An algebraic analysis approach to the simplification of systems of linear functional equations.

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Introduction / Motivation

Motivations

◇ Objective of this work:

Use formal methods (algebraic manipulations) to simplify systems coming from mathematical physics, applied mathematics, engineering sciences or control theory

◇ Interest:

- Simplify the equations of the system
  $\Rightarrow$ simplify the study of its structural properties
- Pre-conditioner to numerical analysis methods

General methodology

1. A linear system is defined by a matrix R with coefficients in a ring D of functional operators:

$$R\,y = 0. \quad (\star)$$

2. To $(\star)$ we associate a left D-module M (finitely presented).
3. There exists a dictionary between the properties of $(\star)$ and M.
4. Homological algebra allows to check the properties of M.
5. Effective algebra (non-commutative Gröbner/Janet bases) gives algorithms.
6. Implementation (Maple, Singular:Plural, GAP4, . . .).

Example

◇ Linearization of the Navier-Stokes $\sim$ a parabolic Poiseuille profile

$$\begin{cases} \partial_t\, u_1 + 4\, y\, (1 - y)\, \partial_x\, u_1 - 4\, (2\, y - 1)\, u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, u_1 + \partial_x\, p = 0, \\ \partial_t\, u_2 + 4\, y\, (1 - y)\, \partial_x\, u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, u_2 + \partial_y\, p = 0, \qquad (*) \\ \partial_x\, u_1 + \partial_y\, u_2 = 0. \qquad \text{(e.g., Vazquez-Krstic, IEEE 07)} \end{cases}$$

◇ Let us introduce the so-called Weyl algebra $A_3(\mathbb{Q}(\nu))$

$$D = \mathbb{Q}(\nu)[t, x, y] \left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right] \left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right] \left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right].$$

$$(\partial_x\, y = y\, \partial_x,\ \partial_x\, x = x\, \partial_x + 1,\ \partial_x\, \partial_y = \partial_y\, \partial_x \ldots):$$

◇ The system $(*)$ is defined by the matrix of PD operators:

$$\begin{pmatrix} \partial_t + 4\, y\, (1 - y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & -4\, (2\, y - 1) & \partial_x \\ 0 & \partial_t + 4\, y\, (1 - y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}.$$

D: ring of functional operators (*e.g.*, non-commutative)

◇ Differential operators: $A = \mathbb{Q},\ \mathbb{Q}[x_1, \ldots, x_n],\ \mathbb{Q}(x_1, \ldots, x_n)$,

$$D = A[\partial_1, \ldots, \partial_n], \quad \partial_i = \frac{\partial}{\partial x_i},$$

$$P = \sum_{0 \leq |\mu| \leq m} a_\mu(x)\, \partial^\mu \in D, \quad \partial^\mu = \partial_1^{\mu_1} \ldots \partial_n^{\mu_n}, \quad a_\mu \in A.$$

◇ Shift operators: $A = \mathbb{Q},\ \mathbb{Q}[n],\ \mathbb{Q}(n)$,

$$D = A[\sigma],$$

$$P = \sum_{i=0}^{m} a_i(n)\, \sigma^i \in D, \quad \sigma(a(n)) = a(n+1).$$

◇ Differential time-delay operators: $A = \mathbb{Q},\ \mathbb{Q}[t],\ \mathbb{Q}(t)$,

$$D = A[\tfrac{d}{dt}, \delta],$$

$$P = \sum_{0 \leq i+j \leq m} a_{ij}(t)\, \frac{d^i}{dt^i}\, \delta^j \in D, \quad \delta(a(t)) = a(t-h).$$

◇ For every monomial order, there exists a Gröbner basis which can be computed by Buchberger algorithm.

Algebraic analysis

◇ D ring of functional operators, $R \in D^{q \times p}$ and $\mathcal{F}$ a left D-module (the functional space):

$$\forall\, P_1,\, P_2 \in D,\; \forall\, \eta_1,\, \eta_2 \in \mathcal{F} :\; P_1\, \eta_1 + P_2\, \eta_2 \in \mathcal{F}$$

◇ Consider the linear system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}$

◇ As in number theory or algebraic geometry, with $\ker_{\mathcal{F}}(R.)$ we associate the left D-module:

$$M := \mathrm{coker}_D(.R) = D^{1 \times p} / (D^{1 \times q}\, R),$$

given by the finite presentation:

$$\begin{array}{cccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\, R & & & & \end{array}$$

◇ **Malgrange's remark**: $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$

Problems considered in this talk

◇ Decomposition problem: $M = M_1 \oplus M_2$

$$\exists\, W \in \mathrm{GL}_p(D),\ V \in \mathrm{GL}_q(D) \text{ s.t. } V\,R\,W = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}?$$

◇ Serre's reduction problem: find a presentation of M defined by less generators and less relations

$$\exists\, W \in \mathrm{GL}_p(D),\ V \in \mathrm{GL}_q(D) \text{ s.t. } V\,R\,W = \begin{pmatrix} I_r & 0 \\ 0 & R_2 \end{pmatrix}?$$

◇ Isomorphism / Equivalence problem: testing whether two linear systems (resp. modules) are isomorphic

$\rightsquigarrow$ Important and difficult issue in system (resp. module) theory

Part 1

Decomposition problem

Homomorphims of finitely presented modules (1)

$\diamond$ Let D be an Ore algebra of functional operators.

