Sur la multiplicité de la valeur spectrale à l'origine pour les systèmes à retard et son lien avec les matrices d'incidence de Birkhoff

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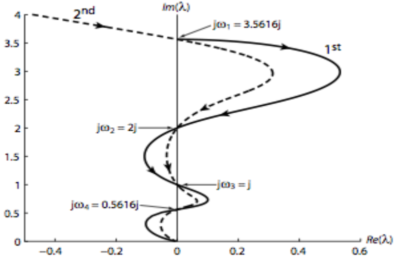
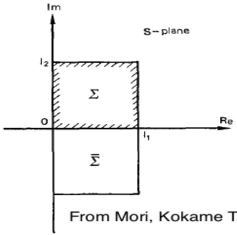
Outlines

- Motivations
  - Center manifold theorem
  - Examples
- Birkhoff interpolation problem
- Main results : Bound for the zero multiplicity
  - Polya-Szegö generic bound
  - Regular case : recovering the generic bound
  - Sparse case
- Conclusion & current investigations

Starting points

- The analysis of time-delay systems mainly relies on detecting and understanding the spectral values bifurcations when crossing the imaginary axis.





From Mori, Kokame TAC 1989

Fig. 1. Existence region of unstable characteristic roots in the S-plane.

- One of the most important type of such singularities is when the zero spectral value is multiple.
- The bound of such a multiplicity was not deeply investigated in the literature

Why we are interested in multiplicity of imaginary roots ?

Theorem

The standard Jordan normal form of any autonomous system of ODE with $\sigma = \sigma_- \cup \sigma_c$

$$\dot{x} = Ax + F(x, y), \qquad \dot{y} = By + G(x, y) \tag{1}$$

where $(x, y) \in \mathbb{R}^c \times \mathbb{R}^s$*,* A *is with* c *eigenvalues with zero real parts,* B *is with* s *eigenvalues with negative real parts.*

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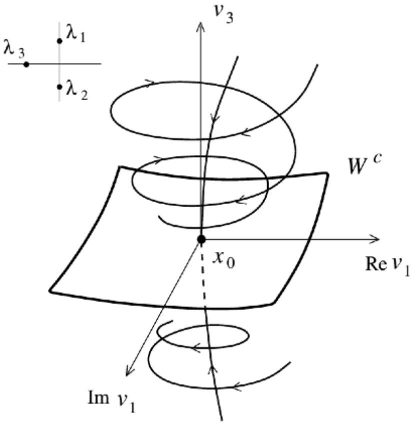
where $(x, y) \in \mathbb{R}^c \times \mathbb{R}^s$, A is with c eigenvalues with zero real parts, B is with s eigenvalues with negative real parts.
There exists a $\delta > 0$ and a function $h \in \mathcal{C}^r(V_\delta(0))$,
$h(0) = 0, Dh(0) = 0$ the local center manifold
$W^c(0) := \{(x, y) \in \mathbb{R}^c \times \mathbb{R}^s | y = h(x) \text{ for } |x| < \delta\}$ and satisfies

$$Dh(x)[Ax + F(x, h(x))] = Bh(x) + G(x, h(x)), \qquad \text{for } |x| < \delta \tag{2}$$

and the flow on the center manifold $W^c(0)$ is defined by

$$\dot{x} = Ax + F(x, h(x)), \qquad \forall x \in \mathbb{R}^c \text{ and } |x| < \delta \tag{3}$$

- $W^c(0)$ is an invariant variety.
- $W^c(0)$ is an attractive variety.



From Kuznetsov 1998

Time-Delay Systems with discrete delays : the general form

Let us consider the general autonomous first order nonlinear system of Neutral type with constant delay where we separate its linear and nonlinear quantities as follow

$$\frac{d}{dt}\mathcal{D}x_t = \mathcal{L}x_t + \mathcal{F}(x_t) \tag{4}$$

where $x_t \in C = C([-r,0],\mathbb{R}^n)$, $x_t(\theta) = x(t+\theta)$, $\mathcal{D}$, $\mathcal{L}$ are bounded linear operators such that $\mathcal{L}\phi = \sum_{k=0}^{n} B_k\,\phi(-\tau_k)$, $\mathcal{D}\phi = \phi(0) + \sum_{k=1}^{n} A_k\,\phi(-\tau_k)$ and $\mathcal{F}$ is sufficiently smooth function mapping C into $\mathbb{R}^n$ with $\mathcal{F}(0) = D\mathcal{F}(0) = 0$.

