^{ij})$ is defined by:

$$\left\{ \begin{array}{l} \epsilon^{11} = \partial_1 \xi^1, \\ 2\,\epsilon^{12} = \partial_2 \xi^1 + \partial_1 \xi^2, \\ \epsilon^{22} = \partial_2 \xi^2. \end{array} \right. \qquad \xi^i : \text{ displacement}$$

- This system with $\epsilon = 0$ corresponds to the integrable connection:

$$\nabla_i = \partial_i I_3 - A_i, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- $D = \mathbb{R}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right]$, $\quad R = (\nabla_1^T \;\; \nabla_2^T)^T$, $\quad M = D^{1\times 3}/(D^{1\times 6}\, R)$.
- $f \in \mathrm{end}_D(M)$ is defined by $(P, \mathrm{diag}(P, P))$, where $P \in \mathbb{R}^{3\times 3}$:

$$\left\{ \begin{array}{l} P\,A_1 - A_1\,P = 0, \\ P\,A_2 - A_2\,P = 0, \end{array} \right. \Leftrightarrow P = \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix}, \; \alpha, \beta, \gamma \in \mathbb{R}.$$

Computation of $\hom_D(M, M')$

- Problem: Given $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$, find $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying the commutation relation $R\,P = Q\,R'$.

- If D is a commutative ring, then $\hom_D(M, M')$ is a D-module.

- The Kronecker product of $E \in D^{q\times p}$ and $F \in D^{r\times s}$ is:

$$E \otimes F = \begin{pmatrix} E_{11}\,F & \dots & E_{1p}\,F \\ \vdots & \vdots & \vdots \\ E_{q1}\,F & \dots & E_{qp}\,F \end{pmatrix} \in D^{(q\,r)\times(p\,s)}.$$

Lemma: If $U \in D^{a\times b}$, $V \in D^{b\times c}$ and $W \in D^{c\times d}$, then we have:

$$U\,V\,W = (V_1 \dots V_b)\,(U^T \otimes W).$$

$$R\,P\,I_{p'} = (P_1 \dots P_p)\,(R^T \otimes I_{p'}), \quad I_q\,Q\,R' = (Q_1 \dots Q_q)\,(I_q \otimes R').$$

We are reduced to compute $\ker_D\left(.\begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix}\right)$.

Computation of $\hom_D(M, M')$

- Problem: Given $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, find $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the commutation relation $R\,P = Q\,R'$.

- If D is a non-commutative ring, then $\hom_D(M, M')$ is an abelian group and generally an infinite-dimensional k-vector space.

$\Rightarrow$ find a k-basis of morphisms with given degrees in x_i and in ∂_j:

1. Take an ansatz for P with chosen degrees.
2. Compute $R\,P$ and a Gröbner basis G of the rows of R'.
3. Reduce the rows of $R\,P$ w.r.t. G.
4. Solve the system on the coefficients of the ansatz so that all the normal forms vanish.
5. Substitute the solutions in P and compute Q by means of a factorization.

Example: Bipendulum

- We consider the Ore algebra $D = \mathbb{R}(g, l)\left[\frac{d}{dt}\right]$.

- We consider the matrix of the bipendulum with $l = l_1 = l_2$:

$$R = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} & 0 & -\frac{g}{l} \\ 0 & \frac{d^2}{dt^2} + \frac{g}{l} & -\frac{g}{l} \end{pmatrix} \in D^{2 \times 3}.$$

- Let us consider the D-module $M = D^{1 \times 3}/(D^{1 \times 2}\, R)$.

- We obtain that $\mathrm{end}_D(M)$ is defined by the matrices:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3\, g \\ \alpha_4 & \alpha_1 + \alpha_2 - \alpha_4 & \alpha_3\, g \\ 0 & 0 & \alpha_3\, l\, \frac{d^2}{dt^2} + \alpha_1 + \alpha_2 + \alpha_3\, g \end{pmatrix},$$

$$Q = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_1 + \alpha_2 - \alpha_4 \end{pmatrix}, \quad \forall\, \alpha_1, \ldots, \alpha_4 \in D.$$

Example: OD system

- Let $D = \mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $M = D^{1\times 4}/(D^{1\times 4} R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}.$$

- $f \in \mathrm{end}_D(M)$ is defined by (P, P) where $P \in \mathbb{Q}[t]^{4\times 4}$ satisfies

$$P = \begin{pmatrix} a_4 - 2\,a_2\,t^2 & a_1 + a_5\,t^2 + a_3\,t^4 & 0 & 0 \\ -4\,a_3 & a_4 + 2\;a_5 + 2\;a_3\,t^2 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix},$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$.

Example: Bomboy's PhD, p. 80

q-dilatation case: $D = \mathbb{R}(q)(x)\,[H]$ where $H(f(x)) = f(q\,x)$ and:

$$R = \begin{pmatrix} H & -1 \\ -\dfrac{1 - q^3\,x^2}{1 - q\,x^2} & \dfrac{x\,(1 - q^2)}{1 - q\,x^2} + H \end{pmatrix} \in D^{2\times 2}.$$

• Searching for endomorphisms with degree 0 in H and 2 in x (both in numerator and denominator), we obtain

$$P = \begin{pmatrix} \dfrac{-a + b\,x\,q - b\,x + a\,q\,x^2}{c\,(-1 + q\,x^2)} & \dfrac{b\,(-1 + x^2)}{c\,(-1 + q\,x^2)} \\ \dfrac{b\,(-1 + q^2\,x^2)}{c\,(-1 + q\,x^2)} & -\dfrac{a + b\,x\,q - b\,x - a\,q\,x^2}{c\,(-1 + q\,x^2)} \end{pmatrix},$$

where, a, b, c are constants or $P = I_2$ (and corresponding Q).

Example: Saint-Venant equations

- Let $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \sigma, 0]$ be the ring of differential time-delay operators and consider the matrix of the tank model:

$$R = \begin{pmatrix} \partial_2^2 & 1 & -2\,\partial_1\,\partial_2 \\ 1 & \partial_2^2 & -2\,\partial_1\,\partial_2 \end{pmatrix} \in D^{2\times 3}.$$

- The endomorphisms of $M = D^{1\times 3}/(D^{1\times 2}\,R)$ are defined by:

$$P_\alpha = \left(\begin{array}{ccc} \alpha_1 & \alpha_2 & 2\,\alpha_3\,\partial_1\,\partial_2 \\ \alpha_2 + 2\,\alpha_4\,\partial_1 + 2\,\alpha_5\,\partial_1\,\partial_2 & \alpha_1 - 2\,\alpha_4\,\partial_1 - 2\,\alpha_5\,\partial_1\,\partial_2 & 2\,\alpha_3\,\partial_1\,\partial_2 \\ \alpha_4\,\partial_2 + \alpha_5 & -\alpha_4\,\partial_2 - \alpha_5 & \alpha_1 + \alpha_2 + \alpha_3\,(\partial_2^2 + 1) \end{array}\right),$$

$$Q_\alpha = \begin{pmatrix} \alpha_1 - 2\,\alpha_4\,\partial_1 & \alpha_2 + 2\,\alpha_4\,\partial_1 \\ \alpha_2 + 2\,\alpha_5\,\partial_1\,\partial_2 & \alpha_1 - 2\,\alpha_5\,\partial_1\,\partial_2 \end{pmatrix}, \quad \forall\, \alpha_1, \ldots, \alpha_5 \in D.$$

Example: Euler-Tricomi equation

- Let us consider the Euler-Tricomi equation (transonic flow):

$$\partial_1^2 \, u(x_1, x_2) - x_1 \, \partial_2^2 \, u(x_1, x_2) = 0.$$

- Let $D = A_2(\mathbb{Q})$, $R = (\partial_1^2 - x_1 \, \partial_2^2) \in D$ and $M = D/(D\,R)$.
  - $\mathrm{end}_D(M)_{1,1}$ is defined by:

$$\begin{cases} P = a_1 + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1, \\ Q = (a_1 + 2\, a_3) + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1, \end{cases}$$

  - $\mathrm{end}_D(M)_{2,0}$ is defined by $P = Q = a_1 + a_2 \, \partial_2 + a_3 \, \partial_2^2$.
  - $\mathrm{end}_D(M)_{2,1}$ is defined by:

$$\begin{cases} P = & a_1 + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1 \\ & + a_4 \, \partial_2^2 + \frac{3}{2} \, a_5 \, x_2 \, \partial_2^2 + a_5 \, x_1 \, \partial_1 \, \partial_2, \\ Q = & (a_1 + 2\, a_3) + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1 \\ & + a_4 \, \partial_2^2 + a_5 \, x_1 \, \partial_1 \, \partial_2 + 2\, a_5 \, \partial_2 + \frac{3}{2} \, a_5 \, x_2 \, \partial_2^2. \end{cases}$$

Example: Beltrami equations

- Let $D = A_2(\mathbb{Q}) = \mathbb{Q}[x, y][\partial_x, \partial_y]$ and $M = D^{1\times 2}/(D^{1\times 2}\, R)$:

$$R = \begin{pmatrix} \partial_x & -x\partial_y \\ \partial_y & x\,\partial_x \end{pmatrix} \in D^{2\times 2}.$$

  - $\mathrm{end}_D(M)_{0,1}$ is defined by $P = Q = a\, I_2,\ a \in \mathbb{Q}$.
  - $\mathrm{end}_D(M)_{1,0}$ is defined by $(a_1, a_2 \in \mathbb{Q})$:

$$P = Q = \begin{pmatrix} a_1 + a_2\,\partial_y & 0 \\ 0 & a_1 + a_2\,\partial_y \end{pmatrix}.$$

  - $\mathrm{end}_D(M)_{1,1}$ is defined by $(a_1, a_2, a_3 \in \mathbb{Q})$:

$$P = \begin{pmatrix} a_3\,(y\,\partial_y + x\,\partial_x - 1) + a_2\,\partial_y + a_1 & 0 \\ -a_3\,\partial_y & a_3\,y\,\partial_y + a_2\,\partial_y + a_1 \end{pmatrix},$$

$$Q = \begin{pmatrix} a_3\,(y\,\partial_y + x\,\partial_x) + a_2\,\partial_y + a_1 & a_3\,x\,\partial_y \\ 0 & a_2\,\partial_y + a_3\,y\,\partial_y + a_1 \end{pmatrix}.$$

III. A few applications: Galois symmetries, quadratic first integrals of motion and conservation laws

Galois transformations

We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array} \qquad (\star)$$

If $\mathcal{F}$ is a left D-module, by applying the functor $\hom_D(\cdot, \mathcal{F})$ to $(\star)$, we then obtain the following commutative exact diagram:

$$\begin{array}{ccccccc} 0 = Q\,(R'\,y) = R\,(P\,y) & \longleftarrow\!\!\shortmid & P\,y & & & & \\ \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\ \uparrow Q. & & \uparrow P. & & \uparrow f^\star & & \\ \mathcal{F}^{q'} & \xleftarrow{R'.} & \mathcal{F}^{p'} & \longleftarrow & \ker_{\mathcal{F}}(R'.) & \longleftarrow & 0. \\ 0 = R'\,y & \longleftarrow\!\!\shortmid & y & & & & \end{array}$$

$\Rightarrow$ $f^\star$ sends $\ker_{\mathcal{F}}(R.')$ to $\ker_{\mathcal{F}}(R.)$.

Example: Linear elasticity

- Consider the Killing operator for the euclidian metric defined by:

$$R = \begin{pmatrix} \partial_1 & 0 \\ \partial_2/2 & \partial_1/2 \\ 0 & \partial_2 \end{pmatrix}.$$

- The system $R\, y = 0$ admits the following general solution:

$$y = \begin{pmatrix} c_1\, x_2 + c_2 \\ -c_1\, x_1 + c_3 \end{pmatrix}, \quad c_1,\, c_2,\, c_3 \in \mathbb{R}. \quad (\star)$$

- We find that $\mathrm{end}_D(D^{1\times 2}/(D^{1\times 3}\, R))$ is defined by:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2\, \partial_1 \\ 0 & 2\, \alpha_3\, \partial_1 + \alpha_1 \end{pmatrix}, \quad \alpha_1,\, \alpha_2,\, \alpha_3 \in D.$$

- Applying P to $(\star)$, we then get the new solution:

$$\overline{y} = P\, y = \begin{pmatrix} \alpha_1\, c_1\, x_2 + \alpha_1\, c_2 - \alpha_2\, c_1 \\ -\alpha_1\, c_1\, x_1 + \alpha_1\, c_3 - 2\, \alpha_3\, c_1 \end{pmatrix}, \text{ i.e., } R\, \overline{y} = 0.$$

Quadratic first integrals of motion

Let us consider a morphism f from $\widetilde{N}$ to M defined by:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p + E^T)} & D^{1\times p} & \xrightarrow{\pi} & \widetilde{N} & \longrightarrow & 0 \\ & & \downarrow .P & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - E)} & D^{1\times p} & \xrightarrow{\pi'} & M & \longrightarrow & 0. \end{array}$$

We then have:

$$\dot{P} + E^T\, P + P\, E = 0.$$

If $V(x) = x^T\, P\, x$, then $\dot{V}(x) = x^T\, (\dot{P} + E^T\, P + P\, E)\, x$ so that:

$$\dot{P} + E^T\, P + P\, E = 0 \Longleftrightarrow V(x) = x^T\, P\, x \ \text{ first integral.}$$

$\Rightarrow$ Morphisms from $\widetilde{N}$ to M give quadratic first integrals.

If E is a skew-symmetric matrix, i.e., $E = -E^T$, then we have:

$$(\partial\, I_p + E^T) = (\partial\, I_p - E), \quad \widetilde{N} = M, \quad \hom_D(\widetilde{N}, M) = \mathrm{end}_D(M).$$

Example: Landau & Lifchitz (p. 117)

- Consider $R = \partial\, I_4 - E$, where $E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & \alpha \end{pmatrix}$.