$\diamond$ Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.

$\diamond$ Let us consider the finitely presented left D-modules:

$$M = D^{1\times p}/(D^{1\times q}\, R), \quad M' = D^{1\times p'}/(D^{1\times q'}\, R').$$

$\diamond$ We are interested in the abelian group $\hom_D(M, M')$ of D-morphisms from M to M':

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

Homomorphims of finitely presented modules (1)

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$$\exists\, f : M \to M' \Longleftrightarrow \exists\, P \in D^{p\times p'},\ Q \in D^{q\times q'} \text{ such that:}$$

$$R\,P = Q\,R'.$$

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda\, P)$, for all $\lambda \in D^{1\times p}$.

Homomorphims of finitely presented modules (1)

◇ Let D be an Ore algebra of functional operators.

◇ Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.

◇ We have the following commutative exact diagram:

$$
\begin{array}{ccccccccc}
 & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & .Z_2 \swarrow & \downarrow .Q & \swarrow .Z & \downarrow .P & & \downarrow f & & \\
D^{1\times r'} & \xrightarrow{.R_2'} & D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}
$$

- $\overline{P} = P + Z\,R'$,
- $\overline{Q} = Q + R\,Z + Z_2\,R_2'$,
- $R\,\overline{P} = \overline{Q}\,R'$.

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda\,P) = \pi'(\lambda\,\overline{P})$, for all $\lambda \in D^{1\times p}$.

Homomorphims of finitely presented modules (2)

◇ $\hom_D(M, M')$ can be written as a quotient: let

$$L = \{P \in D^{p \times p'} \mid \exists Q \in D^{q \times q'} : \; R\,P = Q\,R'\}.$$

We have

$$\hom_D(M, M') \cong L/(D^{p \times q'}\,R').$$

◇ If $M' = M$, then $\mathrm{end}_D(M)$ can be written as a quotient of the ring

$$L = \{P \in D^{p \times p} \mid \exists Q \in D^{q \times q} : \; R\,P = Q\,R\}$$

by the two-sided ideal $D^{p \times q}\,R$:

$$\mathrm{end}_D(M)^{\mathrm{op}} \cong L/(D^{p \times q}\,R).$$

Computation of $\hom_D(M, M')$: commutative case (1)

$\diamond$ If D is a commutative ring, then $\hom_D(M, M')$ is a D-module
$\rightsquigarrow$ We are going to compute a family of generators with their relations

$\diamond$ The Kronecker product of $E \in D^{q \times p}$ and $F \in D^{r \times s}$ is:

$$E \otimes F = \begin{pmatrix} E_{11}\, F & \dots & E_{1p}\, F \\ \vdots & \vdots & \vdots \\ E_{q1}\, F & \dots & E_{qp}\, F \end{pmatrix} \in D^{(q\, r) \times (p\, s)}.$$

Lemma: If $U \in D^{a \times b}$, $V \in D^{b \times c}$ and $W \in D^{c \times d}$, then we have:

$$\mathrm{row}(U\, V\, W) = \mathrm{row}(V)\, (U^T \otimes W).$$

$$\mathrm{row}(R\, P\, I_{p'}) = \mathrm{row}(P)\, (R^T \otimes I_{p'}), \quad \mathrm{row}(I_q\, Q\, R') = \mathrm{row}(Q)\, (I_q \otimes R').$$

We are reduced to computing $\ker_D\left(.\begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix}\right)$.

Computation of $\hom_D(M, M')$: commutative case (2)

◇ This way we get a family $\{P_1, \dots, P_s\}$ of generators of L

◇ Then, we reduce the rows of the P_i with respect to G a Gröbner basis of the rows of R' to get a family of generators $\{f_1, \dots, f_s\}$ of $\hom_D(M, M')$

◇ Using another syzygies computation, we obtain the relations between generators: $\sum_{j=1}^{s} X_{ij}\, f_j = 0,\, i = 1, \dots, \ell$

◇ If $M' = M$, we can also compute the table of multiplication of the generators: $f_i \circ f_j = \sum_{j=1}^{s} \gamma_{ijk}\, f_k,\, i, j = 1, \dots, s$

If $D < F_1, \dots, F_s >$ is the free associated algebra generated by the F_i, then

$$\mathrm{end}_D(M) = D < F_1, \dots, F_s > / I,$$

where I is the two-sided ideal

$$I =< \sum_{j=1}^{s} X_{ij}\, F_j,\, i = 1, \dots, \ell,\, F_i \circ F_j - \sum_{k=1}^{s} \gamma_{ijk}\, F_k,\, i, j = 1, \dots, s >$$

Example: tank model Dubois-Petit-Rouchon, ECC99 (1)