The linearized equation of (4) is given by

$$\frac{d}{dt}\mathcal{D}x_t = \mathcal{L}x_t \tag{5}$$

for which the operator solution $\mathcal{T}(t)$ defined by $\mathcal{T}(t)(\phi) = x_t(.\,,\phi)$ such that $x_t(.\,,\phi)(\theta) = x(t+\theta,\phi)$ for $\theta \in [-r,0]$ is a strongly continuous semigroup with the infinitesimal generator given by $\mathcal{A}\phi = \frac{d\phi}{d\theta}$ with the domain

$$\mathcal{D}om(\mathcal{A}) = \{\phi \in C : \frac{d\phi}{d\theta} \in C, \mathcal{D}\frac{d\phi}{d\theta} = \mathcal{L}\phi\}$$

The spectrum of $\mathcal{A}$, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ consists of complex values $\lambda \in \mathbb{C}$ which are zeros of the characteristic equation.

consider the bilinear form on $C^* \times C$:

$$(\psi, \phi) = \phi(0)\,\psi(0) - \int_{-r}^{0} d[\int_{0}^{\theta} \psi(\tau-\theta) d\mu(\tau)] + \int_{-r}^{0}\int_{0}^{\theta} \psi(\tau-\theta) d\eta(\theta)\phi(\tau) d\tau \tag{6}$$

and let $\mathcal{A}^T$ be the transposed operator of $\mathcal{A}$, i.e., $(\psi, \mathcal{A}\phi) = (\mathcal{A}^T\psi, \phi)$. The following Theorem permits the decomposition of the space C.

Theorem (Hale 1977)

Let Λ be a nonempty finite set of eigenvalues of $\mathcal{A}$ and let $P = span\{\mathcal{M}_\lambda(\mathcal{A}),\, \lambda \in \Lambda\}$ and $P^T = span\{\mathcal{M}_\lambda(\mathcal{A}^T),\, \lambda \in \Lambda\}$. Then P is invariant under $\mathcal{T}(t), t \geq 0$ and there exists a space Q, also invariant under $\mathcal{T}(t)$ such that $C = P \bigoplus Q$. Furthermore, if $\Phi = (\phi_1, \ldots, \phi_m)$ forms a basis of P, $\Psi = col(\psi_1, \ldots, \psi_m)$ is a basis of P^T in C^* such that $(\Phi, \Psi) = Id$, then

$$\begin{aligned} Q &= \{\phi \in C \setminus (\Psi, \phi) = 0\} \text{ and} \\ P &= \{\phi \in C \setminus \exists b \in \mathbb{R}^m :\ \phi = \Phi b\}. \end{aligned} \tag{7}$$

Also, $\mathcal{T}(t)\Phi = \Phi\, e^{Jt}$, where J is an $m \times m$ matrix such that $\sigma(J) = \Lambda$.

Under the above consideration one can write equation (4) as an abstract ODE

$$\dot{x}_t = \tilde{\mathcal{A}} x_t + X_0 \mathcal{F}(x_t), \tag{8}$$

Define the projection $\Pi : BC \to P$ such that $\Pi(\varphi + X_0\alpha) = \Phi[(\Psi, \varphi) + \Psi(0)\alpha]$ and let $x_t = \Phi y(t) + z_t$ where $y(t) \in \mathbb{R}^m$, some intermediate computations lead to split (4) to

$$\begin{aligned} \dot{y} &= Jy + \Psi(0)\mathcal{F}(\Phi y + z_t) \\ \dot{z}_t &= \tilde{\mathcal{A}}_Q + (I - \Pi) X_0 \mathcal{F}(\Phi y + z_t). \end{aligned} \tag{9}$$

The interest will be focused only on the first equation after writing z_t as a function of y. The way to do that is the infinite dimensional version of center manifold theorem.