- We find that the morphisms from $\widetilde{N}$ to M are defined by

$$P = \begin{pmatrix} c_1\,\omega^4 & c_2\,\omega^2 & -\omega^2\,(c_1\,\alpha + c_2) & c_1\,\omega^2 \\ -c_2\,\omega^2 & c_1\,\omega^2 & -c_1\,\omega^2 + c_2\,\alpha & -c_2 \\ -\omega^2\,(c_1\,\alpha - c_2) & -c_1\,\omega^2 - c_2\,\alpha & c_1\,(\alpha^2 + \omega^2) & -c_1\,\alpha + c_2 \\ c_1\,\omega^2 & c_2 & -c_1\,\alpha - c_2 & c_1 \end{pmatrix},$$

which leads to the quadratic first integral $V(x) = x^T\, P\, x$:

$$\begin{aligned} V(x) \;=\; & c_1\,\omega^4\,x_1(t)^2 - 2\,x_1(t)\,\omega^2\,x_3(t)\,c_1\,\alpha + 2\,x_1(t)\,c_1\,\omega^2\,x_4(t) \\ & + x_2(t)^2\,c_1\,\omega^2 - 2\,x_2(t)\,c_1\,x_3(t)\,\omega^2 + c_1\,x_3(t)^2\,\alpha^2 \\ & + c_1\,x_3(t)^2\,\omega^2 - 2\,x_3(t)\,x_4(t)\,c_1\,\alpha + c_1\,x_4(t)^2. \end{aligned}$$

Example: A time-varying system

- Let us consider the following time-varying system:

$$\begin{cases} \dot{x}_1(t) + t\, x_2(t) = 0, \\ \dot{x}_2(t) + t\, x_1(t) = 0. \end{cases}$$

- Let us consider the ring $D = \mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$,

$$R = \begin{pmatrix} \dfrac{d}{dt} & t \\ t & \dfrac{d}{dt} \end{pmatrix}, \quad \widetilde{R} = \begin{pmatrix} -\dfrac{d}{dt} & t \\ t & -\dfrac{d}{dt} \end{pmatrix},$$

$M = D^{1\times 2}/(D^{1\times 2}\, R)$ and $\widetilde{N} = D^{1\times 2}/(D^{1\times 2}\, \widetilde{R})$.

- A morphism $f \in \hom_D(\widetilde{N}, M)$ can be defined by (P, P), where:

$$P = \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \quad \alpha, \beta \in \mathbb{Q}.$$

Example: A time-varying system

- Let us denote by $x = (x_1 \;\; x_2)^T$ and $\lambda = (\lambda_1 \;\; \lambda_2)^T$.
- Using the differential relation

$$\lambda^T \, R \, x = x^T \, \widetilde{R} \, \lambda + \frac{d}{dt}(\lambda^T \, x),$$

and the fact that $\lambda = P \, x$ is a solution of $\widetilde{R} \, \lambda = 0$ whenever $R \, x = 0$, we obtain the quadratic first integral of the system:

$$V = x^T \, P \, x = \alpha \, (x_1^2 - x_2^2).$$

Formal adjoint

- Let $D = A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$ be the ring of differential operators with coefficients in A (e.g., $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$).

- The formal adjoint $\widetilde{R} \in D^{p \times q}$ of $R \in D^{q \times p}$ is defined by:

$$< \lambda, R\,\eta > = < \widetilde{R}\,\lambda, \eta > + \sum_{i=1}^{n} \partial_i\, \Phi_i(\lambda, \eta).$$

- The formal adjoint $\widetilde{R}$ can be defined by $\widetilde{R} = (\theta(R_{ij}))^T \in D^{p \times q}$, where $\theta : D \longrightarrow D$ is the involution defined by:

1. $\forall\ a \in A, \quad \theta(a) = a.$
2. $\theta(\partial_i) = -\partial_i, \quad i = 1, \ldots, n.$

Involution: $\theta^2 = \mathrm{id}_D, \quad \forall\ P_1,\ P_2 \in D: \quad \theta(P_1\, P_2) = \theta(P_2)\, \theta(P_1).$

Conservation laws

- Let us consider the left D-modules:

$$M = D^{1\times p}/(D^{1\times q}\,R), \quad \widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R}).$$

- Let $f : \widetilde{N} \rightarrow M$ be a morphism defined by the matrices P and Q.
- Let $\mathcal{F}$ be a left D-module and the commutative exact diagram:

$$\begin{array}{ccccccc}
\mathcal{F}^p & \xleftarrow{\widetilde{R}.} & \mathcal{F}^q & \longleftarrow & \ker_{\mathcal{F}}(\widetilde{R}.) & \longleftarrow & 0 \\
\uparrow Q. & & \uparrow P. & & \uparrow f^\star & & \\
\mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0.
\end{array}$$

- $\eta \in \mathcal{F}^p$ solution of $R\,\eta = 0 \Rightarrow \lambda = P\,\eta$ is a solution of $\widetilde{R}\,\lambda = 0$.

$$\Rightarrow < P\,\eta, R\,\eta > - < \widetilde{R}\,(P\,\eta), \eta >= \sum_{i=1}^{n} \partial_i\,\Phi_i(P\,\eta, \eta) = 0,$$

i.e., $\Phi = (\Phi_1(P\,\eta, \eta), \ldots, \Phi_n(P\,\eta, \eta))^T$ satisfies $\operatorname{div}\Phi = 0$.

Example: A simple mechanical system

- Let us consider the simple mechanical system:

$$\ddot{x} + \left(\frac{\alpha}{m}\right) \dot{x} + \left(\frac{\beta}{m}\right) x = 0.$$

- $D = \mathbb{Q}(\alpha, \beta, m)\,[\partial]$, $R = \left(\partial^2 + \frac{\alpha}{m}\,\partial + \frac{\beta}{m}\right)$, $M = D/(D\,R)$.

- The formal adjoint of R is $\widetilde{R} = \left(\partial^2 - \frac{\alpha}{m}\,\partial + \frac{\beta}{m}\right) \in D$ and:

$$\lambda\,(R\,x) = (\widetilde{R}\,\lambda)\,x + \partial\left(\lambda\,\dot{x} + \left(-\dot{\lambda} + \frac{\alpha}{m}\,\lambda\right)\,x\right).$$

- If we define $\widetilde{N} = D/(D\,\widetilde{R})$, then $f \in \hom_D(\widetilde{N}, M)$ is defined by means of a pair P and $Q \in D$ satisfying $\widetilde{R}\,P = Q\,R$, i.e.:

$$(P \quad -Q)\,\begin{pmatrix} \widetilde{R} \\ R \end{pmatrix} = 0.$$

Example: A simple mechanical system

- Using the fact that D is an euclidean ring, we obtain:

$$\gcd\left(\partial^2 - \frac{\alpha}{m}\,\partial + \frac{\beta}{m}, \partial^2 + \frac{\alpha}{m}\,\partial + \frac{\beta}{m}\right) = \begin{cases} 1, & \text{if } \alpha \neq 0, \\ \partial^2 - \dfrac{\beta}{m}, & \text{if } \alpha = 0. \end{cases}$$

$$\Rightarrow (P \quad - Q) = \begin{cases} T\,(R \quad - \widetilde{R}), & T \in D, \quad \text{if } \alpha \neq 0, \\ T\,(1 \quad - 1), & T \in D, \quad \text{if } \alpha = 0. \end{cases}$$

- Hence, if x is a smooth solution of $R\,x = 0$, then

$$\lambda = R\,x = 0, \quad \text{if} \quad \alpha \neq 0, \quad \lambda = T\,x, \quad \text{if} \quad \alpha = 0,$$

is a solution of $\widetilde{R}\,\lambda = 0$ and we obtain the quadratic first integral:

$$V(x) = 0, \quad \text{if} \quad \alpha \neq 0, \quad (T\,x)\,\dot{x} - (T\,\dot{x})\,x, \quad \text{if} \quad \alpha = 0.$$

We can take $T = a_1\,\partial + a_2$, where $a_1,\ a_2 \in \mathbb{Q}(\beta, m)$ and get:

$$V(x) = (a_1\,\dot{x} + a_2\,x)\,\dot{x} + (-a_1\,\ddot{x} - a_2\,\dot{x})\,x = a_1\left(\dot{x}^2 + \frac{\beta}{m}\,x^2\right).$$

Example: Linear elasticity

- The following wave equation appears in linear elasticity theory:

$$\rho \, \frac{\partial^2 \xi(x,t)}{\partial t^2} - E \, \frac{\partial^2 \xi(x,t)}{\partial x^2} = 0.$$

ρ: mass per volume, E: Young modulus, ξ: displacement.

- $D = \mathbb{Q}(\rho, E)[\partial_t, \partial_x]$, $R = (\rho \, \partial_t^2 - E \partial_x^2) \in D$, $M = D/(D\, R)$.

- We can check that $\widetilde{R} = R$, i.e., R is self-adjoint and:

$$\lambda \, (R \, \xi) = \xi \, (R \, \lambda) + \partial_t \, (\rho \, (\lambda \, \partial_t \, \xi - \xi \, \partial_t \, \lambda)) + \partial_x (\, E \, (\xi \, \partial_x \, \lambda - \lambda \, \partial_x \, \xi)).$$

- As D is a commutative ring and $R \in D$, we have $\mathrm{end}_D(M) = D$

$\Rightarrow$ if ξ is a solution of the system, then so $\lambda = \frac{1}{2} \, \partial_t \, \xi$.

Thus, we obtain the following conservation law:

$$\partial_t \, \left(\frac{\rho}{2} \, (\partial_t \, \xi)^2 + \frac{E}{2} \, (\partial_x \, \xi)^2 \right) + \partial_x \, (-E \, \partial_t \, \xi \, \partial_x \, \xi) = 0.$$

Example: Linear elasticity

- We have the following conservation law:

$$\partial_t \left(\frac{\rho}{2} (\partial_t \xi)^2 + \frac{E}{2} (\partial_x \xi)^2 \right) + \partial_x (-E\, \partial_t \xi\, \partial_x \xi) = 0.$$

- If we call energy $I(t) = \frac{1}{2} \int_a^b \left(\rho\, (\partial_t \xi)^2 + E\, (\partial_x \xi)^2\right) dx$, we get:

$$\frac{d}{dt} I(t) = E\, (\partial_t \xi(b,t)\, \partial_x \xi(b,t) - \partial_t \xi(a,t)\, \partial_x \xi(a,t)).$$

- If we have the boundary conditions $\xi(a,t) = 0$, $\xi(b,t) = 0$, then we get $\partial_t \xi(a,t) = 0$ and $\partial_t \xi(b,t) = 0$, $dI(t)/dt = 0$ and:

$$I(t) = I(0) = \frac{1}{2} \int_a^b \left(\rho\, (\partial_t \xi)^2(x,0) + E\, (\partial_x \xi)^2(x,0)\right) dx.$$

- If $\xi(x,0) = f(x)$ and $(\partial_t \xi)(x,0) = g(x)$, then we obtain:

$$\forall\, t \geq 0, \quad I(t) = \frac{1}{2} \int_a^b \left(\rho\, g(x)^2 + E\, \dot{f}(x)^2\right) dx.$$

Example: Hydrodynamics

- The movement of an incompressible rotating fluid with a rotation axis lies along the x_3 axis and a small velocity is defined by:

$$\begin{cases} \rho_0 \dfrac{\partial u_1}{\partial t} - 2\,\rho_0\,\Omega_0\,u_2 + \dfrac{\partial p}{\partial x_1} = 0, \\ \rho_0 \dfrac{\partial u_2}{\partial t} + 2\,\rho_0\,\Omega_0\,u_1 + \dfrac{\partial p}{\partial x_2} = 0, \\ \rho_0 \dfrac{\partial u_3}{\partial t} + \dfrac{\partial p}{\partial x_3} = 0, \\ \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3} = 0, \end{cases}$$

$u = (u_1 \;\; u_2 \;\; u_3)^T$: local rate of velocity, p: pressure, ρ_0: constant fluid density, Ω_0: constant angle speed.

- We have: $$R = \begin{pmatrix} \rho_0\,\partial_t & -2\,\rho_0\,\Omega_0 & 0 & \partial_1 \\ 2\,\rho_0\,\Omega_0 & \rho_0\,\partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0\,\partial_t & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix} = -\widetilde{R}.$$

Example: Hydrodynamics

- $\widetilde{R} = -R$ implies that if $(\vec{u}, p)$ is a solution of the system, so is:

$$\lambda_1 = u_1, \quad \lambda_2 = u_2, \quad \lambda_3 = u_3, \quad \lambda_4 = p.$$

- Denote by $\xi = (u_1 \;\; u_2 \;\; u_2 \;\; p)^T$. We have the identity:

$$(\lambda, R\,\xi) = (\xi, \widetilde{R}\,\lambda) + \begin{pmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} \rho_0\,(\lambda_1\,u_1 + \lambda_2\,u_2 + \lambda_3\,u_3) \\ \lambda_1\,p + \lambda_4\,u_1 \\ \lambda_2\,p + \lambda_4\,u_2 \\ \lambda_3\,p + \lambda_4\,u_3 \end{pmatrix}$$

- If we take $\lambda = \xi$, then we get $\widetilde{R}\,\lambda = 0$ and

$$\frac{\partial}{\partial t}\,(\rho_0\,(u_1^2 + u_2^2 + u_3^2)) + \frac{\partial}{\partial x_1}\,(2\,p\,u_1) + \frac{\partial}{\partial x_2}\,(2\,p\,u_2) + \frac{\partial}{\partial x_3}\,(2\,p\,u_3) = 0,$$

i.e., we obtain the conservation of law:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\,\rho_0\,\parallel \vec{u} \parallel^2\right) + \mathrm{div}\,(p\,\vec{u}) = 0.$$

Example: Electromagnetism

- Let us consider the Maxwell equations in the vaccum:

$$\left\{\begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \dfrac{1}{\mu_0}\, \vec{\nabla} \wedge \vec{B} - \epsilon_0\, \dfrac{\partial \vec{E}}{\partial t} = \vec{0}, \end{array}\right.$$

$\vec{B}$ (resp., $\vec{E}$): magnetic (resp., electric) field, μ_0 (resp., ϵ_0): magnetic (resp., electric) constant.