◇ Let $D = \mathbb{Q}[\partial, \delta]$ ($\partial f(t) = \dot{f}(t)$, $\delta f(t) = f(t-h)$) and consider the system matrix

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2\times 3}, \quad M = D^{1\times 3}/(D^{1\times 2}\,R).$$

◇ Applying our algorithm: $\mathrm{end}_D(M)$ generated by the f_{e_i}, $i = 1, \ldots, 4$ defined by $f_\alpha(\pi(\lambda)) = \pi(\lambda\,P_\alpha)$, for all $\lambda \in D^{1\times 3}$, where

$$P_\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & 2\,\alpha_3\,\partial\,\delta \\ \alpha_2 + 2\,\alpha_4\,\partial & \alpha_1 - 2\,\alpha_4\,\partial & 2\,\alpha_3\,\partial\,\delta \\ \alpha_4\,\delta & -\alpha_4\,\delta & \alpha_1 + \alpha_2 + \alpha_3\,(\delta^2 + 1) \end{pmatrix},$$

$$Q_\alpha = \begin{pmatrix} \alpha_1 - 2\,\alpha_4\,\partial & \alpha_2 + 2\,\alpha_4\,\partial \\ \alpha_2 & \alpha_1 \end{pmatrix},$$

$\alpha = (\alpha_1, \ldots, \alpha_4) \in D^{1\times 4}$ and $\{e_j\}_{j=1,\ldots,4}$ the standard basis of $D^{1\times 4}$.

Example: tank model Dubois-Petit-Rouchon, ECC99 (2)

◇ We also obtain that the $\{f_{e_i}\}_{i=1,\ldots,4}$ satisfy the following *D*-linear relations:

$$(\delta^2 - 1)\, f_{e_4} = 0, \quad \delta^2\, f_{e_1} + f_{e_2} - f_{e_3} = 0, \quad f_{e_1} + \delta^2\, f_{e_2} - f_{e_3} = 0.$$

◇ The multiplication table is given by:

$f_{e_c} \circ f_{e_r}$	f_{e_1}	f_{e_2}	f_{e_3}	f_{e_4}
f_{e_1}	f_{e_1}	f_{e_2}	f_{e_3}	f_{e_4}
f_{e_2}	f_{e_2}	f_{e_1}	f_{e_3}	$2\,\partial\, f_{e_1} - 2\,\partial\, f_{e_2} + f_{e_4}$
f_{e_3}	f_{e_3}	f_{e_3}	$(\delta^2 + 1)\, f_{e_3}$	$2\,\partial\, f_{e_1} - 2\,\partial\, f_{e_2} + 2\, f_{e_4}$
f_{e_4}	f_{e_4}	$-f_{e_4}$	0	$-2\,\partial\, f_{e_4}$

◇ $I = <$
$(\delta^2 - 1)\, f_{e_4}, \delta^2\, f_{e_1} + f_{e_2} - f_{e_3}, f_{e_1} + \delta^2\, f_{e_2} - f_{e_3}, \ldots, f_{e_4} \circ f_{e_4} + 2\,\partial\, f_{e_4} >$
We have:

$$\mathrm{end}_D(M) = D < f_{e_1}, f_{e_2}, f_{e_3}, f_{e_4} > / I.$$

Computation of $\hom_D(M, M')$: non-commutative case

$\diamond$ If D is a non-commutative ring, then $\hom_D(M, M')$ is an abelian group and generally an infinite-dimensional k-vector space.

$\Rightarrow$ find a k-basis of morphisms with given degrees in x_i and in ∂_j:

1. Take an ansatz for P with chosen degrees.
2. Compute $R\,P$ and a Gröbner basis G of the rows of R'.
3. Reduce the rows of $R\,P$ w.r.t. G.
4. Solve the system on the coefficients of the ansatz so that all the normal forms vanish.
5. Substitute the solutions in P and compute Q by means of a factorization.

Beltrami equations

◇ Let $D = A_2(\mathbb{Q}) = \mathbb{Q}[x, y][\partial_x, \partial_y]$ and $M = D^{1\times 2}/(D^{1\times 2}\, R)$:

$$R = \begin{pmatrix} \partial_x & -x\partial_y \\ \partial_y & x\,\partial_x \end{pmatrix} \in D^{2\times 2}.$$

- $\mathrm{end}_D(M)_{0,1}$ is defined by $P = Q = a\, I_2$, $a \in \mathbb{Q}$.
- $\mathrm{end}_D(M)_{1,0}$ is defined by $(a_1, a_2 \in \mathbb{Q})$:

$$P = Q = \begin{pmatrix} a_1 + a_2\,\partial_y & 0 \\ 0 & a_1 + a_2\,\partial_y \end{pmatrix}.$$