A Vector Disease Model

A scalar differential equation with one delay representing a biological model proposed in [Cooke 1979] for describing the dynamics of a vector disease model where the infected host population $x(t)$ is governed by :

$$\dot{x}(t) + a_0\,x(t) + a_1\,x(t-\tau) - a_1\,x(t-\tau)\,x(t) = 0, \qquad (10)$$

- $a_1 > 0$ designates the contact rate between infected and uninfected populations.
- The infection of the host recovery proceeds exponentially at a rate $-a_0 > 0$.

The linearized system is given by :

$$\dot{x}(t) + a_0\, x(t) + a_1\, x(t-\tau) = 0, \tag{11}$$

with $(a_0, a_1, \tau) \in \mathbb{R}^2 \times \mathbb{R}_+^*$, then the associated characteristic function Δ is given by :

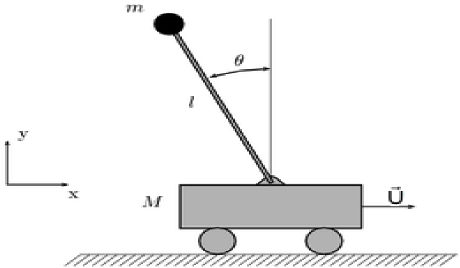
$$\Delta(\lambda) = \lambda + a_0 + a_1\, e^{-\lambda\tau}. \tag{12}$$

Zero is a spectral value for (11) if and only if $a_0 + a_1 = 0$. Computations of the first derivatives of (12) with respect to λ give with the notation $\frac{\partial .}{\partial \lambda} = .'$

$$\Delta'(\lambda,\tau) = 1 - \tau\, a_1\, e^{-\lambda\tau},$$
$$\Delta''(\lambda,\tau) = \tau^2\, a_1\, e^{-\lambda\tau}.$$

The multiplicity of the zero spectral value is **at most** two. The algebraic multiplicity two is insured by $\tau = 1/a_1,\ a_0 = -a_1$.

Inverted Pendulum on a cart



In the dimensionless form, the dynamics of the inverted pendulum on a cart is governed by the following second-order differential equation :

$$\left(1 - \frac{3\epsilon}{4}\cos^2(\theta)\right)\ddot{\theta} + \frac{3\epsilon}{8}\dot{\theta}^2\sin(2\theta) - \sin(\theta) + U\cos(\theta) = 0, \quad (13)$$

where $\epsilon = m/(m+M)$, M the mass of the cart and m the mass of the pendulum and U represents the horizontal driving force.

Effect of the delay

$U = a_{1,0}\,\theta(t-\tau_1) + a_{2,0}\,\theta(t-\tau_2)$
$\lambda_0 = (a_0, b_0, {\tau_1}^*, {\tau_2}^*) = (-7, 8, 1, \frac{7}{8}) \Rightarrow$ the zero spectral value admits an algebraic multiplicity 3 and a geometric multiplicity 1.
Dans la variété du centre $h_6(z)$

$$\dot{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & \gamma \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ u_1^3 \end{bmatrix}.$$

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- Multi Delayed Proportionals $\equiv$ Delayed PD.
- **The multiplicity of the zero root can exceed the number of scalar equations** $\Rightarrow$ The time-delay enriches the nonlinear dynamics.

Vandermonde matrices

To the best of the author knowledge, the first time the Vandermonde matrix appears in a control problem is reported in [Kailath 1998], where the controllability of a finite dimensional dynamical system is guaranteed by the invertibility of such a matrix. Next, in the context of time-delay systems, the use of Vandermonde matrix properties was proposed by [Niculescu & Michiels 2004] when controlling one chain of integrators by delay blocks.

Vandermonde & Birkhoff incidence matrices

Initially, Birkhoff and Vandermonde matrices are derived from the problem of polynomial interpolation of an unknown function g, that can be presented in a general way by describing the interpolation conditions in terms of incidence matrices. For a given integers $n \geq 1$ and $r \geq 0$, the matrix

$$\mathcal{E} = \begin{pmatrix} e_{1,0} & \cdots & e_{1,r} \\ \vdots & & \vdots \\ e_{n,0} & \cdots & e_{n,r} \end{pmatrix},$$

is called an incidence matrix if $e_{i,j} \in \{0,1\}$ for every i and j. Such a matrix contains the data providing the known information about the function g.