- Let us consider $D = \mathbb{Q}(\mu_0, \epsilon_0)[\partial_t, \partial_1, \partial_2, \partial_3]$ and the matrix:

$$R = \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_t & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_t & -\partial_2 & \partial_1 & 0 \\ 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 & -\epsilon_0\, \partial_t & 0 & 0 \\ \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 & 0 & -\epsilon_0\, \partial_t & 0 \\ -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 & 0 & 0 & -\epsilon_0\, \partial_t \end{pmatrix}.$$

Example: Electromagnetism

- We have:

$$\widetilde{R} = \begin{pmatrix} -\partial_t & 0 & 0 & 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 \\ 0 & -\partial_t & 0 & \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 \\ 0 & 0 & -\partial_t & -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 \\ 0 & -\partial_3 & \partial_2 & \epsilon_0\,\partial_t & 0 & 0 \\ \partial_3 & 0 & -\partial_1 & 0 & \epsilon_0\,\partial_t & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & \epsilon_0\,\partial_t \end{pmatrix}.$$

- $\xi = (B_1 \;\; B_2 \;\; B_3 \;\; E_1 \;\; E_2 \;\; E_3)^T, \quad \lambda = (C_1 \;\; C_2 \;\; C_3 \;\; F_1 \;\; F_2 \;\; F_3)^T.$

- We have the differential relation:

$$(\lambda, R\,\xi) = (\xi, \widetilde{R}\,\lambda) + \partial_t \left(\sum_{i=1}^3 C_i\,B_i - \epsilon_0 \sum_{i=1}^3 F_i\,E_i\right)$$

$$+\vec{\nabla}. \begin{pmatrix} C_3\,E_2 - C_2\,E_3 + (F_3\,B_2 - F_2\,B_3)/\mu_0 \\ C_1\,E_3 - C_3\,E_1 + (F_1\,B_3 - F_3\,B_1)/\mu_0 \\ C_2\,E_1 - C_1\,E_2 + (F_2\,B_1 - F_1\,B_2)/\mu_0 \end{pmatrix}.$$

Example: Electromagnetism

- Let us consider $M = D^{1\times 6}/(D^{1\times 6}\, R)$ and $\widetilde{N} = D^{1\times 6}/(D^{1\times 6}\, \widetilde{R})$.
- A morphism $f \in \hom_D(\widetilde{N}, M)$ can be defined by:

$$P = \begin{pmatrix} 1/\mu_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mu_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q = -P.$$

- If ξ is a solution of the system, then $\lambda = P\,\xi$, i.e.,

$$C_i = B_i/\mu_0, \quad F_i = -E_i, \quad i = 1, 2, 3,$$

is a solution of $\widetilde{R}\,\lambda = 0$. Then, we obtain the conservation law:

$$\partial_t \left(\frac{1}{\mu_0} \parallel \vec{B} \parallel^2 + \epsilon_0 \parallel \vec{E} \parallel^2 \right) + \mathrm{div} \left(\frac{1}{\mu_0} (\vec{E} \wedge \vec{B}) \right) = 0.$$

- $\omega = \parallel \vec{B} \parallel^2 /\mu_0 + \epsilon_0 \parallel \vec{E} \parallel^2$: electromagnetic energy,
- $\Pi = (\vec{E} \wedge \vec{B})/\mu_0$: Poynting vector.

A Lax pair for the KdV equation

- Let us consider the differential ring $\mathbb{Q}\{u\}$ formed by differential polynomials in u, the prime differential ideal of $\mathbb{Q}\{u\}$ defined by

$$\mathfrak{p} = \left\{ \frac{\partial u}{\partial t} - 6\, u \left(\frac{\partial u}{\partial x} \right) + \frac{\partial^3 u}{\partial x^3} \right\},$$

the differential ring $L = \mathbb{Q}\{u\}/\mathfrak{p}$ and $K = \{n/d \mid 0 \neq d,\, n \in L\}$, i.e., the differential field defined by the KdV equation.

- Let us consider the rings $A = K\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$, $D = A\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]$,

$$\left\{ \begin{array}{l} E = -4\, \partial_x^3 + 6\, u\, \partial_x + 3 \left(\dfrac{\partial\, u}{\partial\, x} \right) \in D, \\ R = \partial_t - E \in D, \end{array} \right. \qquad M = D/(D\, R).$$

- The Schrödinger operator $P = -\partial_x^2 + u$ with potential u satisfies:

$$R\, P - P\, R = \partial_t\, P - E\, P + P\, E = \frac{\partial u}{\partial t} - 6\, u \left(\frac{\partial u}{\partial x} \right) - \frac{\partial^3 u}{\partial x^3} = 0.$$

A Lax pair for the KdV equation

In the inverse scattering theory, a key point is that the smooth one-parameter family of differential operators

$$t \longmapsto -\partial_x^2 + u(x,t)$$

defines an isospectral flow on the solutions of $\partial_t \, \eta = E \, \eta$:

$$(-\partial_x^2 + u(x,0)) \, \psi(x) = \lambda \, \psi(x),$$

$$\begin{cases} \partial_t \, \eta(x,t) = E \, \eta(x,t), \quad E = -4 \, \partial_x^3 + 6 \, u \, \partial_x + 3 \left(\dfrac{\partial \, u}{\partial \, x} \right), \\ \eta(x,0) = \psi(x), \end{cases}$$

$$\Rightarrow (-\partial_x^2 + u(x,t)) \, \eta(x,t) = \lambda \, \eta(x,t),$$

$\Rightarrow$ the inverse scattering method proves that the KdV equation is completely integrable.

IV. Factorization of linear functional systems

Kernel and factorization

$$\begin{array}{ccccccc} & & \lambda & \longmapsto & y & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\ \exists\, \mu & \longmapsto & \mu\, R' = \lambda\, P & \longmapsto & 0 & & \end{array}$$

- $\ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = D^{1\times r}\,(S \quad -T)$

$$\Rightarrow \{\lambda \in D^{1\times p} \,|\, \lambda\, P \in D^{1\times q}\, R'\} = D^{1\times r}\, S$$

$$\Rightarrow \ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R).$$

- $(D^{1\times q}\,(R \quad -Q)) \in \ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) \Rightarrow (D^{1\times q}\, R) \subseteq (D^{1\times r}\, S).$

$$\exists\, V \in D^{q\times r}: \quad R = V\, S.$$

Kernel and factorization

We have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \ker f & & \\
 & & & & \downarrow i & & \\
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .V & & \| & & \downarrow \kappa & & \\
D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\pi'} & M/\ker f & \longrightarrow & 0. \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Example: Linearized Euler equations

- Let $R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix}$ over $D = \mathbb{R}[\partial_1, \partial_2, \partial_3, \partial_t]$.
- Let us consider $f \in \mathrm{end}_D(M)$ defined by:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \partial_3^2 & -\partial_2\,\partial_3 & 0 \\ 0 & -\partial_2\,\partial_3 & \partial_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Computing $\ker_D\left(.\begin{pmatrix} P \\ R \end{pmatrix}\right)$ and factorizing R by S, we obtain:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & 0 \\ 0 & -\partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \partial_1 & 1 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & -1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix}.$$

Example: Linearized Euler equations

- We have $R = V\,S$ where:

$$\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} = \begin{pmatrix} \partial_1 & 1 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & -1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & 0 \\ 0 & -\partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The solutions of $S\,y = 0$ are particular solutions of $R\,y = 0$.

$\Rightarrow$ Integrating S, we obtain the following solutions of $R\,y = 0$:

$$y(x_1, x_2, x_3, t) = \begin{pmatrix} 0 \\ -\frac{\partial}{\partial x_3}\,\xi(x_1,\, x_2,\, x_3) \\ \frac{\partial}{\partial x_2}\,\xi(x_1,\, x_2,\, x_3) \\ 0 \end{pmatrix}, \quad \forall\, \xi \in C^{\infty}(\Omega).$$

Ker f, im f, coim f and coker f

• Proposition: Let $M = D^{1\times p}/(D^{1\times q}\,R)$, $M' = D^{1\times p'}/(D^{1\times q'}\,R')$ and $f : M \longrightarrow M'$ be a morphism defined by $R\,P = Q\,R'$.

Let us consider the matrices $S \in D^{r\times p}$, $T \in D^{r\times q'}$, $U \in D^{s\times r}$ and $V \in D^{q\times r}$ satisfying $R = V\,S$, $\ker_D(.S) = D^{1\times s}\,U$ and:

$$\ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = D^{1\times r}\,(S \quad -T).$$

Then, we have:

- $\ker f = (D^{1\times r}\,S)/(D^{1\times q}\,R) \cong D^{1\times l}/\left(D^{1\times(q+s)}\begin{pmatrix} U \\ V \end{pmatrix}\right)$,
- $\operatorname{coim} f \triangleq M/\ker f = D^{1\times p}/(D^{1\times r}\,S)$,
- $\operatorname{im} f = D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}/(D^{1\times q}\,R) \cong D^{1\times p}/(D^{1\times r}\,S)$,
- $\operatorname{coker} f \triangleq M'/\operatorname{im} f = D^{1\times p'}/\left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right)$.

Equivalence of systems

- Corollary: Let us consider $f \in \hom_D(M, M')$. Then, we have:

1. f is injective iff one of the assertions holds:
   - There exists $L \in D^{r \times q}$ such that $S = L\, R$.
   - $\begin{pmatrix} U \\ V \end{pmatrix}$ admits a left-inverse.
2. f is surjective iff $\begin{pmatrix} P \\ R' \end{pmatrix}$ admits a left-inverse.
3. f is an isomorphism, i.e., $M \cong M'$, iff 1 and 2 are satisfied.

Pommaret's example

- Equivalence of the systems defined by the following R and R'?

$$R = \begin{pmatrix} \partial_1^2\,\partial_2^2 - 1 & -\partial_1\,\partial_2^3 - \partial_2^2 \\ \partial_1^3\,\partial_2 - \partial_1^2 & -\partial_1^2\,\partial_2^2 \end{pmatrix}, \quad R' = (\partial_1\,\partial_2 - 1 \quad -\partial_2^2).$$

- We find a morphism given by $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 + \partial_1\,\partial_2 \\ \partial_1^2 \end{pmatrix}$.

- $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 + \partial_1\,\partial_2 \\ \partial_1^2 \end{pmatrix}$ admits the left-inverse $(1 - \partial_1\,\partial_2 \quad \partial_2^2)$.

- $\begin{pmatrix} P \\ R' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1\,\partial_2 - 1 & -\partial_2^2 \end{pmatrix}$ admits the left-inverse $(I_2 \quad 0)$.

$$\Rightarrow M = D^{1\times 2}/(D^{1\times 2}\,R) \cong M' = D^{1\times 2}/(D\,R').$$

Free modules & similarity transformations

- **Definition**: A left D-module M is free if there exists $l \in \mathbb{Z}_+$ s.t.:

$$M \cong D^{1 \times l}.$$

- Proposition: Let $P \in D^{p \times p}$. We have the equivalences:

1. $\ker_D(.P)$ and $\mathrm{coim}_D(.P)$ are free left D-modules of rank p and $p - m$.

2. There exists a unimodular matrix $U \in D^{p \times p}$, i.e., $U \in \mathrm{GL}_p(D)$, such that:

$$J \triangleq U\, P\, U^{-1} = \begin{pmatrix} 0 \\ J_2 \end{pmatrix}, \quad J_2 \in D^{(p-m) \times p}.$$

$$\Rightarrow U = (U_1^T \quad U_2^T)^T, \quad \begin{cases} \ker_D(.P) = D^{1 \times m}\, U_1 \\ \mathrm{coim}_D(.P) = \pi'(D^{1 \times (p-m)}\, U_2). \end{cases}$$

A useful proposition

- Proposition: Let $R \in D^{q \times p}$ and $P \in D^{p \times p}$, $Q \in D^{q \times q}$ be two matrices satisfying:

$$R\,P = Q\,R.$$

Let $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that

$$\begin{cases} P = U^{-1}\,J_P\,U, \\ Q = V^{-1}\,J_Q\,V, \end{cases}$$

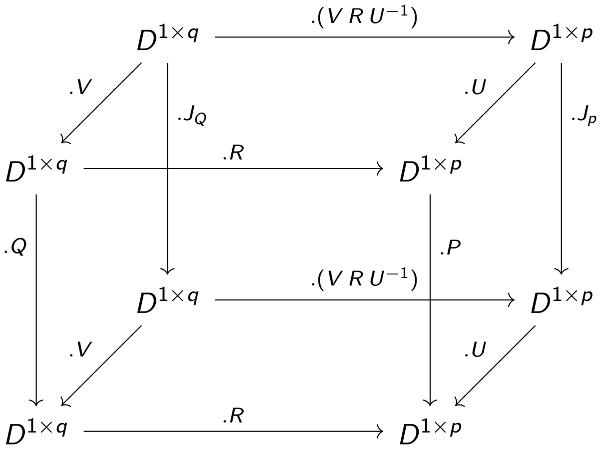
for certain $J_P \in D^{p \times p}$ and $J_Q \in D^{q \times q}$.

Then, the matrix $\overline{R} = V\,R\,U^{-1}$ satisfies:

$$\overline{R}\,J_P = J_Q\,\overline{R}.$$

A commutative diagram

The following commutative diagram



implies $(V\,R\,U^{-1})\,J_p = J_Q\,(V\,R\,U^{-1})$.

Block triangular decomposition

- Theorem: Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying $R\,P = Q\,R$.

If the left D-modules

$$\ker_D(.P), \quad \mathrm{coim}_D(.P), \quad \ker_D(.Q), \quad \mathrm{coim}_D(.Q)$$

are free of rank m, $p-m$, l, $q-l$, then there exist two matrices $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$ and $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$ such that

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ V_2\,R\,W_1 & V_2\,R\,W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Example: OD system

- Let $D = k[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $M = D^{1\times 4}/(D^{1\times 4}\, R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}.$$

- An endomorphism f of M is defined by:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

- We can prove that the left D-modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$ are free of rank 2.