- $\mathrm{end}_D(M)_{1,1}$ is defined by $(a_1, a_2, a_3 \in \mathbb{Q})$:

$$P = \begin{pmatrix} a_3\,(y\,\partial_y + x\,\partial_x - 1) + a_2\,\partial_y + a_1 & 0 \\ -a_3\,\partial_y & a_3\,y\,\partial_y + a_2\,\partial_y + a_1 \end{pmatrix},$$

$$Q = \begin{pmatrix} a_3\,(y\,\partial_y + x\,\partial_x) + a_2\,\partial_y + a_1 & a_3\,x\,\partial_y \\ 0 & a_2\,\partial_y + a_3\,y\,\partial_y + a_1 \end{pmatrix}.$$

Decomposition theorem - *C.-Q., 2008*

Theorem

Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ *and* $f \in \mathrm{end}_D(M)$ *defined by* P *and* Q *satisfying*

$$P^2 = P, \quad Q^2 = Q \quad (\textit{idempotent matrices}) \quad \Rightarrow f^2 = f.$$

If the left D*-modules*

$$\ker_D(.P),\ \mathrm{im}_D(.P),\ \ker_D(.Q),\ \mathrm{im}_D(.Q)$$

are free, then there exist $U \in \mathrm{GL}_p(D)$, $V \in \mathrm{GL}_q(D)$ *such that*

$$V\, R\, U^{-1} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \in D^{q\times p}.$$

Dirac equations (1)

◇ Let us consider the following complex matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

◇ The Dirac equation has the form $\sum_{i=1}^4 \gamma^i \, \partial y / \partial x_i = 0$:

$$\begin{cases} d_4 \, y_1 - i \, d_3 \, y_3 - (i \, d_1 + d_2) \, y_4 = 0, \\ d_4 \, y_2 - (i \, d_1 - d_2) \, y_3 + i \, d_3 \, y_4 = 0, \\ i \, d_3 \, y_1 + (i \, d_1 + d_2) \, y_2 - d_4 \, y_3 = 0, \\ (i \, d_1 - d_2) \, y_1 - i \, d_3 \, y_2 - d_4 \, y_4 = 0, \end{cases} \qquad d_i = \partial / \partial x_i.$$

Dirac equations (2)

$\diamond$ Let us consider $D = \mathbb{Q}(i)[d_1, d_2, d_3, d_4]$, the matrix

$$R = \begin{pmatrix} d_4 & 0 & -i\, d_3 & -(i\, d_1 + d_2) \\ 0 & d_4 & -i\, d_1 + d_2 & i\, d_3 \\ i\, d_3 & i\, d_1 + d_2 & -d_4 & 0 \\ i\, d_1 - d_2 & -i\, d_3 & 0 & -d_4 \end{pmatrix} \in D^{4 \times 4},$$

and the finitely presented D-module $M = D^{1 \times 4}/(D^{1 \times 4}\, R)$.

$\diamond$ Computing idempotents of $\mathrm{end}_D(M)$, we obtain a idempotent f defined by the pair of matrices:

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$\diamond$ We have $P^2 = P$ and $Q^2 = Q$, i.e., the D-modules $\ker_D(.P)$, $\mathrm{im}\,(.P)$, $\ker_D(.Q)$ and $\mathrm{im}\,(.Q)$ are projective, i.e., free.

Dirac equation

$\diamond$ Computing bases for these modules, we then get:

$$\Rightarrow U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix},$$

$\diamond$ The matrix R is then equivalent to the block-diagonal one:

$$\Rightarrow V\,R\,U^{-1} = \begin{pmatrix} i\,d_3 - d_4 & -i\,d_1 - d_2 & 0 & 0 \\ i\,d_1 - d_2 & i\,d_3 + d_4 & 0 & 0 \\ 0 & 0 & i\,d_3 + d_4 & i\,d_1 + d_2 \\ 0 & 0 & i\,d_1 - d_2 & -i\,d_3 + d_4 \end{pmatrix}.$$

Part 2

Serre's reduction

String with an interior mass (Fliess et al, COCV 98)

$$\left\{\begin{array}{l}\phi_1(t)+\psi_1(t)-\phi_2(t)-\psi_2(t)=0,\\ \dot{\phi}_1(t)+\dot{\psi}_1(t)+a\,\phi_1(t)-a\,\psi_1(t)-b\,\phi_2(t)+b\,\psi_2(t)=0,\\ \phi_1(t-2\,h_1)+\psi_1(t)-u(t-h_1)=0,\\ \phi_2(t)+\psi_2(t-2\,h_2)-v(t-h_2)=0.\end{array}\right. \qquad (\star)$$

$$\partial f(t)=\dot{f}(t), \quad \sigma_1 f(t)=f(t-h_1), \quad \sigma_2 f(t)=f(t-h_2).$$

$$V\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0\\ \partial+a & \partial-a & -b & b & 0 & 0\\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0\\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} W$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \partial_1+a+b & a\,\sigma_1 & b\,\sigma_2 \end{pmatrix}.$$

$$(\star) \;\Leftrightarrow\; \dot{z}_1(t)+(a+b)\,z_1(t)+a\,z_2(t-h_1)+b\,z_3(t-h_2)=0.$$

String with an interior mass (Fliess et al, COCV 98)

The unimodular matrices V and W are defined by:

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \sigma_1^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ (a - b - \partial)\,\sigma_1^2 + \partial + a & -1 & \partial - a + b & -2\,b \end{pmatrix} \in \mathrm{GL}_4(D),$$

$$W = \begin{pmatrix} 1 & 0 & 0 & -1 & -\sigma_1 & 0 \\ 0 & -1 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \sigma_2 \\ 0 & -1 & -1 & -1 & 0 & -\sigma_2 \\ 0 & 0 & 0 & -\sigma_1 & -\sigma_1^2 + 1 & 0 \\ 0 & -\sigma_2 & -\sigma_2 & -\sigma_2 & 0 & -\sigma_2^2 + 1 \end{pmatrix} \in \mathrm{GL}_6(D).$$

Serre's reduction isomorphism - *Boudellioua-Q., 2010*

◇ Let $R \in D^{q \times p}$ be a full row rank matrix, $0 \leq r \leq q-1$, and $\Lambda \in D^{q \times (q-r)}$ such that there exists $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying:

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

◇ Let us denote by

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \in \mathrm{GL}_{p+q-r}(D),$$

where:

$$S_1 \in D^{p \times q}, \; S_2 \in D^{(q-r) \times q}, \; Q_1 \in D^{p \times (p-r)}, \; Q_2 \in D^{(q-r) \times (p-r)}.$$

◇ Then, we have:

$$M = D^{1 \times p}/(D^{1 \times q}\, R) \;\cong\; L = D^{1 \times (p-r)}/(D^{1 \times (q-r)}\, Q_2)$$

◇ The converse result also holds. These results only depend on:

$$\rho(\Lambda) \in \mathrm{ext}_D^1(M, D^{1 \times (q-r)}) \triangleq D^{q \times (q-r)}/(R\, D^{p \times (q-r)}).$$

Ring conditions

◇ Proposition: Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^{q \times (q-r)}$ such that $P = (R \quad -\Lambda) \in D^{q \times (p+q-r)}$ admits a right-inverse over D. Moreover, if D is either a

1. principal left ideal domain,
2. commutative polynomial ring with coefficients in a field,
3. Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - r \geq 2$,
4. ring of OD operators $A\langle\partial\rangle$, where $A = k[\![t]\!]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, and $p - r \geq 2$,

then there exists $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying that $P\,U = (I_q \quad 0)$.

◇ The matrix U can be obtained by means of:

- a Jacobson form (Jacobson),
- the Quillen-Suslin theorem (QuillenSuslin),
- Stafford's theorem (Stafford).

Serre's reduction equivalence - *Boudellioua-Q., 2010*

◇ If $\Lambda \in D^{q \times (q-r)}$ admits a left-inverse $\Gamma \in D^{(q-r) \times q}$, $\Gamma \, \Lambda = I_{q-r}$, then Q_1 admits the left-inverse $T_1 + T_2 \, \Gamma \, R \in D^{(p-r) \times p}$ and the left D-module $\ker_D(.Q_1)$ is stably free of rank r:

$$\ker_D(.Q_1) \oplus D^{1 \times (p+1-q)} \cong D^{1 \times p}.$$

◇ If the left D-module $\ker_D(.Q_1)$ is free, then $\exists \, Q_3 \in D^{p \times r}$ s.t.:

$$V = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D).$$

◇ Then, we have $W = (R \, Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$,

$$W^{-1} = \begin{pmatrix} Y_3 \, S_1 \\ -S_2 + Q_2 \, Y_1 \, S_1 \end{pmatrix},$$

with $V^{-1} = (Y_3^T \quad Y_1^T)^T$, $Y_3 \in D^{r \times p}$, $Y_1 \in D^{(p-r) \times p}$ and:

$$W^{-1} \, R \, V = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Example: Wind tunnel model

◇ The wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{cases}$$

◇ Let us consider $D = \mathbb{Q}(a, k, \omega, \zeta)[\partial, \delta]$, the system matrix

$$R = \begin{pmatrix} \partial + a & -k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3\times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 3}\,R)$.

Example: Wind tunnel model

- Let us consider $\Lambda = (1 \quad 0 \quad 0)^T$ and $P = (R \quad -\Lambda)$.
- The matrix P admits the following right-inverse S:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{\partial + 2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} \\ -1 & 0 & 0 \end{pmatrix} \in D^{5\times 3}.$$

- According to Quillen-Suslin theorem, $E = D^{1\times 5}/(D^{1\times 3}\,P)$ is a free D-module of rank 2.

Example: Wind tunnel model

◇ Computing a basis of E, we obtain that $U \in \mathrm{GL}_5(D)$,

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & 0 & \omega^2\,\partial \\ 0 & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \partial^2+2\,\zeta\,\omega\,\partial+\omega^2 \\ -1 & 0 & 0 & -(\partial+a) & -\omega^2\,k\,a\,\delta \end{pmatrix},$$

satisfies that $P\,U = (I_3 \quad 0)$ (OreModules, QuillenSuslin).