Birkhoff interpolation problem

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_1 < \ldots < x_n$, the problem of determining a polynomial $\hat{P} \in \mathbb{R}[x]$ with degree less or equal to r that interpolates g at $(x, \mathcal{E})$, i.e. which satisfies the conditions :

$$\hat{P}^{(j)}(x_i) = g^{(j)}(x_i)$$

is known as the **Birkhoff interpolation problem**.
An incidence matrix $\mathcal{E}$ is said to be **regular (or poised)** if such a polynomial $\hat{P}$ is unique.

An example

Consider the incidence matrix

$$\mathcal{E} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{14}$$

for which the associated **Birkhoff matrix** is given by

$$\Upsilon_{\mathcal{E}}^{T} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_2 \\ 0 & 1 & 2x_3 & 3x_3^2 \end{pmatrix}.$$

The interpolation problem is solvable if and only if

$$12\, x_3\, x_2 + 6\, x_1^2 - 12\, x_2\, x_1 - 6\, x_3^2$$

does not vanish for all values of x such that $x_1 < x_2 < x_3$.

Vandermonde a particular Birkhoff matrix

- When the first column of $\mathcal{E}$ is $\mathbf{1}$ and the remaining coefficients of $\mathcal{E}$ are zeros then we are dealing with Vandermonde case.
- When the first column of $\mathcal{E}$ is $\mathbf{1}$ and there are no zeros between two ones in the same row of $\mathcal{E}$ then we are dealing with Confluent-Vandermonde case.
- $$\Upsilon_{\mathcal{E}}^{T} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_2 \\ 0 & 1 & 2x_3 & 3x_3^2 \end{pmatrix} \leftrightarrow \mathcal{V}_{\mathcal{E}} = (x_1, x_1, \star, \star, x_2, \star, x_3)$$

The problem formulation

Consider a the general time-delay system with discrete delays :

$$\dot{x} = \sum_{k=0}^{N} A_k x(t - \tau_k) \tag{15}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ denotes the state-vector, under appropriate initial conditions belonging to the Banach space of continous functions $\mathcal{C}([-\tau_N, 0], \mathbb{R}^n)$. Here τ_j, $j = 1 \ldots N$ are strictly increasing positive constant delays with $\tau_0 = 0$ and $0 < \tau_1 < \tau_2 < \ldots < \tau_N$, the matrices $A_j \in \mathcal{M}_n(\mathbb{R})$ for $j = 0 \ldots N$.

System (15) has a characteristic function $\Delta : \mathbb{C} \times \mathbb{R}^N{}_+ \to \mathbb{C}$ of the form :

$$\Delta(\lambda, \tau) = \det\left(\lambda\, I - A_0 - \sum_{k=1}^{N} A_k\, e^{-\tau_k \lambda}\right) \tag{16}$$

or shorter, denoted $\Delta(\lambda)$, which gives

$$\Delta(\lambda) = P_0(\lambda) + \sum_{M^k \in S_{N,n}} P_{M^k}(\lambda)\, e^{\sigma_{M^k}\,\lambda} = P_0(\lambda) + \sum_{k=1}^{\tilde{N}_{N,n}} P_{M^k}(\lambda)\, e^{\sigma_k\,\lambda} \tag{17}$$

where $\sigma_{M^k} = -M^k\,\tau^T$, $\tau = (\tau_1, \ldots, \tau_N)$ is the delays vector and $S_{N,n}$ is the set of all the possible row vectors $M^k = (M_1^k, \ldots, M_N^k)$ belonging to $\mathbb{R}^{N^*}$ such that $1 \leq M_1^k + \ldots + M_N^k \leq n$ and $\tilde{N}_{N,n} = \#(S_{N,n})$. For instance,

$$\begin{aligned} S_{3,2} = \{&(1,0,0),\ (0,1,0),\ (0,0,1),\ (2,0,0),\ (1,1,0),\\ &(1,0,1),\ (0,2,0),\ (0,1,1),\ (0,0,2)\}, \end{aligned}$$

is ordered first by increasing sums ($\sum_{i=1}^{N} M_i^k$) then by lexicographical order, in this case one has :

$$M^2 = (0,1,0) \quad \text{and} \quad \tilde{N}_{3,2} = 9.$$

Proposition (Pólya-Szegö, 1972)