Example: OD system

- We obtain:

$$\begin{cases} U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t-1 & -t \end{pmatrix}, & V_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \end{cases}$$

$$\Rightarrow U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t-1 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

we then obtain that R is equivalent to:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} -\partial & 1 & 0 & 0 \\ t\,\partial - t & -\partial - t & 0 & 0 \\ \partial + t & \partial - 1 & \partial & -t \\ -\partial & 1 & 0 & \partial \end{pmatrix}.$$

Example: Tank model

- We consider $D = \mathbb{Q}\left[\frac{d}{dt}, \delta\right]$ and:

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\frac{d}{dt}\,\delta \\ 1 & \delta^2 & -2\,\frac{d}{dt}\,\delta \end{pmatrix}.$$

- A endomorphism f of M is defined by:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 2\,\frac{d}{dt}\,\delta & -2\,\frac{d}{dt}\,\delta & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 2\,\frac{d}{dt}\,\delta & -2\,\frac{d}{dt}\,\delta \end{pmatrix}.$$

- We can check that $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$ are free D-modules of rank respectively 2, 1, 1, 1.

$$\Rightarrow \left\{ \begin{array}{ll} U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\,\frac{d}{dt}\,\delta \end{pmatrix}, & U_2 = (0 \quad 0 \quad 1), \\ V_1 = (1 \quad 0), & V_2 = (0 \quad 1). \end{array} \right.$$

Example: Tank model

- If we denote by

$$U=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\,\frac{d}{dt}\,\delta \\ 0 & 0 & 1 \end{pmatrix}, \quad V=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we then obtain

$$U^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\,\frac{d}{dt}\,\delta \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D),$$

and the matrix R is equivalent to:

$$\overline{R}=V\,R\,U^{-1}=\begin{pmatrix} \delta^2 & -1 & 0 \\ 1 & -\delta^2 & 2\,\frac{d}{dt}\,\delta\,(\delta^2-1) \end{pmatrix}.$$

Exemple: Electromagnetism

$$\sigma\,\partial_t\,\vec{A}+\frac{1}{\mu}\,\vec{\nabla}\wedge\vec{\nabla}\,\vec{A}-\sigma\,\vec{\nabla}\,V=0$$

$$\Rightarrow R=\begin{pmatrix} \sigma\,\partial_t-\frac{1}{\mu}\,(\partial_2^2+\partial_3^2) & \frac{1}{\mu}\,\partial_1\,\partial_2 & \frac{1}{\mu}\,\partial_1\,\partial_3 & -\sigma\,\partial_1 \\ \frac{1}{\mu}\,\partial_1\,\partial_2 & \sigma\,\partial_t-\frac{1}{\mu}\,(\partial_1^2+\partial_3^2) & \frac{1}{\mu}\,\partial_2\,\partial_3 & -\sigma\,\partial_2 \\ \frac{1}{\mu}\,\partial_1\,\partial_3 & \frac{1}{\mu}\,\partial_2\,\partial_3 & \sigma\,\partial_t-\frac{1}{\mu}\,(\partial_1^2+\partial_2^2) & -\sigma\,\partial_3 \end{pmatrix}.$$

- Let $D=\mathbb{Q}[\partial_t,\partial_1,\partial_2,\partial_3]$ and $M=D^{1\times 4}/(D^{1\times 3}\,R)$.

$$P=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma\,\mu\,\partial_t & 0 & -\sigma\,\mu\,\partial_2 \\ 0 & 0 & \sigma\,\mu\,\partial_t & -\sigma\,\mu\,\partial_3 \\ 0 & \partial_t\,\partial_2 & \partial_t\,\partial_3 & -(\partial_2^2+\partial_3^2) \end{pmatrix},$$

$$Q=\begin{pmatrix} 0 & 0 & 0 \\ -\partial_1\,\partial_2 & \sigma\,\mu\,\partial_t-\partial_2^2 & -\partial_2\,\partial_3 \\ -\partial_1\,\partial_3 & -\partial_2\,\partial_3 & \sigma\,\mu\,\partial_t-\partial_3^2 \end{pmatrix},$$

satisfy $R\,P=Q\,R$ and define an morphism $f\in\mathrm{end}_D(M)$.

Exemple: Electromagnetism

- The modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$, $\mathrm{coim}_D(.Q)$ are free D-modules and:

$$\begin{cases} \ker_D(.P) = D^{1\times 2}\, U_1, \quad U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & -\sigma\,\mu \end{pmatrix}, \\ \mathrm{coim}_D(.P) = D^{1\times 2}\, U_2, \quad U_2 = \dfrac{1}{\sigma\,\mu} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \ker_D(.Q) = D^{1\times 2}\, V_1, \quad V_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \\ \mathrm{coim}_D(.Q) = D^{1\times 2}\, V_2, \quad V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{cases}$$

- The matrix R is then equivalent to $\overline{R} = V\,R\,U^{-1}$ defined by:

$$\overline{R} = \begin{pmatrix} \sigma\,\partial_t - \frac{1}{\mu}\,(\partial_2^2 + \partial_3^2) & \frac{1}{\mu}\,\partial_1 & 0 & 0 \\ \frac{1}{\mu}\,\partial_1\,\partial_2 & \frac{1}{\mu}\,\partial_2 & \sigma\,(\sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) & 0 \\ \frac{1}{\mu}\,\partial_1\,\partial_3 & \frac{1}{\mu}\,\partial_3 & 0 & \sigma\,(\sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) \end{pmatrix}.$$

V. Decomposition of linear functional systems

Projectors of $\mathrm{end}_D(M)$

• Lemma: An endomorphism f of $M = D^{1\times p}/(D^{1\times q}\, R)$, defined by the matrices P and Q, is a projector, i.e., $f^2 = f$, iff there exist $Z \in D^{p\times q}$ and $Z' \in D^{q\times t}$ such that

$$\begin{cases} P^2 = P + Z\, R, \\ Q^2 = Q + R\, Z + Z'\, R_2, \end{cases}$$

where $R_2 \in D^{t\times q}$ satisfies $\ker_D(.R) = D^{1\times t}\, R_2$.

• Some projectors of $\mathrm{end}_D(M)$ can be computed when a family of endomorphisms of M is known.

• **Example:** $D = A_1(\mathbb{Q})$, $R = (\partial^2 \quad -t\,\partial - 1)$, $M = D^{1\times 2}/(D\, R)$.

$$P = \begin{pmatrix} -(t+a)\,\partial + 1 & t^2 + a\,t \\ 0 & 1 \end{pmatrix}, \quad P^2 = P + \begin{pmatrix} (t+a)^2 \\ 0 \end{pmatrix} R.$$

Projectors of $\mathrm{end}_D(M)$ & Idempotents

- Particular case: $(R_2 = 0 \text{ and } P^2 = P) \Longrightarrow Q^2 = Q$.

- Lemma: Let us suppose that $R_2 = 0$ and $P^2 = P + Z\, R$. If there exists a solution $\Lambda \in D^{p \times q}$ of the Riccati equation

$$\Lambda\, R\, \Lambda + (P - I_p)\, \Lambda + \Lambda\, Q + Z = 0, \quad (\star)$$

then the matrices $\overline{P} = P + \Lambda\, R$ and $\overline{Q} = Q + R\, \Lambda$ satisfy:

$$R\, \overline{P} = \overline{Q}\, R, \quad \overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}.$$

- **Example:** $\Lambda = (a\, t \quad a\, \partial - 1)^T$ is a solution of $(\star)$

$$\Rightarrow \overline{P} = \begin{pmatrix} a\, t\, \partial^2 - (t + a)\, \partial + 1 & t^2\, (1 - a\, \partial) \\ (a\, \partial - 1)\, \partial^2 & -a\, t\, \partial^2 + (t - 2\, a)\, \partial + 2 \end{pmatrix}, \overline{Q} = 0,$$

then satisfy $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$.

- Proposition: f is a projector of $\mathrm{end}_D(M)$, i.e., $f^2 = f$, iff there exists a matrix $X \in D^{p \times s}$ such that $P = I_p - X\,S$ and we have the following commutative exact diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \ker f & & \\
 & & & & & & \downarrow i & & \\
 & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0\\
 & & .T \uparrow\downarrow .V & & .P \uparrow\downarrow .I_p & & f \uparrow\downarrow \kappa & & \\
D^{1\times s} & \xrightarrow{.U} & D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\pi'} & M/\ker f & \longrightarrow & 0.\\
 & & & \xleftarrow{.X} & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

$$\Rightarrow M \cong \ker f \oplus \operatorname{im} f \quad \& \quad S - S\,X\,S = T\,R. \quad (\star)$$

- Corollary: If $\ker_D(.S) = 0$, then $R = V\,S$ satisfies:

$$S\,X - T\,V = I_r.$$

Decomposition of solutions

• Corollary: Let us suppose that $\mathcal{F}$ is an injective left D-module. Then, we have the following commutative exact diagram:

$$\begin{array}{ccccccccccc}
 & & & & V\,z = 0 = R\,y & \longleftarrow\!\shortmid & y & & & & \\
 & & & & \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\
 & & & & \uparrow V. & & \| & & \uparrow f^\star & & \\
\mathcal{F}^s & \xleftarrow{U.} & & & \mathcal{F}^r & \underset{\xrightarrow{X.}}{\xleftarrow{S.}} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(S.) & \longleftarrow & 0. \\
0 = U\,z & \longleftarrow\!\shortmid & & & z = S\,y & \longleftarrow\!\shortmid & y & & & &
\end{array}$$

Moreover, we have: $\quad R\,y = 0 \quad \Leftrightarrow \quad \begin{pmatrix} U \\ V \end{pmatrix} z = 0, \quad S\,y = z.$

General solution: $y = u + X\,z$ where $S\,u = 0$ and $\begin{pmatrix} U \\ V \end{pmatrix} z = 0.$

Example: OD system

- Let $D = k[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $M = D^{1\times 4}/(D^{1\times 4}\, R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}.$$

- We obtain the following idempotent:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in k^{4\times 4} : \quad P^2 = P.$$

- We obtain the factorization $R = V\, S$, where:

$$S = \begin{pmatrix} \partial & -t & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & t & \partial \\ 1 & t & \partial & -1 \\ 1 & 0 & \partial + t & \partial - 1 \\ 1 & 1 & t & \partial \end{pmatrix}.$$

Example

- Using the fact that we must have $I_p - P = X\,S$, we then obtain:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R\,y = 0 \Leftrightarrow y = u + X\,z : \quad V\,z = 0, \quad S\,u = 0.$$

- The solution of $S\,u = 0$ is defined by:

$$u_1 = \frac{1}{2}\,C_1\,t^2 + C_2, \quad u_2 = C_1, \quad u_3 = 0, \quad u_4 = 0.$$

- The solution of $V\,z = 0$ is defined by: $z_1 = 0$, $z_2 = 0$ and

$$z_3(t) = C_3\,\mathrm{Ai}(t) + C_4\,\mathrm{Bi}(t), \quad z_4(t) = C_3\,\partial\,\mathrm{Ai}(t) + C_4\,\partial\,\mathrm{Bi}(t).$$

- The general solution of $R\,y = 0$ is then given by:

$$y = u + X\,z = \left(\frac{1}{2}\,C_1\,t^2 + C_2 \quad C_1 \quad z_3(t) \quad z_4(t)\right)^T.$$

Idempotents & Projective modules

- **Definition:** A left D-module M is projective if there exists a left D-module N and $l \in \mathbb{Z}_+$ such that $M \oplus N \cong D^{1 \times l}$.

- Lemma: If $P \in D^{p \times p}$ is an idempotent, i.e., $P^2 = P$, then:
  - $\ker_D(.P)$ and $\mathrm{im}_D\,(.P)$ are projective left D-modules of rank m and $p - m$.
  - $\mathrm{im}_D(.P) = \ker_D(.(I_p - P))$.

- Proposition: Let $P \in D^{p \times p}$ be an idempotent. $1 \Leftrightarrow 2$:
  1. $\ker_D(.P)$ and $\mathrm{im}_D\,(.P)$ are free modules of rank m and $p - m$.
  2. $\exists\, U \in \mathrm{GL}_p(D)$ satisfying $U\,P\,U^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix}$

$$\Rightarrow U = (U_1^T \quad U_2^T)^T, \quad \begin{cases} \ker_D(.P) = D^{1 \times m}\, U_1, \\ \mathrm{im}_D(.P) = D^{1 \times (p-m)}\, U_2. \end{cases}$$

Block diagonal decomposition

- Theorem: Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying:

$$P^2 = P, \quad Q^2 = Q.$$

If the left D-modules

$$\ker_D(.P), \quad \mathrm{im}_D(.P), \quad \ker_D(.Q), \quad \mathrm{im}_D(.Q)$$

are free of rank m, $p - m$, l, $q - l$, then there exist two matrices $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$ and $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$ such that

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} V_1\, R\, W_1 & 0 \\ 0 & V_2\, R\, W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Example: Two pendulum on a car

- Let us consider the linearized equations of two pendulum:

$$\begin{cases} \ddot{y}_1 + \dfrac{g}{l_1}\, y_1 - \dfrac{g}{l_1}\, u = 0, \\ \ddot{y}_2 + \dfrac{g}{l_2}\, y_2 - \dfrac{g}{l_2}\, u = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(g, l_1, l_2)\left[\frac{d}{dt}\right]$, the system matrix

$$R = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l_1} & 0 & -\frac{g}{l_1} \\ 0 & \frac{d^2}{dt^2} + \frac{g}{l_2} & -\frac{g}{l_2} \end{pmatrix} \in D^{2\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$.