◇ The wind tunnel model is equivalent to the sole equation:

$$(\partial+a)\,\zeta_1+\omega^2\,k\,a\,\delta\,\zeta_2=0$$

$$\Leftrightarrow \quad \dot{\zeta}_1(t)+a\,\zeta_1(t)+\omega^2\,k\,a\,\zeta_2(t-h)=0.$$

Example: Wind tunnel model

◇ The vector $\Lambda = (1 \quad 0 \quad 0)^T$ admits the left-inverse $\Gamma = \Lambda^T$.

◇ We compute $Q_3 \in D^{2\times 2}$ such that $V = (Q_3^T \quad Q_1^T) \in \mathrm{GL}_4(D)$:

$$V = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & \omega^2\,\partial \\ -\frac{1}{\omega^2} & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & 0 & \partial^2+2\,\zeta\,\omega\,\partial+\omega^2 \end{pmatrix}.$$

◇ We have $W = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(D)$ and:

$$W^{-1}\,R\,V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(\partial+a) & -\omega^2\,k\,a\,\delta \end{pmatrix}.$$

Part 3

Isomorphism / Equivalence problem

Problem considered

$\diamond$ Isomorphism / Equivalence problem: testing whether two linear systems (resp. modules) are isomorphic.

$\rightsquigarrow$ Important issue in system (resp. module) theory.

Explicit characterization of isomorphic f. p. modules

Theorem

Let $M_1 = D^{1\times p}/(D^{1\times q}\, R_1)$, $M_2 = D^{1\times t}/(D^{1\times s}\, Q_2)$.

Then $M_1 \cong M_2$ *iff there exist matrices* $R_2 \in D^{q\times s}$, $Q_1 \in D^{p\times t}$, $S_1 \in D^{p\times q}$, $S_2 \in D^{s\times q}$, $T_1 \in D^{t\times p}$, $T_2 \in D^{t\times s}$, $V_1 \in D^{q\times l}$, $V_2 \in D^{t\times l}$, $W_1 \in D^{p\times m}$, *and* $W_2 \in D^{s\times m}$ *such that*

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t} + \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_{p+s} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \begin{pmatrix} 0 & P_2 \end{pmatrix},$$

where $P_1 \in D^{l\times q}$ *and* $P_2 \in D^{m\times s}$ *are defined by:*

$$\ker_D(.R_1) = D^{1\times l}\, P_1, \quad \ker_D(.Q_2) = D^{1\times m}\, P_2.$$

Consequence, links with Serre's reduction

Corollary

With the notations and the assumptions of the previous theorem, let $q+t=p+s=:u$*.*

1. *Then, we have:*

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_u \Leftrightarrow \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_u. \tag{1}$$

2. ◇ *If either* R_1 *or* Q_2 *has full row rank, then (1) holds.*
   ◇ *Equivalently, if* R_1 *or* Q_2 *has full row rank, then* $M_1 \cong M_2$ *is equivalent to the existence of* $R_2, Q_1, S_1, S_2, T_1, T_2$ *such that*
   $$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t}.$$

Isomorphisms from the unimodular completion problem

Theorem

Let p, q, s, t be 4 non-negative integers s.t. $q+t=p+s:=u$, *and* $R_1 \in D^{q\times p}$, $R_2 \in D^{q\times s}$, $Q_1 \in D^{p\times t}$, $Q_2 \in D^{s\times t}$, $S_1 \in D^{p\times q}$, $S_2 \in D^{s\times q}$, $T_1 \in D^{t\times p}$, *and* $T_2 \in D^{t\times s}$ *matrices such that:*

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_u, \quad \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_u.$$

Then, we have:

$$\left\{ \begin{array}{l} \mathrm{coker}_D(.R_1) \cong \mathrm{coker}_D(.Q_2),\ \ker_D(.R_1) \cong \ker_D(.Q_2), \\ \mathrm{coker}_D(.S_1) \cong \mathrm{coker}_D(.T_2),\ \ker_D(.S_1) \cong \ker_D(.T_2), \\ \mathrm{coker}_D(.Q_1) \cong \mathrm{coker}_D(.R_2),\ \ker_D(.Q_1) \cong \ker_D(.R_2), \\ \mathrm{coker}_D(.T_1) \cong \mathrm{coker}_D(.S_2),\ \ker_D(.T_1) \cong \ker_D(.S_2). \end{array} \right.$$

Consequences of the theorem

1. R_1 has full row rank iff so is Q_2.
2. R_2 admits a left inv. iff so is Q_1. More precisely:
   - $Z_2 \in D^{s \times q}$ left inv. of $R_2 \Longrightarrow T_1 - T_2\, Z_2\, R_1$ left inv. of Q_1.
   - $Y_1 \in D^{t \times p}$ left inv. of $Q_1 \Longrightarrow S_2 - Q_2\, Y_1\, S_1$ left inv. of R_2.
3. If R_2 or Q_1 admits a left inv., then $\ker_D(.R_2) \cong \ker_D(.Q_1)$ is a stably free left D-module of rank $q - s = p - t$.
4. $\ker_D(.R_2)$ is a free left D-module of rank r iff so is $\ker_D(.Q_1)$:
   - If $B_2 \in D^{r \times q}$ is a basis of $\ker_D(.R_2)$ (*i.e., B_2 has full row rank and satisfies* $\ker_D(.R_2) = D^{1 \times r}\, B_2$), then $C_2 := B_2\, R_1 \in D^{r \times p}$ is a basis of $\ker_D(.Q_1)$.
   - If $C_1 \in D^{r \times p}$ is a basis of $\ker_D(.Q_1)$, then $B_1 := C_1\, S_1 \in D^{r \times q}$ is a basis of $\ker_D(.Q_1)$.