Let $\tau_1 < \tau_2 < \ldots < \tau_N$, denote real numbers and $d_1, \ldots, d_N$ positive integers satisfying

$$d_1 \geq 1,\ d_2 \geq 1\ \ldots\ d_N \geq 1, \quad d_1 + d_2 + \ldots + d_N = D + N.$$

Let $f_{i,j}(s)$ stands for the function $f_{i,j}(s) = s^{j-1}\, e^{\tau_i\, s}$, for $1 \leq j \leq d_i$ and $1 \leq i \leq N$.
Let $\sharp$ be the number of zeros of the function

$$f(s) = \sum_{1 \leq i \leq N,\ 1 \leq j \leq d_i} c_{i,j}\, f_{i,j}(s),$$

that are contained in the horizontal strip $\alpha \leq \mathcal{I}(z) \leq \beta$.
Assuming that $\sum_{1 \leq k \leq d_1} |c_{1,k}| > 0,\ \ldots,\ \sum_{1 \leq k \leq d_N} |c_{N,k}| > 0$, then

$$\frac{(\tau_N - \tau_1)\,(\beta - \alpha)}{2\,\pi} - D + 1 \leq \sharp \leq \frac{(\tau_N - \tau_1)\,(\beta - \alpha)}{2\,\pi} + D + N - 1.$$

The regular case (confluent Vandermonde)

When dealing with (generic) complete polynomials.

Proposition (1)

The multiplicity of the zero root for the generic quasipolynomial function (17) *cannot be larger than Pólya-Szegö bound* $D + \tilde{N}_{N,n}$, *(where* D *is the degree of the quasipolynomial and* $\tilde{N}_{N,n} + 1$ *the number of the associated polynomials). Moreover, such a bound is reached if and only if the parameters of* (17) *satisfy simultaneously for* $0 \leq k \leq D + \tilde{N}_{N,n} - 1$ *:*

$$a_{0,k} = - \sum_{i \in S_{N,n}} \left(a_{i,k} + \sum_{l=0}^{k-1} \frac{a_{i,l} {\sigma_i}^{k-l}}{(k-l)!} \right). \qquad (18)$$

Sketch of the proof

Lemma

Zero is a root of $\Delta^{(k)}(\lambda)$ for $k \geq 0$ if and only if the coefficients of P_{M^j} for $0 \leq j \leq \tilde{N}_{N,n}$ satisfy the following assertion

$$a_{0,k} = -\sum_{i \in S_{N,n}} \left[a_{i,k} + \sum_{l=0}^{k-1} \frac{a_{i,l} {\sigma_i}^{k-l}}{(k-l)!} \right]. \tag{A.1}$$

Interpretation of the Lemma

- The n first equations gives the coefficients of the delay-free polynomial as functions of the coefficients of the polynomials associated with the delays.
- The $(n+1)$-th equation gives $n! =$ linear combination in the coefficients of the polynomials associated with the delays.
- From the $n+2$-th equation we select a square generalized Birkhoff matrix.
- If the determinant of the obtained Birkhoff matrix is non singular then we have the bound.

Consider the quasipolynomial function :

$$\begin{aligned}\Delta(\lambda) =& \lambda^2 + a_{0,0,1}\lambda + a_{0,0,0} + (a_{1,0,0} + a_{1,0,1}\lambda)\, e^{\lambda\sigma_{1,0}} \\ &+ (a_{0,1,0} + a_{0,1,1}\lambda)\, e^{\lambda\sigma_{0,1}} \\ &+ a_{2,0,0}e^{\lambda\sigma_{2,0}} + a_{1,1,0}e^{\lambda\sigma_{1,1}} + a_{0,2,0}e^{\lambda\sigma_{0,2}}.\end{aligned} \tag{19}$$

$\Upsilon_1\, a = a_0$, $\nabla_2(0) = 0$ and $\Upsilon_2\, a = 0$ where $a_0 = (a_{0,0,0}, a_{0,0,1})^T$:

$$\Upsilon_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ \sigma_{1,0} & 1 & \sigma_{0,1} & 1 & \sigma_{2,0} & \sigma_{1,1} & \sigma_{0,2} \end{bmatrix},$$

$$\nabla_2(0) - 2 = \begin{bmatrix} {\sigma_{1,0}}^2 & 2\,\sigma_{1,0} & {\sigma_{0,1}}^2 & 2\,\sigma_{0,1} & {\sigma_{2,0}}^2 & {\sigma_{1,1}}^2 & {\sigma_{0,2}}^2 \end{bmatrix} a$$