Example: Two pendulum on a car

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents $Q = 0$,

$$P = \frac{1}{g\,(l_2 - l_1)} \begin{pmatrix} -l_1 \left(l_2 \frac{d^2}{dt^2} + g\right) & l_2 \left(l_2 \frac{d^2}{dt^2} + g\right) & 0 \\ -l_1 \left(l_1 \frac{d^2}{dt^2} + g\right) & l_2 \left(l_1 \frac{d^2}{dt^2} + g\right) & 0 \\ -\frac{l_1}{g} \left(l_2 \frac{d}{dt}^2 + g\right) \left(l_1 \frac{d}{dt}^2 + g\right) & \frac{l_2}{g} \left(l_2 \frac{d}{dt}^2 + g\right) \left(l_1 \frac{d}{dt}^2 + g\right) & 0 \end{pmatrix}.$$

- The modules $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are free with bases:

$$\begin{cases} \ker_D(.P) = D^{1\times 2}\, U_1, \quad U_1 = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l_1} & 0 & -\frac{g}{l_1} \\ 0 & \frac{d^2}{dt^2} + \frac{g}{l_2} & -\frac{g}{l_2} \end{pmatrix}, \\ \mathrm{im}_D(.P) = D\, U_2, \qquad U_2 = (l_1 \quad -l_2 \quad 0), \end{cases}$$

$$\Rightarrow R\, U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_3(D).$$

Example: Two pendulum of the same length on a car

- Let us consider the linearized equations of two pendulum:

$$\begin{cases} \ddot{y}_1 + \dfrac{g}{l}\, y_1 - \dfrac{g}{l}\, u = 0, \\ \ddot{y}_2 + \dfrac{g}{l}\, y_2 - \dfrac{g}{l}\, u = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(g, l)\left[\frac{d}{dt}\right]$, the system matrix

$$R = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} & 0 & -\frac{g}{l} \\ 0 & \frac{d^2}{dt^2} + \frac{g}{l} & -\frac{g}{l} \end{pmatrix} \in D^{2\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$.

Example: Two pendulum of the same length on a car

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

- $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free modules:

$$\left\{ \begin{array}{ll} \ker_D(.P) = D^{1\times 2}\, U_1, & U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathrm{im}_D(.P) = D\, U_2, & U_2 = (1 \quad -1 \quad 0), \end{array} \right. \quad U = (U_1^T \quad U_2^T)^T,$$

$$\left\{ \begin{array}{ll} \ker_D(.Q) = D\, V_1, & V_1 = (1 \quad 0), \\ \mathrm{im}_D(.Q) = D\, V_2, & V_2 = (1 \quad -1), \end{array} \right. \quad V = (V_1^T \quad V_2^T)^T,$$

$$\Rightarrow \overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} & -\frac{g}{l} & 0 \\ 0 & 0 & \frac{d^2}{dt^2} + \frac{g}{l} \end{pmatrix}.$$

Example: OD system

- Let us consider the matrix again:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix}.$$

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

i.e., P and Q satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

Example: OD system

- Computing bases of the left D-modules

$$\ker_D(.P), \quad \mathrm{im}_D(.P), \quad \ker_D(.P), \quad \mathrm{im}_D(.Q),$$

we obtain the unimodular matrices:

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -t & -1 & 1 & t \\ t+1 & 1 & -1 & -t \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

- R is then equivalent to the following block diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial & -1 & 0 & 0 \\ t & \partial & 0 & 0 \\ 0 & 0 & \partial & -t \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

Example: Cauchy-Riemann equations

- Let us consider the Cauchy-Riemann equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0, \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

- $D = \mathbb{Q}(i)[\partial_x, \partial_y]$, $R = \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}$, $M = D^{1\times 2}/(D^{1\times 2}\, R)$.

- The matrices P and Q defined by $P = Q = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

satisfy $R\,P = P\,R$ and $P^2 = P$, i.e., define a projector.

$$\begin{cases} \ker_{\mathbb{Q}(i)}(.P) = \mathbb{Q}(i)\,(1 \quad -i), \\ \mathrm{im}_{\mathbb{Q}(i)}(.P) = \mathbb{Q}(i)\,(1 \quad i), \end{cases} \Rightarrow U = V = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in \mathrm{GL}_2(D).$$

$$\Rightarrow \overline{R} = U\,R\,U^{-1} = \begin{pmatrix} \partial_x - i\,\partial_y & 0 \\ 0 & \partial_x + i\,\partial_y \end{pmatrix} = 2\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}.$$

Example: A classical system of PDEs

- Let us consider the following important system of PDEs:

$$\begin{cases} \dfrac{\partial y_1}{\partial x} + a\,\dfrac{\partial y_2}{\partial t} = 0, \\ \dfrac{\partial y_1}{\partial t} + b\,\dfrac{\partial y_2}{\partial x} = 0. \end{cases}$$

- Acoustic wave: $y_1 = u$, $y_2 = p$, $a = 1/\rho$, $b = \rho\, c^2$.
- LC transmission line: $y_1 = v$, $y_2 = i$, $a = L$, $b = 1/C$.
- $D = \mathbb{Q}(a, b)[\partial_x, \partial_t]$, $R = \begin{pmatrix} \partial_x & a\,\partial_t \\ \partial_t & b\,\partial_x \end{pmatrix}$, $M = D^{1\times 2}/(D^{1\times 2}\, R)$.
- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents

$$P = \frac{1}{2} \begin{pmatrix} 1 & 2\,a\,b\,\alpha \\ 2\,\alpha & 1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 2\,a\,\alpha \\ 2\,b\,\alpha & 1 \end{pmatrix},$$

where α satisfies $4\,a\,b\,\alpha^2 - 1 = 0$.

Example: A classical system of PDEs

- Let us denote by $D' = \mathbb{Q}(a, b, \alpha)/(4\, a\, b\, \alpha^2 - 1)[\partial_x, \partial_t]$.
- $\ker_{D'}(.P)$, $\mathrm{im}_{D'}(.P)$, $\ker_{D'}(.Q)$ and $\mathrm{im}_{D'}(.Q)$ are free with bases:

$$\begin{cases} \ker_{D'}(.P) = D'\, U_1, & U_1 = (-2\,\alpha \quad 1)\,, \\ \mathrm{im}_{D'}(.P) = D'\, U_2, & U_2 = (2\,\alpha \quad 1)\,. \end{cases}$$

$$\begin{cases} \ker_{D'}(.Q) = D'\, V_1, & V_1 = (2\, b\, \alpha \quad -1)\,, \\ \mathrm{im}_{D'}(.Q) = D'\, V_2, & V_2 = (2\, b\, \alpha \quad 1)\,. \end{cases}$$

- $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_2(D')$, $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_2(D')$.
- The matrix R is equivalent to $(1/(2\,\alpha) = \sqrt{a\, b})$:

$$\overline{R} = V\, R\, U^{-1} = \left(\begin{array}{cc} -b\, \partial_x + \frac{1}{2\,\alpha}\, \partial_t & 0 \\ 0 & b\, \partial_x + \frac{1}{2\,\alpha} \partial_t \end{array} \right).$$

Example: Transmission line

- Let us consider the following transmission line:

$$\begin{cases} \dfrac{\partial V}{\partial x} + L\,\dfrac{\partial I}{\partial t} + R\,I = 0, \\ C\,\dfrac{\partial V}{\partial t} + G\,V + \dfrac{\partial I}{\partial x} = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(C, G, L, R)[\partial_x, \partial_t]$,

$$S = \begin{pmatrix} \partial_x & L\,\partial_t + R \\ C\,\partial_t + G & \partial_x \end{pmatrix}, \quad M = D^{1\times 2}/(D^{1\times 2}\,S).$$

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotent

$$P = \frac{1}{C\,R - L\,G} \begin{pmatrix} C\,L\,\partial_t - \alpha\,\partial_x + C\,R & L\,\partial_x - \alpha\,L\,\partial_t - \alpha\,R \\ \alpha\,C\,\partial_t - C\,\partial_x + \alpha\,G & \alpha\,\partial_x - C\,L\,\partial_t - L\,G \end{pmatrix},$$

where α satisfies $\alpha^2 - L\,C = 0$ and $S\,P = Q\,S$.

Example: Transmission line

- Let us denote by $D' = \mathbb{Q}(C, G, L, R, \alpha)/(\alpha^2 - L\,C)[\partial_x, \partial_t]$.
- $\ker_{D'}(.P)$, $\mathrm{im}_{D'}(.P)$, $\ker_{D'}(.Q)$ and $\mathrm{im}_{D'}(.Q)$ are free with bases:

$$\begin{cases} \mathrm{im}_{D'}(.P) = D'\,U_2, \quad U_2 = (C \quad -\alpha), \\ \ker_{D'}(.P) = D'\,U_1, \end{cases}$$

$$U_1 = (C\,\partial_x - \alpha\,C\,\partial_t - \alpha\,G \quad C\,L\,\partial_t - \alpha\,\partial_x + C\,R),$$

$$\begin{cases} \ker_{D'}(.Q) = D'\,V_1, \quad V_1 = (C \quad -\alpha), \\ \mathrm{im}_{D'}(.Q) = D'\,V_2, \end{cases}$$

$$V_2 = (-C\,\partial_x - \alpha\,C\,\partial_t - \alpha\,G \quad \alpha\,\partial_x + C\,L\,\partial_t + C\,R).$$

- $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_2(D')$, $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_2(D')$.
- The matrix R is then equivalent to:

$$\Rightarrow \overline{S} = V\,S\,U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (R + L\,\partial_t)\,(G + C\,\partial_t) - \partial_x^2 \end{pmatrix}.$$

Example: Transmission line

$$\left\{\begin{array}{l}\overline{V} = C\,\partial_x\,V - \alpha\,C\,\partial_t\,V - \alpha\,G\,V + C\,L\,\partial_t\,I - \alpha\,\partial_x\,I + C\,R\,I,\\ \overline{I} = C\,V - \alpha\,I,\end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} V = \frac{1}{C\,(C\,R - L\,G)}\,(\alpha\,\overline{V} + (C\,L\,\partial_t\,\overline{I} - \alpha\,\partial_x\,\overline{I} + C\,R\,\overline{I})),\\ I = \frac{1}{C\,(C\,R - L\,G)}\,(C\,\overline{V} + \alpha\,C\,\partial_t\,\overline{I} - C\,\partial_x\,\overline{I} + \alpha\,G\,\overline{I}).\end{array}\right.$$

- We have the following equivalence of systems:

$$(1)\left\{\begin{array}{l}\dfrac{\partial V}{\partial x} + L\,\dfrac{\partial I}{\partial t} + R\,I = 0,\\ C\,\dfrac{\partial V}{\partial t} + G\,V + \dfrac{\partial I}{\partial x} = 0,\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\overline{V} = 0,\\ ((R + L\,\partial_t)\,(G + C\,\partial_t) - \partial_x^2)\,\overline{I} = 0.\ (2)\end{array}\right.$$

- The solutions of (1) can be written as ($\overline{I}$: cyclic vector):

$$\left\{\begin{array}{l} V = \frac{1}{C\,(C\,R - L\,G)}\,((C\,L\,\partial_t\,\overline{I} - \alpha\,\partial_x\,\overline{I} + C\,R\,\overline{I})),\\ I = \frac{1}{C\,(C\,R - L\,G)}\,(\alpha\,C\,\partial_t\,\overline{I} - C\,\partial_x\,\overline{I} + \alpha\,G\,\overline{I}),\end{array}\right. \quad \text{where } \overline{I} \text{ satisfies (2).}$$

Example: Beltrami equations

- Let us consider the Beltrami equations:

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial x} - x \, \dfrac{\partial v}{\partial y} = 0, \\ \dfrac{\partial u}{\partial y} + x \, \dfrac{\partial v}{\partial x} = 0, \end{array} \right. \quad \Rightarrow R = \begin{pmatrix} \partial_x & -x \, \partial_y \\ \partial_y & x \, \partial_x \end{pmatrix}.$$

- The matrices P and Q defined by

$$P = \begin{pmatrix} 1 - x \, \partial_x + i \, x \, \partial_y & x^2 \, (\partial_y + i \, \partial_x) \\ \partial_y + i \, \partial_x & 1 + x \, \partial_x - i \, x \, \partial_y \end{pmatrix},$$

$$Q = \begin{pmatrix} -x \, \partial_x - i \, x \, \partial_y & -x \, \partial_y + i \, (1 + x \, \partial_x) \\ -x \, \partial_y + i \, x \, \partial_x & 1 + x \, \partial_x + i \, x \, \partial_y \end{pmatrix},$$

satisfy

$$P \, R = Q \, R, \quad P^2 = P, \quad Q^2 = Q,$$

i.e., define a projector f of $\mathrm{end}_D(M)$ ($f^2 = f$).