New formulation of Serre's reduction equivalence

Theorem

If $R \in D^{q \times p}$ (not necessarily full row rank), then the following assertions are equivalent:

1. *There exist $\Lambda \in D^{q \times (q-r)}$ such that:*
   - *there exists $\Gamma \in D^{(q-r) \times q}$ satisfying $\Gamma \Lambda = I_q$,*
   - *the stably free left D-module $\ker_D(.\Lambda)$ is free of rank r, i.e., there exists a full row rank matrix $B \in D^{r \times q}$ such that $\ker_D(.\Lambda) = D^{1 \times r} B$,*
   - *there exists a matrix $U \in \mathrm{GL}_{p+q-r}(D)$ such that $(R \quad -\Lambda)\, U = (I_q \quad 0)$.*
2. *There exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$, $0 \leq r \leq q-1$, and $R_2 \in D^{(q-r) \times (p-r)}$ such that:*

$$V\, R\, W = \begin{pmatrix} I_r & 0 \\ 0 & R_2 \end{pmatrix}.$$

Theorem (H. Fitting, 1936)

Two matrices presenting isomorphic left D-modules can be inflated with blocks of 0 *and* I *to get equivalent matrices.*

Example: theory of 2D linear elasticity

$\diamond$ $(S) \begin{cases} \partial_1 \, \xi_1 = 0, \\ \frac{1}{2} \, (\partial_2 \, \xi_1 + \partial_1 \, \xi_2) = 0, \\ \partial_2 \, \xi_2 = 0, \end{cases}$ $\quad (S') \begin{cases} \partial_1 \, \zeta_1 = 0, \\ \partial_2 \, \zeta_1 - \zeta_2 = 0, \\ \partial_1 \, \zeta_2 = 0, \\ \partial_1 \, \zeta_3 + \zeta_2 = 0, \\ \partial_2 \, \zeta_3 = 0, \\ \partial_2 \, \zeta_2 = 0. \end{cases}$

$\diamond$ $D = \mathbb{Q}[\partial_1, \partial_2]$. We have:

$$(S) \Leftrightarrow R \, (\xi_1 \quad \xi_2)^T = 0, \quad (S') \Leftrightarrow R' \, (\zeta_1 \quad \zeta_2 \quad \zeta_3)^T = 0,$$

where

$$R = \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2} \, \partial_2 & \frac{1}{2} \, \partial_1 \\ 0 & \partial_2 \end{pmatrix}, \quad R' = \begin{pmatrix} \partial_1 & \partial_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & \partial_1 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & 0 \end{pmatrix}^T.$$

$\diamond$ We associate $M = D^{1 \times 2} / (D^{1 \times 3} \, R)$ and $M' = D^{1 \times 3} / (D^{1 \times 6} \, R')$.

Example: theory of 2D linear elasticity (continued)

◇ Algorithms computing homomorphisms are implemented in:

1. the Maple package OreMorphisms based on OreModules (F. Chyzak, A. Quadrat, D. Robertz):
   http://www.ensil.unilim.fr/~cluzeau/OreMorphisms/
2. GAP4/homalg (M. Barakat et al):
   http://wwwb.math.rwth-aachen.de/homalg/

Here, we find that the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

satisfy

$$R\,P = Q\,R',$$

and then define a homomorphism between M and M'.

Characterization of an isomorphism

◇ $f \in \hom_D(M, M')$ given by P and Q such that $R\,P = Q\,R'$.

◇ Let $S \in D^{r \times p}$, $T \in D^{r \times q'}$ be such that

$$\ker_D\left(.(P^T \quad R'^T)^T\right) = D^{1 \times r}\,(S \quad -T),$$

and $L \in D^{q \times r}$, $S_2 \in D^{r_2 \times r}$ s.t. $R = L\,S$ and $\ker_D(.S) = D^{1 \times r_2}\,S_2$.

$$\ker f = (D^{1 \times r}\,S)/(D^{1 \times q}\,R) \cong D^{1 \times r}/\left(D^{1 \times (q+r_2)}\,\left(L^T \quad S_2^T\right)^T\right).$$

◇ Moreover: $\operatorname{coker} f = D^{1 \times p'}/\left(D^{1 \times (p+q')}\,(P^T \quad R'^T)^T\right)$.

◇ Then f is an isomorphism iff the matrices $(L^T \quad S_2^T)^T$ and $(P^T \quad R'^T)^T$ admit a left inverse.