$$\Upsilon_2 = \begin{bmatrix} {\sigma_{1,0}}^3 & 3\,{\sigma_{1,0}}^2 & {\sigma_{0,1}}^3 & 3\,{\sigma_{0,1}}^2 & {\sigma_{2,0}}^3 & {\sigma_{1,1}}^3 & {\sigma_{0,2}}^3 \\ {\sigma_{1,0}}^4 & 4\,{\sigma_{1,0}}^3 & {\sigma_{0,1}}^4 & 4\,{\sigma_{0,1}}^3 & {\sigma_{2,0}}^4 & {\sigma_{1,1}}^4 & {\sigma_{0,2}}^4 \\ {\sigma_{1,0}}^5 & 5\,{\sigma_{1,0}}^4 & {\sigma_{0,1}}^5 & 5\,{\sigma_{0,1}}^4 & {\sigma_{2,0}}^5 & {\sigma_{1,1}}^5 & {\sigma_{0,2}}^5 \\ {\sigma_{1,0}}^6 & 6\,{\sigma_{1,0}}^5 & {\sigma_{0,1}}^6 & 6\,{\sigma_{0,1}}^5 & {\sigma_{2,0}}^6 & {\sigma_{1,1}}^6 & {\sigma_{0,2}}^6 \\ {\sigma_{1,0}}^7 & 7\,{\sigma_{1,0}}^6 & {\sigma_{0,1}}^7 & 7\,{\sigma_{0,1}}^6 & {\sigma_{2,0}}^7 & {\sigma_{1,1}}^7 & {\sigma_{0,2}}^7 \\ {\sigma_{1,0}}^8 & 8\,{\sigma_{1,0}}^7 & {\sigma_{0,1}}^8 & 8\,{\sigma_{0,1}}^7 & {\sigma_{2,0}}^8 & {\sigma_{1,1}}^8 & {\sigma_{0,2}}^8 \\ {\sigma_{1,0}}^9 & 9\,{\sigma_{1,0}}^8 & {\sigma_{0,1}}^9 & 9\,{\sigma_{0,1}}^8 & {\sigma_{2,0}}^9 & {\sigma_{1,1}}^9 & {\sigma_{0,2}}^9 \end{bmatrix}.$$

Theorem

Given the generalized confluent Vandermonde matrix, the unique LU-factorization with unitary diagonal elements $L_{i,i} = 1$ is given by the formulae :

$$\begin{cases} L_{i,1} = x_1^{i-1} \quad \textit{for} \quad 1 \leq i \leq \delta, \\ U_{1,j} = \Upsilon_{1,j} \quad \textit{for} \quad 1 \leq j \leq \delta, \\ L_{i,j} = L_{i-1,j-1} + L_{i-1,j}\, \xi_j \quad \textit{for} \quad 2 \leq j \leq i, \\ U_{i,j} = (\varkappa(j) - 1)\, U_{i-1,j-1} + U_{i-1,j} \left(x_{\varrho(j)} - \xi_{i-1} \right) \quad \textit{for} \quad 2 \leq i \leq j. \end{cases} \tag{20}$$

Corollary

The diagonal elements of the matrix U associated with the generalized confluent Vandermonde matrix Υ are obtained as follows :

$$\begin{cases} U_{1,1} = x_1^{n+1}, \\ U_{j,j} = x_{k+1}^{n+1} \displaystyle\prod_{l=1}^{k} (x_{k+1} - x_l)^{d_l} \text{ when } j = 1 + d_k \text{ for } 1 \leq k \leq M-1, \\ U_{j,j} = (j - 1 - d_k)\, U_{j-1,j-1} \text{ when } d_k + 1 < j \leq d_{k+1} \text{ for } 1 \leq k \leq M \end{cases}$$

Moreover, the generalized confluent Vandermonde matrix Υ is invertible if and only if $\forall\ 1 \leq i \neq j \leq \delta$ we have $x_i \neq 0$ and $x_i \neq x_j$.