• The left D-module $\ker_D(.P)$, $\mathrm{im}_D(.P) = \ker_D(.(I_2 - P))$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q) = \ker_D(.(I_2 - Q))$ are free with bases:

$$\begin{cases} \ker_D(.P) = D\,(-\partial_x + i\,\partial_y \quad x\,(\partial_y + i\,\partial_x)), \\ \mathrm{im}_D(.P) = D\,(i \quad x), \\ \ker_D(.Q) = D\,(-1 \quad i), \\ \mathrm{im}_D(.Q) = D\,(-x\,(\partial_y - i\,\partial_x) \quad (1 + x\,\partial_x) + i\,x\,\partial_y). \end{cases}$$

• If we form the following unimodular matrices

$$U = \begin{pmatrix} -\partial_x + i\,\partial_y & x\,(\partial_y + i\,\partial_x) \\ i & x \end{pmatrix} \in \mathrm{GL}_2(D),$$

$$V = \begin{pmatrix} -1 & i \\ -x\,(\partial_y - i\,\partial_x) & (1 + x\,\partial_x) + i\,x\,\partial_y \end{pmatrix} \in \mathrm{GL}_2(D),$$

the matrix R is then equivalent to:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & x\,\Delta - i\,\partial_y \end{pmatrix}.$$

- Let us define $(\overline{u} \quad \overline{v})^T = U\,(u \quad v)^T$, i.e.:

$$\left\{ \begin{array}{l} \overline{u} = -(\partial_x\, u - i\, \partial_y\, u) + i\, x\, (\partial_x\, v - i\, \partial_y\, v), \\ \overline{v} = i\, u + x\, v, \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} u = -x\, \overline{u} + i\, x\, (\partial_x\, \overline{v} - i\, \partial_y\, \overline{v}) - i\, \overline{v}, \\ v = i\, \overline{u} + (\partial_x\, \overline{v} - i\, \partial_y\, \overline{v}). \end{array} \right.$$

- We have the following equivalence of systems:

$$(\star) \quad \left\{ \begin{array}{l} \dfrac{\partial u}{\partial x} - x\, \dfrac{\partial v}{\partial y} = 0, \\ \dfrac{\partial u}{\partial y} + x\, \dfrac{\partial v}{\partial x} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \overline{u} = 0, \\ x\, \Delta\, \overline{v} - i\, \partial_y\, \overline{v} = 0. \end{array} \right.$$

- The solutions of $(\star)$ can be written as ($\overline{v}$: cyclic vector):

$$\left\{ \begin{array}{l} u = i\, x\, (\partial_x\, \overline{v} - i\, \partial_y\, \overline{v}) - i\, \overline{v}, \\ v = \partial_x\, \overline{v} - i\, \partial_y\, \overline{v}, \end{array} \right. \quad \text{where} \quad x\, \Delta\, \overline{v} - i\, \partial_y\, \overline{v} = 0.$$

Example: Dirac equation

- Let us consider the following complex matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The Dirac equation has the form $\sum_{i=1}^4 \gamma^i \, \partial \, y / \partial x_i = 0$:

$$\left\{ \begin{array}{l} d_4 \, y_1 - i \, d_3 \, y_3 - (i \, d_1 + d_2) \, y_4 = 0, \\ d_4 \, y_2 - (i \, d_1 - d_2) \, y_3 + i \, d_3 \, y_4 = 0, \\ i \, d_3 \, y_1 + (i \, d_1 + d_2) \, y_2 - d_4 \, y_3 = 0, \\ (i \, d_1 - d_2) \, y_1 - i \, d_3 \, y_2 - d_4 \, y_4 = 0, \end{array} \right. \qquad d_i = \partial / \partial x_i.$$

Example: Dirac equation

- Let us consider $D = \mathbb{Q}(i)[d_1, d_2, d_3, d_4]$, the matrix

$$R = \left(\begin{array}{cccc} d_4 & 0 & -i\, d_3 & -(i\, d_1 + d_2) \\ 0 & d_4 & -i\, d_1 + d_2 & i\, d_3 \\ i\, d_3 & i\, d_1 + d_2 & -d_4 & 0 \\ i\, d_1 - d_2 & -i\, d_3 & 0 & -d_4 \end{array} \right) \in D^{4 \times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 4}\, R)$.

- Computing projectors of $\mathrm{end}_D(M)$, we obtain a projector f defined by the pair of matrices:

$$P = \frac{1}{2} \left(\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right), \quad Q = \frac{1}{2} \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right).$$

- We have $P^2 = P$ and $Q^2 = Q$, i.e., the D-modules $\ker_D(.P)$, $\mathrm{im}\,(.P)$, $\ker_D(.Q)$ and $\mathrm{im}\,(.Q)$ are free.

Example: Dirac equation

- Computing bases for these modules, we then get:

$$\left\{\begin{array}{ll} \ker_D(.P) = D^{1\times 2}\, U_1, & U_1 = \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right), \\ \mathrm{im}\,(.P) = D^{1\times 2}\, U_2, & U_2 = \left(\begin{array}{cccc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right), \\ \ker_D(.Q) = D^{1\times 2}\, V_1, & V_1 = \left(\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right), \\ \mathrm{im}\,(.Q) = D^{1\times 2}\, V_2, & V_2 = \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right). \end{array}\right.$$

- Let us form the unimodular matrices:

$$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_4(D), \quad V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_4(D).$$

Example: Dirac equation

- The matrix R is then equivalent to the block-diagonal one:

$$V\,R\,U^{-1} = \begin{pmatrix} i\,d_3 - d_4 & -i\,d_1 - d_2 & 0 & 0 \\ i\,d_1 - d_2 & i\,d_3 + d_4 & 0 & 0 \\ 0 & 0 & i\,d_3 + d_4 & i\,d_1 + d_2 \\ 0 & 0 & i\,d_1 - d_2 & -i\,d_3 + d_4 \end{pmatrix}.$$

- If we denote by $z = U\,y$, we obtain that the Dirac equation is then equivalent to the decoupled system of PDEs:

$$\begin{cases} (i\,d_3 - d_4)\,z_1 - (i\,d_1 + d_2)\,z_2 = 0, \\ (i\,d_1 - d_2)\,z_1 + (i\,d_3 + d_4)\,z_2 = 0, \\ (i\,d_3 + d_4)\,z_3 + (i\,d_1 + d_2)\,z_4 = 0, \\ (i\,d_1 - d_2)\,z_3 + (-i\,d_3 + d_4)\,z_4 = 0. \end{cases}$$

Example: 2-D rotational isentropic flow

- We consider the linearized approximation of the steady two-dimensional rotational isentropic flow (Courant-Hilbert)

$$\begin{cases} u\,\rho \dfrac{\partial\,\omega}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial x} = 0, \\ u\,\rho\,\dfrac{\partial\,\lambda}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial y} = 0, \\ \rho\,\dfrac{\partial\,\omega}{\partial x} + \rho\,\dfrac{\partial\,\lambda}{\partial y} + u\,\dfrac{\partial \sigma}{\partial x} = 0, \end{cases}$$

where u is a constant velocity parallel to the x-axis, ρ a constant density and c the sound speed.

- Let us consider $D = \mathbb{Q}(u, \rho, c)[\partial_x, \partial_y]$, the system matrix

$$R = \begin{pmatrix} u\,\rho\,\partial_x & c^2\,\partial_x & 0 \\ 0 & c^2\,\partial_y & u\,\rho\,\partial_x \\ \rho\,\partial_x & u\,\partial_x & \rho\,\partial_y \end{pmatrix} \in D^{3\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 3}\,R)$.

Example: 2-D rotational isentropic flow

- If α satisfies $1 + 4\,(c^2 - u^2)\,\alpha^2 = 0$ and we denote by

$$D' = (\mathbb{Q}(u, \rho, c, \alpha)/(1 + 4\,(c^2 - u^2)\,\alpha^2))[\partial_x, \partial_y],$$

$$U = \begin{pmatrix} 0 & 2\,\alpha\,c\,(c^2 - u^2) & u\,\rho \\ 0 & 2\,\alpha\,c\,(c^2 - u^2) & -u\,\rho \\ u\,\rho & c^2 & 0 \end{pmatrix} \in \mathrm{GL}_3(D'),$$

$$V = \begin{pmatrix} 2\,\alpha\,c & 1 & -2\,\alpha\,c\,u \\ 2\,\alpha\,c & -1 & -2\,\alpha\,c\,u \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(D'),$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial_x - 2\,\alpha\,c\,\partial_y & 0 & 0 \\ 0 & \partial_x + 2\,\alpha\,c\,\partial_y & 0 \\ 0 & 0 & \partial_x \end{pmatrix}.$$

- We have $M \cong M_1 \oplus M_2 \oplus M_3$, where $M_3 = D'/(D'\,\partial_x)$ and:

$$M_1 = D'/(D'\,(\partial_x - 2\,\alpha\,c\,\partial_y)), \quad M_2 = D'/(D'\,(\partial_x + 2\,\alpha\,c\,\partial_y)).$$

Example: Fluid in rotation with a small velocity

- Let us consider an incompressible fluid in rotation with a small fluid velocity (Landau & Lifchitz, Mécanique des fluides, p. 62):

$$\begin{cases} \rho_0 \dfrac{\partial \vec{v}}{\partial t} + 2\,\rho_0\,\vec{\Omega} \wedge \vec{v} + \vec{\nabla}\, p = 0, \\ \vec{\nabla}.\vec{v} = 0. \end{cases}$$

- If we consider $\vec{\Omega} = (0, 0, \Omega_0/2)$, we then the system matrix

$$R = \begin{pmatrix} \rho_0\,\partial_t & -\rho_0\,\Omega_0 & 0 & \partial_1 \\ \rho_0\,\Omega_0 & \rho_0\,\partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0\,\partial_t & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix} \in D^{4\times 4},$$

where $D = \mathbb{Q}(\rho, \Omega_0)[\partial_t, \partial_1, \partial_2, \partial_3]$.

- We consider the D-module $M = D^{1\times 4}/(D^{1\times 4}\, R)$.

Example: Fluid in rotation with a small velocity

- The matrices P and Q defined by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{\Omega_0}\,\partial_t & 0 & 0 & \frac{1}{\rho_0\,\Omega_0}\,\partial_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{\Omega_0}\,\partial_t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\rho_0\,\Omega_0}\,\partial_2 & 0 & 0 & 1 \end{pmatrix},$$

satisfy the relations $R\,P = Q\,R$, $P^2 = P$ and $Q^2 = Q$.

- The D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free of rank 1, 3, 1 and 3 and we obtain the unimodular matrices:

$$U = \begin{pmatrix} \rho_0\,\partial_t & -\rho_0\,\partial_t & 0 & \partial_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{\Omega_0}\,\partial_t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\rho_0\,\Omega_0}\,\partial_2 & 0 & 0 & 1 \end{pmatrix}$$

Example: Fluid in rotation with a small velocity

- We obtain that the matrix

$$R = \begin{pmatrix} \rho_0 \, \partial_t & -2 \, \rho_0 \, \Omega & 0 & \partial_1 \\ 2 \, \rho_0 \, \Omega & \rho_0 \, \partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0 \, \partial_t & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix}$$

is then equivalent to the following matrix:

$$V \, R \, U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\rho_0}{\Omega_0} \, (\partial_t^2 + \Omega_0^2) & 0 & \frac{1}{\Omega_0} \, (\partial_t \, \partial_1 + \Omega_0 \, \partial_2) \\ 0 & 0 & \rho_0 \, \partial_t & \partial_3 \\ 0 & \frac{1}{\Omega_0} \, (\partial_t \, \partial_2 + \Omega_0 \, \partial_1) & \partial_3 & \frac{1}{\rho_0 \, \Omega_0} \, \partial_1 \, \partial_2 \end{pmatrix}.$$

Example: A linear system of two second order equations

• We consider the system of PDEs (Handbook of Mathematics for Engineers and Scientists, Polyanin-Manzhirov, p. 1341):

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\,\dfrac{\partial^2 u}{\partial x^2} - b_1\,u - c_1\,w = 0, \\ \dfrac{\partial w}{\partial t} - a\,\dfrac{\partial^2 w}{\partial x^2} - b_2\,u - c_2\,w = 0. \end{cases}$$

• We consider the ring $D = \mathbb{Q}(a, b_1, b_2, c_1, c_2)[\partial_t, \partial_x]$, the matrix

$$R = \begin{pmatrix} \partial_t - a\,\partial_x^2 - b_1 & -c_1 \\ -b_2 & \partial_t - a\,\partial_x^2 - c_2 \end{pmatrix} \in D^{2\times 2},$$

and the D-module $M = D^{1\times 2}/(D^{1\times 2}\,R)$.

Example: A linear system of two second order equations

- Let α be the algebraic number satisfying:

$$((b_1 - b_2)^2 + 4\, c_1\, b_2)\, \alpha^2 - 1 = 0.$$

A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotent matrices:

$$P = Q = \frac{1}{2} \begin{pmatrix} (b_1 - b_2)\, \alpha + 1 & 2\, c_1\, \alpha \\ 2\, b_2\, \alpha & (b_2 - b_1)\, \alpha + 1 \end{pmatrix}$$

- $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are free D-modules with bases:

$$\begin{cases} \ker_D(.P) = D\, U_1, & U_1 = (2\, b_2\, \alpha \quad (b_2 - b_1)\, \alpha - 1), \\ \mathrm{im}_D(.P) = D\, U_2, & U_2 = (-2\, b_2\, \alpha \quad (b_1 - b_2)\, \alpha - 1). \end{cases}$$

- Let us denote by $U = V = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_2(D)$.
- R is equivalent to the matrix $\overline{R} = V\, R\, U^{-1}$ defined by:

$$\overline{R} = \begin{pmatrix} \partial_t - a\, \partial_x^2 - \frac{(b_2 + b_1)}{2} + \frac{1}{2\, \alpha} & 0 \\ 0 & \partial_t - a\, \partial_x^2 - \frac{(b_2 + b_1)}{2} - \frac{1}{2\, \alpha} \end{pmatrix}.$$

Example: A linear system of two second order equations

• We consider the system of PDEs (Handbook of Mathematics for Engineers and Scientists, Polyanin-Manzhirov, p. 1342):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - k\,\dfrac{\partial^2 u}{\partial x^2} - a_1\,u - b_1\,w = 0, \\ \dfrac{\partial^2 w}{\partial t^2} - k\,\dfrac{\partial^2 w}{\partial x^2} - a_2\,u - b_2\,w = 0. \end{cases}$$

• We consider the ring $D = \mathbb{Q}(k, a_1, a_2, b_1, b_2)[\partial_t, \partial_x]$, the matrix

$$R = \begin{pmatrix} \partial_t^2 - k\,\partial_x^2 - a_1 & -b_1 \\ -a_2 & \partial_t^2 - k\,\partial_x^2 - b_2 \end{pmatrix} \in D^{2\times 2},$$

and the D-module $M = D^{1\times 2}/(D^{1\times 2}\,R)$.