◇ All matrices can be computed from P and Q using Gröbner bases techniques (OreMorphisms, GAP4/homalg, ...).

Example: theory of 2D linear elasticity (continued)

◇ Using this result and, for example OREMORPHISMS, we can easily check that the homomorphism defined by the matrices P and Q given before is an isomorphism so that for our example

$$M \cong M'.$$

◇ Fitting's theorem $\Rightarrow$ R and R' can be inflated with blocks of 0 and I to get equivalent matrices.

◇ Goal of the following: construct this equivalence of matrices.

Explicit formula for the inverse of an isomorphism

◇ $f \in \hom_D(M, M')$ given by P and Q such that $R\,P = Q\,R'$.

◇ $f \in \mathrm{iso}_D(M, M')$ iff there exist $P' \in D^{p' \times p}$, $Q' \in D^{q' \times q}$, $Z \in D^{p \times q}$ and $Z' \in D^{p' \times q'}$ satisfying the following relations:

$$\left\{\begin{array}{c} R'\,P' = Q'\,R, \\ P\,P' + Z\,R = I_p, \quad P'\,P + Z'\,R' = I_{p'}. \end{array}\right.$$

Then, there exist $Z_2 \in D^{q \times r}$ and $Z_2' \in D^{q' \times r'}$ satisfying:

$$Q\,Q' + R\,Z + Z_2\,R_2 = I_q, \quad Q'\,Q + R'\,Z' + Z_2'\,R_2' = I_{q'},$$

where $R_2 \in D^{r \times q}$ (resp., $R_2' \in D^{r' \times q'}$) is s.t. $\ker_D(.R) = D^{1 \times r}\,R_2$ (resp., $\ker_D(.R') = D^{1 \times r'}\,R_2'$).

◇ Particular case of an isom.: equivalence $(P' = P^{-1},\; Q' = Q^{-1})$.

The case of full row rank matrices

◇ Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ and $f \in \mathrm{iso}_D(M, M')$.

◇ We have:

1. The matrices

$$U = \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\,P \end{pmatrix}, \; V = \begin{pmatrix} I_p - P\,P' & -P \\ P' & I_{p'} \end{pmatrix}$$

   are unimodular and satisfy $V = U^{-1}$.

2. $\mathrm{diag}(R, I_{p'})\,U = W\,\mathrm{diag}(I_p, R')$ where

$$W = \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \in D^{(q+p') \times (p+q')}$$

3. If R and R' have full row rank, then $q + p' = p + q'$,

$$W \in \mathrm{GL}_{(q+p')}(D), \quad W^{-1} = \begin{pmatrix} Z & -P \\ Q' & R' \end{pmatrix} \text{ and}$$

$$\mathrm{diag}(I_p, R') = W^{-1}\,\mathrm{diag}(R, I_{p'})\,U.$$

Example: theory of linear elasticity (continued)

◇ Here, we get:

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & -\partial_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_5(D).$$

◇ Moreover, the matrix W has the form:

$$W = \begin{pmatrix} \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{6\times 8}.$$

◇ Then $\mathrm{diag}(R, I_3)\, U = W\, \mathrm{diag}(I_2, R')$ but W is not unimodular.

The general case

Theorem

Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ and $f \in \mathrm{iso}_D(M, M')$. With the previous notations and $s := q + p' + p + q'$, we have

$$L' = Y^{-1}\, L\, X \quad \Leftrightarrow \quad L = Y\, L'\, X^{-1}$$

where $$L = \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{s \times (p+p')}, \quad L' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{s \times (p+p')},$$

$$X = \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\, P \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} I_p - P\, P' & -P \\ P' & I_{p'} \end{pmatrix},$$

$$Y = \begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P\, Z' - Z\, Q \\ -Q' & -R' & 0 & Z_2'\, R_2' \end{pmatrix}, \; Y^{-1} = \begin{pmatrix} Z_2\, R_2 & 0 & -R & -Q \\ P'\, Z - Z'\, Q' & 0 & P' & -Z' \\ Z & -P & I_p & 0 \\ Q' & R' & 0 & I_{q'} \end{pmatrix}.$$

Example: theory of 2D linear elasticity (continued)

- Here, we get

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{pmatrix} = Y^{-1} \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X,$$

where $X \in \mathrm{GL}_5(D)$ and $Y \in \mathrm{GL}_{14}(D)$ are given by:

Example: theory of 2D linear elasticity (continued)

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & -\partial_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$Y =$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\partial_2 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & -1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2\,\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1
\end{pmatrix}.$$

Work in progress

◇ Reduction of the size of 0 and I blocks using stable range hypotheses (constructive version of Warfield's results).

The OreMorphisms package

◇ Algorithms are implemented in a Maple package called OreMorphisms based on the library OreModules developed by Q. et Robertz:

http://wwwb.math.rwth-aachen.de/OreModules

◇ It is freely available with a library of examples at:

http://www.ensil.unilim.fr/~cluzeau/OreMorphisms