The sparse case

Proposition (2)

Consider a quasipolynomial function (17) *containing one or several incomplete polynomials, for which we associate an incidence matrix* $\mathcal{E}$*. When the associated generalized Birkhoff matrix* $\Upsilon_{\tilde{\mathcal{E}}}$ *is nonsingular then the multiplicity of the zero root for the quasipolynomial function* (17) *cannot be larger than n plus the number of nonzero coefficients of the polynomial family* $(P_{M^k})_{M^k \in S_{N,n}}$.

Remark

Obviously, the number of non-zero coefficients of a given quasipolynomial function is bounded by its degree plus its number of polynomials. Thus, the bound elaborated in Proposition 3 is sharper than the one established in Proposition 2 , even in the generic case, that is when all the parameters of the quasipolynomial are left free, these two bounds are equal.

Example of results in the sparse case

In all generality, a nondegenerate generalized Birkhoff matrix with incidence vector

$$\mathcal{V}_{\mathcal{E}} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{x_2, \ldots, x_2}_{d_2^-}, \underbrace{\star, \ldots, \star}_{d_*}, \underbrace{x_2, \ldots, x_2}_{d_2^+})$$

is $LU-$factorizable where the associated L and U matrices are with rational coefficients in the variables x_1 and x_2. Nevertheless, there exists a unique configuration in which L and U conserve their polynomial structure (as in the regular case), which occurs when $d_2^+ = 1$.

Theorem

Given the generalized Birkhoff matrix with incidence vector

$$\mathcal{V}_{\mathcal{E}} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{x_2, \ldots, x_2}_{d_2^-}, \underbrace{\star, \ldots, \star}_{d_*}, x_2),$$

the unique LU-factorization with unitary diagonal elements $L_{i,i} = 1$ is given by the formulae :

$$L_{i,1} = x_1^{i-1} \quad \text{for} \quad 1 \le i \le d_1 + d_2^- + 1, \tag{21}$$

$$U_{1,j} = \Upsilon_{1,j} \quad \text{for} \quad 1 \le j \le d_1 + d_2^- + 1, \tag{22}$$

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}\, \xi_j \quad \text{for} \quad 2 \le j \le i \le d_1 + d_2^- + 1, \tag{23}$$

$$U_{i,j} = (\varkappa(j) - 1)\, U_{i-1,j-1} + U_{i-1,j}\left(x_{\varrho(j)} - \xi_{i-1}\right) \quad \text{for} \quad 2 \le i \le j \le d_1 + d_2^-, \tag{24}$$

$$U_{i,j} = \Upsilon_{i,j} - (i-1)\int_0^{x_1} U_{i-1,j}(y, x_2)\, dy \quad \text{for} \quad j = d_1 + d_2^- + 1 \text{ and } 2 \le i \le d_1 + 1, \tag{25}$$

$$U_{i,j} = (j + d^* - (i-1))\int_0^{x_2} U_{i-1,j}(x_1, y)\, dy \quad \text{for} \quad j = d_1 + d_2^- + 1 \text{ and } d_1 + 2 \le i \le j, \tag{26}$$

where $\xi = (\underbrace{x_1, \ldots, x_1}_{d_1+1})$

Sketch of the proof

Lemma

- (25) *is equivalent to :*

$$U_{i,j} = \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^l \, x_1^l \, \Upsilon_{i-l,j}$$

$$\textit{for} \quad j = d_1 + d_2^- + 1 \textit{ and } 2 \leq i \leq d_1 + 1.$$

- $$\Upsilon_{i+1,j} = x_2 \, \Upsilon_{i,j} + (d_2^- + d^*) \int_0^{x_2} \Upsilon_{i,j}$$

$$\textit{for} \quad j = d_1 + d_2^- + 1 \textit{ and } 1 \leq i \leq d_1 + d_2^- .$$

Conclusion

- By this talk we give an adaptive sharp bound for multiplicity of the zero spectral value.
- We identify the link with Birkhoff interpolation problem.
- Under the hypothesis :

$$\Delta(i\omega) = 0 \Rightarrow \omega = 0 \tag{H}$$

that is all the imaginary roots are located at the origin, then the dimension of the projected state on the center manifold associated with zero singularity for equation (17) is less or equal to its number of nonzero coefficients minus one.

Future works and difficulties

- General formulas for LU-factorization for Birkhoff matrices in the sparse case (with rational L and U).

Thank you for your attention !
Questions ?