Example: A linear system of two second order equations

- Let α be the algebraic number satisfying:

$$((a_1 - b_2)^2 + 4\, a_2\, b_1)\, \alpha^2 - 1 = 0.$$

A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotent matrices:

$$P = Q = \frac{1}{2} \begin{pmatrix} (b_2 - a_1)\, \alpha + 1 & 2\, b_1\, \alpha \\ 2\, a_2\, \alpha & (b_2 - b_1)\, \alpha + 1 \end{pmatrix}$$

- $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are free D-modules with bases:

$$\begin{cases} \ker_D(.P) = D\, U_1, & U_1 = (2\, a_2\, \alpha \quad (b_2 - a_1)\, \alpha - 1), \\ \mathrm{im}_D(.P) = D\, U_2, & U_2 = (-2\, a_2\, \alpha \quad (a_1 - b_2)\, \alpha - 1). \end{cases}$$

- Let us denote by $U = V = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_2(D)$.
- R is equivalent to the matrix $\overline{R} = V\, R\, U^{-1}$ defined by:

$$\overline{R} = \begin{pmatrix} \partial_t^2 - k\, \partial_x^2 - \frac{(a_1+b_2)}{2} + \frac{1}{2\,\alpha} & 0 \\ 0 & \partial_t^2 - k\, \partial_x^2 - \frac{(a_1+b_2)}{2} - \frac{1}{2\,\alpha} \end{pmatrix}.$$

Pommaret's example I

- Let us consider the system of PDEs (Pommaret, LNCIS 311):

$$\begin{cases} \partial_2\, y_2(x) - \partial_1\, y_2(x) + \partial_2\, y_3(x) - \partial_1\, y_3(x) - a\, y_3(x) = 0, \\ \partial_2\, y_1(x) - \partial_1\, y_2(x) - \partial_2\, y_3(x) - \partial_1\, y_3(x) - a\, y_3(x) = 0, \\ \partial_1\, y_1(x) - \partial_1\, y_2(x) - 2\, \partial_1\, y_3(x) = 0. \end{cases}$$

- Let us consider the ring $D = \mathbb{Q}(a)[\partial_1, \partial_2, \partial_3]$, the matrix

$$R = \begin{pmatrix} 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 - a \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 - a \\ \partial_1 & -\partial_1 & -2\, \partial_1 \end{pmatrix}$$

and the D-module $M = D^{1\times 3}/(D^{1\times 3}\, R)$.

- A projector $f \in \mathrm{end}_D(M)$ is defined by the pair of idempotents:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Pommaret's example I

- $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free with bases:

$$\begin{cases} \ker_D(.P) = D^{1\times 2}\, U_1, & U_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \\ \mathrm{im}_D(.P) = D\, U_2, & U_2 = (1 \quad -1 \quad -2)\,. \end{cases}$$

$$\begin{cases} \ker_D(.Q) = D\, V_1, & V_1 = (0 \quad 1 \quad -1)\,, \\ \mathrm{im}_D(.Q) = D^{1\times 2}\, V_2, & V_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}. \end{cases}$$

- $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_3(D)$, $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_3(D)$.
- The matrix R is equivalent to:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \partial_1 - \partial_2 & a & 0 \\ 0 & 0 & \partial_1 \\ 0 & 0 & \partial_2 \end{pmatrix}.$$

Pommaret's example II

- Let us consider the system of PDEs (Pommaret, PDCT, p. 807):

$$\left\{\begin{array}{l} -2\,\partial_1\,y_2+\partial_3\,y_3-2\,\partial_2\,y_3-\partial_1\,y_3-y_4=0,\\ \partial_3\,y_2-2\,\partial_1\,y_2+2\,\partial_2\,y_3-3\,\partial_1\,y_3+y_4=0,\\ \partial_3\,y_1-6\,\partial_1\,y_2-2\,\partial_2\,y_3-5\,\partial_1\,y_3-y_4=0,\\ \partial_2\,y_2-\partial_1\,y_2+\partial_2\,y_3-\partial_1\,y_3=0,\\ \partial_2\,y_1-\partial_1\,y_2-\partial_2\,y_3-\partial_1\,y_3=0,\\ \partial_1\,y_1-\partial_1\,y_2-2\,\partial_1\,y_3=0. \end{array}\right.$$

- Let us consider the ring $D=\mathbb{Q}[\partial_1,\partial_2,\partial_3]$, the matrix

$$R=\left(\begin{array}{llll} 0 & -2\,\partial_1 & \partial_3-2\,\partial_2-\partial_1 & -1\\ 0 & \partial_3-2\,\partial_1 & 2\,\partial_2-3\,\partial_1 & 1\\ \partial_3 & -6\,\partial_1 & -2\,\partial_2-5\,\partial_1 & -1\\ 0 & \partial_2-\partial_1 & \partial_2-\partial_1 & 0\\ \partial_2 & -\partial_1 & -\partial_2-\partial_1 & 0\\ \partial_1 & -\partial_1 & -2\,\partial_1 & 0 \end{array}\right)$$

and the D-module $M=D^{1\times 4}/(D^{1\times 6}\,R)$.

Pommaret's example II

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents:

$$P = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 2 & 1 & -1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 2 & -2 & 3 \\ 0 & 0 & 0 & -2 & 2 & 5 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

- $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free with bases:

$$\left\{ \begin{array}{ll} \ker_D(.P) = D^{1\times 3}\, U_1, & U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathrm{im}_D(.P) = D\, U_2, & U_2 = (1 \quad -1 \quad -2 \quad 0). \end{array} \right.$$

Pommaret's example II

$$\left\{\begin{array}{l} \ker_D(.Q) = D^{1\times 3}\, V_1, \quad V_1 = \begin{pmatrix} 0 & 2 & 0 & 0 & 4 & -5 \\ 0 & 0 & 2 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}, \\ \mathrm{im}_D(.Q) = D^{1\times 3}\, V_2 \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -2 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}. \end{array}\right.$$

- $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_4(D)$, $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_6(D)$.
- The matrix R is equivalent to:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} 4\,\partial_2 - 5\,\partial_1 & 2\,\partial_3 - 3\,\partial_1 & 2 & 0 \\ 2\,\partial_3 - 4\,\partial_2 - 3\,\partial_1 & -5\,\partial_1 & -2 & 0 \\ \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_3 \end{pmatrix}.$$

Pommaret's example II

- Let us consider the first block of $\overline{R}$

$$S = \begin{pmatrix} 4\,\partial_2 - 5\,\partial_1 & 2\,\partial_3 - 3\,\partial_1 & 2 \\ 2\,\partial_3 - 4\,\partial_2 - 3\,\partial_1 & -5\,\partial_1 & -2 \\ \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 \end{pmatrix} \in D^{3\times 3},$$

and the D-module $L = D^{1\times 3}/(D^{1\times 3}\,S)$.

- A projector $g \in \mathrm{end}_D(L)$ is defined by the idempotents:

$$X = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ -2 & 0 & 0 \\ -20\,\partial_2 + 25\,\partial_1 & -20\,\partial_2 + 25\,\partial_1 & 2 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & -1 & -20 \\ 0 & 1 & 20 \\ 0 & 0 & 0 \end{pmatrix}.$$

Pommaret's example II

- $\ker_D(.X)$, $\mathrm{im}_D(.X)$, $\ker_D(.Y)$ and $\mathrm{im}_D(.Y)$ are free with bases:

$$\left\{\begin{array}{ll} \ker_D(.X) = D\, W_1, & W_1 = (1 \quad 1 \quad 0)\,, \\ \mathrm{im}_D(.X) = D^{1\times 2}\, W_2, & W_2 = \left(\begin{smallmatrix} -23\,\partial_1 + 16\,\partial_2 + 2\,\partial_3 & -25\,\partial_1 + 20\,\partial_2 & -2 \\ -1 & 0 & 0 \end{smallmatrix}\right). \end{array}\right.$$

$$\left\{\begin{array}{ll} \ker_D(.Y) = D^{1\times 2}\, Z_1, & Z_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \\ \mathrm{im}_D(.Y) = D\, Z_2, & Z_2 = (0 \quad 1 \quad 20)\,. \end{array}\right.$$

- $W = (W_1^T \quad W_2^T)^T \in \mathrm{GL}_3(D)$, $Z = (Z_1^T \quad Z_2^T)^T \in \mathrm{GL}_3(D)$.
- The matrix R is equivalent to:

$$\overline{S} = Z\, S\, W^{-1} = \begin{pmatrix} \partial_3 - 4\,\partial_1 & 0 & 0 \\ \partial_2 - \partial_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Pommaret's example II

- $W' = \text{diag}(W, 1) \in \text{GL}_4(D)$, $Z' = \text{diag}(Z, I_3) \in \text{GL}_6(D)$.
- The matrix R is equivalent to $\overline{\overline{R}} = (Z'\, V)\, R\, (W'\, U)^{-1}$:

$$\overline{\overline{R}} = \begin{pmatrix} \partial_3 - 4\,\partial_1 & 0 & 0 & 0 \\ \partial_2 - \partial_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_3 \end{pmatrix}.$$

- The solutions of $\overline{\overline{R}}\, z = 0$ is then:

$$z = \begin{pmatrix} f(x_3 + \frac{1}{4}\,(x_1 + x_2)) \\ 0 \\ g(x_1, x_2, x_3) \\ c \end{pmatrix},$$

$$\forall\, f \in C^{\infty}(\mathbb{R}),\ \forall\, g \in C^{\infty}(\mathbb{R}^3),\ \forall\, c \in \mathbb{R}.$$

Example: Wind tunnel model

- Let us consider the wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\,\zeta\,\omega\, x_3(t) - \omega^2\, u(t) = 0. \end{cases}$$

- Let us consider the algebra $D = \mathbb{Q}(a, k, \omega, \zeta)\left[\frac{d}{dt}, \delta\right]$ of differential time-delay operators and the system matrix:

$$R = \begin{pmatrix} \frac{d}{dt} + a & -k\, a\, \delta & 0 & 0 \\ 0 & \frac{d}{dt} & -1 & 0 \\ 0 & \omega^2 & \frac{d}{dt} + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3 \times 4}.$$

Example: Wind tunnel model

- If we define the following unimodular matrices

$$U = \begin{pmatrix} \omega^2 & \frac{d}{dt} & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \omega^2 \left(\frac{d}{dt} + a\right) & -\omega^2 \,(k\, a\, \delta + 1) & -\left(\frac{d}{dt} + 2\,\omega\,\zeta\right) & \omega^2 \\ 0 & \frac{d}{dt} & -1 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \omega^2 & \frac{d}{dt} + a & 0 \\ \omega^2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

the matrix R is then equivalent to:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \frac{d}{dt} + a & -a\, k\, \omega^2\, \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example: Stirred tank

- Let us consider the stirred tank model considered in Kwakernaak, Sivan, Linear Optimal Control Systems, Wiley, 1972:

$$\begin{cases} \dot{x}_1(t) + \dfrac{1}{2\,\theta}\, x_1(t) - u_1(t) - u_2(t) = 0, \\ \dot{x}_2(t) + \dfrac{1}{\theta}\, x_2(t) - \dfrac{(c_1 - c_0)}{V_0}\, u_1(t-\tau) - \dfrac{(c_2 - c_0)}{V_0}\, u_2(t-\tau) = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(\theta, c_0, c_1, c_2, V_0)\left[\frac{d}{dt}, \delta\right]$, the system matrix

$$R = \begin{pmatrix} \frac{d}{dt} + \frac{1}{2\,\theta} & 0 & -1 & -1 \\ 0 & \frac{d}{dt} + \frac{1}{\theta} & -\frac{(c_1 - c_0)}{V_0}\,\delta & -\frac{(c_2 - c_0)}{V_0}\,\delta \end{pmatrix} \in D^{2\times 4},$$

and the D-module $M = D^{1\times 4}/(D^{1\times 2}\, R)$.

Example: Stirred tank

- The matrices $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ 0 & 0 & \beta & \beta \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

where $\alpha = \frac{c_0 - c_2}{c_1 - c_2}$ and $\beta = \frac{c_1 - c_0}{c_1 - c_2}$, satisfy $R\,P = Q\,R$, $P^2 = P$ and $Q^2 = Q$, i.e., define a projector of $\mathrm{end}_D(M)$.

- The D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free and we obtain the following unimodular matrices:

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & c_0 - c_1 & c_0 - c_2 \\ \frac{d}{dt} + \frac{1}{2\,\theta} & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \frac{d}{dt} + \frac{1}{\theta} & \frac{1}{V_0}\,\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Example: Tank model I

- We consider $D = \mathbb{Q}\left[\frac{d}{dt}, \delta\right]$ and the system matrix

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\frac{d}{dt}\,\delta \\ 1 & \delta^2 & -2\,\frac{d}{dt}\,\delta \end{pmatrix}$$

considered in Dubois, Petit, Rouchon, ECC99.

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

i.e., P and Q satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

Example: Tank model I

$$\left\{\begin{array}{l} U_1 = \ker_D(.P) = \left(\begin{array}{ccc} 1 & -1 & 0 \end{array}\right), \\ U_2 = \mathrm{im}_D(.P) = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \\ V_1 = \ker_D(.Q) = \left(\begin{array}{cc} 1 & -1 \end{array}\right), \\ V_2 = \mathrm{im}_D(.Q) = \left(\begin{array}{cc} 1 & 1 \end{array}\right), \end{array}\right.$$

and we obtain the following two unimodular matrices:

$$U = \left(\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad V = \left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right).$$

- We easily check that we have the following block diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \left(\begin{array}{ccc} \delta^2 - 1 & 0 & 0 \\ 0 & 1 + \delta^2 & -4\,\frac{d}{dt}\,\delta \end{array}\right).$$

Example: Tank model II

- Model of a one-dimensional tank containing a fluid subjected to an horizontal move (Petit, Rouchon, IEEE TAC, 2002):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-2\,h) + \alpha\,\ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t-2\,h) - \dot{y}_2(t) + \alpha\,\ddot{y}_3(t-h) = 0, \end{cases} \qquad \alpha \in \mathbb{R}, \quad h \in \mathbb{R}_+.$$

- Let us consider $D = \mathbb{Q}(\alpha)\left[\frac{d}{dt}, \delta\right]$, the system matrix

$$R = \begin{pmatrix} \frac{d}{dt} & -\frac{d}{dt}\,\delta^2 & \alpha\,\frac{d^2}{dt^2}\,\delta \\ \frac{d}{dt}\,\delta^2 & -\frac{d}{dt} & \alpha\,\frac{d^2}{dt^2}\,\delta \end{pmatrix} \in D^{2\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$.

- The matrices $P = \begin{pmatrix} 1 & 0 & 0 \\ \delta^2 & 0 & \alpha\,\frac{d}{dt}\,\delta \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -\delta^2 \\ 0 & 0 \end{pmatrix}$

satisfy $\quad R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$

Example: Tank model II

- $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free with bases:

$$\begin{cases} \ker_D(.P) = D \begin{pmatrix} \delta^2 & -1 & \alpha \, \frac{d}{dt} \, \delta \end{pmatrix}, & \ker_D(.Q) = D \, (0 \quad 1), \\ \mathrm{im}_D(.P) = D^{1\times 2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathrm{im}_D(.Q) = D \, (-1 \quad \delta^2). \end{cases}$$

- If we denote by

$$U = \begin{pmatrix} \delta^2 & -1 & \alpha \, \frac{d}{dt} \, \delta \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 0 & 1 \\ -1 & \delta^2 \end{pmatrix} \in \mathrm{GL}_2(D),$$

then R is equivalent to the following block-diagonal matrix:

$$V \, R \, U^{-1} = \begin{pmatrix} \frac{d}{dt} & 0 & 0 \\ 0 & \frac{d}{dt} \, (\delta - 1) \, (\delta + 1) \, (\delta^2 + 1) & \alpha \, \frac{d^2}{dt^2} \, \delta \, (\delta - 1) \, (\delta + 1) \end{pmatrix}.$$

Example: Tank model II

• Another projector of $\mathrm{end}_D(M)$ can be defined by the idempotents P' and Q' defined by:

$$P' = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q' = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

• Using linear algebra, we obtain

$$U' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(D),$$

and R is equivalent to the following block-diagonal matrix:

$$V' \, R \, U'^{-1} = \begin{pmatrix} \frac{d}{dt} \, (1 - \delta) \, (\delta + 1) & 0 & 0 \\ 0 & \frac{d}{dt} \, (\delta^2 + 1) & 2 \, \alpha \, \frac{d^2}{dt^2} \, \delta \end{pmatrix}.$$

Example: A controlled string with an interior mass

• We consider the model of a string with an interior mass considered by Mounier, Rudolph, Fliess & Rouchon (COCV 98)

$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\, \phi_1(t) - \eta_1\, \psi_1(t) - \eta_2\, \phi_2(t) + \eta_2\, \psi_2(t) = 0, \\ \phi_1(t - 2\, h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\, h_2) - v(t - h_2) = 0, \end{cases}$$

where $h_1, h_2 \in \mathbb{R}_+$ is s.t. $\mathbb{Q}\, h_1 + \mathbb{Q}\, h_2$ is a 2-dim. $\mathbb{Q}$-vector space.

• Let us consider $D = \mathbb{Q}(\eta_1, \eta_2)\left[\frac{d}{dt}, \sigma_1, \sigma_2\right]$, $M = D^{1\times 6}/(D^{1\times 4}\, R)$,

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \dfrac{d}{dt} + \eta_1 & \dfrac{d}{dt} - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} \in D^{4\times 6}.$$

Example: A controlled string with an interior mass

- Computing projectors of $\text{end}_D(M)$, we obtain:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\sigma_1^2 & 0 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & -\sigma_2^2 & 0 & \sigma_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -\dfrac{d}{dt} + \eta_1 & \eta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- We have $P^2 = P$ and $Q^2 = Q$, i.e., the D-modules $\ker_D(.P)$, $\text{im}_D(.P)$, $\ker_D(.P)$ and $\text{im}_D(.P)$ are free of rank 2, 4, 2, 2.

Example: A controlled string with an interior mass

- Computing bases of these D-modules, we obtain:

$$
\left\{
\begin{array}{l}
\ker_D(.P) = D^{1\times 2}\, U_1, \quad U_1 = \begin{pmatrix} \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix}, \\
\mathrm{im}_D(.P) = D^{1\times 4}\, U_2, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
\ker_D(.Q) = D^{1\times 2}\, V_1, \quad V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathrm{im}_D(.Q) = D^{1\times 2}\, V_2, \quad V_2 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & \dfrac{d}{dt} - \eta_1 & -\eta_2 \end{pmatrix}.
\end{array}
\right.
$$

- We form the unimodular matrices:

$$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_6(D), \quad V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_4(D).$$

Example: A controlled string with an interior mass

- R is then equivalent to the block-diagonal matrix:

$$\overline{R} = V\, R\, U^{-1} = \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\sigma_1^2 & \sigma_2^2-1 & \sigma_1 & -\sigma_2 \\ 0 & 0 & \sigma_1^2\left(\frac{d}{dt}-\eta_1\right)-\left(\frac{d}{dt}+\eta_1\right) & -\eta_2\,(\sigma_2^2+1) & -\sigma_1\left(\frac{d}{dt}+\eta_1\right) & \eta_2\,\sigma_2 \end{array}\right).$$

- Let us now consider the second block-diagonal matrix:

$$S = \left(\begin{array}{cccc} 1-\sigma_1^2 & \sigma_2^2-1 & \sigma_1 & -\sigma_2 \\ \sigma_1^2\left(\frac{d}{dt}-\eta_1\right)-\left(\frac{d}{dt}+\eta_1\right) & -\eta_2\,(\sigma_2^2+1) & -\sigma_1\left(\frac{d}{dt}+\eta_1\right) & \eta_2\,\sigma_2 \end{array}\right),$$

and the D-module $N = D^{1\times 4}/(D^{1\times 2}\, S)$.

- A projector $g \in \mathrm{end}_D(N)$ is defined by the idempotent matrices:

$$P' = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad Q' = \frac{1}{2}\left(\begin{array}{cc} \sigma_2^2+1 & (\sigma_2^2-1)/\eta_2 \\ -\eta_2\,(\sigma_2^2+1) & -\sigma_2^2+1 \end{array}\right),$$

$$\left\{\begin{array}{l} a = (\sigma_1^2\,\left(\frac{d}{dt}-(\eta_1+\eta_2)\right)-\frac{d}{dt}+(\eta_2-\eta_1))/(2\,\eta_2), \\ b = -\sigma_1\,(\frac{d}{dt}-(\eta_1+\eta_2))/(2\,\eta_2). \end{array}\right.$$

Example: A controlled string with an interior mass

- The D-modules $\ker_D(.P)$, $\mathrm{im}\,(.P)$, $\ker_D(.Q)$ and $\mathrm{im}\,(.Q)$ are free

$\ker_D(.P') = D\,U_1'$, $\mathrm{im}_D(.P') = D^{1\times 3}\,U_2'$, $\ker_D(.Q') = D\,V_1'$, $\mathrm{im}_D(.Q') = D\,V_2'$

$$U_1' = \left(\sigma_1^2\left(\frac{d}{dt} - \eta_1 - \eta_2\right) - \left(\frac{d}{dt} + \eta_1 - \eta_2\right) : -2\,\eta_2 : -\sigma_1\left(\frac{d}{dt} - \eta_1 - \eta_2\right) : 0\right),$$

$$U_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\sigma_1 & 0 & 1 & 0 \\ \sigma_1^2\,\sigma_2\,(d - \eta_1 - \eta_2) - \sigma_2\,(d + \eta_1 - \eta_2) & 0 & -\sigma_1\,\sigma_2\,(d - \eta_1 - \eta_2) & -2\,\eta_2 \end{pmatrix},$$

$$V_1' = (\eta_2 \quad 1), \quad V_2' = (\eta_2\,(\sigma_2^2 + 1) \quad \sigma_2^2 - 1).$$

- Let $U' = (U_1'^T \;\; U_2'^T)^T \in \mathrm{GL}_4(D)$, $V' = (V_1'^T \;\; V_2'^T)^T \in \mathrm{GL}_2(D)$.

$$\Rightarrow \overline{S} = V'\,S\,U'^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{d}{dt} + \eta_1 + \eta_2 & \sigma_1\left(\frac{d}{dt} + \eta_2 - \eta_1\right) & \sigma_2 \end{pmatrix}.$$

- Let $U'' = \mathrm{diag}(I_2, U')$, $V'' = \mathrm{diag}(I_2, V')$. We finally obtain:

$$\overline{\overline{R}} = (V''\,V)\,R\,(U''\,U)^{-1} = \mathrm{diag}(I_2, \overline{S}).$$

Example: A neutral differential time-delay system

- Let us consider the following neutral differential time-delay system (Logemann, SCL87):

$$\left\{ \begin{array}{l} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\, x_2(t) = 0, \end{array} \right. \qquad a \in \mathbb{R}.$$

- Let us consider $D = \mathbb{Q}(a)\left[\frac{d}{dt}, \delta\right]$, the system matrix

$$R = \left(\begin{array}{ccc} \frac{d}{dt} + 1 & 0 & -1 \\ -1 & \frac{d}{dt}\,(1-\delta) + a & 0 \end{array} \right) \in D^{2 \times 3},$$

and the D-module $M = D^{1 \times 3}/(D^{1 \times 2}\, R)$.

Example: A neutral differential time-delay system

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{d}{dt} - 1 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

- Computing bases of the free D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$, we obtain the unimodular matrices

$$U = \begin{pmatrix} -1 & \frac{d}{dt}(1-\delta) + a & 0 \\ \frac{d}{dt} + 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}_3(D).$$

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(D),$$

$$\Rightarrow \overline{R} = V\, R\, U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Example: Flexible rod

- Flexible rod (Mounier, Rudolph, Petitot, Fliess ECC95):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases}$$

$$\Rightarrow R = \begin{pmatrix} \frac{d}{dt} & -\frac{d}{dt}\,\delta & -1 \\ 2\,\frac{d}{dt}\,\delta & -\frac{d}{dt}\,\delta^2 - \frac{d}{dt} & 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1+\delta^2 & -\frac{1}{2}\,\delta\,(1+\delta^2) & 0 \\ 2\,\delta & -\delta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -\frac{1}{2}\,\delta \\ 0 & 0 \end{pmatrix},$$

$$\Rightarrow U = \begin{pmatrix} -2\,\delta & \delta^2+1 & 0 \\ 2\,\frac{d}{dt}\,(1-\delta^2) & \frac{d}{dt}\,\delta\,(\delta^2-1) & -2 \\ -1 & \frac{1}{2}\,\delta & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\delta \end{pmatrix},$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \frac{d}{dt} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example: Network model

- Let us consider the network model (Fliess-Mounier, IFAC98)

$$\begin{cases} \dot{x}_1(t) + u_1(t) - u_2(t - h_1) = 0, \\ \dot{x}_2(t) - u_1(t - h_2) = 0, \end{cases}$$

- Let us consider $D = \mathbb{Q}\left[\frac{d}{dt}, \delta_1, \delta_2\right]$, the system matrix

$$R = \begin{pmatrix} \frac{d}{dt} & 0 & 1 & -\delta_1 \\ 0 & \frac{d}{dt} & -\delta_2 & 0 \end{pmatrix} \in D^{2 \times 4},$$

and the D-module $M = D^{1 \times 4}/(D^{1 \times 2}\, R)$.

Example: Network model

- A projector $f \in \mathrm{end}_D(M)$ is defined by the idempotents

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ \delta_2 & 1 \end{pmatrix},$$

- Computing bases of the free D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$, we obtain the unimodular matrices

$$U = \begin{pmatrix} \frac{d}{dt} & 0 & 1 & -\delta_1 \\ 1 & 0 & 0 & 0 \\ \delta_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ \delta_2 & 1 \end{pmatrix},$$

$$\Rightarrow \overline{R} = V\, R\, U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{dt} & -\delta_1\,\delta_2 \end{pmatrix}.$$

Corollary

- Corollary: Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$ and $f \in \mathrm{end}_D(M)$ be defined by P and Q and satisfying $P^2 = P$ and $Q^2 = Q$. Let us suppose that one of the conditions holds:

1. $D = A[\partial; \sigma, \delta]$, where A is a field and σ is injective,
2. $D = k[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ is a commutative Ore algebra,
3. $D = A[\partial_1; \mathrm{id}, \delta_1] \ldots [\partial_n; \mathrm{id}, \delta_n]$, where $A = k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$ and k is a field of characteristic 0, and:

$$\mathrm{rank}_D(\ker_D(.P)) \geq 2, \quad \mathrm{rank}_D(\mathrm{im}_D\,(.P)) \geq 2,$$
$$\mathrm{rank}_D(\ker_D(.Q)) \geq 2, \quad \mathrm{rank}_D(\mathrm{im}_D\,(.Q)) \geq 2.$$

Then, there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that $\overline{R} = V\,R\,U^{-1}$ is a block diagonal matrix.

V. Implementation: the Maple MORPHISMS package

The MORPHISMS package

- The algorithms have been implemented in a Maple package called MORPHISMS based on the library OREMODULES developed by Chyzak, Q. and Robertz:

http://wwwb.math.rwth-aachen.de/OreModules

- List of functions:
  - Morphisms, MorphismsConst, MorphismsRat, MorphimsRat1.
  - Projectors, ProjectorsConst, ProjectorsRat, Idempotents.
  - KerMorphism, ImMorphism, CokerMorphism, CoimMorphism.
  - TestSurj, TestInj, TestBij.
  - QuadraticFirstIntegralConst...

- It also uses the following OREMODULES packages:

QUILLENSUSLIN & STAFFORD.

- It will be soon available with a library of examples.

Conclusion

- Contributions:
  - We use constructive homological algebra to provide algorithms for studying general LFSs (e.g., factoring or decomposing).
  - We apply the obtained results in mathematical physics and control theory.

- Work in progress:

T. Cluzeau, A. Quadrat, *Using morphism computations for factoring and decomposing general linear functional systems*, proceedings of MTNS 2006, Kyoto (Japan), INRIA report 5942.

A. Quadrat, Systems & Structures, Habilitation, to appear.

- Open questions:
  - Bounds in the general case.
  - Criteria for choosing the right P.
  - Existence of a solution to the Riccati equation...